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FUNCTIONAL ANALYTIC AXIOMS AND SET THEORY

CHAIM SAMUEL HÖNIG

Instituto de Matemática e Estatística
Universidade de São Paulo
Cx. Postal 20570 - Ag. Jardim Paulistano
01452-990 Sao Paulo, Brasil

To the Memory of Leopoldo Nachbin

ABSTRACT: *In analysis the Axiom of Choice and some of its consequence are used to obtain non-measurable subsets of* $\mathbb{R}$, *non-continuous linear functionals on Banach spaces, and to prove that on every Banach space* $E \neq \{0\}$ *there exists non-zero linear continuous functionals. These examples cannot be obtained in a constructive way. The weaker axiom of dependent choice is sufficient for most results of analysis. We show that using the weaker axiom we may add some very simple new axioms (incompatible with the axiom of choice but consistent with the other axioms of Zermelo-Fränkel) which imply the following properties: (a) Every linear functional on a Banach space is continuous; (b) Every linear operator from a Banach space into another is continuous; (c) On the Banach space* ℓ_∞/c_0 *zero is the only linear continuous functional. These properties follow, respectively, from the following axioms: (A) Every linear functional on* ℓ_1 *is continuous; (B) Every norm (or seminorm) on* ℓ_1 *is continuous; (C)* $\ell'_\infty = \ell_1$ *(i.e., every linear continuous functional on* ℓ_∞ *is defined by an element of* ℓ_1*). We mention several open problems.*

KEY WORDS: Axiom of choice, functional analysis.

RESUMO: *AXIOMAS FUNCIONAL ANALÍTICOS E TEORIA DOS CONJUNTOS. O axioma da escolha e algumas de suas consequências são utilizados em análise para se obter subconjuntos não-mensuráveis da reta, funcionais lineares não-contínuos sobre espaços de Banach, e para se demonstrar que em todo espaço de Banach não-trivial existem funcionais lineares contínuos não-triviais. Estes exemplos não podem ser obtidos de maneira construtiva. O axioma da escolha dependente, mais fraco que o axioma da escolha, é suficiente para a maioria dos resultados da análise.*

PALAVRAS-CHAVE: Axioma da escolha, análise funcional.

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1. INTRODUCTION

There have always been objections against the use of the axiom of choice, AC, since by its non-constructive nature it is very different from the other axioms of set theory and it gives way to strange beings. So, in analysis we use AC to prove the existence of non-measurable subsets of $\mathbf{R}$, the existence of linear non-continuous functionals on Banach spaces of infinite dimension and, mainly, the Banach-Tarski paradox that allows us to decompose the unit ball of $\mathbf{R}^3$ into a finite number of pieces and to regroup them into two unit balls.

On the other hand, it was difficult to find adequate axioms to replace AC: the axiom of countable choice, AC^ω, is not sufficient for many results in analysis: it does not allow, for instance, to prove the Baire Theorem (which says that in a complete metric space the intersection of a sequence of dense open subsets is dense). Indeed, the axiom of dependent choice, DC, (which implies AC^ω) was proved by Blair in [1] to be equivalent in ZF to the Baire theorem (see App. A for ZF and DC). We recall that this theorem is used in the proof of the closed graph theorem, the uniform boundedness principle, the open mapping theorem and the Banach-Steinhaus theorem which are of fundamental importance in Functional Analysis. In their 3 volume treatise of Constructive Functional Analysis, [4], the authors assert that they use only AC^ω (see p. v of their introduction) but the result of Blair proves that they really use and need DC (indeed: see, for instance, the proof on p. 109 of Vol. I: "De proche en proche on détermine ainsi une suite $\cdots$"). There arises the question of the formulations in $ZF + DC$ of other *natural* axioms that contradict AC. In what follows we bring 6 such axioms ($AD, LM, BP, \ell_1^* = \ell_1'$, every norm on ℓ_1 is continuous, $\ell_\infty' = \ell_1$) and some of their relations.

One candidate in the Theory of Games, the Axiom of Determinacy, AD (see App. A), contradicts AC (but not AC^ω nor DC) since it implies that every subset of $\mathbf{R}$ is measurable, [9, Th. 102]. AD has been proved for borelian subsets $A \subset \{0,1\}^{\mathbf{N}}$ (but not for analytic subsets (see App. A) of $\{0,1\}^{\mathbf{N}}$) nor has the consistency of $ZF + DC + AD$ (if ZF is consistent) been proved. Let us recall that Gödel proved that if ZF is consistent so is ZFC (see App. A) and Paul Cohen proved that the axiom AC is independent of ZF (they also proved analogous results for the continuum hypothesis).

Another candidate to replace AC is

$$LM: \text{ every subset of } \mathbf{R} \text{ is measurable.}$$

It has been proved that analytic subsets of $\mathbf{R}$ are measurable [9, Th. 94]. Solovay proved that if $ZFC + IC$ (see App. A) is consistent so is $ZF + DC + LM$ and Shelah proved that the reverse implication, [11, p.14], [12, p. 209]. However the axiom IC on the existence of inacessible cardinals cannot be proved in ZFC. We have even more:

1) one cannot prove that IC is consistent with ZFC, [9, Th.27].
2) If one ever proves that in ZF inacessible cardinals do not exist, then in $ZF + DC$ one can prove the existence of non-measurable subsets of $\mathbf{R}$, [12, p. 209].

Garnir in [5] proved that in $ZF + DC + LM$ all Banach and Frechet spaces are "good", i.e., every seminorm on them is continuous and that every operator from a "good" space into a normed or more generally, a locally convex space, is continuous. Later on Gilioli proved in [7] (see also [6]) that the "good" spaces are precisely the ultrabornological ones (see App. B).

Another candidate to replace AC is BP (see App. A); Borel and analytic sets have the Baire property, [9, Th. 94], and Shelah proved that the consistency of ZF implies the consistency of $ZF + DC + BP$ *without needing the axiom* IC, [11, Th. 5.1]. Working in $ZF + DC + BP$ Wright in [13] proved that any linear mapping from a Banach or Frechet space into a *separable* normed or locally convex space (see App. B) is continuous. More generally we will prove this result without the restriction of separability (see Theorem 2 and 3), one of our main results.

2. FUNCTIONAL ANALYTIC AXIOMS

(See App. B for the notions not defined in this item). From the results of Wright and Shelah of 1, it follows that if ZF is consistent, so is the system

$$ZF + DC + \text{“}\ell_1^* = \ell_1'\text{”}.$$

Theorem 1. *In* $ZF + DC + \text{“}\ell_1^* = \ell_1'\text{”}$ *we have* $E^* = E'$ *for every ultrabornological space* E *(hence in particular for all Banach and Frechet spaces).*

Proof.

(a) From $\ell_1^* = \ell_1'$ it follows that $\ell_1(I)^* = \ell_1(I)'$ for any set I: indeed, if there exists an $f \in \ell_1(I)^*$ that is not continuous then for every $n \in \mathbf{N}$ there exists an $x^n \in \ell_1(I)$ with $||x^n||_1 = 1$ and $|f(x^n)| \geq n$. We define $I_n = \{i \in I \mid (x^n)_i \neq 0\}$ then $A = \cup_{n\in\mathbf{N}} I_n$ is countable and $f|_{\ell_1(A)}$ would be non-continuous.

(b) $E^* = E'$ for every Banach space: by B_3 of App. B we have $E = \ell_1(I)/\varphi^{-1}(0)$ with $\varphi : \ell_1(I) \to E$. Given $f \in E^*$ we have $f \circ \varphi \in \ell_1(I)^* = \ell_1(I)'$. Since the quotient topology is final it follows from B_1 that f is continuous.

(c) $E^* = E'$ for every ultrabornological space: $E = \text{final}_{\alpha\in A}(E_\alpha, f_\alpha)$ where the E_α are Banach spaces and $f_\alpha \in \mathcal{L}(E_\alpha, E)$. If $f \in E^*$ then $f \circ f_\alpha \in E_\alpha^* = E_\alpha'$ for every $\alpha \in A$ hence $f \in E'$ by B_1. ■

Theorem 2. *In* $ZF + DC + BP$ *every seminorm* p *on* ℓ_1 *is continuous.*

Proof.

(i) If p is not continuous then for every $k \in \mathbf{N}$ we have

$$E_k = \left\{ y \in \ell_1 \mid ||y||_1 \leq \frac{1}{2^k}, \, p(y) > k \right\} \neq \emptyset;$$

if we take $y_k \in E_k$ we have

$$\sum_{k=1}^{\infty} \|y_k\|_1 < \infty \tag{1}$$

and

$$\lim_{k\to\infty} p(y_k) = \infty. \tag{2}$$

(ii) Let K denote the Cantor set, i.e., the set of all $\alpha \in [0,1]$ that have a ternary representation $0,\alpha_1\alpha_2\cdots\alpha_j\cdots$ with $\alpha_j = 0$ or 2 for every $j \in \mathbf{N}$. We may also write $K = \{0,2\}^{\mathbf{N}}$ and if we define the addition coordinatewise (i.e., $0+0=0$, $2+2=0$, $0+2=2+0=2$) we have a compact abelian group since we have

$$\text{the topology of } K \subset [0,1] \text{ coincides with the product topology of } \{0,2\}^{\mathbf{N}}. \tag{3}$$

(iii) For every $\alpha \in K$ it follows from (1) that the series $\sum_{j=1}^{\infty} \alpha_j y_j$ is convergent in ℓ_1 (since we have $\alpha_j = 0$ or 2). For every $k \in \mathbf{N}$ we define

$$B_k = \left\{ \alpha \in K \mid p\left(\sum_{j=1}^{\infty} \alpha_j y_j\right) \le k \right\};$$

we have $K = \cup_{k\in\mathbf{N}} B_k$.

(iv) By the hypothesis BP for every B_k there exists an open set $O_k \subset K$ (we may have $O_k = \emptyset$) such that $B_k \Delta O_k$ is meager; since K is compact there exist $m \in \mathbf{N}$ such that $O_m \neq \emptyset$ (if not, all B_k, hence K, would be meager). Then by a theorem on topological groups (see for instance Theorem 5.1 of [3]) $B_m + B_m (= B_m - B_m)$ is a neighbourhood of $0 \in K$, i.e. by (3), there exists $\varepsilon > 0$ such that

$$B_m + B_m \supset K \cap \{x \in [0,1] \mid 0 \le x < \varepsilon\}.$$

Hence for $k \in \mathbf{N}$ sufficiently large we have $\frac{2}{3^k} \in B_m + B_m$, i.e., there are $\alpha^{(k)}, \beta^{(k)} \in B_m$ such that $\alpha^{(k)} + \beta^{(k)} = \frac{2}{3^k}$; if we consider the components $\alpha_j^{(k)}$, $\beta_j^{(k)}$, $j = 1,2,\cdots$ we have $\alpha_j^{(k)} + \beta_j^{(k)} = \left(\frac{2}{3^k}\right)_j = 0$ if $j \neq k$ and $\alpha_k^{(k)} + \beta_k^{(k)} = \left(\frac{2}{3^k}\right)_k = 2$; hence in ℓ_1 we have

$$\sum_{j=1}^{\infty} \alpha_j^{(k)} y_j + \sum_{j=1}^{\infty} \beta_j^{(k)} y_j = 2y_k;$$

since $\alpha^{(k)}, \beta^{(k)} \in B_m$ it follows that, in contradiction to (2),

$$p(2y_k) \le p\left(\sum_{j=1}^{\infty} \alpha_j^{(k)} y_j\right) + p\left(\sum_{j=1}^{\infty} \beta_j^{(k)} y_j\right) \le 2m.$$

■

Remark. The above proof is based on an analogous one in $ZF + DC + LM$ given by Garnir, [5] (using a trick discovered by J.P.R. Christensen). Dealing only with ℓ_1 also allows one to simplify the proof of the Theorem of Garnir (the Haar measure on K is the product measure on $\{0, 2\}^{\mathbb{N}}$ etc.).

Theorem 3. *In $ZF + DC + BC$ the following properties are true and equivalent.*

(1) *Every seminorm on ℓ_1 is continuous.*

(2) *For every set I every seminorm on $\ell_1(I)$ is continuous.*

(3) *On any Banach space every seminorm is continuous.*

(4) *On any ultrabornological space every seminorm is continuous.*

(5) *For every Banach, Frechet or ultrabornological space E and any normed or locally convex space F, all linear mappings $f : E \to F$ are continuuos.*

Proof. In Theorem 2 we proved (1). (1) $\Rightarrow$ (2): If p is a non-continuous seminorm on $\ell_1(I)$ then for every $n \in \mathbb{N}$ there exists an $x^n \in \ell_1(\mathbb{N})$ with $\|x^n\|_1 = 1$ and $p(x^n) \geq n$. We define $I_n = \{i \in I \mid (x^n)_i \neq 0\}$ and the proof follows as in (a) of Theorem 1, and analogously for the proofs (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4). (4) $\Rightarrow$ (5): Take $f \in \mathcal{L}(E.F)$; then for every continuous seminorm q on F, $p = q \circ f$ is a seminorm on E, hence continuous by (4), hence f is continuous. (5) $\Rightarrow$ (1) is obvious. ■

From the result of Garnir and Theorem 2 it follows that if $ZF + DC + LM$ or $ZF + DC + BP$ is consistent so is the system

$$ZF + DC + \text{“every seminorm on } \ell_1 \text{ is continuous”}.$$

In $ZF + DC$, “every seminorm on ℓ_1 is continuous” does not imply LM (or BP) since by Theorem 2 (respectively, by the Theorem of Garnir) it would imply BP (respectively, LM) contradicting a result of Shelah, see [11, p.21], that in $ZF + DC$, LM and BP do not imply each other.

Problem 1. In $ZF + DC +$ “$\ell_1^* = \ell_1'$” do we have that “every seminorm on ℓ_1 is continuous”?

Theorem 4. *In $ZF + DC +$ “$\ell_1^* = \ell_1'$” the following properties are equivalent.*

(1) *Every seminorm on ℓ_1 is continuous.*

(2) *For every norm p on the vector space $\ell_1(\mathbb{N})$ the normed space $(\ell_1(\mathbb{N}))_p$ is separable.*

Proof. (1) $\Rightarrow$ (2): Let p be a norm on ℓ_1; since p is continuous it follows that $\ell_1 \hookrightarrow (\ell_1(\mathbb{N}))_p$ is continuous, hence $(\ell_1(\mathbb{N}))_p$ is separable as the continuous image of the separable space

ℓ_1. (2) $\Rightarrow$ (1): Let p be a seminorm on ℓ_1; taking $p(x) + ||x||_1$, we may suppose that $p(x) \geq ||x||_1$. Let B_p be the unit ball of $\ell_1(\mathbb{N})_p$. We recall that the proof of the Hahn-Banach theorem for separable spaces does not use AC nor AC^ω; its application to the separable space $\ell_1(\mathbb{N})_p$ implies that B_p is the intersection of p-closed semispaces defined by elements $f \in (\ell_1(\mathbb{N})_p)' \subset \ell_1^* = \ell_1'$. Since we have $f \in \ell_1'$, B_p is a ℓ_1-barrel, hence a ℓ_1-neighbourhood of zero by B_4 of App. B, i.e., p is ℓ_1-continuous. ■

Theorem 5. *In $ZF + AC^\omega$ the following properties are equivalent.*

(a) $\ell_\infty' = \ell_1$.

(b) $(\ell_\infty/c_0)' = \{0\}$.

(c) *For every $f \in \ell_\infty'$ such that $f|_{c_0} = 0$ we have $f = 0$.*

Proof. It is obvious that (b) $\Leftrightarrow$ (c). (c) $\Rightarrow$ (a): Take $f \in \ell_\infty'$ and define $g = f|_{c_0}$; since $c_0' = \ell_1$ it follows that g is given by a $y \in \ell_1$, i.e., $g(x) = \langle x, y\rangle$ for every $x \in c_0$. We may define $f_0 \in \ell_\infty'$ by $f_0(x) = \langle x, y\rangle$ for every $x \in \ell_\infty$, hence we have $(f - f_0)|_{c_0} = 0$ and so by (c) we have $f - f_0 = 0$, i.e., $f = f_0$ and so $f(x) = \langle x, y\rangle$ for every $x \in \ell_\infty$, i.e., f is given by $y \in \ell_1$. (a) $\Rightarrow$ (c): Let us take $f \in \ell_\infty' = \ell_1$. Hence there exists $y \in \ell_1$ such that $f(x) = \langle x, y\rangle$ for every $x \in \ell_\infty$. If $f|_{c_0} = 0$ it follows that $y = 0$, i.e., $f = 0$. ■

Problem 2. $ZF + DC + LM \Rightarrow \ell_\infty' = \ell_1$?

Problem 3. $ZF + DC + BP \Rightarrow \ell_\infty' = \ell_1$?

Problem 4. $ZF + DC$: "every seminorm on ℓ_1 is continuous" $\Rightarrow \ell_\infty' = \ell_1$?

Problem 5. $ZF + DC$: "$\ell_\infty' = \ell_1$" $\Rightarrow$ "every seminorm on ℓ_1 is continuous"?

Problem 6. Does the consistency of $ZF + DC + LM$ imply the consistency of $ZF + DC +$ "$\ell_\infty' = \ell_1$"?

Remark. We get Problems 7 and 8 if in Problem 6 we replace LM by BP or by "every seminorm on ℓ_1 is continuous", respectively.

Related to the result of Blair we may ask if in $ZF + AC^\omega$ the closed graph theorem implies the theorem of Baire; more precisely.

Problem 9. In $ZF + AC^\omega$, if for all Banach spaces F the closed graph theorem is true in $\mathcal{L}(\ell_1, F)$ then do we have DC, or equivalently, is the theorem of Baire true?

Theorem 6. *Let Y be a normed or a locally convex space. In $ZF + AC^{\omega}$ we have: if the closed graph theorem is valid for all linear mappings $\ell_1 \rightarrow Y$ then it is valid for all linear mappings $E \rightarrow Y$, where E is any Banach, Frechet or ultrabornological space.*

Proof. Following the steps of the proof of Theorem 1, we show successively that the result is true if $E = \ell_1(I)$, if E is a Banach space and if E is a ultrabornological space. ■

APPENDIX A: Definitions of Set Theory

We denote by ZFC the axioms of Zermelo-Fränkel with the axiom of choice, AC; ZF denotes the same axioms without AC. By AC^{ω} we denote the axiom of countable choice. DC denotes the *axiom of dependent choice:* given a non-empty set X and $R \subset X \times X$ such that for every $x \in X$ we have $\{y \in X \mid (x, y) \in R\} \neq \emptyset$, then for every $\overline{x} \in X$ there exists a sequence $(x_n)_{n \in \mathbf{N}}$ of X such that $x_1 = \overline{x}$ and $(x_n, x_{n+1}) \in R$ for every $n \in \mathbf{N}$.

The *axiom of determinacy*, AD, is an axiom that has its origin in the theory of games. We need some definitions: we take a non-empty subset $A \subset \{0,1\}^{\mathbf{N}}$ and define the following game between two players I and II: they take alternatively an element $a_n \in \{0,1\}$, I being the first player; if $(a_n)_{n \in \mathbf{N}} \in A$ then I wins; if $(a_n)_{n \in \mathbf{N}} \notin A$ then II wins. We say that player I has a winning strategy if he can make his choices in such a way to assure that $(a_n)_{n \in \mathbf{N}} \in A$ whatever may be the choices of player II. The axiom AD says that for any non-empty subset $A \subset \{0,1\}^{\mathbf{N}}$ one of the players has a winning strategy.

The axiom IC says that there exist an *inacessible cardinal*, i.e., a cardinal number $K > \aleph_0$ that satisfies the equivalent properties.

- I_1- Given sets A and X_α, $\alpha \in A$, with $|A| < K$ and $|X_\alpha| < K$ for every $\alpha \in A$ then we have $|\prod_{\alpha \in A} X_\alpha| < K$.
- I_2- If $|X| < K$ then $|\mathcal{P}(X)| < K$ and given any sets A and X_α, $\alpha \in A$, with $|A| < K$ and $|X_\alpha| < K$ for every $\alpha \in A$ then we have $|\bigcup_{\alpha \in A} X_\alpha| < K$.

We say that a subset A of a topological space E has the *Baire property* if there is an open set $U \subset E$ such that $U \Delta A$ is meager (= of first category, i.e., contained in the union of a sequence of closed sets with empty interior). The axiom BP says that in a complete separable metric space every subset has the Baire property.

In a complete separable metric space a set that is the continuous image of a Borel set of a complete separable metric space is called *analytic* or suslinian.

APPENDIX B: Definitions from Functional Analysis

Let I be any set; $\ell_\infty(I)$ denotes the Banach space of all families $x = (x_i)_{i \in I} \in \mathbf{R}^I$ such that $\|x\| = \sup_{i \in I} |x_i| < \infty$; we write $\ell_\infty = \ell_\infty(\mathbf{N})$ and c_0 denotes the subspace of all sequences $x = (x_n)$ such that $\lim_{n\to\infty} x_n = 0$. $\ell_1(I)$ denotes the Banach space of all families $x = (x_i)_{i\in I} \in \mathbf{R}^I$ such that $||x||_1 = \sum_{i\in I} |x_i| < \infty$; we write $\ell_1 = \ell_1(\mathbf{N})$. A seminorm on a vector space E is a function $p : E \to \mathbf{R}_+$, such that $p(x+y) \le p(x) + p(y)$ and $p(\lambda x) = |\lambda| p(x)$ for all x, $y \in E$, $\lambda \in \mathbf{R}$. A locally convex space, LCS, is a vector space with a topology defined by a family of seminorms. A Frechet space is a LCS that is metrisable and complete. If E, F are LCS we denote by $\mathcal{L}(E,F)$ $[L(E,F)]$ the space of all linear [continuous] mappings $f : E \to F$; if $F = \mathbf{R}$ we write E^* $[E']$.

Given a vector space E and a family $(E_\alpha, f_\alpha)_{\alpha\in A}$ where E_α is a LCS and $f_\alpha \in \mathcal{L}(E_\alpha, E)$, the *final* locally convex topology on E defined by $(E_\alpha, f_\alpha)_{\alpha\in A}$ is the finest locally convex topology on E for which all the mappings $f_\alpha : E_\alpha \to E$ are continuous. We denote this LCS by $E = \text{final}_{\alpha\in A}(E_\alpha, f_\alpha)$. If $E = \text{final}_{\alpha\in A}(E_\alpha, f_\alpha)$ we have (see [2]):

B_1- *If F is a LCS then $f \in L(E,F)$ iff $f \circ f_\alpha \in L(E_\alpha, F)$ for all $\alpha \in A$.*

B_2- *A seminorm p on E is continuous iff the seminorms $p \circ f_\alpha$ (on E_α) are continuous for all $\alpha \in A$.*

A quotient space is a particular instance of a final LCS.

B_3- *Every Banach space E is the quotient of a $\ell_1(I)$.*

Proof. We take $I = \{i \in E \mid ||i|| = 1\}$ and

$$\varphi : x = (x_i)_{i\in I} \in \ell_1(I) \mapsto \varphi(x) = \sum_{i\in I} x_i i \in E.$$

We have $||\varphi(x)|| \le \sum_{i\in I} |x_i| = ||x||_1$ and for every $y \in E$, $y \ne 0$, we have $||y|| = \inf\{||x|| \mid x \in \ell_1(I), \varphi(x) = y\}$ (take $i = y/||y||$ and $x = ||y|| i$). ∎

A *barrel* in a LCS E is an absolutely convex closed subset $T \subset E$ that absorbs all points of E, i.e., $\cup_{n\in\mathbf{N}} nT = E$. We say that a LCS E is *barreled* if every barrel of E is a neighbourhood of zero.

B_4- *A LCS that is a Baire space is barreled, hence so are the Banach and Frechet spaces* (see [2, Chap. III, §4]).

B_5- *If $E = \text{final}_{\alpha\in A}(E_\alpha, f_\alpha)$ where the E_α are barreled, so is E* (see [2, Cap.III, §4]).

For barreled spaces we have the Banach-Steinhaus theorem: Let E be a barreled space and F a separated LCS; if the $f_n \in L(E,F)$, $n \in \mathbf{N}$, are such that for every $x \in E$ there exists $f(x) = \lim_{n\to\infty} f_n(x)$ then $f \in L(E,F)$ ([2], Chap. III, §4).

We say that a bounded absolutely convex subset B of a separated LCS E is a *Banach disc* if $E_B = \cup_{n\in\mathbf{N}} nB$ is a Banach space when endowed with the norm p_B defined by the Minkowski functional of B : $p_B(x) = \inf\{\lambda \ge 0 \mid x \in \lambda B\}$. A separated LCS E is *ultrabornological* if it satisfies the following equivalent properties.

U_1- Every absolutely convex set $U \subset E$ that absorbs all Banach discs is a neighbourhood of zero.

U_2- $E = \mathrm{final}_{\alpha \in A}(E_\alpha, f_\alpha)$ where the E_α, $\alpha \in A$, are Banach spaces (see [10]).

From U_2, B_4 and B_5 it follows that every ultrabornological space is barreled. It is immediate that if E is a ultrabornological space and if F is a Frechet space then every linear mapping $f : E \to F$ with closed graph is continuous.

The space $K([a,b])$ of equivalence classes of functions $f : [a,b] \to \mathbf{R}$ that are Kurzweil-Henstock-Denjoy-Perron integrable is ultrabornological when endowed with the norm

$$||f||_A = \sup_{a \le t \le b} \left| \int_a^t f(s)\, ds \right|$$

(see [7, p.472]) but is not complete and it is even meager. It does not allow a natural Banach or Frechet space topology (see [8]). We have a similar result for the classical space $R_1([a,b])$ of all equivalence classes of functions $f : [a,b] \to \mathbf{R}$ that are Riemann integrable in the improper sense, with a finite number of singularities (also with the norm $|| \; ||_A$, see [6] or [7]).

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REFERENCES

[1] CH.E. BLAIR, *The Baire category theorem implies the principle of dependent choices*, Bull. Acad. Polon. Sciences, XXV (1977), pp. 933–934.
[2] N. BOURBAKI, *Espaces Vectoriels Topologiques*, Masson, 1981.
[3] J.P.R. CHRISTENSEN, *Topology and Borel Structure*, North-Holland Math. Studies, 1974.
[4] H.G. GARNIR, M. DE WILDE, and J. SCHMETZ, *Analyse Fonctionelle*, Birkhäuser Verlag, 1968.
[5] H.G. GARNIR, *Solovay's axiom and functional analysis*, Springer Lecture Notes in Mathematics, 399 (1974), pp. 189–204.
[6] A. GILIOLI, *The ultrabornologicalness of the space of Denjoy-Perron-Kurzweil integrable functions and of other natural non-complete subspaces of* $C([a,b],X)$ *and of* ΠE_j, 31º Sem. Bras. Análise (1990), pp. 323–367.
[7] A. GILIOLI, *Natural ultrabornological, non-complete, normed function spaces*, Arch. Math., 61 (1993), pp. 465–477.
[8] C.S. HÖNIG, *There is no natural Banach space norm on the space of Kurzweil-Henstock-Denjoy-Perron integrable functions*, 30º Sem. Bras. Análise (1989), pp. 387–397.
[9] T. JECH, *Set Theory*, Academic Press, 1978.
[10] G. KÖTHE, *Topological Vector Spaces II*, Springer Verlag, 1979.
[11] J. STERN, *Le problème de la mésure*, Séminaire Bourbaki, 632 (1983-1984).
[12] St. WAGON, *The Banach-Tarski Paradox*, Cambridge University Press, 1985.
[13] J.D.M. WRIGHT, *All operators on a Hilbert space are bounded*, Bull. of the Amer. Math. Soc., 79 (1973), pp. 1247–1250.