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Abstract

Several classes of irreducible orthogonal representations of compact Lie groups that are of importance in Differential Geometry have the property that the second osculating spaces of all of their nontrivial orbits coincide with the representation space. We say that representations with this property are of class $\mathcal{O}^2$. Our approach in the present paper will be to find restrictions on the class $\mathcal{O}^2$ and then apply them to classify taut irreducible representations up to orbit equivalence. The classification of variationally complete representations up to orbit equivalence is an easy corollary.

1 Introduction

Several classes of irreducible orthogonal representations of compact Lie groups that are of importance in Differential Geometry have the property that the second osculating spaces of all of their nontrivial orbits coincide with the representation space. We say that representations with this property are of class $\mathcal{O}^2$. Our approach in the present paper will be to find restrictions on the class $\mathcal{O}^2$ and then apply them to classify taut irreducible representations up to orbit equivalence. The classification of variationally complete representations up to orbit equivalence is an easy corollary. We refer to Section 2 for the definitions of concepts used in this introduction.

We next state our main theorem to be proved in Section 6.

1.1 Theorem *A taut irreducible representation ρ of a compact connected Lie group G is either orbit equivalent to the isotropy representation of a symmetric space or it is one of the following orthogonal representations ($n \geq 2$):*

G	ρ
$\mathbf{SO}(2) \times \mathbf{Spin}(9)$	*(standard)* $\otimes_{\mathbf{R}}$ *(spin)*
$\mathbf{U}(2) \times \mathbf{Sp}(n)$	*(standard)* $\otimes_{\mathbf{C}}$ *(standard)*
$\mathbf{SU}(2) \times \mathbf{Sp}(n)$	*(standard)*^{3} $\otimes_{\mathbf{H}}$ *(standard)*

Bott and Samelson proved in [3] that isotropy representations of symmetric spaces are variationally complete and that almost all distance functions of orbits of variationally complete representations are perfect Morse functions. In our terminology, this implies that isotropy representations of symmetric spaces are taut. The three representations in the table of Theorem 1.1 are also taut. Their tautness was proved by the authors in [15]; see also [16] for a different proof. Notice that these representations are precisely the representations of cohomogeneity three that are not orbit equivalent to the isotropy representation of a symmetric space (see [35, 12]). Hence the taut irreducible

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representations are precisely those that are either orbit equivalent to the isotropy representation of a symmetric space or of cohomogeneity three.

The three representations in the table of Theorem 1.1 are not variationally complete by Theorem 3.8 in [15] (see also [16]). Since a variationally complete representation is taut, it now follows from Theorem 1.1 that an irreducible variationally complete representation is orbit equivalent to the isotropy representation of a symmetric space. Also, it is not difficult to see that an arbitrary variationally complete representation is orbit equivalent to the outer direct sum of irreducible variationally complete representations. This proves the following corollary which is a converse of the result of Bott and Samelson in [3] that isotropy representations of symmetric spaces are variationally complete.

1.2 Corollary *A variationally complete representation of a compact connected Lie group is orbit equivalent to the isotropy representation of a symmetric space.*

After [16] had been circulated, Di Scala and Olmos gave in [11] a very short, direct proof of the fact that a variationally complete representation is polar. Their result together with Dadok's classification of polar representations in [10] can then be used to give a different proof of Corollary 1.2.

In fact one can do more with the method of this paper and get a complete classification of variationally complete and taut irreducible representations (up to image equivalence; see the beginning of Section 5 for this concept) independent of [12]. As a consequence one gets a new proof of the classification of polar representations due to Dadok [10] since these are variationally complete (see [7]) and isotropy representations of symmetric spaces are easily seen to be polar. It follows that an orthogonal representation is variationally complete if and only if it is polar, and it is polar if and only if it is orbit equivalent to the isotropy representation of a symmetric space. We do not go into the details of how these results can be proved in a unified way and refer to [16] instead. The complete classification of variationally complete representations also follows from combining results in [10, 12, 1, 11].

Some other known classification results can also easily be proved with our methods as shown in [16]. These include the classification of cohomogeneity one representations as well as the classification of cohomogeneity two representations due to Hsiang and Lawson [20]. The cohomogeneity two representations are polar and therefore included in Dadok's classification, but the point here is that it is very easy to see directly that they belong to class $\mathcal{O}^2$ without referring to tautness.

Kuiper observed in [21] that the second osculating space of a taut submanifold in a Euclidean space V coincides with V if the submanifold is not contained in a proper affine subspace. In fact, he proved this more generally for tight submanifolds, but this is unimportant for us since an orbit is tight if and only if it is taut. Since the classes of representations we are dealing with are all taut, it follows from this observation of Kuiper that they belong to class $\mathcal{O}^2$ if they are irreducible. The class $\mathcal{O}^2$ is much more tractable than the other classes of representations we are dealing with since it involves an infinitesimal condition. The technique of Dadok [10], notably his invariant $k(\lambda)$, turns out to be an extremely powerful tool to reduce the class $\mathcal{O}^2$ in size so that the remaining cases are accessible to the geometric methods developed in Section 3.

Previous to this paper taut representations were studied in [14] and [9]. It is decided in [14], with some exceptions, which representations of $\mathrm{SU}(n)$ and $\mathrm{U}(n)$ can be taut. In [9] it is proved among other things that a compact group admitting an almost faithful taut representation can have at most four simple factors.

The present paper is an abridged version of the preprint [16] which we have already quoted in this introduction, and it is organized as follows. In Section 2 we bring definitions and explain some known results. In Sections 3 and 4 we collect preliminary material that is needed for the classification in Sections 5 and 6.

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2 A review of basic definitions and results

Let G be a compact Lie group acting on a Riemannian manifold M by isometries. A geodesic γ in M is called *G-transversal* if it is orthogonal to the G-orbit through $\gamma(t)$ for every t. One can show that a geodesic γ is G-transversal if there is a point t_0 such that $\dot\gamma(t_0)$ is orthogonal to $G\gamma(t_0)$. A Jacobi field along a geodesic in M is called *G-transversal* if it is the variational vector field of a variation through G-transversal geodesics. The action of G on M is called *variationally complete* if every G-transversal Jacobi field J in M that is tangent to the G-orbits at two different parameter values is the restriction of a Killing field on M induced by the G-action (see [2] and [3]). It is proved in [3] on p. 974 that instead of requiring tangency at two different points in the definition of variational completeness it is equivalent to require tangency at one point and vanishing at another point.

Let ρ be a variationally complete reducible representation and $\tilde\rho$ a summand of ρ. Then it is easy to see that $\tilde\rho$ is also variationally complete. Notice though that it is not true that the direct sum of two variationally complete representations of a compact Lie group G is variationally complete as can be seen by taking the direct sum of of two copies of $\mathbf{SO}(2)$ acting on $\mathbf{R}^2$.

Let N be a properly embedded submanifold of a Euclidean space V and let x be some point in V. Then we define the *distance function* $L_x : N \to \mathbf{R}$ *from* x *to* N by setting $L_x(p) = ||p - x||^2$. It follows that L_x is a non-negative proper function since N is properly embedded. Hence it is possible to apply Morse theory to L_x. We say that L_x is *perfect with respect to a field* F if it is a Morse function and the Morse inequalities for L_x with respect to F are equalities. Furthermore we say that N is *F-taut* or simply *taut* if L_x is perfect with respect to F whenever L_x is a Morse function, see [6]. The concept of tautness can be extended to submanifolds of complete Riemannian manifolds, see [33], but we will not need that here. We will say that an orthogonal representation $\rho : G \to \mathbf{O}(V)$ of a compact Lie group G is *F-taut* or simply *taut* if the the orbits of G are F-taut submanifolds of V.

If ρ is a reducible taut representation and $\tilde\rho$ one of its factors, then $\tilde\rho$ is clearly taut. Notice that it is not true that direct sums of taut representations of a compact Lie group G are also taut. An example of this is the direct sum of n copies of $\mathbf{SU}(n)$ acting on $\mathbf{C}^n$. This representation is not taut since one can compute that $\mathbf{SU}(n)$ is not taut in the matrix space $M(n;\mathbf{C})$.

The following theorem mentioned in the introduction was proved by Bott and Samelson more generally for variationally complete actions on complete Riemannian manifolds. In fact it is an immediate corollary of Theorem I in [3]. Notice that Bott and Samelson do not use the concept of a taut submanifold which was introduced later.

2.1 Theorem (Bott-Samelson) *A variationally complete representation of a compact connected Lie group on an Euclidean space is $\mathbf{Z}_2$-taut.*

Bott proved in [2], Propositions 7.1 and 11.6, that the action of a compact Lie group G with a bi-invariant metric on itself by conjugations and the adjoint representation of G on its Lie algebra $\mathfrak{g}$ are variationally complete. This was generalized by Bott and Samelson in [3], Theorem II as follows. Let (L, G) be a symmetric pair and $\mathfrak{l} = \mathfrak{g} \oplus \mathfrak{p}$ the corresponding Cartan decomposition. Then the action of $L \times L$ on L, the action of G on L/G and the action of $\mathrm{Ad}_L(G)$ restricted to $\mathfrak{p}$ are variationally complete. The action of $\mathrm{Ad}_L(G)$ on $\mathfrak{p}$ is equivalent to the isotropy representation of G on the tangent space $T_p(L/G)$ where p denotes the coset G. Therefore we have the following theorem mentioned in the introduction.

2.2 Theorem (Bott-Samelson) *The isotropy representation of a symmetric space is variationally complete.*

Conlon considered in [7] actions of a Lie group G on a complete Riemannian manifold M with the property that there is a connected submanifold Σ of M that meets all orbits of G in such a way that the intersections between Σ and the orbits of G are all orthogonal. Such a submanifold is called a *section* and an action admitting a section is now usually called *polar* if Σ is properly embedded. Notice that Conlon does not assume in [7] that Σ is properly embedded, but it is usually required in the recent literature on the subject. It is easy to see that a section Σ is totally geodesic in M. An action admitting a section that is flat in the induced metric is called *hyperpolar*. There is clearly no difference between polar and hyperpolar representations since totally geodesic submanifolds of a Euclidean space are affine subspaces. Moreover, the question whether Σ should be required to be properly embedded or not becomes redundant. Conlon proved the following theorem in [7].

2.3 Theorem (Conlon) *A hyperpolar action of a compact Lie group on a complete Riemannian manifold is variationally complete.*

Polar representations were classified by Dadok in [10]. We recall that two representations $\rho_1 : G_1 \to \mathrm{O}(V_1)$ and $\rho_2 : G_2 \to \mathrm{O}(V_2)$ are said to be *orbit equivalent* if there is an isometry $A : V_1 \to V_2$ under which the orbits of G_1 and G_2 correspond. As a consequence of his classification he obtained the following result.

2.4 Theorem (Dadok) *A polar representation of a compact connected Lie group is orbit equivalent to the isotropy representation of a symmetric space.*

We can summarize this discussion as follows. Let G be a compact connected Lie group. Denote by $\mathcal{I}$ the representations of G that are isotropy representations of symmetric spaces, by $\mathcal{P}$ those that are polar, by $\mathcal{V}$ those that are variationally complete, and by $\mathcal{T}$ those that are taut. Then

$$\mathcal{I} \subset \mathcal{P} \subset \mathcal{V} \subset \mathcal{T}.$$

The starting point in the present paper will be to consider a class of representations that is a priori larger than $\mathcal{T}$, but easier to deal with, see [9]. We recall that the *second osculating space* $O^2_p(N)$ at a point p of a submanifold N in a Euclidean space V is the vector space spanned by the first and second derivatives at p of the inclusion of N into V. It is easy to see that

$$O^2_p(N) = T_pN \oplus \langle\{\alpha(X,Y) \mid X, Y \in T_pN\}\rangle.$$

where α denotes the second fundamental form of N and $\langle S\rangle$ stands for the linear hull of the set S. The following is a corollary of the discussion of Kuiper in [21].

2.5 Theorem (Kuiper) *Let N be a taut submanifold of a Euclidean space. Then the affine hull of N coincides with $p + O^2_p(N)$ for every point p in N.*

We let $\mathcal{O}^2$ denote the class of representations of a compact connected Lie group G such that the representation space coincides with $O^2_p(Gp)$ for all nonzero p. The representations in $\mathcal{O}^2$ are clearly irreducible. If a taut representation is irreducible, then Theorem 2.5 implies that it belongs to class $\mathcal{O}^2$. Thus we have the inclusions

$$\mathcal{I}_i \subset \mathcal{P}_i \subset \mathcal{V}_i \subset \mathcal{T}_i \subset \mathcal{O}^2,$$

where $\mathcal{I}_i$, $\mathcal{P}_i$, $\mathcal{V}_i$, and $\mathcal{T}_i$ are the the subclasses of $\mathcal{I}$, $\mathcal{P}$, $\mathcal{V}$, and $\mathcal{T}$ consisting of irreducible representations.

3 Taut representations

We will first show that the slice representation of a taut representation is also taut and then prove an important proposition about taut reducible representations. It also follows immediately from our methods that a taut representation does not have exceptional orbits. Here we follow the terminology of Bredon in [5], pp. 180-181, and distinguish between three types of orbits, namely principal (regular), exceptional and singular. Recall that orbits are called *exceptional* if they have maximal dimension without being principal, and *singular* if their dimension is not maximal. We also call points *regular* when they belong to principal orbits.

3.1 Proposition *Let G be a compact connected Lie group with a taut representation $\rho : G \to \mathrm{O}(V)$. Let $p \in V$ and let $\nu_p : G_p \to \mathrm{O}(N_p(Gp))$ be the slice representation of ρ at p. Then ν_p is taut.*

Proof. Let $N^\epsilon(Gp)$ denote the bundle of normal vectors of length less than ϵ over the orbit Gp. Let $\epsilon > 0$ be so small that $T = \exp(N^\epsilon(Gp))$ is a tubular neighborhood around the orbit Gp. Let $v \in N_p(Gp)$. We want to show that G_pv is taut in the normal space $N_p(Gp)$. Let $\alpha > 0$ be a number that is so small that $w = \alpha v$ has length less than ϵ. It is clear that G_pv is taut in $N_p(Gp)$ if and only if G_pw is taut in $N_p(Gp)$. It is also clear that G_pw is taut in $N_p(Gp)$ if and only if $\exp_p(G_pw) = G_p \exp_p(w)$ is taut in $\exp_p(N_p(Gp))$ since $\exp_p$ is an isometry. We set $q = \exp_p(w)$. A submanifold in an affine subspace A of V is taut in A if and only if it is taut in V. Hence G_pv is taut in $N_p(Gp)$ if and only if G_pq is taut in V.

Now let $L_p : Gq \to \mathbf{R}$ be the distance function. The segment $\overline{pq}$ is orthogonal to Gp and hence also to Gq. It follows that q is a critical point of L_p. Since q is in the tubular neighborhood T and the length of the segment $\overline{pq}$ is smaller than ϵ it follows that L_p takes on its minimum in q. Let C be the subset of Gq on which L_p takes on its minimum value. Clearly $C = Gq \cap \exp_p(N_p^\epsilon(Gp))$ since a geodesic segment between Gp and Gq that is orthogonal to Gp is orthogonal to Gq and vice versa. By the slice theorem $G_pq = Gq \cap \exp_p(N_p^\epsilon(Gp))$. Hence $G_pq = C$. The set of critical points of a distance function to a taut submanifold is a union over taut submanifolds by a theorem of Ozawa, see [26]. It follows that G_pq is taut and hence that the slice representation ν_p is taut. □

3.2 Remark We are assuming in Proposition 3.1 that G is connected. Hence its orbits are connected and it follows from their tautness that the set of points where a distance function takes on its minimal value is connected. The proof of Proposition 3.1 now implies that the orbits of G_p are connected even if G_p is not connected.

It is well-known that the isotropy representation of a symmetric space does not have an exceptional orbit. The methods used in the proof of Proposition 3.1 immediately give that this is also the case for taut representations, as we show in the next proposition.

3.3 Proposition *A taut representation $\rho : G \to \mathrm{O}(V)$ of a compact connected Lie group G does not have exceptional orbits.*

Proof. Assume the orbit through p in V is exceptional. Then the slice representation of G_p at p has a disconnected orbit. Arguing exactly as in the proof of Proposition 3.1, we see that the distance function from p to a principal orbit through some regular point close to p assumes its minimum value on a disconnected set, contradicting tautness. Thus ρ cannot have an exceptional orbit. □

We close this subsection with a discussion of taut reducible representations.

3.4 Proposition *Let ρ_1 and ρ_2 be representations of a compact connected Lie group G with representation spaces V_1 and V_2 respectively. Assume that $\rho_1 \oplus \rho_2$ is F-taut. Then the restriction of ρ_2 to the isotropy group G_{v_1} is taut for every $v_1 \in V_1$.*

Furthermore, we have that $p(G(v_1,v_2);F) = p(Gv_1;F)\,p(G_{v_1}v_2;F)$, where $p(M;F)$ denotes the Poincaré polynomial of M with respect to the field F. In particular, $G_{v_1}v_2$ is connected and

$$b_1(G(v_1,v_2);F) = b_1(Gv_1;F) + b_1(G_{v_1}v_2;F),$$

where $b_1(M;F)$ denotes the first Betti number of M with respect to F.

Proof. We can work with height functions instead of distance functions in this proof since a submanifold contained in a round sphere in a Euclidean space is taut if and only if all height functions are perfect Morse functions, see [27]. Furthermore, in this situation the set of critical points of a distance function will also occur as the set of critical points of a height function, and vice versa. Fix $(v_1,v_2) \in V_1 \oplus V_2$. Let $a \in V_1$ be such that the height function $h_a : Gv_1 \to \mathbf{R}$ defined by $h_a(v) = \langle a, v\rangle$ is a Morse function. We define the height function $h_{(a,0)} : G(v_1,v_2) \to \mathbf{R}$ similarly. The point (u_1,u_2) is a critical point of $h_{(a,0)}$ with index i if and only if u_1 is a critical point of h_a with index i. Hence the critical set C on the critical level $h_{(a,0)}(u_1,u_2) = h_a(u_1)$ is

$$C = \{(w_1,w_2) \in G(v_1,v_2) \mid w_1 = u_1\} = \{(u_1,w_2) \mid w_2 \in G_{u_1}v_2\}.$$

Ozawa proves in [26] that the set of critical points of a distance function on a taut submanifold is a union over nondegenerate critical submanifolds that are again taut submanifolds. It follows that C is taut. The projection of C into V_2, which coincides with $G_{u_1}v_2$, is then also taut. We can choose a such that h_a is a Morse function of which v_1 is a critical point. It follows that $G_{v_1}v_2$ is taut for every $v_1 \in V_1$ and $v_2 \in V_2$ and hence that the restriction of ρ_2 to G_{v_1} is taut.

Now fix again a point (v_1,v_2). If $C \subset G(v_1,v_2)$ is an arbitrary critical submanifold of $h_{(a,0)}$, then it is diffeomorphic to $G_{v_1}v_2$. To see this let $\pi_1 : G(v_1,v_2) \to Gv_1$ be the projection onto the first factor. Then one easily sees that

$$\pi_1^{-1}(gv_1) = G_{gv_1}gv_2 = gG_{v_1}g^{-1}(gv_2) = gG_{v_1}v_2.$$

Hence we have that π_1 is a G-equivariant fibration. The critical submanifolds of $h_{(a,0)}$ are fibers of π_1 and hence diffeomorphic to each other.

Since the orbit $G(v_1,v_2)$ is taut we have by [26] that the Morse-Bott inequalities are equalities. Let $C_1,\ldots,C_k$ be the critical manifolds of $h_{(a,0)}$. Let $i(C_j)$ be the index of C_j for $j = 1,\ldots,k$. Then

$$p(G(v_1,v_2)) = \sum_{j=1}^{k} p(C_j)t^{i(C_j)} = p(G_{v_1}v_2)\sum_{j=1}^{k} t^{i(C_j)},$$

since the critical submanifolds are all diffeomorphic to $G_{v_1}v_2$. We have

$$p(Gv_1) = \sum_{j=1}^{k} t^{i(C_j)}$$

since h_a is a perfect Morse functions with critical points $p_1 = \pi_1(C), \dots, p_k = \pi_1(C_k)$ and the index of p_j is equal to $i(C_j)$. It follows that

$$p(G(v_1, v_2)) = p(Gv_1)\, p(G_{v_1} v_2).$$

Multiplying out the Poincaré polynomials gives $b_0(G_{v_1} v_2) = 1$ and

$$b_1(G(v_1, v_2)) = b_1(Gv_1) + b_1(G_{v_1} v_2).$$

This finishes the proof of the proposition. □

4 A necessary condition for a representation to be of class $\mathcal{O}^2$

Let π be a complex representation of the compact connected Lie group G on a finite-dimensional vector space. We say that π is of *real type* if it comes from a representation of G on a real vector space by extension of scalars, and we say that π is of *quaternionic type* if it comes from a representation of G on a quaternionic vector space by restriction of scalars. If π is neither of real type nor of quaternionic type, we say that π is of *complex type*

Now it is known that the finite-dimensional real irreducible representations ρ of G fall into one of the following disjoint classes:

(a) the complexification ρ^c is irreducible and $\rho^c = \pi$ is a complex representation of real type;

(b) the complexification ρ^c is reducible and $\rho^c = \pi \oplus \pi$ where π is a complex irreducible representation of quaternionic type;

(c) the complexification ρ^c is reducible and $\rho^c = \pi \oplus \pi^*$ where π is a complex irreducible representation of complex type and π^* is not equivalent to π (where π^* denotes the dual representation of π).

The relation between ρ and π is that ρ is a real form of π in the first case ($\rho^c = \pi$), but ρ is π viewed as a real representation in the other two cases ($\rho = \pi^r$). We shall call ρ of *real, quaternionic* or *complex type* according to whether the associated π is of real, quaternionic or complex type. Note also that π is self-dual precisely in the first two cases.

Suppose now that G is semisimple, let $\mathfrak{g}$ denote its Lie algebra and $\mathfrak{g}^c$ its complexification. Write Δ for the root system of $\mathfrak{g}^c$ with respect to a chosen Cartan subalgebra, Δ^+ for the positive root system with respect to an ordering of the roots, and $S = \{\alpha_1, \dots, \alpha_r\}$ for the corresponding simple root system. Let $\lambda_1, \dots, \lambda_r$ be the fundamental highest weights defined by the relations $2(\lambda_i, \alpha_j)/(\alpha_j, \alpha_j) = \delta_{ij}$ where $(\,,\,)$ is the Cartan-Killing form. The Theorem of the Highest Weight of É. Cartan states that the complex irreducible representations of $\mathfrak{g}^c$ are parametrized by their highest weights, and these are exactly the linear combinations $\lambda = \sum_{i=1}^r m_i \lambda_i$ for $m_i \in \{0, 1, 2, \dots\}$. The following proposition is a useful criterium of Dadok to decide for which λ the corresponding representation π_λ is of real, quaternionic or complex type. Recall that the roots $\alpha, \beta \in \Delta$ are called *strongly orthogonal* if $\alpha \pm \beta$ is not a root.

4.1 Proposition ([10]) *There is a maximal subset $B = \{\beta_1, \dots, \beta_s\} \subset \Delta^+$ of strongly orthogonal roots such that:*

(a) we have $s_0 = s_{\beta_1} \cdots s_{\beta_s}$ is the Weyl group element that maps the positive Weyl chamber into its negative;

(b) *the representation π_λ is of complex type if and only if λ does not belong to the real span of $\mathcal{B}$;*

(c) *the representation π_λ is of real type (resp. quaternionic type) if and only if λ belongs to the real span of $\mathcal{B}$ and*

$$k(\lambda) = \sum_{i=1}^{s} \frac{(\lambda, \beta_i) - (s_0\lambda, \beta_i)}{(\beta_i, \beta_i)}$$

is an even (resp. odd) integer.

4.2 Remark For a simple Lie algebra the set $\mathcal{B}$ can be constructed as follows, as is explained in [10] and will be assumed throughout our paper. Let β_1 be the highest root. The root system $\{\alpha \in \Delta : (\alpha, \beta_1) = 0\}$ is either irreducible or equals $\{\pm\zeta_1\} \cup \Delta_1$, with Δ_1 being irreducible and $\zeta_1 \in \Delta^+$. In the former case set β_2 equal to the highest root of Δ_1 (with the inherited order from Δ), and proceed by induction. In the latter case set $\beta_2 = \zeta_1$ and β_3 equal to the highest root of Δ_1, and proceed by induction.

We continue to assume that G is semisimple. Let H_α, $\alpha \in \Delta$, be the coroots of $\mathfrak{g}^c$. It is possible to choose root vectors X_α for $\mathfrak{g}^c$, $\alpha \in \Delta$, such that the compact real form $\mathfrak{g}$ is spanned by

(4.3) $$iH_\alpha, \quad X_\alpha - X_{-\alpha}, \quad i(X_\alpha + X_{-\alpha}), \qquad \text{where } \alpha \in \Delta^+.$$

Now if π is a complex representation of G, we have that the adjoint map $\pi(X_\alpha)^* = \pi(X_{-\alpha})$ with respect to any G-invariant Hermitian product on the representation space. Moreover, if π is of real (resp. quaternionic) type and ϵ is an invariant real (resp. quaternionic) structure on the representation space, then $\pi(X_\alpha)\epsilon = \epsilon\pi(X_{-\alpha})$.

By compactness of G, any real representation is equivalent to an orthogonal one. The following proposition, which is stated as a remark in [10], p. 128, and the ensuing lemma, which is a refinement of a result in [9], are precisely the ingredients we need to establish a necessary condition for an orthogonal representation of G to be of class $\mathcal{O}^2$.

4.4 Proposition ([10]) *Let $\mathcal{U}^k(\mathfrak{g}^c)$ be the kth level in the natural filtration of the universal enveloping algebra of $\mathfrak{g}^c$. Fix π_λ, the irreducible representation of $\mathfrak{g}^c$ with highest weight λ, and fix v_λ, a highest weight vector. Let*

$$n_i = \frac{(\lambda, \beta_i) - (s_0\lambda, \beta_i)}{(\beta_i, \beta_i)},$$

for $i = 1, \ldots, s$. Then the element $s_0 v_\lambda$ is in $\mathcal{U}^{k(\lambda)}(\mathfrak{g}^c)v_\lambda$, but not in $\mathcal{U}^l(\mathfrak{g}^c)v_\lambda$ for any $l < k(\lambda)$, where $s_0 v_\lambda$ is given as

$$s_0 v_\lambda = \pi(X_{-\beta_1})^{n_1} \cdots \pi(X_{-\beta_s})^{n_s} v_\lambda.$$

There is no proof of the above proposition in [10]. In [8], p. 270, there is an attempt to prove the proposition which in our view contains serious gaps.

Proof. It is enough to consider the case where $\mathfrak{g}^c$ is a complex simple Lie algebra. We first claim that the element $s_0 v_\lambda$ is not zero. In fact, consider the complex subalgebra $\mathfrak{k} \subset \mathfrak{g}^c$ generated by the root spaces of $\mathfrak{g}^c$ corresponding to $\pm\beta_1, \ldots, \pm\beta_s$. Then $\mathfrak{k}$ is isomorphic to the direct product of s copies of $\mathfrak{sl}(2, \mathbf{C})$. We restrict π_λ to $\mathfrak{k}$ and let U_λ be the unique irreducible $\mathfrak{k}$-module generated by v_λ. Since $\lambda - s_0\lambda = \sum_{i=1}^{s} n_i\beta_i$, it is now clear that U_λ is the representation space of $\mathfrak{k}$: $\overset{n_1}{\circ} \otimes \cdots \otimes \overset{n_s}{\circ}$

and that $s_0 v_\lambda \in U_\lambda$. Therefore, our claim is reduced to the case of the $(n+1)$-dimensional complex irreducible representation of $\mathfrak{sl}(2,\mathbb{C})$, namely $\overset{n}{\circ}$, which is immediate to verify.

It is obvious that $s_0 v_\lambda \in \mathcal{U}^{k(\lambda)}(\mathfrak{g}^c)v_\lambda$. We next prove that $s_0 v_\lambda \notin \mathcal{U}^l(\mathfrak{g}^c)v_\lambda$ for $l < k(\lambda)$ by contradiction. In fact, enumerate the positive roots $\Delta^+ = \{\alpha_1, \dots, \alpha_t\}$ and suppose $s_0 v_\lambda \in \mathcal{U}^l(\mathfrak{g}^c)v_\lambda$ for $l < k(\lambda)$. It follows from the Poincaré-Birkhoff-Witt theorem that we can write

$$s_0 v_\lambda = \sum_{p_1,\dots,p_t \geq 0} c_{p_1\cdots p_t} X_{-\alpha_1}^{p_1} \cdots X_{-\alpha_t}^{p_t} v_\lambda,$$

for some complex constants $c_{p_1\cdots p_t}$. Now, using that $X_\alpha^* = X_{-\alpha}$,

$$\begin{aligned} 0 &\neq (s_0 v_\lambda, s_0 v_\lambda) \\ &= \sum_{p_1,\dots,p_t \geq 0} c_{p_1\cdots p_t} (X_{-\alpha_1}^{p_1} \cdots X_{-\alpha_t}^{p_t} v_\lambda, X_{-\beta_1}^{n_1} \cdots X_{-\beta_s}^{n_s} v_\lambda) \\ &= \sum_{p_1,\dots,p_t \geq 0} c_{p_1\cdots p_t} (X_{\beta_s}^{n_s} \cdots X_{\beta_1}^{n_1} X_{-\alpha_1}^{p_1} \cdots X_{-\alpha_t}^{p_t} v_\lambda, v_\lambda). \end{aligned}$$

Recall that the highest weight space is one-dimensional; hence we can write

$$X_{\beta_s}^{n_s} \cdots X_{\beta_1}^{n_1} X_{-\alpha_1}^{p_1} \cdots X_{-\alpha_t}^{p_t} v_\lambda = c v_\lambda,$$

for some nonnegative integers $p_1, \dots, p_t$ such that $k(\lambda) = n_1 + \cdots + n_s > p_1 + \cdots + p_t = l$ and for some nonzero complex constant c. Note that $n_1\beta_1 + \cdots + n_s\beta_s = p_1\alpha_1 + \cdots + p_t\alpha_t$. The contradiction we are aiming at now follows from the following claim.

Claim 1 *If $N > M$ then*

$$X_{\delta_N} \cdots X_{\delta_1} X_{-\gamma_1} \cdots X_{-\gamma_M} v_\lambda = 0,$$

where $\delta_1, \dots, \delta_N \in \mathcal{B}$, and $\gamma_1, \dots, \gamma_M \in \Delta^+$.

We proceed by induction on the integer N. The cases $N = 1$ and $N = 2$ are trivial (use that v_λ is a highest weight vector for $N = 1$ and that the sum of two roots in $\mathcal{B}$ is not a root for $N = 2$). Assume the assertion is true for some $N - 1 \geq 2$ and let us prove it for N. Since $X_{\beta_1}, \dots, X_{\beta_s}$ pairwise commute, we may assume that $\delta_1 = \beta_i$ where $i \leq j$ for every $j = 1, \dots, s$ which satisfies $\beta_j = \delta_k$ for some $k = 1, \dots, N$.

Claim 2 *For each $j = 1, \dots, M$, if $\delta_1 - \gamma_j$ is a root then it is a positive root.*

Indeed we have $\delta_1 = \beta_i$ where i is as above. If $i = 1$, then we are done because β_1 is the highest root. If not, $0 = (\beta_1, \delta_1 + \cdots + \delta_N) = (\beta_1, \gamma_1 + \cdots + \gamma_M)$. Now $(\beta_1, \gamma_1) = -(\beta_1, \gamma_2 + \cdots + \gamma_M)$, and since β_1 is the highest root we have $(\beta_1, \gamma_1) \geq 0, \dots, (\beta_1, \gamma_M) \geq 0$. It follows that $(\beta_1, \gamma_1) = 0$ and $(\beta_1, \gamma_2 + \cdots + \gamma_M) = 0$. An easy induction argument shows that $(\beta_1, \gamma_j) = 0$ for $j = 1, \dots, M$. Consider now the root system $\{\alpha \in \Delta : (\alpha, \beta_1) = 0\}$. The first case occurs when it is irreducible. Then β_2 is by definition its highest root so that $\beta_2 \geq \gamma_j$ for $j = 1, \dots, M$. If $i = 2$ we are done. If not, we see that $(\beta_2, \gamma_j) = 0$ for $j = 1, \dots, M$ and we proceed by induction. The second case occurs when the above root system is of the form $\{\pm\zeta_1\} \cup \Delta_1$, with Δ_1 being irreducible and $\zeta_1 \in \Delta^+$. Here $\beta_2 = \zeta_1$ and β_3 is the highest root of Δ_1. If $\gamma_j \in \Delta_1$ for $j = 1, \dots, M$ then $(\beta_2, \gamma_j) = 0$ for $j = 1, \dots, M$ so that $i \geq 3$ and we proceed by induction as above. On the other hand, if $\zeta_1 = \gamma_{j_0}$ for some j_0, then $i = 2$. In this case, for each $j = 1, \dots, M$ either $\beta_2 = \gamma_j$ or $-\gamma_j \in \Delta_1$. So $\beta_2 - \gamma_j$ is never a root. This completes the proof of Claim 2.

CLAIM 3 *We have*

$$X_{\delta_N}\cdots X_{\delta_2}X_{-\gamma_1}\cdots X_{-\gamma_{j-1}}H_\zeta X_{-\gamma_{j+1}}\cdots X_{-\gamma_M}v_\lambda=0,$$

where H_ζ is the coroot vector corresponding to $\zeta\in\Delta^+$ and $j=1,\dots,M$.

In order to prove Claim 3, proceed by induction on j. The initial case is $j=M$, which follows from the induction hypothesis on N, since $H_\zeta v_\lambda=(\zeta,\lambda)v_\lambda$. Next write

$$\begin{aligned}(4.5)\qquad & X_{\delta_N}\cdots X_{\delta_2}X_{-\gamma_1}\cdots X_{-\gamma_{j-1}}H_\zeta X_{-\gamma_{j+1}}\cdots X_{-\gamma_M}v_\lambda=\\ & \quad X_{\delta_N}\cdots X_{\delta_2}X_{-\gamma_1}\cdots X_{-\gamma_{j-1}}X_{-\gamma_{j+1}}H_\zeta\cdots X_{-\gamma_M}v_\lambda\\ & + X_{\delta_N}\cdots X_{\delta_2}X_{-\gamma_1}\cdots X_{-\gamma_{j-1}}[H_\zeta,X_{-\gamma_{j+1}}]\cdots X_{-\gamma_M}v_\lambda.\end{aligned}$$

The first summand on the right hand side of (4.5) is zero by the induction hypothesis on j, and the second summand is zero because $[H_\zeta,X_{-\gamma_{j+1}}]=-(\zeta,\gamma_{j+1})X_{-\gamma_{j+1}}$ so that we can use the induction hypothesis on N. This proves Claim 3.

CLAIM 4 *We have*

$$X_{\delta_N}\cdots X_{\delta_2}X_{-\gamma_1}\cdots X_{-\gamma_{j-1}}X_\zeta X_{-\gamma_{j+1}}\cdots X_{-\gamma_M}v_\lambda=0,$$

where X_ζ is the root vector corresponding to $\zeta\in\Delta^+$ and $j=1,\dots,M$.

In order to prove Claim 4, proceed by induction on j. The initial case is $j=M$, which is trivial because v_λ is a highest weight vector. Next write

$$\begin{aligned}(4.6)\qquad & X_{\delta_N}\cdots X_{\delta_2}X_{-\gamma_1}\cdots X_{-\gamma_{j-1}}X_\zeta X_{-\gamma_{j+1}}\cdots X_{-\gamma_M}v_\lambda=\\ & \quad X_{\delta_N}\cdots X_{\delta_2}X_{-\gamma_1}\cdots X_{-\gamma_{j-1}}X_{-\gamma_{j+1}}X_\zeta\cdots X_{-\gamma_M}v_\lambda\\ & + X_{\delta_N}\cdots X_{\delta_2}X_{-\gamma_1}\cdots X_{-\gamma_{j-1}}[X_\zeta,X_{-\gamma_{j+1}}]\cdots X_{-\gamma_M}v_\lambda.\end{aligned}$$

The first summand on the right hand side of (4.6) is zero by the induction hypothesis on j. The second summand is also zero for the following reasons. If $\zeta=\gamma_{j+1}$ we use Claim 3. If not, then either $\zeta-\gamma_{j+1}$ is not a root and then $[X_\zeta,X_{-\gamma_{j+1}}]=0$, or else $\zeta-\gamma_{j+1}$ is a root. In the latter case, if it is a positive root we can use the induction hypothesis on j, and if it is a negative root then we can use the induction hypothesis on N. This completes the proof of Claim 4. We finally turn to the proof of Claim 1. We can write

$$\begin{aligned}(4.7)\qquad & X_{\delta_N}\cdots X_{\delta_1}X_{-\gamma_1}\cdots X_{-\gamma_M}v_\lambda=\\ & \sum_{j=1}^{M}X_{\delta_N}\cdots X_{\delta_2}X_{-\gamma_1}\cdots X_{-\gamma_{j-1}}[X_{\delta_1},X_{-\gamma_j}]X_{-\gamma_{j+1}}\cdots X_{-\gamma_M}v_\lambda\\ & + X_{\delta_N}\cdots X_{\delta_2}X_{-\gamma_1}\cdots X_{-\gamma_M}X_{\delta_1}v_\lambda.\end{aligned}$$

The second summand in the right hand side of (4.7) is zero because v_λ is a highest weight vector. The first summand is also zero because if $\delta_1=\gamma_j$ we can apply Claim 3. Otherwise, either $\delta_1-\gamma_j$ is not a root and then $[X_{\delta_1},X_{-\gamma_j}]=0$ or else $\delta_1-\gamma_j$ is a root and then it is a positive root by Claim 2 so that we can apply Claim 4. This completes the proof of Claim 1 and the proof of the proposition. □

4.8 Lemma *Let π_λ be the complex irreducible representation of G with highest weight λ and representation space V_λ. Let μ be a weight of π_λ and fix a weight vector v_μ. Let ρ denote a real form of π_λ in case π_λ is of real type, or the realification of π_λ in case it is either of quaternionic or of complex type. Suppose that ρ is of class $\mathcal{O}^2$.*

(a) If π_λ is of real type, then

$$\mathcal{U}^2(\mathfrak{g}^c)v_\mu + \mathcal{U}^2(\mathfrak{g}^c)\epsilon(v_\mu) = V_\lambda,$$

where ϵ is the real structure on V_λ defined by ρ. In particular, by taking $\mu = \lambda$, we find that

$$\mathcal{U}^2(\mathfrak{g}^c)v_\lambda + \mathcal{U}^2(\mathfrak{g}^c)\epsilon(v_\lambda) = V_\lambda. \qquad (C_2)$$

(b) If π_λ is of complex or quaternionic type, then $\mathcal{U}^2(\mathfrak{g}^c)v_\mu = V_\lambda$.

Proof. (a) Consider the real vector $p = v_\mu + \epsilon v_\mu$. The second osculating space $\mathcal{O}^2_p(Gp)$ is spanned over $\mathbf{R}$ by Xp, XYp, where X, $Y \in \mathfrak{g}$. Taking linear combinations with complex coefficients of these vectors, we can write that the complexification

$$\mathcal{O}^2_p(Gp)^c \subset \mathcal{U}^2(\mathfrak{g}^c)v_\mu + \mathcal{U}^2(\mathfrak{g}^c)\epsilon(v_\mu) \subset V_\lambda.$$

But ρ of class $(\mathcal{O}^2)$ implies that $\mathcal{O}^2_p(Gp)^c = V_\lambda$.

(b) Let $p = v_\mu$. We have that $\mathcal{O}^2_p(Gp) \subset \mathcal{U}^2(\mathfrak{g}^c)v_\mu \subset V_\lambda$ as real vector spaces and the proof is similar as in (a). □

We now state the main result of this section. Notice that the same result is claimed in [8], p. 271, with an attempt of a proof which in our opinion is not satisfactory.

4.9 Proposition *Let ρ be a real (orthogonal) irreducible representation of a compact connected semisimple Lie group G with complexified Lie algebra $\mathfrak{g}^c$, and let π_λ be the associated complex irreducible representation. Suppose that ρ is of class $\mathcal{O}^2$.*

(a) If ρ is of quaternionic type, then $k(\lambda) = 1$.

(b) If ρ is of complex type, then $k(\lambda) = 1, 2$.

(c) If ρ is of real type, then $k(\lambda) = 2, 4$.

Proof. First consider ρ to be of quaternionic or complex type, i. e. $\rho^c = \pi_\lambda \oplus \pi^*_\lambda$. Then ρ of class $\mathcal{O}^2$ forces $\mathcal{U}^2(\mathfrak{g}^c)v_\lambda = V_\lambda$ (Lemma 4.8) and then $s_0 v_\lambda \in \mathcal{U}^2(\mathfrak{g}^c)v_\lambda$ implies that $k(\lambda) \leq 2$ (Proposition 4.4). From this follow (a) and (b).

Now take ρ to be of real type. Let v_λ be a highest weight vector of $\rho^c = \pi_\lambda$. Since π_λ is self-dual,

$$\text{lowest weight of } \pi_\lambda = -\text{highest weight of } \pi^*_\lambda = -\lambda = s_0\lambda.$$

Also, there exists a $\mathbf{C}$-conjugate linear, G-invariant involution ϵ of V_λ and $\epsilon(v_\lambda)$ is a lowest weight vector. We shall assume $k(\lambda) \geq 6$ and derive a contradiction. Since $k(\lambda)$ is even, it is possible to write

$$\frac{k(\lambda)}{2} = n_1 + \ldots + n_{i_0} + m$$

for some integers $0 \leq i_0 < s$, $0 \leq m < n_{i_0+1}$, where n_i is as in Proposition 4.4. Set

$$\mu = \lambda - n_1\beta_1 - \ldots - n_{i_0}\beta_{i_0} - m\beta_{i_0+1}.$$

Note that μ is a weight: a μ-weight vector is

$$u = X^{n_1}_{-\beta_1} \ldots X^{n_{i_0}}_{-\beta_{i_0}} X^{m}_{-\beta_{i_0+1}} v_\lambda.$$

Use Lemma 4.8 to decompose $u = u_1 + u_2$ where $u_1 \in \mathcal{U}^2(\mathfrak{g}^c)v_\lambda$ and $u_2 \in \mathcal{U}^2(\mathfrak{g}^c)\epsilon(v_\lambda)$. First, assume that $u_2 \neq 0$. Since $\frac{k(\lambda)}{2} \geq 3$, we have

$$u_2 = u - u_1 \in \mathcal{U}^{\frac{k(\lambda)}{2}}(\mathfrak{g}^c)v_\lambda + \mathcal{U}^2(\mathfrak{g}^c)v_\lambda \subset \mathcal{U}^{\frac{k(\lambda)}{2}}(\mathfrak{g}^c)v_\lambda. \tag{4.10}$$

Moreover, it is clear that u_2 may be assumed to be a μ-weight vector (because a component of u_2 in a different weight space has to cancel with the corresponding component of u_1 in the same weight space) and that

$$u_2 = \sum_{\gamma,\delta\in\Delta^+} c_{\gamma,\delta} X_{-\gamma} X_{-\delta} \epsilon(v_\lambda)$$

for some complex constants $c_{\gamma,\delta}$ (we do not need to consider terms of first order in the sum because $\lambda - \mu$ cannot be a root). Therefore,

$$0 \neq (u_2, u_2) = (u_2, \sum_{\gamma,\delta\in\Delta^+} c_{\gamma,\delta} X_{-\gamma} X_{-\delta} \epsilon(v_\lambda)) = (\sum_{\gamma,\delta\in\Delta^+} \bar{c}_{\gamma,\delta} X_\delta X_\gamma u_2, \epsilon(v_\lambda)).$$

This shows that

$$\sum_{\gamma,\delta\in\Delta^+} \bar{c}_{\gamma,\delta} X_\delta X_\gamma u_2$$

is a nonzero multiple of the lowest weight vector $\epsilon(v_\lambda)$. This, combined with (4.10), gives that $\epsilon(v_\lambda)$ is in $\mathcal{U}^{\frac{k(\lambda)}{2}+2}(\mathfrak{g}^c)v_\lambda$. But $\frac{k(\lambda)}{2} + 2 < k(\lambda)$, contradicting Proposition 4.4.

In case $u_2 = 0$ we have that $u = u_1 \in \mathcal{U}^2(\mathfrak{g}^c)v_\lambda$. Since

$$X_{-\beta_{i_0+1}}^{n_{i_0+1}-m} X_{-\beta_{i_0+2}}^{n_{i_0+2}} \cdots X_{-\beta_s}^{n_s} u = s_0 v_\lambda$$

is a nonzero multiple of the lowest weight vector $\epsilon(v_\lambda)$ and $n_{i_0+1} - m + n_{i_0+2} + \ldots + n_s = \frac{k(\lambda)}{2}$, again $\epsilon(v_\lambda) \in \mathcal{U}^{\frac{k(\lambda)}{2}+2}(\mathfrak{g}^c)v_\lambda$, contradicting the same proposition. □

5 The candidates to a position in class $\mathcal{O}^2$

In this section we want to elaborate a list of possibilities for real irreducible representations of class $\mathcal{O}^2$ which we shall use later in Section 6 to classify taut irreducible representations. Our point of view, as is usually the case in other papers on the subject, is to classify orthogonal representations up to the following equivalence relation: we call two representations $\rho : G \to \mathrm{O}(V)$, $\rho' : G' \to \mathrm{O}(V')$ *image equivalent* if there exists an isometry $\psi : V \to V'$ such that $\mathrm{O}(\psi)(\rho(G)) = \rho'(G')$, where $\mathrm{O}(\psi) : \mathrm{O}(V) \to \mathrm{O}(V')$ is the induced conjugation map[1]. In particular, the image equivalence class of ρ always contains its dual ρ^* (in fact any representation obtained from ρ by an automorphism of the Dynkin diagram), as well as the pull back representation $\tilde{\rho} = \rho \circ p$ of a covering group $p : \tilde{G} \to G$.

Let G be a compact connected Lie group. We have that G is finitely covered by $T^n \times G_s$, where T^n is an n-dimensional torus and G_s is a compact connected semisimple Lie group (which may also be assumed to be simply-connected, whenever convenient), and any representation of G pulls back

[1]This is finer than the notion of orbit equivalence which is explained in Section 2, but of course not as fine as the usual notion of equivalence for representations.

to a representation of $T^n \times G_s$. In view of image equivalence, in order to study representations of G, we may assume that $G = T^n \times G_s$ and restrict to almost faithful representations. Next we observe that any complex irreducible representation of T^n is a character (hence of complex type). Therefore a complex irreducible representation of G can be of real type or of quaternionic type only if it is trivial on T^n, and in case it is of complex type, then it has an $(n-1)$-dimensional kernel on T^n. Now we have come to our working hypothesis (for a circle group S^1, let x^n denote the nth power representation of that circle which is a complex representation of complex type):

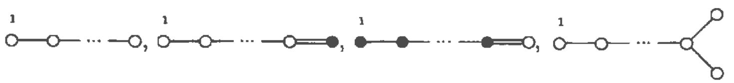
> Let ρ be a real irreducible representation of a compact connected Lie group G on a finite-dimensional real vector space V.
>
> - If ρ is of real type, we assume that $G = G_s$ is a compact connected semisimple Lie group, and ρ is a real form of a complex irreducible representation $\pi = \pi_\lambda$ of G of real type on the complex vector space V_λ for some highest weight λ.
> - If ρ is of quaternionic type, we assume that $G = G_s$, where G_s is as above, and ρ is the realification of a complex irreducible representation $\pi = \pi_\lambda$ of G of quaternionic type on the complex vector space V_λ for some highest weight λ.
> - If ρ is of complex type, we assume that $G = G_s$ or $G = S^1 \times G_s$, where G_s is as above, and ρ is the realification of a complex irreducible representation π of G of complex type. According to the form of G, we write $\pi = \pi_\lambda$ or $\pi = x \otimes \pi_\lambda$ as a representation on the complex vector space $V_\lambda = \mathbf{C} \otimes_{\mathbf{C}} V_\lambda$ for some highest weight λ.

Isotropy representations of symmetric spaces and, more generally, representations orbit equivalent to those of course are taut and belong to class $\mathcal{O}^2$. So, when analysing possibilities for ρ, we will be disregarding all representations orbit equivalent to the isotropy representation of a symmetric space that we encounter, according to [34] (Tables 8.11.2 and 8.11.5) and [12] (main theorem). Therefore, in addition to our previous hypotheses, we assume throughout this section:

> The real irreducible representation ρ is of class $\mathcal{O}^2$ but it is not orbit equivalent to the isotropy representation of a symmetric space.

The results of our investigation in this section (namely, Propositions 5.1, 5.2, 5.14, 5.15, 5.16, 5.17 and 5.18) finally imply: the possibilities for ρ are those collected in Tables B.1, B.2, B.3 and B.4 in Appendix B.

We need some amount of notation (cf. [34], p. 237). Let $\mathbf{A}_n = \mathbf{SU}(n+1)$, $\mathbf{B}_n = \mathbf{Spin}(2n+1)$, $\mathbf{C}_n = \mathbf{Sp}(n)$, $\mathbf{D}_n = \mathbf{Spin}(2n)$, $\mathbf{G}_2$, $\mathbf{F}_4$, $\mathbf{E}_6$, $\mathbf{E}_7$ and $\mathbf{E}_8$ refer to É. Cartan's classification types of simple Lie groups and Lie algebras. For the complex semisimple Lie algebra $\mathfrak{g}^c$, we use the following notation for its complex irreducible representation π_λ: if the integer $2(\lambda, \alpha_i)/(\alpha_i, \alpha_i) \neq 0$, then we write it next to the vertex of the Dynkin diagram of $\mathfrak{g}^c$ which corresponds to α_i (recall that $S = \{\alpha_1, \dots, \alpha_r\}$ is the simple root system). For example,



denote respectively the vector representations of $\mathbf{A}_n$, $\mathbf{B}_n$, $\mathbf{C}_n$, $\mathbf{D}_n$. Note that we are using the dot convention: if there are two lengths of roots, then the short roots are black in the Dynkin diagram. And a real irreducible representation is denoted by the diagram of its complexification.

We add that we shall repeatedly refer to Table A.1 in Appendix A, which contains the values of the invariant $k(\lambda)$ for the fundamental representations of the complex simple Lie algebras. Additional data about the root systems of the complex simple Lie algebras can be found in the tables of [4].

5.1 The case where ρ is of quaternionic type

Here there is nothing:

5.1 Proposition *The representation ρ cannot be of quaternionic type.*

Proof. If ρ is of class $\mathcal{O}^2$ and quaternionic type, Proposition 4.9 says that $k(\lambda) = 1$. A glance at Table A.1 now shows that $G = \mathbf{Sp}(n)$ and π_λ is the vector representation. But then ρ is of cohomogeneity one. □

5.2 The case where ρ is of complex type

Here the result is:

5.2 Proposition *Let ρ be of complex type. Then ρ is one of the following:*

G	ρ	*Conditions*
$\mathbf{SU}(n)\times\mathbf{Sp}(m)$	$(\overset{1}{\circ}-\cdots-\circ\oplus\circ-\cdots-\overset{1}{\circ})\otimes\overset{1}{\bullet}-\cdots-\bullet\Leftarrow\circ$	$n\geq 3,\ m\geq 2$
$S^1\times\mathbf{SU}(n)\times\mathbf{Sp}(m)$	$(x\otimes\overset{1}{\circ}-\cdots-\circ\oplus x^{-1}\otimes\circ-\cdots-\overset{1}{\circ})\otimes\overset{1}{\bullet}-\cdots-\bullet\Leftarrow\circ$	$m\geq 2$
$\mathbf{SO}(2)\times\mathbf{Spin}(9)$	$(x\oplus x^{-1})\otimes\circ-\circ-\circ\Rightarrow\overset{1}{\bullet}$	–

The proof of Proposition 5.2 will be given after the proof of the following two lemmas.

5.3 Lemma *Let π_λ be a complex irreducible representation of a compact connected semisimple Lie group G_s and suppose that the realification of the tensor product representation $\pi = x\otimes\pi_\lambda$ of $G = S^1\times G_s$ is of class $\mathcal{O}^2$. We have:*

(a) if π_λ is of real type and G_s is simple, then $G_s = \mathbf{Spin}(7)$ or $G_s = \mathbf{Spin}(9)$ and π_λ is the respective spin representation, or $G_s = \mathbf{G}_2$ and π_λ is the 7-dimensional representation, or $G_s = \mathbf{SO}(m)$ for $m\neq 2, 4$ and π_λ is the vector representation;

(b) if π_λ is of real type and G_s is not simple, then $G_s = \mathbf{Sp}(1)\times\mathbf{Sp}(m)$ and π_λ is the tensor product of the vector representations of each of the factors;

(c) if π_λ is of quaternionic type or complex type, then $k(\lambda) = 1, 2$.

Proof. (a) and (b) Let V_λ be the representation space of π_λ and V be the real subspace where a real form acts. Note that the representation space for π is still V_λ, but π is of complex type. Let

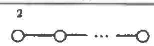
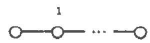
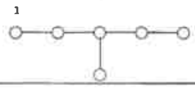
$p = 1 \otimes a \in \mathbf{C} \otimes_{\mathbf{R}} V = V^c = V_\lambda$, n the dimension of V and q the codimension of $G_s a$ in V. Since the realification of π is of class $\mathcal{O}^2$, we have that $\mathcal{O}^2_p(Gp) = V^{cr} \cong V \oplus iV$. But

$$\mathcal{O}^2_p(Gp) \subset 1 \otimes [\mathbf{R}a + \mathcal{O}^2_a(G_s a)] + i \otimes [\mathbf{R}a + T_a(G_s a)],$$

so by comparing real and imaginary parts we get that $V = \mathbf{R}a + T_a(G_s a)$. Therefore $q = 1$ and G_s acts with cohomogeneity one on V. Now (a) and (b) follow from the classification of transitive linear actions of compact Lie groups on spheres.

(c) Consider the point $p = v_\lambda \in V_\lambda$. We have that $\mathcal{O}^2_p(Gp) \subset \mathcal{U}^2(\mathfrak{g}^c)v_\lambda \subset V_\lambda$ as real vector spaces. Since the realification of π is of class $\mathcal{O}^2$, we have that $\mathcal{U}^2(\mathfrak{g}^c)v_\lambda = V_\lambda$. But $\mathcal{U}^2(\mathfrak{g}_s^c)v_\lambda = \mathcal{U}^2(\mathfrak{g}^c)v_\lambda$, since the circle subgroup S^1 preserves the weight spaces of π_λ. Therefore $\mathcal{U}^2(\mathfrak{g}_s^c)v_\lambda = V_\lambda$ and we can apply Proposition 4.4 as in the first paragraph of the proof of Proposition 4.9. □

5.4 Lemma *The realifications of the following complex irreducible representations π_λ of complex type of the compact connected semisimple Lie group G_s are* not *of class $\mathcal{O}^2$:*

G_s	π_λ	*Conditions*
$\mathbf{SU}(n)$		$n \geq 3$
$\mathbf{SU}(n)$		$n = 2p \geq 6$
$\mathbf{SU}(n) \times \mathbf{SU}(n)$		$n \geq 3$
$\mathbf{E}_6$		–

Proof. Each one of these representations π_λ has the property that the realification of $\pi = x \otimes \pi_\lambda$ is the isotropy representation of a compact irreducible Hermitian symmetric space $X = L_0/K_0$, where K_0 is locally isomorphic to $S^1 \times G_s$. We refer to [18], Chapter VIII, §7, for results about Hermitian symmetric spaces that we will be using in the following. Let $\mathfrak{l}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ be the decomposition of the Lie algebra $\mathfrak{l}_0$ of L_0 into the ± 1-eigenspaces of the symmetry. Let $\mathfrak{c}_0$ be the Lie algebra of S^1, and let $\mathfrak{t}_0$ be some Cartan subalgebra of the Lie algebra $\mathfrak{g}_s$ of G_s. Then $\mathfrak{h}_0 = \mathfrak{c}_0 + \mathfrak{t}_0$ is a Cartan subalgebra of $\mathfrak{k}_0$ and of $\mathfrak{l}_0$. Let $\mathfrak{l}$ be the complexification of $\mathfrak{l}_0$ and let $\mathfrak{c}$, $\mathfrak{t}$, $\mathfrak{h}$, $\mathfrak{k}$, $\mathfrak{p}$ be the complex subspaces of $\mathfrak{l}$ spanned by $\mathfrak{c}_0$, $\mathfrak{t}_0$, $\mathfrak{h}_0$, $\mathfrak{k}_0$, $\mathfrak{p}_0$. Let Δ denote the root system of $(\mathfrak{l}, \mathfrak{h})$ and consider the root space decomposition $\mathfrak{l} = \mathfrak{h} + \sum_{\alpha \in \Delta} \mathfrak{l}_\alpha$. For each $\alpha \in \Delta$, we have that either $\mathfrak{l}_\alpha \subset \mathfrak{k}$ or $\mathfrak{l}_\alpha \subset \mathfrak{p}$, in which cases the root α is called respectively *compact* or *noncompact*. A root is compact if and only if it vanishes on $\mathfrak{c}$. Let Δ_c and Δ_n denote respectively the subsets of Δ of compact and noncompact roots. Then we have decompositions $\mathfrak{k} = \mathfrak{h} + \sum_{\alpha \in \Delta_c} \mathfrak{l}_\alpha$, $\mathfrak{p} = \sum_{\gamma \in \Delta_n} \mathfrak{l}_\gamma$. Each root is real-valued on $i\mathfrak{h}_0$. We introduce a lexicographic ordering on the dual of $i\mathfrak{h}_0$ that takes $i\mathfrak{c}_0$ before $i\mathfrak{t}_0$. Let Δ^+, Δ_c^+ and Δ_n^+ be the set of positive roots in Δ, Δ_c and Δ_n, and define

$$\mathfrak{p}^+ = \sum_{\gamma \in \Delta_n^+} \mathfrak{l}_\gamma, \quad \mathfrak{p}^- = \sum_{-\gamma \in \Delta_n^+} \mathfrak{l}_\gamma.$$

Then $\mathfrak{p} = \mathfrak{p}^+ + \mathfrak{p}^-$ is an $\mathrm{ad}_{\mathfrak{k}}$-invariant decomposition. The $\mathrm{ad}_{\mathfrak{k}_0}$-invariant complex structure J_0 on $\mathfrak{p}_0$ is given by ad_{H_0}, where $H_0 \in \mathfrak{c}_0$ is determined by $\gamma(H_0) = i$ for $\gamma \in \Delta_n^+$. The $\mathbf{C}$-linear extension J of J_0 to $\mathfrak{p}$ has $\mathfrak{p}^\pm$ as $\pm i$-eigenspaces. Let $\Gamma = \{\gamma_1, \ldots, \gamma_s\} \subset \Delta_n^+$ be a maximal subset of strongly orthogonal roots. Then $\mathfrak{a}_0 = \sum_{j=1}^s \mathbf{R}(X_{\gamma_j} - X_{-\gamma_j})$ is a maximal Abelian subspace of $\mathfrak{p}_0$. We view

π^r as the adjoint action of K_0 on $\mathfrak{p}_0$, and then π^r is polar and $\mathfrak{a}_0$ is a section for π^r. We also view π as the the adjoint action of K_0 on $\mathfrak{p}^+$, and then the weight system of π is Δ_n^+. Now the root system of $(\mathfrak{g}_s^c, \mathfrak{t})$ is

$$\Delta_0 = \{\alpha|_{\mathfrak{t}} : \alpha \in \Delta_c\} \quad (\text{also } \Delta_0^+ = \{\alpha|_{\mathfrak{t}} : \alpha \in \Delta_c^+\}),$$

and the weight system of π_λ is

$$\Phi = \{\gamma|_{\mathfrak{t}} : \gamma \in \Delta_n^+\}.$$

It is also useful to remark that any $\mathrm{ad}_{\mathfrak{k}_0}$-invariant inner product $<\cdot,\cdot>$ on $\mathfrak{p}_0$ is given by a negative multiple of the restriction of the Killing form of $\mathfrak{l}_0$; fix one and extend it to a Hermitian product $\ll\cdot,\cdot\gg$ on $\mathfrak{p}$. Then $\ll\cdot,\cdot\gg$ is $\mathrm{ad}_{\mathfrak{k}_0}$-invariant and J is a skew-Hermitian operator on $\mathfrak{p}$ with respect to $\ll\cdot,\cdot\gg$.

We will verify directly that the second osculating space at the point

$$p = \sum_{j=1}^{s} X_{\gamma_j} - X_{-\gamma_j} \in \mathfrak{a}_0 \subset \mathfrak{p}_0$$

is not of maximal dimension. More precisely, we shall see that

$$<\mathrm{ad}_X p, J_0 p> = <\mathrm{ad}_Y \mathrm{ad}_X p, J_0 p> = 0, \tag{5.5}$$

for all X, $Y \in \mathfrak{g}_s$. Notice that $J_0 p = \sum_{j=1}^{s} i(X_{\gamma_j} + X_{-\gamma_j}) \in \mathfrak{p}_0$.

It suffices to prove the identities (5.5) for the basis elements (4.3) of $\mathfrak{g}_s$. We have $<\mathrm{ad}_X p, J_0 p> = \Re\ll\mathrm{ad}_X p, Jp\gg$ and $<\mathrm{ad}_Y \mathrm{ad}_X p, J_0 p> = -\Re\ll\mathrm{ad}_X p, \mathrm{ad}_Y Jp\gg$. The only cases where the real parts of the Hermitian products are not obviously zero are the following (α, $\beta \in \Delta_0^+$):

$$\begin{aligned} \ll\mathrm{ad}_{iH_\alpha} p, Jp\gg &= \sum_{j,k=1}^{s} \ll\mathrm{ad}_{H_\alpha}(X_{\gamma_j} - X_{-\gamma_j}), X_{\gamma_k} + X_{-\gamma_k}\gg \\ &= \sum_{j,k=1}^{s} \gamma_j(H_\alpha)\ll X_{\gamma_j} + X_{-\gamma_j}, X_{\gamma_k} + X_{-\gamma_k})\gg \\ &= 2\sum_{j=1}^{s} \gamma_j(H_\alpha) = 2(\sum_{j=1}^{s} \gamma_j, \alpha), \end{aligned} \tag{5.6}$$

assuming the root vectors to have length one. We will show below that the last expression in (5.6) vanishes for the representations we are interested.

$$\begin{aligned} \ll\mathrm{ad}_{i(X_\alpha + X_{-\alpha})} p, Jp\gg &= \ll\mathrm{ad}_{X_\alpha + X_{-\alpha}} p, \textstyle\sum_{k=1}^{s}(X_{\gamma_k} + X_{-\gamma_k})\gg \\ &= \sum_{j,k=1}^{s} \ll\mathrm{ad}_{X_\alpha + X_{-\alpha}}(X_{\gamma_j} - X_{-\gamma_j}), X_{\gamma_k} + X_{-\gamma_k}\gg \\ &= 0, \end{aligned} \tag{5.7}$$

because $\mathrm{ad}_{X_\alpha} X_{\gamma_j}$ is either a $(\gamma_j + \alpha)$-root vector or 0; $\pm\gamma_j \pm \gamma_k \notin \Delta$; and root spaces corresponding to distinct roots are orthogonal.

$$\begin{aligned}\ll \mathrm{ad}_{iH_\alpha}p\,,\mathrm{ad}_{X_\beta - X_{-\beta}}Jp)\gg &= \sum_{j,k=1}^{s} \gamma_j(H_\alpha) \ll (X_{\gamma_j} + X_{-\gamma_j})\,, \mathrm{ad}_{X_\beta - X_{-\beta}}(X_{\gamma_k} + X_{-\gamma_k})\gg \\ &= 0,\end{aligned}$$

for the same reason why (5.7) is zero.

$$\begin{aligned}\Re\ll \mathrm{ad}_{X_\alpha - X_{-\alpha}}Jp\,,\mathrm{ad}_{i(X_\beta+X_{-\beta})}p\gg &= \Re\ll \mathrm{ad}_{X_\alpha}Jp\,, i\mathrm{ad}_{X_\beta}p\gg \\ &\quad -\Re\ll \mathrm{ad}_{X_{-\alpha}}Jp\,, i\mathrm{ad}_{X_{-\beta}}p\gg \\ &\quad +\Re\ll \mathrm{ad}_{X_\alpha}Jp\,, i\mathrm{ad}_{X_{-\beta}}p\gg \\ &\quad -\Re\ll \mathrm{ad}_{X_{-\alpha}}Jp\,, i\mathrm{ad}_{X_\beta}p\gg \\ &= \Re\ll Jp\,, i\mathrm{ad}_{[X_{-\beta},X_\alpha]}p\gg \\ &\quad +\Re\ll Jp\,, i\mathrm{ad}_{[X_\beta,X_\alpha]}p\gg, \end{aligned} \tag{5.8}$$

because, for instance,

$$\begin{aligned}\ll \mathrm{ad}_{X_\alpha}Jp\,, i\mathrm{ad}_{X_\beta}p\gg &= \ll \mathrm{ad}_{X_{-\beta}}\mathrm{ad}_{X_\alpha}Jp\,, ip\gg \\ &= \ll J\mathrm{ad}_{X_{-\beta}}\mathrm{ad}_{X_\alpha}p\,, ip\gg \\ &= -\ll \mathrm{ad}_{X_{-\beta}}\mathrm{ad}_{X_\alpha}p\,, Jip\gg \\ &= -\ll \mathrm{ad}_{X_{-\beta}}\mathrm{ad}_{X_\alpha}p\,, iJp\gg \\ &= \ll i\mathrm{ad}_{X_{-\beta}}\mathrm{ad}_{X_\alpha}p\,, Jp\gg \\ &= \overline{\ll Jp\,, i\mathrm{ad}_{X_{-\beta}}\mathrm{ad}_{X_\alpha}p\gg}.\end{aligned}$$

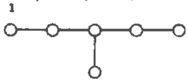
Now (5.8) shows that everything amounts to checking that (5.6) vanishes. Next we do that in each one of the four cases. Let $\gamma = \sum_{j=1}^{s}\gamma_j$. Then the vanishing of (5.6) is equivalent to

$$(\gamma, \Delta_0^+) = 0. \tag{5.9}$$

Let $G_s = \mathbf{SU}(n)$, π_λ : $\overset{2}{\circ}$—···—○. Then L_0 is $\mathbf{Sp}(n)$, $\Delta_0^+ = \{\theta_i - \theta_j : 1 \le i < j \le n\}$, $\Phi = \{\theta_i + \theta_j : 1 \le i < j \le n\} \cup \{2\theta_i : 1 \le i \le n\}$, $\Gamma = \{2\theta_i : 1 \le i \le n\}$ and $\gamma = 2\theta_1 + \ldots + 2\theta_n$.

Let $G_s = \mathbf{SU}(n)$, n even, π_λ : ○—$\underset{1}{\circ}$—···—○. Then L_0 is $\mathbf{SO}(2n)$, $\Delta_0^+ = \{\theta_i - \theta_j : 1 \le i < j \le n\}$, $\Phi = \{\theta_i + \theta_j : 1 \le i < j \le n\}$, $\Gamma = \{\theta_{2i-1} + \theta_{2i} : 1 \le i \le \frac{n}{2}\}$ and $\gamma = \theta_1 + \theta_2 + \ldots + \theta_n$.

Let $G_s = \mathbf{SU}(n) \times \mathbf{SU}(n)$, π_λ : $\underset{1}{\circ}$—···—○ ⊗ $\underset{1}{\circ}$—···—○. Then L_0 is $\mathbf{SU}(2n)$, $\Delta_0^+ = \{\theta_i - \theta_j : 1 \le i < j \le n$ or $n+1 \le i < j \le 2n\}$, $\Phi = \{\theta_i - \theta_j : 1 \le i \le n$ and $n+1 \le j \le 2n\}$, $\Gamma = \{\theta_i - \theta_{2n+1-i} : 1 \le i \le n\}$ and $\gamma = \theta_1 + \theta_2 + \ldots + \theta_n - \theta_{n+1} - \theta_{n+2} - \ldots - \theta_{2n}$.

Let $G_s = \mathbf{E}_6$, π_λ :  . Then $L_0 = \mathbf{E}_7$,

$$\Delta_0^+ = \{\pm\theta_i + \theta_j : 1 \le i < j \le 5\} \cup \{\frac{1}{2}(\theta_8 - \theta_7 - \theta_6 + \sum_{i=1}^{5}\epsilon_i\theta_i) : \Pi_{i=1}^{5}\epsilon_i = +1\},$$

$$\Phi = \{\theta_8 - \theta_7\} \cup \{\pm\theta_i + \theta_6 : 1 \le i \le 5\} \cup \{\frac{1}{2}(\theta_8 - \theta_7 + \theta_6 + \sum_{i=1}^{5}\epsilon_i\theta_i) : \Pi_{i=1}^{5}\epsilon_i = -1\},$$

$\Gamma = \{\theta_8 - \theta_7, \theta_6 + \theta_5, \theta_6 - \theta_5\}$ and $\gamma = \theta_8 - \theta_7 + 2\theta_6$.

It is immediate to see that (5.9) holds in each case, and this completes the proof of (b). □

Proof of Proposition 5.2. We know from Proposition 4.9 and Lemma 5.3(c) that $k(\lambda) = 1, 2$. In particular, G_s can have at most two simple factors.

Suppose $k(\lambda) = 1$. This implies G_s is simple. In the case $G = G_s$, we have π is of complex type, so $G = \mathbf{SU}(n)$ and π is the vector representation. In the case $G = S^1 \times G_s$, we have π_λ is of quaternionic or of complex type, so $G_s = \mathbf{Sp}(n)$ or $G_s = \mathbf{SU}(n)$ and π_λ is the respective vector representation. This only gives representations with cohomogeneity one.

Next suppose $k(\lambda) = 2$. The first case is $G_s = G_1 \times G_2$, where G_i is a simple group. Here π_λ decomposes as an outer tensor product $\pi_{\mu_1} \otimes \pi_{\mu_2}$, where π_{μ_i} is the representation of G_i of highest weight μ_i with $k(\mu_i) = 1$. This forces each G_i to be either $\mathbf{Sp}(n)$ or $\mathbf{SU}(n)$ for some n, and π_{μ_i} to be the corresponding vector representation. Using Lemmas 5.3(b) and 5.4 and eliminating the representations orbit equivalent to the isotropy representation of a symmetric space, we get the first two representations in the table.

The second case is G_s is a simple group. For $G = G_s$, an inspection of Table A.1 reveals four possibilities for π_λ which are all either discarded by Lemma 5.4 or orbit equivalent to the isotropy representation of a symmetric space.

For $G = S^1 \times G_s$, if π_λ is of complex type, we only get representations associated to symmetric spaces, and if π_λ is of real type, Lemma 5.3(a) gives the last representation in our table, for the other possibilities are associated to symmetric spaces. □

5.3 The case where ρ is of real type

The real case is much more involved than the previous two. We know from Proposition 4.9 that $k(\lambda) = 2, 4$. In particular, G can have at most four simple factors.

As a first step towards the classification in the case ρ is of real type, we introduce the following variations of condition (C_2) of Lemma 4.8 for a representation π_λ with representation space V_λ and highest weight vector v_λ:

$$
\begin{aligned}
\mathcal{U}^2(\mathfrak{g}^c)v_\lambda + \mathcal{U}^1(\mathfrak{g}^c)\epsilon(v_\lambda) &= V_\lambda \qquad (C_{1\frac{1}{2}}) \\
\mathcal{U}^1(\mathfrak{g}^c)v_\lambda + \mathcal{U}^2(\mathfrak{g}^c)\epsilon(v_\lambda) &= V_\lambda \qquad (C'_{1\frac{1}{2}}) \\
\mathcal{U}^1(\mathfrak{g}^c)v_\lambda + \mathcal{U}^1(\mathfrak{g}^c)\epsilon(v_\lambda) &= V_\lambda \qquad (C_1) \\
\mathcal{U}^1(\mathfrak{g}^c)v_\lambda + \mathbf{C}\epsilon(v_\lambda) &= V_\lambda \qquad (C_{\frac{1}{2}}) \\
\mathbf{C}v_\lambda + \mathcal{U}^1(\mathfrak{g}^c)\epsilon(v_\lambda) &= V_\lambda \qquad (C'_{\frac{1}{2}})
\end{aligned}
$$

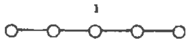
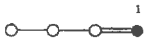
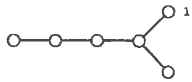
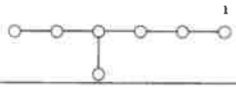
It follows from the identity $\epsilon X_\alpha = X_{-\alpha}\epsilon$ that conditions $(C_{1\frac{1}{2}})$ and $(C'_{1\frac{1}{2}})$ are equivalent, and that conditions $(C_{\frac{1}{2}})$ and $(C'_{\frac{1}{2}})$ are equivalent.

5.10 Lemma *The self-dual complex irreducible representations of compact connected* simple *Lie groups satisfying condition* $(C_{\frac{1}{2}})$ *are precisely*[2]*: the vector representation of* $\mathrm{SO}(m)$ *for* $m \neq 2, 4$, *the vector representation of* $\mathrm{Sp}(m)$, *the 7-dimensional representation of* G_2, *and the spin representation of* $\mathrm{Spin}(7)$.

We will prove Lemma 5.10 together with the next lemma.

[2]The half-spin representations of Spin(8) are image equivalent to the vector representation.

5.11 Lemma *The self-dual complex irreducible representations of compact connected* simple *Lie groups satisfying condition* (C_1) *are,* besides those listed in Lemma 5.10, *precisely the following:*

G	π_λ	$k(\lambda)$
$\mathbf{SU}(2)$		3
$\mathbf{SU}(6)$		3
$\mathbf{Spin}(9)$		2
$\mathbf{Spin}(11)$		3
$\mathbf{Spin}(12)$		3
$\mathbf{Sp}(3)$		3
$\mathbf{E}_7$		3

Proof of Lemmas 5.10 and 5.11. It is obvious that condition $(C_{\frac{1}{2}})$ implies condition (C_1). So we consider (C_1). It implies the following:

(a) $k(\lambda) \leq 3$.

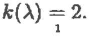
(b) The dimension of π_λ is bounded by $\dim \mathfrak{g}^c - \operatorname{rank} \mathfrak{g}^c + 2$.

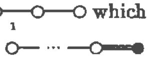
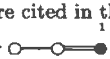
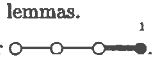
(c) If 0 is a weight, then its multiplicity is one. In particular, if π_λ is the adjoint representation, then the rank is at most one.

In fact, (a) follows from the fact that $k(\lambda) \geq 4$ contradicts Proposition 4.4. For (b) notice that the dimension of the left hand side of (C_1) is bounded by 2(number of positive roots) + 2. Finally, if 0 is a weight, then the 0-weight space is contained in $\mathcal{U}^1(\mathfrak{g}^c)v_\lambda$ (because $\epsilon X_\alpha = X_{-\alpha}\epsilon$), and from this we see that λ is a root and the 0-weight space is in fact spanned by $X_{-\lambda}v_\lambda$, which implies (c).

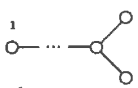
We will use the above observations to complete the proof of the lemmas.

Let π_λ be of real type. Then $k(\lambda) = 2$. The adjoint representations of $\mathbf{A}_n$, $\mathbf{B}_n$, $\mathbf{C}_n$, $\mathbf{D}_n$ with $n \geq 2$, and of $\mathbf{G}_2$, $\mathbf{F}_4$, $\mathbf{E}_6$, $\mathbf{E}_7$, $\mathbf{E}_8$ are immediately excluded. We run through all the remaining cases of representations of real type with $k(\lambda) = 2$.

Let G be of $\mathbf{A}_n$-type. Then π_λ is or  which are cited in the lemmas.

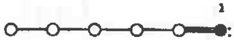
Let G be of $\mathbf{B}_n$-type, $n \geq 2$. Then π_λ is  or  or . All of them are cited in the lemmas.

Let G be of $\mathbf{C}_n$-type, $n \geq 3$. Then π_λ is , which cannot occur because the multiplicity of the 0-weight is $n - 1$.

Let G be of $\mathbf{D}_n$-type, $n \geq 4$. Then π_λ is , which is cited in the lemmas.

Let G be of exceptional type. Then π_λ is  or . Only the former is cited in the lemmas; the latter has 0 as a weight of multiplicity 2.

Now let π_λ be of quaternionic type. Then $k(\lambda) = 1$ or $k(\lambda) = 3$. The case $k(\lambda) = 1$ gives the vector representation of $\mathbf{Sp}(m)$, which is cited in the lemmas. In the case $k(\lambda) = 3$, the only possibilities not cited in the lemmas are the following four representations, which we next eliminate:

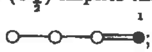
- $G = \mathbf{B}_6$, π_λ : : here $\lambda = \frac{1}{2}(\theta_1 + \theta_2 + \theta_3 + \theta_4 + \theta_5 + \theta_6)$, $\mu = \frac{1}{2}(\theta_1 + \theta_2 + \theta_3 - \theta_4 - \theta_5 - \theta_6)$ is a weight and $\lambda \pm \mu$ are not roots which violates condition (C_1).

In the remaining cases we make use of the bound on the dimension of the representation (see e. g. Table 5 in the Reference Chapter of [25]):

- $G = \mathbf{C}_n$, π_λ : : $\dim \pi_\lambda = \frac{2}{3}n(n+1)(2n+1) > 2n^2 + 2 = \dim \mathbf{C}_n - n + 2$, for $n \geq 2$;

- $G = \mathbf{C}_n$, π_λ : : $\dim \pi_\lambda = \frac{8}{3}n(n-1)(n+1) > 2n^2 + 2 = \dim \mathbf{C}_n - n + 2$, for $n \geq 2$;

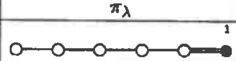
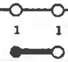
- $G = \mathbf{C}_n$, π_λ : : $\dim \pi_\lambda = \frac{2}{3}n(n-2)(2n+1) > 2n^2 + 2 = \dim \mathbf{C}_n - n + 2$, for $n \geq 4$;

On the other hand, the representations listed in the statements of the Lemmas 5.10 and 5.11 do satisfy (C_1), as can be easily checked using the fact that each of them has all weights with multiplicities one. This concludes the classification of self-dual representations that satisfy (C_1). As for condition $(C_{\frac{1}{2}})$, we observe that $(C_{\frac{1}{2}})$ implies that $k(\lambda) \leq 2$. This eliminates all representations in the table of Lemma 5.11 except ; that one can be eliminated by a direct check. Finally it is easy to verify that the representations cited in Lemma 5.10 do satisfy $(C_{\frac{1}{2}})$. □

5.12 Lemma *The only complex irreducible representation of real type of a compact connected semisimple* nonsimple *Lie group G satisfying condition (C_1) is the tensor product of the vector representations of* $\mathrm{Sp}(1)$ *and* $\mathrm{Sp}(m)$.

Proof. We already know that $k(\lambda) = 2$ (cf. proof of Lemmas 5.10 and 5.11), so $G = G_1 \times G_2$, where G_i is a simple group, and $\pi_\lambda = \pi_{\mu_1} \otimes \pi_{\mu_2}$ with $k(\mu_i) = 1$. Since each π_{μ_i} is self-dual, it must be of quaternionic type. Therefore, each $G_i = \mathrm{Sp}(n_i)$ and the corresponding π_{μ_i} is just the vector representation. Now one of the n_i must be 1, for otherwise condition (C_1) for π_λ is violated by consideration of a weight vector of type $v_1 \otimes v_2$, where v_i is an *intermediate* (that is, neither highest nor lowest) weight vector for π_{μ_i}. □

5.13 Lemma *The complex irreducible representations π_λ of quaternionic type of compact connected simple Lie groups G satisfying $k(\lambda) = 3$ and condition $(C_{1\frac{1}{2}})$,* but not condition (C_1), *are the following:*

G	π_λ
Spin(13)	
Sp(2)	

We postpone the proof of Lemma 5.13 to the end of Subsection 5.3.5, since the methods used to prove it better belong there.

5.3.1 The case where G has four simple factors

In this case we have:

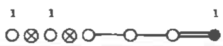
5.14 Proposition *Let ρ be of real type and $G = G_1 \times G_2 \times G_3 \times G_4$, where G_i is a simple group. Then $G = \mathbf{SU}(2) \times \mathbf{SU}(2) \times \mathbf{SU}(2) \times \mathbf{Sp}(n)$, $n \geq 2$, and π is the tensor product of the vector representations of each of the factors.*

Proof. Set $\pi_\lambda = \pi_{\mu_1} \otimes \pi_{\mu_2} \otimes \pi_{\mu_3} \otimes \pi_{\mu_4}$, where π_{μ_i} is the representation of G_i of highest weight μ_i. We have $k(\mu_i) = 1$ for all i, and since each π_{μ_i} is self-dual, it must be of quaternionic type. Therefore, each $G_i = \mathbf{Sp}(n_i)$ and the corresponding π_{μ_i} is just the vector representation. Notice that $n_i = 1$ for all i gives the isotropy representation of a symmetric space. Suppose now that for two indices, say 3 and 4, we have $n_3 > 1$ and $n_4 > 1$. Take weight vectors v_i of π_{μ_i} such that: v_1 is the highest weight vector of π_{μ_1}, v_2 is the lowest weight vector of π_{μ_2}, and v_3 and v_4 are intermediate weight vectors of π_{μ_3} and π_{μ_4}, respectively. Then $v_1 \otimes v_2 \otimes v_3 \otimes v_4$ is a weight vector of π_λ which cannot be contained in $\mathcal{U}^2(\mathfrak{g}^c)v_\lambda + \mathcal{U}^2(\mathfrak{g}^c)\epsilon(v_\lambda)$, violating the condition (C_2) from Lemma 4.8. □

5.3.2 The case where G has three simple factors

The classification is:

5.15 Proposition *Let ρ be of real type and $G = G_1 \times G_2 \times G_3$, where G_i is a simple group. Then ρ is one of the following:*

G	ρ	*Conditions*
$\mathbf{SU}(2) \times \mathbf{Sp}(n) \times \mathbf{SO}(2m)$		$m \geq 3, \quad n \geq 2$
$\mathbf{SU}(2) \times \mathbf{Sp}(n) \times \mathbf{SO}(2m+1)$		$n \geq 2$
$\mathbf{SU}(2) \times \mathbf{Sp}(n) \times \mathbf{G}_2$		–
$\mathbf{SU}(2) \times \mathbf{Sp}(n) \times \mathbf{Spin}(7)$		–
$\mathbf{SU}(2) \times \mathbf{SU}(2) \times \mathbf{Spin}(9)$		–

Proof. Let $\pi_\lambda = \pi_{\mu_1} \otimes \pi_{\mu_2} \otimes \pi_{\mu_3}$, where π_{μ_i} is the representation of G_i of highest weight μ_i, say with $k(\mu_1) = k(\mu_2) = 1$, $k(\mu_3) = 2$. Then π_{μ_1}, π_{μ_2} are each of quaternionic type and π_{μ_3} is of real type. So each of π_{μ_1}, π_{μ_2} is the vector representation of $\mathbf{Sp}(m)$ for some m. Moreover, one of them must have $m = 1$, say π_{μ_1}, for otherwise condition (C_2) for π_λ is violated by consideration of a weight vector of type $v_1 \otimes v_2 \otimes v_3$, where v_i is an intermediate weight vector for π_{μ_i}. Now we have $G_1 = \mathbf{SU}(2)$, $G_2 = \mathbf{Sp}(n)$, and π_{μ_1}, π_{μ_2} the respective vector representations. Next take a weight vector $v_1 \otimes v_2 \otimes v_3$ for π_λ such that v_1 is highest, v_2 is lowest and v_3 is intermediate. Condition (C_2) for π_λ forces π_{μ_3} to satisfy condition (C_1). If $n \geq 2$, we could also take v_2 to be intermediate, and then condition (C_2) for π_λ implies something stronger, namely that π_{μ_3} must satisfy condition $(C_{\frac{1}{2}})$. Our table now follows from Lemmas 5.10 and 5.11. □

5.3.3 The case where G has two simple factors and ρ has two factors of real type

Here the result is:

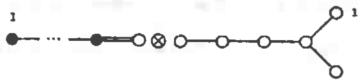
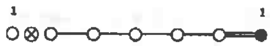
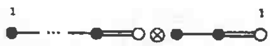
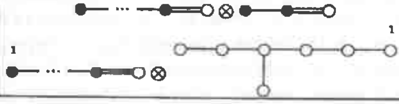
5.16 Proposition *Let ρ be of real type and $G = G_1 \times G_2$, where G_i is a simple group. Write $\rho^c = \pi_\lambda$, decompose $\pi_\lambda = \pi_{\mu_1} \otimes \pi_{\mu_2}$ and suppose that each π_{μ_i} is of real type. Then each π_{μ_i} is one of the following representations: the vector representation of* $\mathbf{SO}(m)$ *for $m \neq 2$, 4, or the 7-dimensional representation of* $\mathbf{G}_2$*, or the spin representation of* $\mathbf{Spin}(7)$*, or the spin representation of* $\mathbf{Spin}(9)$*; but the cases where $G = \mathbf{SO}(m_1) \times \mathbf{SO}(m_2)$, $\mathbf{SO}(3) \times \mathbf{Spin}(7)$ or $\mathbf{Spin}(9) \times \mathbf{Spin}(9)$ are excluded.*

Proof. Here $k(\mu_1) = k(\mu_2) = 2$. Let $v_1 \otimes v_2$ be a weight vector for π_λ such that each v_i is an intermediate weight vector for π_{μ_i}. Condition (C_2) for π_λ applied to $v_1 \otimes v_2$ forces each π_{μ_i} to satisfy (C_1). In fact, one of the π_{μ_i} must even satisfy $(C_{\frac{1}{2}})$. The claim follows from Lemmas 5.10 and 5.11. Notice that we exclude $\mathbf{SO}(m_1) \times \mathbf{SO}(m_2)$ since it is the isotropy representation of a symmetric space and $\mathbf{SO}(3) \times \mathbf{Spin}(7)$ since it is orbit equivalent to such a representation. □

5.3.4 The case where G has two simple factors and ρ has two factors of quaternionic type

The classification is:

5.17 Proposition *Let ρ be of real type and $G = G_1 \times G_2$, where G_i is a simple group. Write $\rho^c = \pi_\lambda$, decompose $\pi_\lambda = \pi_{\mu_1} \otimes \pi_{\mu_2}$ and suppose that each π_{μ_i} is of quaternionic type. Then ρ is one of the following:*

G	ρ	*Conditions*
$\mathbf{Sp}(n) \times \mathbf{SU}(2)$		$n \geq 2$
$\mathbf{Sp}(n) \times \mathbf{SU}(6)$		$n \geq 2$
$\mathbf{Sp}(n) \times \mathbf{Spin}(11)$		–
$\mathbf{Sp}(n) \times \mathbf{Spin}(12)$		$n \geq 2$
$\mathbf{Sp}(1) \times \mathbf{Spin}(13)$		–
$\mathbf{Sp}(1) \times \mathbf{Sp}(2)$		–
$\mathbf{Sp}(n) \times \mathbf{Sp}(3)$		$n \geq 2$
$\mathbf{Sp}(n) \times \mathbf{E}_7$		$n \geq 2$

Proof. If $k(\lambda) = 2$, then $k(\mu_1) = k(\mu_2) = 1$ and each π_{μ_i} must be the vector representation of $\mathbf{Sp}(n_i)$ for some n_i, so we get a representation associated to a symmetric space. In the case $k(\lambda) = 4$ we can write $k(\mu_1) = 1$ and $k(\mu_2) = 3$. Now π_{μ_1} must be the vector representation of $\mathbf{Sp}(n)$ and there are two cases to consider:

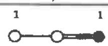
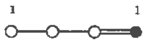
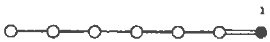
- $n = 1$: here π_{μ_2} must satisfy condition $(C_{1\frac{1}{2}})$;
- $n \geq 2$: here π_{μ_2} must satisfy the stronger condition (C_1).

The claim now follows from Lemmas 5.11 and 5.13. □

5.3.5 The case where G is a simple group

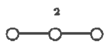
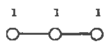
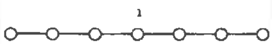
The representations of real type of simple groups that are candidates to a position in class $\mathcal{O}^2$ are:

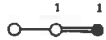
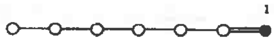
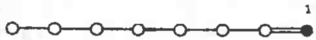
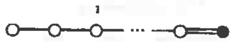
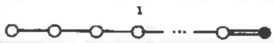
5.18 Proposition *Let ρ be of real type and suppose G is a simple group. Then ρ is one of the following:*

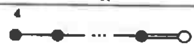
G	ρ
Spin(7)	
Spin(9)	
Spin(15)	
Spin(17)	

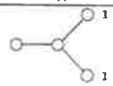
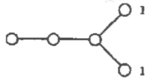
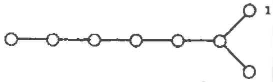
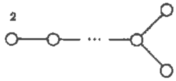
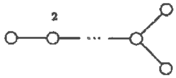
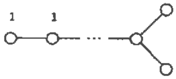
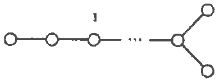
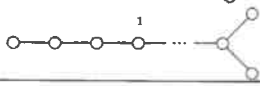
We will accomplish the proof of the above proposition by running through all the cases of representations of real type of a simple group with $k(\lambda) \leq 4$ given in Table A.1. In fact, according to that table, all representations with $k(\lambda) = 2$ are orbit equivalent to isotropy representations of symmetric spaces. So we only need to worry about the case $k(\lambda) = 4$. A careful inspection of Table A.1 shows:

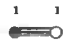
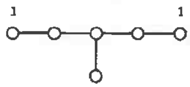
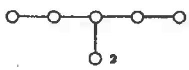
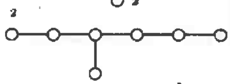
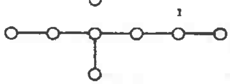
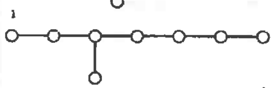
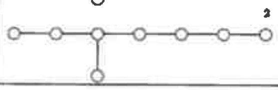
5.19 Lemma *Let π_λ be a representation of real type of a compact connected simple Lie group G with $k(\lambda) = 4$. Then (G, π_λ) is one of the following:*

	G	π_λ	*Conditions*
1	$\mathbf{A}_1$		–
2	$\mathbf{A}_3$		–
3	$\mathbf{A}_3$		–
4	$\mathbf{A}_7$		–
5	$\mathbf{A}_n$		$n \geq 2$
6	$\mathbf{A}_n$		$n \geq 4$

	G	π_λ	*Conditions*
7	$\mathbf{B}_3$		–
8	$\mathbf{B}_3$		–
9	$\mathbf{B}_3$		–
10	$\mathbf{B}_4$		–
11	$\mathbf{B}_4$		–
12	$\mathbf{B}_4$		–
13	$\mathbf{B}_7$		–
14	$\mathbf{B}_8$		–
15	$\mathbf{B}_n$		$n \geq 2$
16	$\mathbf{B}_n$		$n \geq 3$
17	$\mathbf{B}_n$		$n \geq 3$
18	$\mathbf{B}_n$		$n \geq 4$
19	$\mathbf{B}_n$		$n \geq 5$

	G	π_λ	*Conditions*
20	$\mathbf{C}_n$		$n \geq 2$
21	$\mathbf{C}_n$		$n \geq 2$
22	$\mathbf{C}_n$		$n \geq 3$
23	$\mathbf{C}_n$		$n \geq 3$
24	$\mathbf{C}_n$		$n \geq 4$

	G	π_λ	*Conditions*
25	$\mathbf{D}_4$		–
26	$\mathbf{D}_5$		–
27	$\mathbf{D}_8$		–
28	$\mathbf{D}_n$		$n \geq 4$
29	$\mathbf{D}_n$		$n \geq 4$
30	$\mathbf{D}_n$		$n \geq 4$
31	$\mathbf{D}_n$		$n \geq 5$
32	$\mathbf{D}_n$		$n \geq 6$

	G	π_λ
33	G_2	
34	G_2	
35	G_2	
36	F_4	
37	F_4	
38	F_4	
39	F_4	
40	E_6	
41	E_6	
42	E_7	
43	E_7	
44	E_8	
45	E_8	

To begin with, we note the representations listed in the tables of Lemma 5.19 associated to a symmetric space, namely, numbers 1, 2, 4, 15, 24 ($n = 4$), 27 and 28.

5.20 Lemma *Let π_λ be a representation of real type with $k(\lambda) = 4$. Assume that $\lambda = \frac{1}{2}(3\beta_i + \beta_j)$ where β_i, $\beta_j \in B$ are distinct long roots and that there exists a Weyl group involution s such that $s\beta_i = \beta_j$. Then a real form of π_λ is not of class $\mathcal{O}^2$.*

Proof. We have that $\mu = ss_{\beta_i}(\lambda) = \frac{1}{2}(\beta_i - 3\beta_j)$ is a weight and $\lambda - \mu = \beta_i + 2\beta_j$, $\lambda + \mu = 2\beta_i - \beta_j$. Since β_i and β_j are orthogonal, we get that $||\lambda \pm \mu||^2 = 5$ (normalizing the length of a long root to be 1). On the other hand, for α', $\alpha'' \in \Delta$ we have that $||\alpha' + \alpha''||^2 \le (||\alpha'|| + ||\alpha''||)^2 \le 4$, so $\lambda \pm \mu$ is neither a root nor a sum of two roots. Thus a μ-weight vector is not in $\mathcal{U}^2(\mathfrak{g}^c)v_\lambda + \mathcal{U}^2(\mathfrak{g}^c)\epsilon(v_\lambda)$, violating condition (C_2). □

One can use the last lemma together with the tables in [4] to get rid of the representations listed in Lemma 5.19 under the numbers 3, 12, 17, 21, 30 and 37. The following lemma is similar in spirit.

5.21 Lemma *Representation number 9 is not of class $\mathcal{O}^2$.*

Proof. The highest weight is $\lambda = \frac{1}{2}(3\theta_1 + 3\theta_2 + \theta_3)$. Now $\mu = \frac{1}{2}(3\theta_1 - 3\theta_2 + \theta_3)$ is a weight and $\lambda + \mu = 3\theta_1 + \theta_3$, $\lambda - \mu = 3\theta_2$ are neither a root nor a sum of two roots, so we are done as in the proof of Lemma 5.20. $\square$

We next eliminate the representations π_λ with $\lambda = 2(\text{highest root}) = 2\beta_1 = \frac{1}{2}(4\beta_1)$, using the following observations.

5.22 Lemma *Let π_λ be a representation of real type with $\lambda = 2\beta_1$ and assume there is a nonzero weight μ such that $(\lambda, \mu) = 0$. Then a real form of π_λ is not of class $\mathcal{O}^2$.*

Proof. We check that condition (C_2) is violated. In fact, $||\lambda \pm \mu||^2 = ||\lambda|| + ||\mu||^2 > 4||\beta_1||^2 = 4$ (normalizing the length of a long root to be 1), but α', $\alpha'' \in \Delta$ implies that $||\alpha' + \alpha''||^2 \leq (||\alpha'|| + ||\alpha''||)^2 \leq 4$, and we are done as in the proof of Lemma 5.20. $\square$

5.23 Corollary *Let π_λ be a representation of real type with $\lambda = 2\beta_1$. If there is a long root $\beta_j \in B$ with $\beta_j \neq \beta_1$, then a real form of π_λ is not of class $\mathcal{O}^2$.*

Proof. For in this case $-2\beta_1$, $-\beta_1$, 0, β_1, $2\beta_1$ is the maximal β_1-string through $2\beta_1$, so β_1 is a weight, β_j is a weight, too (β_1 and β_j are long roots, hence in the same Weyl orbit, see e. g. [4], Ch. VI, § 1, no. 1.3, Proposition 11) and $(\beta_1, \beta_j) = 0$. $\square$

We use Corollary 5.23 to discard the representations listed in the tables of Lemma 5.19 under the numbers 5 ($n \geq 3$), 16, 20, 29, 36, 41, 42 and 45. And we use Lemma 5.22 with $\mu = -\alpha_1 + \alpha_2$ (resp. $\mu = \beta_2 = \alpha_1$) to eliminate representation number 5 for $n = 2$ (resp. number 34). There still remain 19 representations to be eliminated in the tables, namely numbers 6, 7, 10, 18, 19, 22, 23, 24 ($n \geq 5$), 25, 26, 31, 32, 33, 35, 38, 39, 40, 43 and 44, which in the following we go on to analyze case by case, but before that we want to reformulate an argument that has already been used and that will frequently be used in the sequel.

5.24 Lemma *Let π_λ be a representation of real type and suppose that its highest weight λ can be written as a root or as a sum of two roots in N different ways (where the order of the summands is not important). If 0 is a weight and its multiplicity is bigger than N, then a real form of π_λ is not of class $\mathcal{O}^2$.*

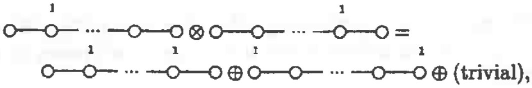
Proof. Write v_λ for a highest weight vector of π_λ. If condition (C_2) is satisfied, then the 0-weight space must be contained in $\mathcal{U}^2(\mathfrak{g}^c)v_\lambda + \mathcal{U}^2(\mathfrak{g}^c)\epsilon(v_\lambda)$. The identity $\epsilon X_\alpha = X_{-\alpha}\epsilon$ shows that the 0-weight space must in fact be contained in $\mathcal{U}^2(\mathfrak{g}^c)v_\lambda$. But the complex dimension of the intersection of the 0-weight space with $\mathcal{U}^2(\mathfrak{g}^c)v_\lambda$ is at most N. $\square$

5.25 Lemma *Representation number 6 is not of class $\mathcal{O}^2$.*

Proof. We label the simple roots of $\mathbf{A}_n$ as $\overset{\alpha_1}{\circ}\!-\!\overset{\alpha_2}{\circ}\!-\cdots-\!\overset{\alpha_n}{\circ}$, where $\alpha_i = \theta_i - \theta_{i+1}$. Then the root system is $\Delta = \{\pm(\theta_i - \theta_j) : 1 \leq i < j \leq n+1\}$ and the highest weight of $\circ\!-\!\underset{1}{\circ}\!-\cdots-\!\underset{1}{\circ}\!-\!\circ$ is $\lambda = \lambda_2 + \lambda_{n-1} = \theta_1 + \theta_2 - \theta_n - \theta_{n+1}$, so it can decomposed as a sum of two roots in exactly two ways:

$$\begin{aligned} \lambda &= (\theta_1 - \theta_{n+1}) + (\theta_2 - \theta_n) \\ &= (\theta_1 - \theta_n) + (\theta_2 - \theta_{n+1}). \end{aligned}$$

On the other hand, we find that:



from where we deduce that the multiplicity of the 0-weight in π_λ is $\frac{n(n+1)}{2} - n - 1 = \frac{n^2-n-2}{2} > 2$, so we can apply Lemma 5.24. □

Similarly as was done in Lemma 5.25, Lemma 5.24 can be used to show that representations numbers 10, 18, 19, 22, 23, 24 (if $n \geq 5$), 26, 32, 35, 38, 39, 40 and 43 are not of class $\mathcal{O}^2$ by computing for each one of them the multiplicity of 0 as a weight.

5.26 Lemma *Representation number* 44 *is not of class* $\mathcal{O}^2$.

Proof. Here we slightly change the argument used in the previous lemmas. The highest weight $\lambda = 2\theta_8$ and $(\lambda, \theta_8 - \theta_7) \neq 0$, so $\nu = \lambda - (\theta_8 - \theta_7) = \theta_8 + \theta_7$ is a weight. Now $\theta_8 + \theta_7$ and $\theta_6 + \theta_5$ are long roots of $\mathbf{E}_8$, hence in the same Weyl orbit. It follows that $\mu = \theta_6 + \theta_5$ is a weight, too. Observe that each of $\lambda \pm \mu$ is not a root and can be decomposed as a sum of two roots in only one way:

$$\begin{aligned} \lambda + \mu &= (\theta_8 + \theta_6) + (\theta_8 + \theta_5), \\ \lambda - \mu &= (\theta_8 - \theta_6) + (\theta_8 - \theta_5). \end{aligned}$$

The proof will be complete if we show that the multiplicity of μ as a weight is greater than 2, since then we will have that condition $(\mathcal{C}_2)$ is violated.

In fact, the multiplicity of μ is the same as the multiplicity of ν. We use Freudenthal's formula as it is stated in [13], namely, the multiplicity of ν as a weight of π_λ is given by the following formula:

$$m_\nu = \frac{2}{c(\nu)} \sum_{\alpha\in\Delta^+} \sum_{k\geq 1} (\nu + k\alpha, \alpha) m_{\nu+k\alpha},$$

where $c(\nu) = ||\lambda + \rho||^2 - ||\nu + \rho||^2$, $\rho = \frac{1}{2}\sum_{\alpha\in\Delta^+} \alpha$ and $m_{\nu+k\alpha}$ is the multiplicity of $\nu + k\alpha$ as a weight of π_λ.

We content ourselves with an estimate. Using that $\rho = \theta_2 + 2\theta_3 + 3\theta_4 + 4\theta_5 + 5\theta_6 + 6\theta_7 + 23\theta_8$, we compute that $c(\nu) = 36||\theta_1||^2$. Since $\nu > 0$, all terms in Freudenthal's formula are nonnegative. Consider the roots $\alpha = \frac{1}{2}(\theta_8 - \theta_7 + \sum_{i=1}^6 \epsilon_i\theta_i)$, $\beta = \frac{1}{2}(\theta_8 - \theta_7 - \sum_{i=1}^6 \epsilon_i\theta_i)$, where $\Pi_{i=1}^6 \epsilon_i = -1$. Note that $\alpha + \beta = \theta_8 - \theta_7$. Since $(\lambda, \alpha) \neq 0$, we have that $\lambda - \alpha$ is a weight. Now we have two strings of weights starting at ν, namely

$$\begin{aligned} &\nu, \quad \nu + (\theta_8 - \theta_7) = \lambda \qquad \text{and} \\ &\nu, \quad \nu + \beta = \lambda - \alpha. \end{aligned}$$

Note that in fact there are 32 different possible choices of the signs $\epsilon_i = \pm 1$, each of which gives rise to a different string of weights of the type ν, $\nu + \beta$. Freudenthal's formula gives

$$\begin{aligned} m_\nu &\geq \frac{2}{36||\theta_1||^2}((\lambda, \theta_8 - \theta_7)m_\lambda + 32(\lambda - \alpha, \beta)m_{\lambda-\alpha}) \\ &= \frac{1}{18}(2 \cdot 1 + 32 \cdot 2 \cdot 1) = \frac{33}{9} > 3, \end{aligned}$$

so $m_\nu \geq 4$ and we are done. □

5.27 Lemma *Representations numbers 25 and 31 are not of class $\mathcal{O}^2$.*

Proof. Here we will use the sharper version of condition (C_2) given by Lemma 4.8, item (a). These representations are just the third exterior power of the vector representation of $\mathbf{SO}(2n)$ for $n \geq 4$. The vector representation has weights $\pm\theta_i$, $1 \leq i \leq n$; let $\{e_i, e_{n+i} : 1 \leq i \leq n\}$ be a basis of weight vectors such that e_i (resp., e_{n+i}) corresponds to the θ_i-weight (resp., $-\theta_i$-weight) weight and $\epsilon(e_i) = e_{n+i}$, where ϵ is a invariant real structure on the representation space.

We have that $v_\mu = e_1 \wedge e_3 \wedge e_{n+3}$ is a $(\mu = \theta_1)$-weight vector for π_λ and $\epsilon(v_\mu) = e_{n+1} \wedge e_{n+3} \wedge e_3$. Now one can easily show that condition (a) of Lemma 4.8 is violated by proving that the θ_2-weight vector $e_2 \wedge e_4 \wedge e_{n+4}$ is not in the complex span of

$$v_\mu, \quad \epsilon(v_\mu), \quad X_\alpha v_\mu, \quad X_\alpha \epsilon(v_\mu), \quad X_\beta X_\alpha(v_\mu), \quad X_\beta X_\alpha(\epsilon(v_\mu)),$$

for α, $\beta \in \Delta$. □

5.28 Lemma *Representation number 7 is not of class $\mathcal{O}^2$.*

Proof. We use an argument somewhat similar to the one used in the previous lemma. The representation is $\pi_\lambda : \Lambda^3(\overset{1}{\circ}\!\!-\!\!\circ\!\!=\!\!\bullet)$, where $\overset{1}{\circ}\!\!-\!\!\circ\!\!=\!\!\bullet$ is the vector representation. Let e_1, e_2, e_3, $e_4 = \epsilon(e_1)$, $e_5 = \epsilon(e_2)$, $e_6 = \epsilon(e_3)$, $e_7 = \epsilon(e_7)$ be weight vectors of the vector representation corresponding to the weights θ_1, θ_2, θ_3, $-\theta_1$, $-\theta_2$, $-\theta_3$, 0, respectively, where ϵ is the invariant real structure on the representation space.

We have that $v_\mu = e_1 \wedge e_4 \wedge e_7$ is a $(\mu = 0)$-weight vector for π_λ and $\epsilon(v_\mu) = -v_\mu$. Now condition (a) of Lemma 4.8 is violated because the θ_2-weight vector $e_2 \wedge e_3 \wedge e_6$ is not in the complex span of v_μ, $X_\alpha v_\mu$, $X_\beta X_\alpha(v_\mu)$, for α, $\beta \in \Delta$. □

5.29 Lemma *Representation number 33 is not of class $\mathcal{O}^2$.*

Proof. We start with a description of the Lie algebra of $\mathbf{G}_2$ and of its 7-dimensional representation, as it is done in [13]. Let α_1, α_2 be the simple roots. We label the other positive roots as $\alpha_3 = \alpha_1 + \alpha_2$, $\alpha_4 = 2\alpha_1 + \alpha_2$, $\alpha_5 = 3\alpha_1 + \alpha_2$, $\alpha_6 = 3\alpha_1 + 2\alpha_2$. Choose root vectors X_i, Y_i for α_i, $-\alpha_i$, respectively, $i = 1$, 2, such that $[H_i, X_i] = 2Y_i$, $[H_i, Y_i] = -2X_i$, where $H_i = [X_i, Y_i]$, $i = 1, 2$. Next define $X_3 = [X_1, X_2]$, $Y_3 = -[Y_1, Y_2]$, $X_4 = \frac{1}{2}[X_1, X_3]$, $Y_4 = -\frac{1}{2}[Y_1, Y_3]$, $X_5 = -\frac{1}{3}[X_1, X_4]$, $Y_5 = \frac{1}{3}[Y_1, Y_4]$, $X_6 = -[X_2, X_5]$, $Y_6 = [Y_2, Y_5]$, and $H_i = [X_i, Y_i]$ for $i = 1, \ldots, 6$. Then $[H_i, X_i] = 2Y_i$, $[H_i, Y_i] = -2X_i$ for $i = 3, \ldots, 6$, and X_i, Y_i are root vectors for α_i, $-\alpha_i$, respectively, $i = 1, \ldots, 6$.

The 7-dimensional representation of $\mathbf{G}_2$ has weights α_1, $-\alpha_1$, α_3, $-\alpha_3$, α_4, $-\alpha_4$, 0 with respective weight vectors v_1, w_1, v_3, w_3, v_4, w_4, u, such that the action of the basis vectors of the Lie algebra of $\mathbf{G}_2$ is described in the following table:

	H_1	H_2	X_1	X_2	X_3	X_4	X_5	X_6	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6
u	0	0	$2v_1$	0	$2v_3$	$2v_4$	0	0	$2w_1$	0	$2w_3$	$2w_4$	0	0
v_4	v_4	0	0	0	0	0	0	0	v_3	0	$-v_1$	u	w_1	w_3
v_1	$2v_1$	$-v_1$	0	$-v_3$	$-v_4$	0	0	0	u	0	0	w_3	$-w_4$	0
v_3	$-v_3$	v_3	v_4	0	0	0	0	0	0	$-v_1$	u	$-w_1$	0	$-w_4$
w_4	$-w_4$	0	$-w_3$	0	w_1	u	$-v_1$	$-v_3$	0	0	0	0	0	0
w_1	$-2w_1$	w_1	u	0	0	$-v_3$	v_4	0	0	w_3	w_4	0	0	0
w_3	w_3	$-w_3$	0	w_1	u	v_1	0	v_4	$-w_4$	0	0	0	0	0

Now we have the equation

$$S^2($$  $$) =$$  $$\oplus \text{(trivial)},$$

from which we learn three things. First, π_λ :  is a subrepresentation of the symmetric square of the 7-dimensional representation, so we can read it off the table above. Second, the multiplicity of the weights of π_λ; in particular, α_4 is a weight of multiplicity 2. And third, the trivial summand above is spanned by $-\frac{1}{2}u^2 + 2v_1w_1 + 2v_3w_3 + 2v_4w_4$, so that an invariant real structure on the representation space of the 7-dimensional representation is given by $\epsilon(v_i) = w_i$, $i = 1, 2, 3$ and $\epsilon(u) = -u$.

Let $v_\mu = 2u^2 - 8v_1w_1 + 6v_3w_3 + 6v_4w_4$, which is a real, ($\mu = 0$)-weight vector for π_λ. We next show that condition (a) of Lemma 4.8 is violated because the α_4-weight space is not contained in the complex span of

(5.30) $$v_\mu, \quad X_\alpha v_\mu, \quad X_\beta X_\alpha(v_\mu),$$

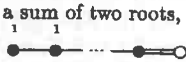
for α, $\beta \in \Delta$. In fact, $\alpha_4 - 0$ can be written as a root or as a sum of two roots as follows:

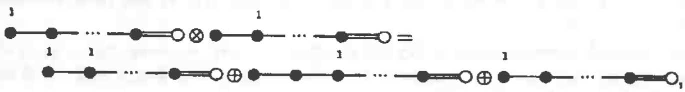
$$\begin{aligned} \alpha_4 - 0 &= 2\alpha_1 + \alpha_2 \\ &= \alpha_1 + (\alpha_1 + \alpha_2) \\ &= -\alpha_1 + (3\alpha_1 + \alpha_2) \\ &= (3\alpha_1 + 2\alpha_2) - (\alpha_1 + \alpha_2) \end{aligned}$$

Now applying the corresponding root vectors to v_μ the only nonzero vector we get is $X_4(v_\mu) = 14(uv_4 + v_1v_3)$. This shows that the intersection of the complex span of (5.30) with the α_4-weight space has dimension 1; but the weight space itself has dimension 2. □

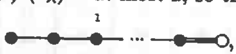
Proof of Lemma 5.13. We eliminate the possibilities given by Table A.1 and not listed in the table.

Let $G = \mathbf{C}_n$, π_λ : , $n \geq 2$. Then $\lambda = 3\theta_1$, $\mu = -3\theta_2$ is a weight and $\lambda \pm \mu$ are neither roots nor a sum of two roots, so π_λ does not even satisfy (C_2).

Let $G = \mathbf{C}_n$, π_λ : , $n \geq 3$. We have



from where we deduce that the multiplicity of $\mu = \theta_3$ as a weight of  is $2n-2 \geq 4$. Now $\lambda - \mu$ is not a root and can be written as a sum of two roots in two ways, and $\lambda + \mu = 2\theta_1 + \theta_2 + \theta_3$ is not a root. It follows that the dimension of the intersection of the μ-weight space with $\mathcal{U}^2(\mathfrak{g}^c)v_\lambda + \mathcal{U}^1(\mathfrak{g}^c)\epsilon(v_\lambda)$ is at most 2, so that condition $(C_{1\frac{1}{2}})$ is not satisfied.

Let $G = \mathbf{C}_n$, π_λ : , $n \geq 4$. Then $\lambda = \theta_1 + \theta_2 + \theta_3$ and this representation is a subrepresentation of the cubic exterior power of the vector representation. Also, $\mu = -\theta_1$ is a weight of multiplicity $n - 2 \geq 2$. But one can check by direct computation that the dimension of the intersection of the μ-weight space with $\mathcal{U}^2(\mathfrak{g}^c)v_\lambda + \mathcal{U}^1(\mathfrak{g}^c)\epsilon(v_\lambda)$ is 1, so condition $(C_{1\frac{1}{2}})$ is not satisfied. □

6 The main theorem

In this section we classify taut irreducible representations up to orbit equivalence. We first exclude many irreducible representations in class $\mathcal{O}^2$ from being taut by making use of Propositions 3.1 and 3.4 and a reduction principle. Then we only need to observe that the remaining representations are already known to be taut by the main result of [15] (see also Proposition 7.12 in [16]). We finally get the following theorem.

6.1 Theorem *A taut irreducible representation ρ of a compact connected Lie group G is either orbit equivalent to the isotropy representation of a symmetric space or it is one of the following orthogonal representations ($n \geq 2$):*

G	ρ
$\mathbf{SO}(2) \times \mathbf{Spin}(9)$	$(x \oplus x^{-1}) \otimes \circ - \circ - \circ \Rightarrow \overset{1}{\bullet}$
$\mathbf{U}(2) \times \mathbf{Sp}(n)$	$(x \oplus x^{-1}) \otimes \overset{1}{\circ} \otimes \overset{1}{\bullet} - \cdots - \bullet \Leftarrow \circ$
$\mathbf{SU}(2) \times \mathbf{Sp}(n)$	$\overset{3}{\circ} \otimes \overset{1}{\bullet} - \cdots - \bullet \Leftarrow \circ$

6.2 Corollary *A taut irreducible representation of a compact connected* simple *Lie group is orbit equivalent to the isotropy representation of a symmetric space.*

6.3 Remark According to [35], see also [32], the representations given in the table of Theorem 6.1 are precisely the representations of compact connected Lie groups which have cohomogeneity three and are not orbit equivalent to the isotropy representation of a symmetric space.

Recall that the irreducible representations which according to the results in Section 5 can belong to class $\mathcal{O}^2$ without being orbit equivalent to the isotropy representation of a symmetric space are listed in the tables of Propositions 5.1, 5.2, 5.14, 5.15, 5.16, 5.17 and 5.18. For the sake of convenience, these representations are rearranged in a more systematic way in Tables B.1, B.2, B.3 and B.4 in Appendix B. Now the proof of Theorem 6.1 follows from Propositions 6.4, 6.5, 6.6 below and the main result of [15].

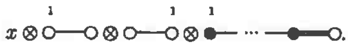
6.4 Proposition *The representations listed in Table B.2 are not taut.*

Proof. Let π_λ be a representation of the compact connected Lie group G on V_λ which is in the table. Notice that π_λ is of real type. Let ρ be a real form acting on $V \subset V_\lambda$ and let ϵ be the invariant real structure on V_λ. We shall go case by case and show that the slice representation of ρ at $p = v_\lambda + \epsilon(v_\lambda) \in V$, where v_λ is a highest weight vector for π_λ, is not taut, and then use Proposition 3.1 to conclude that ρ is not taut.

Let $G = \mathbf{Sp}(n) \times \mathbf{SU}(6)$. Then the isotropy subalgebra at p is $\mathfrak{u}(1) + \mathfrak{su}(3) + \mathfrak{su}(3) + \mathfrak{sp}(n-1)$, $n \geq 2$, and the complexified slice representation at p is given by:

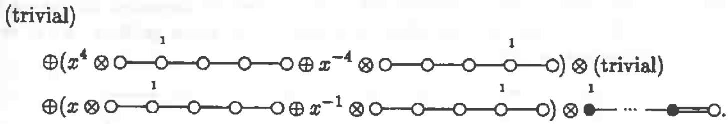
$$\begin{aligned}
&(\text{trivial})\\
&\oplus(x^4 \otimes \overset{1}{\circ} - \circ \otimes \circ - \overset{1}{\circ} \oplus x^{-4} \otimes \circ - \overset{1}{\circ} \otimes \overset{1}{\circ} - \circ) \otimes (\text{trivial})\\
&\oplus(x \otimes \overset{1}{\circ} - \circ \otimes \circ - \overset{1}{\circ} \oplus x^{-1} \otimes \circ - \overset{1}{\circ} \otimes \overset{1}{\circ} - \circ) \otimes \overset{1}{\bullet} - \cdots - \bullet \Leftarrow \circ.
\end{aligned}$$

Therefore the slice representation contains as a summand the realification of



But this is a representation of complex type with Dadok invariant 3. By Proposition 4.9, it cannot be of class $\mathcal{O}^2$. In particular, it is not taut.

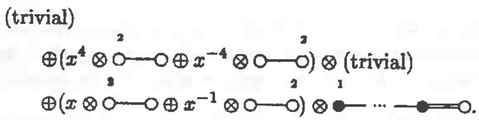
Let $G = \mathrm{Sp}(n) \times \mathrm{Spin}(12)$. Then the isotropy subalgebra at p is $\mathfrak{u}(1) + \mathfrak{su}(6) + \mathfrak{sp}(n-1)$, $n \geq 2$, and the complexified slice representation at p is given by:



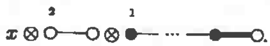
Therefore the slice representation contains as a summand the realification of

But this is a representation of complex type with Dadok invariant 3. By Proposition 4.9, it cannot be of class $\mathcal{O}^2$. In particular, it is not taut.

Let $G = \mathrm{Sp}(n) \times \mathrm{Sp}(3)$. Then the isotropy subalgebra at p is $\mathfrak{u}(1) + \mathfrak{su}(3) + \mathfrak{sp}(n-1)$, $n \geq 2$, and the complexified slice representation at p is given by:

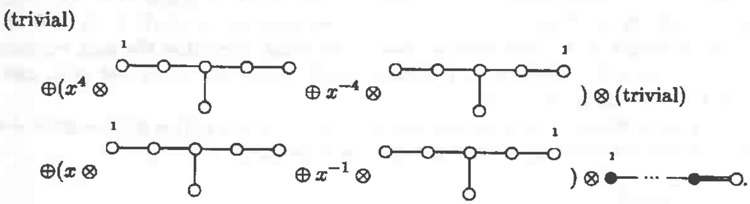


Therefore the slice representation contains as a summand the realification of

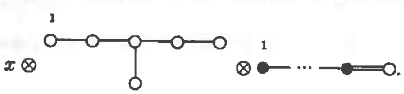


But this is a representation of complex type with Dadok invariant 3. By Proposition 4.9, it cannot be of class $\mathcal{O}^2$. In particular, it is not taut.

Let $G = \mathrm{Sp}(n) \times \mathrm{E}_7$. Then the isotropy subalgebra at p is $\mathfrak{u}(1) + \mathfrak{e}_6 + \mathfrak{sp}(n-1)$, $n \geq 2$, and the complexified slice representation at p is given by:



Therefore the slice representation contains as a summand the realification of



But this is a representation of complex type with Dadok invariant 3. By Proposition 4.9, it cannot be of class $\mathcal{O}^2$. In particular, it is not taut.

Let $G = \mathbf{Sp}(1) \times \mathbf{Spin}(13)$. Then the isotropy subalgebra at p is $\mathfrak{u}(1) + \mathfrak{su}(6)$, and the complexified slice representation at p is given by:

$$\begin{aligned} &(\text{trivial}) \\ &\oplus(x^5 \otimes \overset{1}{\circ}-\circ-\circ-\circ-\circ \oplus x^{-5} \otimes \circ-\circ-\circ-\circ-\overset{1}{\circ}) \\ &\oplus(x^4 \otimes \circ-\overset{1}{\circ}-\circ-\circ-\circ \oplus x^{-4} \otimes \circ-\circ-\circ-\overset{1}{\circ}) \\ &\oplus(x^3 \oplus x^{-3}) \otimes \circ-\circ-\overset{1}{\circ}-\circ-\circ. \end{aligned}$$

Therefore the slice representation contains as a summand the realification of

$$x^3 \otimes \circ-\circ-\overset{1}{\circ}-\circ-\circ.$$

But this is a representation of complex type with Dadok invariant 3. By Proposition 4.9, it cannot be of class $\mathcal{O}^2$. In particular, it is not taut.

Let $G = \mathbf{Sp}(1) \times \mathbf{Sp}(2)$. Then the isotropy subalgebra at p is $\mathfrak{u}(1) + \mathfrak{u}(1)$ and the slice representation at p is given by the realification of:

$$\begin{aligned} &(\text{trivial}) \\ &\oplus(x^{-5} \oplus x^{-2} \oplus x^1) \otimes x^2 \\ &\oplus(x^5 \oplus x^{-4} \oplus 2x^{-1} \oplus 2x^2) \otimes x^4 \\ &\oplus x^3 \otimes x^6. \end{aligned}$$

Therefore the slice representation contains as a summand the realification of $x^{-2} \otimes x^2 \oplus x^{-4} \otimes x^4$ which is orbit equivalent to the circle action $x^2 \oplus x^4$ on $\mathbf{R}^2 \oplus \mathbf{R}^2$. But the orbit of the circle action through $((1,0),(0,1))$ is a torus-knot which is not taut.

Let $G = \mathbf{Spin}(15)$. Then the isotropy subalgebra at p is $\mathfrak{su}(7)$ and the complexified slice representation at p is given by:

$$(\text{trivial}) \oplus \circ-\circ-\overset{1}{\circ}-\circ-\circ-\circ \oplus \circ-\circ-\circ-\overset{1}{\circ}-\circ-\circ$$

Therefore the slice representation contains as a summand the realification of

$$\circ-\circ-\overset{1}{\circ}-\circ-\circ-\circ$$

But this is a representation of complex type with Dadok invariant 3. By Proposition 4.9, it cannot be of class $\mathcal{O}^2$. In particular, it is not taut.

Let $G = \mathbf{Spin}(17)$. Then the isotropy subalgebra at p is $\mathfrak{su}(8)$, and the complexified slice representation at p is given by:

$$\begin{aligned} &(\text{trivial}) \oplus \circ-\circ-\overset{1}{\circ}-\circ-\circ-\circ-\circ \\ &\oplus \circ-\circ-\circ-\overset{1}{\circ}-\circ-\circ-\circ \oplus \circ-\circ-\circ-\circ-\overset{1}{\circ}-\circ-\circ \end{aligned}$$

Therefore the slice representation contains as a summand the realification of

$$\circ-\circ-\overset{1}{\circ}-\circ-\circ-\circ$$

But this is a representation of complex type with Dadok invariant 3. By Proposition 4.9, it cannot be of class $\mathcal{O}^2$. In particular, it is not taut.

Let $G = \mathbf{Spin}(7)$. Then the isotropy subalgebra at p is $\mathfrak{u}(1) + \mathfrak{su}(2)$ and the slice representation at p is given by:

$$\begin{aligned}&(\text{trivial})\\ &\oplus(2(x^4 \oplus x^{-4}) \oplus (x^2 \oplus x^{-2})) \otimes (\text{trivial})\\ &\oplus(2(x^1 \oplus x^{-1}) \oplus (x^7 \oplus x^{-7})) \otimes \overset{1}{\circ}\\ &\oplus((x^2 \oplus x^{-2}) \oplus (x^4 \oplus x^{-4}))) \otimes \overset{2}{\circ}\end{aligned}$$

Therefore the slice representation contains as a summand the realification of $x^4 \oplus x^2$ acting on $\mathbf{R}^2 \oplus \mathbf{R}^2$. But the orbit through $((1,0),(0,1))$ is a torus-knot which is not taut.

Let $G = \mathbf{Spin}(9)$. Then the isotropy subalgebra at p is $\mathfrak{u}(1) + \mathfrak{su}(3)$ and the complexified slice representation at p is given by:

$$\begin{aligned}&(\text{trivial})\\ &\oplus((x^3 \oplus x^{-3}) \oplus 2(x^2 \oplus x^{-2}) \oplus (x \oplus x^{-1})) \otimes (\text{trivial})\\ &\oplus((x \oplus x^{-1}) \oplus (x^2 \oplus x^{-2})) \otimes \overset{1}{\circ}-\overset{1}{\circ}\\ &\oplus((3x \oplus 2x^0 \oplus x^{-3}) \otimes \overset{1}{\circ}-\circ \oplus (3x^{-1} \oplus 2x^0 \oplus x^3) \otimes \circ-\overset{1}{\circ})\\ &\oplus((x^0 \oplus x^{-1}) \otimes \overset{2}{\circ}-\circ \oplus (x^0 \oplus x^1) \otimes \circ-\overset{2}{\circ})\end{aligned}$$

Therefore the slice representation contains as a summand the realification of $x \otimes \overset{1}{\circ}-\overset{1}{\circ}$, but this representation cannot be of class $\mathcal{O}^2$ (because $\overset{1}{\circ}-\overset{1}{\circ}$ is of real type and we use Lemma 5.3(a)). In particular, it is not taut.

Let $G = \mathbf{Sp}(n) \times \mathbf{Spin}(11)$. For $n \geq 2$, the isotropy subalgebra at p is $\mathfrak{u}(1) + \mathfrak{su}(5) + \mathfrak{sp}(n-1)$ and the complexified slice representation at p is given by:

$$\begin{aligned}&(\text{trivial})\\ &\oplus(x^8 \otimes \overset{1}{\circ}-\circ-\circ-\circ \oplus x^{-8} \otimes \circ-\circ-\circ-\overset{1}{\circ}) \otimes (\text{trivial})\\ &\oplus(x^6 \otimes \circ-\overset{1}{\circ}-\circ-\circ \oplus x^{-6} \otimes \circ-\circ-\overset{1}{\circ}-\circ) \otimes (\text{trivial})\\ &\oplus(x^3 \otimes \overset{1}{\circ}-\circ-\circ-\circ \oplus x^{-3} \otimes \circ-\circ-\circ-\overset{1}{\circ}) \otimes \overset{1}{\bullet}-\cdots-\bullet\Leftarrow\circ\\ &\oplus(x \otimes \circ-\overset{1}{\circ}-\circ-\circ \oplus x^{-1} \otimes \circ-\circ-\overset{1}{\circ}-\circ) \otimes \overset{1}{\bullet}-\cdots-\bullet\Leftarrow\circ\end{aligned}$$

Therefore the slice representation contains as a summand the realification of

$$x \otimes \circ-\overset{1}{\circ}-\circ-\circ \otimes \overset{1}{\bullet}-\cdots-\bullet\Leftarrow\circ.$$

But this is a representation of complex type with Dadok invariant 3. By Proposition 4.9, it cannot be of class $\mathcal{O}^2$. In particular, it is not taut.

For $n = 1$ we need a different argument. In this case the connected component K of the isotropy group at p is locally isomorphic to $\mathbf{U}(5)$, and the slice representation at p contains as a summand the realification of

$$x^8 \otimes \overset{1}{\circ}-\circ-\circ-\circ \oplus x^6 \otimes \circ-\overset{1}{\circ}-\circ-\circ.$$

Denote by V_1, V_2 the respective representation spaces of the above summands. Then $V_1 = \mathbf{C}^5$ and $V_2 = \Lambda^2\mathbf{C}^5$. Choose an orthonormal basis $\{e_1, \ldots, e_5\}$ for $\mathbf{C}^5$. Let $q_2 = ae_1 \wedge e_2 + be_3 \wedge e_4 \in V_2$ be a regular point, where a, b are distinct positive real numbers. We have that the isotropy K_{q_2} is locally isomorphic to $\mathbf{SU}(2) \times \mathbf{SU}(2) \times \mathbf{U}(1)$ sitting diagonally in K. Let $q_1 = e_1 + e_3 \in V_1$. Now the orbit $K_{q_2}(q_1)$ is diffeomorphic to $S^3 \times S^3$, whereas the orbit $K(q_1, q_2)$ is diffeomorphic to $\mathbf{SU}(5)$. It is known that the third Betti number of a compact connected simple Lie group is one. It follows from Proposition 3.4 that the slice representation at p is not taut. □

6.5 Proposition *The representations listed in Table B.4 are not taut.*

Proof. We shall go case by case and show that the slice representation of ρ at some point is not taut, and then use Proposition 3.1 to conclude that ρ is not taut.

Let $G = \mathbf{G}_2 \times \mathbf{Spin}(7)$. Then ρ is the tensor product of the 7-dimensional representation of $\mathbf{G}_2$ on $\mathbf{R}^7$ and the spin representation of $\mathbf{Spin}(7)$ on $\mathbf{R}^8$. Let $p = v_1 \otimes v_2 \in \mathbf{R}^7 \otimes \mathbf{R}^8$. The slice representation at p minus the trivial component $\mathbf{R}p$ is given by the tensor product of the vector representation of $\mathfrak{su}(3)$ on $\mathbf{R}^6$ and the 7-dimensional representation of $\mathfrak{g}_2$ on $\mathbf{R}^7$. Since this representation is of complex type and Dadok invariant 3, it follows from Proposition 4.9 that it cannot be of class $\mathcal{O}^2$. In particular, it is not taut.

Let $G = \mathbf{Spin}(7) \times \mathbf{Spin}(9)$. Then ρ is the tensor product of the spin representation of $\mathbf{Spin}(7)$ on $\mathbf{R}^8$ and the spin representation of $\mathbf{Spin}(9)$ on $\mathbf{R}^{16}$. Let $p = v_1 \otimes v_2 \in \mathbf{R}^8 \otimes \mathbf{R}^{16}$. The isotropy subalgebra at p is $\mathfrak{g}_2 + \mathfrak{spin}(7)$. The slice representation at p minus the trivial component $\mathbf{R}p$ is given by the real tensor product of the 7-dimensional representation of $\mathfrak{g}_2$ and the vector representation of $\mathfrak{spin}(7)$ plus the real tensor product of the 7-dimensional representation of $\mathfrak{g}_2$ and the spin representation of $\mathfrak{spin}(7)$. But we already know that the second summand is not taut (see the last paragraph).

Let $G = \mathbf{G}_2 \times \mathbf{Spin}(9)$. Then ρ is the tensor product of the 7-dimensional representation of $\mathbf{G}_2$ on $\mathbf{R}^7$ and the spin representation of $\mathbf{Spin}(9)$ on $\mathbf{R}^{16}$. Let $p = v_1 \otimes v_2 \in \mathbf{R}^7 \otimes \mathbf{R}^{16}$. The isotropy subalgebra at p is $\mathfrak{su}(3) + \mathfrak{spin}(7)$. The slice representation at p minus the trivial component $\mathbf{R}p$ is given by the real tensor product of the vector representations of $\mathfrak{su}(3)$ and $\mathfrak{spin}(7)$ plus the real tensor product of the vector representation of $\mathfrak{su}(3)$ and the spin representation of $\mathfrak{spin}(7)$. Since the second summand is a representation of complex type and Dadok invariant 3, it follows from Proposition 4.9 that it cannot be of class $\mathcal{O}^2$. In particular, it is not taut.

Consider[3] $G = \mathbf{Sp}(1) \cdot \mathbf{Sp}(n) \times \mathbf{G}_2$, $n \geq 2$. Let τ_n be a real form of $\overset{1}{\circ}\otimes\overset{1}{\circ}\!-\!\!-\!\circ\!-\!\!-\cdots-\!\!-\!\circ$. Then τ_n can be realized as the representation of $\mathbf{Sp}(1) \cdot \mathbf{Sp}(n)$ on $\mathbf{R}^{4n}$ given by $\tau_n(q, A)x = Axq^{-1}$, where $q \in \mathbf{Sp}(1)$, $A \in \mathbf{Sp}(n)$, $x \in \mathbf{H}^n$ and we identify $\mathbf{H}^n \cong \mathbf{R}^{4n}$. Now ρ is the real tensor product of τ_n and the 7-dimensional representation of $\mathbf{G}_2$. Let $p = v_1 \otimes v_2 \in \mathbf{R}^{4n} \otimes \mathbf{R}^7$. The isotropy subalgebra at p is isomorphic to $\mathfrak{sp}(1) + \mathfrak{sp}(n-1) + \mathfrak{su}(3)$. The slice representation at p minus the trivial component $\mathbf{R}p$ is given by the real tensor product of τ_{n-1} and the vector representation of $\mathfrak{su}(3)$ plus the real tensor product of the adjoint representation of $\mathfrak{sp}(1)$ and the the vector representation of $\mathfrak{su}(3)$. Since the second summand is a representation of complex type and Dadok invariant 3, it follows from Proposition 4.9 that it cannot be of class $\mathcal{O}^2$. In particular, it is not taut.

Consider $G = \mathbf{Sp}(1) \cdot \mathbf{Sp}(n) \times \mathbf{Spin}(7)$, $n \geq 2$. Let τ_n be as above. Now ρ is the tensor product of τ_n and the spin representation of $\mathbf{Spin}(7)$. Let $p = v_1 \otimes v_2 \in \mathbf{R}^{4n} \otimes \mathbf{R}^8$. The isotropy subalgebra at p is isomorphic to $\mathfrak{sp}(1) + \mathfrak{sp}(n-1) + \mathfrak{g}_2$. The slice representation at p minus the trivial component $\mathbf{R}p$ is given by the real tensor product of τ_{n-1} and the 7-dimensional representation of $\mathfrak{g}_2$ plus the real tensor product of the adjoint representation of $\mathfrak{sp}(1)$ and the the 7-dimensional representation

[3]We use the convention that $G \cdot H$ refers to a quotient of $G \times H$ by a finite central subgroup.

of $\mathfrak{g}_2$. But the first summand is not a taut representation by the above (for $n = 2$ we refer to the case of $G = \mathbf{SO}(4) \times \mathbf{G}_2$ which will be dealt with in Proposition 6.6).

Let $G = \mathbf{Spin}(7) \times \mathbf{Spin}(7)$. Then ρ is the tensor product of the spin representations of each of the factors on $\mathbf{R}^8$. Let $p = v_1 \otimes v_2 \in \mathbf{R}^8 \otimes \mathbf{R}^8$. The isotropy subalgebra at p is $\mathfrak{g}_2 + \mathfrak{g}_2$. The slice representation at p minus the trivial component $\mathbf{R}p$ is given by the real tensor product of the 7-dimensional representations of each of the factors. But this representation is not taut (see next case).

Consider $G = \mathbf{G}_2 \times \mathbf{G}_2$. Then ρ is the tensor product of the 7-dimensional representations of each of the factors on $\mathbf{R}^7$. Let $\{e_1, \ldots, e_7\}$ be an orthonormal basis of $\mathbf{R}^7$ and take $p = e_1 \otimes e_1 + \ldots + e_7 \otimes e_7$. Then the isotropy subalgebra at p is

$$\mathfrak{g}_p = \{(X, X) : X \in \mathfrak{g}_2\}$$

which is isomorphic to $\mathfrak{g}_2$. Now the restriction of ρ to $\mathfrak{g}_p$ is a real form of

$$\mathfrak{g}_2 : \overset{1}{\bullet}\!\Rrightarrow\!\circ \otimes (\overset{1}{\bullet}\!\Rrightarrow\!\circ)^* = \overset{1}{\bullet}\!\Rrightarrow\!\circ \otimes \overset{1}{\bullet}\!\Rrightarrow\!\circ$$

acting on $\mathbf{R}^{49}$, and $\mathbf{R}^{49} = T_pG(p) \oplus N_pG(p)$ is a $\mathfrak{g}_p$-invariant decomposition. We have the following decomposition into irreducible components:

$$\begin{aligned} \mathfrak{g}_2 : \otimes^2(\overset{1}{\bullet}\!\Rrightarrow\!\circ) &= S^2(\overset{1}{\bullet}\!\Rrightarrow\!\circ) \oplus \Lambda^2(\overset{1}{\bullet}\!\Rrightarrow\!\circ) \\ &= (\overset{2}{\bullet}\!\Rrightarrow\!\circ \oplus \text{(trivial)}) \oplus (\bullet\!\Rrightarrow\!\overset{1}{\circ} \oplus \overset{1}{\bullet}\!\Rrightarrow\!\circ). \end{aligned}$$

Since $T_pG(p)$ is 14-dimensional, it follows that the 27-dimensional real representation $\overset{2}{\bullet}\!\Rrightarrow\!\circ$ must be a component of the slice representation at p. But we saw in Lemma 5.29 that it is not of class $\mathcal{O}^2$. □

6.6 Proposition *The representations listed in Table B.3 are not taut.*

Proof. Let $G = \mathbf{SO}(m) \times \mathbf{G}_2$, $m \geq 4$. Then ρ is the tensor product of the vector representation of $\mathbf{SO}(m)$ on $\mathbf{R}^m$ and the 7-dimensional representation of $\mathbf{G}_2$ on $\mathbf{R}^7$. Let $p = v_1 \otimes v_2 \in \mathbf{R}^m \otimes \mathbf{R}^7$. The isotropy subalgebra at p is $\mathfrak{so}(m-1) + \mathfrak{su}(3)$. The slice representation at p minus the trivial component $\mathbf{R}p$ is given by the tensor product of the vector representations of $\mathfrak{so}(m-1)$ and $\mathfrak{su}(3)$ on $\mathbf{R}^{m-1}$ and $\mathbf{R}^6$, respectively. Since this is a representation of complex type and Dadok invariant 3, it follows from Proposition 4.9 that it cannot be of class $\mathcal{O}^2$. In particular, it is not taut.

The proof for $G = \mathbf{SO}(3) \times \mathbf{G}_2$ will be done in Lemma 6.11 below, since this case is more involved.

Let $G = \mathbf{SO}(m) \times \mathbf{Spin}(7)$, $m \geq 4$. Then ρ is the tensor product of the vector representation of $\mathbf{SO}(m)$ on $\mathbf{R}^m$ and the spin representation of $\mathbf{Spin}(7)$ on $\mathbf{R}^8$. Let $p = v_1 \otimes v_2 \in \mathbf{R}^m \otimes \mathbf{R}^8$. The isotropy subalgebra at p is $\mathfrak{so}(m-1) + \mathfrak{g}_2$. The slice representation at p minus the trivial component $\mathbf{R}p$ is given by the real tensor product of the vector representation of $\mathfrak{so}(m-1)$ and the 7-dimensional representation of $\mathfrak{g}_2$. But this representation is not taut by the above.

Let $G = \mathbf{SO}(m) \times \mathbf{Spin}(9)$, $m \geq 4$. Then ρ is the tensor product of the vector representation of $\mathbf{SO}(m)$ on $\mathbf{R}^m$ and the spin representation of $\mathbf{Spin}(9)$ on $\mathbf{R}^{16}$. Let $p = v_1 \otimes v_2 \in \mathbf{R}^m \otimes \mathbf{R}^{16}$. The isotropy subalgebra at p is $\mathfrak{so}(m-1) + \mathfrak{spin}(7)$. The slice representation at p minus the trivial component $\mathbf{R}p$ is given by the real tensor product of the vector representations of $\mathfrak{so}(m-1)$ and $\mathfrak{spin}(7)$ plus the real tensor product of the vector representation of $\mathfrak{so}(m-1)$ and the spin representation of $\mathfrak{spin}(7)$. If $m \geq 5$ then the second summand is not a taut representation by the above. In the case $m = 4$ we give a special argument. Here we have that the connected component of the isotropy subgroup at p is $K = \mathbf{SO}(3) \times \mathbf{Spin}(7)$. Let ν_1, ν_2 be respectively the irreducible summands of dimensions 21, 24 of the slice representation at p, and denote with V_1, V_2 the representation spaces. Note that both ν_1 and ν_2 are taut representations of K, since ν_1 is the isotropy representation of a real Grassmann manifold and ν_2 is orbit equivalent to the isotropy representation of a real Grassmann manifold. Let $q_1 \in V_1$ be a regular point. Then the isotropy group K_{q_1} is isomorphic to $\mathbf{Z}_2^2 \times \mathbf{Spin}(4)$. We can choose $q_2 \in V_2$ so that the isotropy $(K_{q_1})_{q_2}$ is contained in $\{1\} \times \mathbf{Spin}(4)$. Hence the orbit $K_{q_1}(q_2)$ is disconnected. Since the orbit $K(q_1, q_2)$ is connected, Proposition 3.4 implies that the slice representation at p is not taut.

We postpone the proof for the case $\mathbf{SO}(3) \times \mathbf{Spin}(9)$ to Lemma 6.7 below.

Let $G = \mathbf{SU}(n) \times \mathbf{Sp}(m)$, $n \geq 3$ and $m \geq 2$. Then ρ is the realification of the tensor product of the vector representations of $\mathbf{SU}(n)$ on $\mathbf{C}^n$ and of $\mathbf{Sp}(m)$ on $\mathbf{C}^{2m}$. Consider complex bases $\{e_1, \ldots, e_n\}$ for $\mathbf{C}^n$ and $\{f_1, \ldots, f_m, f_{m+1}, \ldots, f_{2m}\}$ for $\mathbf{C}^{2m}$ where $f_{m+j} = \epsilon f_j$, ϵ the quaternionic structure on $\mathbf{C}^{2m}$. Let $p = e_1 \otimes f_1$. Then the isotropy $K = G_p$ is isomorphic to $T_0 \cdot \mathbf{SU}(n-1) \times \mathbf{Sp}(m-1)$, where T_0 is the circle group generated by

$$\begin{aligned}\varphi_0(t) = (&\mathrm{diag}(e^{(n-1)it}, e^{-it}, \ldots, e^{-it}),\\ &\mathrm{diag}(e^{-(n-1)it}, 1, \ldots, 1; e^{(n-1)it}, 1, \ldots, 1)) \in \mathbf{SU}(n) \times \mathbf{Sp}(m),\end{aligned}$$

and the slice representation ν_p minus the trivial component corresponding to the radial direction $\mathbf{R}p$ decomposes into irreducible components as $\nu_1 \oplus \nu_2$, where ν_1 restricted to $\mathbf{SU}(n-1) \times \mathbf{Sp}(m-1)$ is the realification of the complex tensor product of the vector representations of the factors and ν_2 is the realification of the vector representation on $\mathbf{SU}(n-1)$ and it is trivial on $\mathbf{Sp}(m-1)$. Denote the representation spaces by V_1, V_2. Then V_1 is spanned by $e_\alpha \otimes f_\beta$ and $i(e_\alpha \otimes f_\beta)$, for $2 \leq \alpha \leq n$, $2 \leq \beta \leq m$ or $m+2 \leq \beta \leq 2m$, and V_2 is spanned by $e_\alpha \otimes f_{m+1}$ and $i(e_\alpha \otimes f_{m+1})$, for $2 \leq \alpha \leq n$. Let $q_1 = e_2 \otimes f_2 \in V_1$. Then the isotropy $K_1 = K_{q_1}$ is isomorphic to $T_1 \cdot T_2 \cdot \mathbf{SU}(n-2) \times \mathbf{Sp}(m-2)$, where T_1, T_2 are circle groups respectively generated by

$$\begin{aligned}\varphi_1(t) = (&\mathrm{diag}(1, e^{(n-2)it}, e^{-it}, \ldots, e^{-it}),\\ &\mathrm{diag}(1, e^{-(n-2)it}, 1, \ldots, 1; 1, e^{(n-2)it}, 1, \ldots, 1)) \in \mathbf{SU}(n) \times \mathbf{Sp}(m),\end{aligned}$$

and

$$\begin{aligned}\varphi_2(t) = (&\mathrm{diag}(e^{(n-1)it}, e^{-it}, e^{-it}, \ldots, e^{-it}),\\ &\mathrm{diag}(e^{-(n-1)it}, e^{it}, 1, \ldots, 1; e^{(n-1)it}, e^{-it}, 1, \ldots, 1)) \in \mathbf{SU}(n) \times \mathbf{Sp}(m).\end{aligned}$$

(Here we have used the hypothesis $n \geq 3$ and $m \geq 2$). Finally let $q_2 = e_2 \otimes f_{m+1} \in V_2$. Then the isotropy $K_2 = (K_{q_1})_{q_2}$ is isomorphic to $T_3 \cdot \mathbf{SU}(n-2) \times \mathbf{Sp}(m-2)$, where T_3 is the circle subgroup of $T_1 \cdot T_2$ generated by $\varphi_3(t) = \varphi_1(-t)\varphi_2(t)$. It is easy to see that K_1/K_2 is diffeomorphic to S^1, whereas K/K_2 is simply-connected. Therefore Proposition 3.4 implies that ν_p is not taut.

The case where $G = \mathbf{U}(n) \times \mathbf{Sp}(m)$, $n \geq 3$ and $m \geq 2$, is completely analogous to the previous case.

Let $G = \mathbf{SO}(m) \times \mathbf{Sp}(1) \cdot \mathbf{Sp}(n)$, $m \geq 3$ and $n \geq 2$. Then ρ is the tensor product of the vector representation of $\mathbf{SO}(m)$ on $\mathbf{R}^m$ and τ_n, where τ_n is a representation on $\mathbf{R}^{4n}$ as in Proposition 6.5. We choose an orthonormal basis $\{e_1, \ldots, e_m\}$ for $\mathbf{R}^m$ and an orthonormal basis

$$\{f_1, \ldots, f_n, f_1 i, \ldots, f_n i, f_1 j, \ldots, f_n j, f_1 k, \ldots, f_n k\}$$

for $\mathbf{R}^{4n}$. Let $p = e_1 \otimes f_1$. The connected component of the isotropy subgroup $K = G_p^0$ is isomorphic to $\mathbf{SO}(m-1) \times \mathbf{Sp}(1)' \cdot \mathbf{Sp}(n-1)$, where

$$\mathbf{Sp}(1)' = \left\{ \left(q, \begin{pmatrix} q & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \right) : q \in \mathbf{Sp}(1) \right\} \subset \mathbf{Sp}(1) \cdot \mathbf{Sp}(n).$$

The slice representation at p minus the trivial component $\mathbf{R}p$ decomposes as $\nu_1 \oplus \nu_2$, where ν_1 is the real tensor product of the vector representation of $\mathbf{SO}(m-1)$ and τ_{n-1}, and ν_2 is the real tensor product of the vector representation of $\mathbf{SO}(m-1)$ and the adjoint representation of $\mathbf{Sp}(1)$ (ν_2 is trivial on $\mathbf{Sp}(n-1)$). Note that ν_2 is the isotropy representation of a real Grassmann manifold. Let V_1, V_2 denote the representation spaces of ν_1, ν_2. Then V_1 is spanned by $e_\alpha \otimes f_\beta$, $e_\alpha \otimes f_\beta i$, $e_\alpha \otimes f_\beta j$, $e_\alpha \otimes f_\beta k$, for $2 \leq \alpha \leq m$, $2 \leq \beta \leq n$, and V_2 is spanned by $e_\alpha \otimes f_1 i$, $e_\alpha \otimes f_1 j$, $e_\alpha \otimes f_1 k$, for $2 \leq \alpha \leq m$. First consider the case $m \geq 4$. Let $q_2 \in V_2$ be a regular point, say $q_2 = a e_2 \otimes f_1 i + b e_3 \otimes f_1 j + c e_4 \otimes f_1 k$, where a, b, c are pairwise distinct positive real numbers. Then the isotropy group K_{q_2} is isomorphic to $\mathbf{SO}(m-4) \times Q \cdot \mathbf{Sp}(n-1)$, where Q is the quaternion subgroup $\{\pm 1, \pm i, \pm j, \pm k\}$ of $\mathbf{Sp}(1)$. Consider the restriction of ν_1 to K_{q_2}. We can choose $q_1 \in V_1$ so that the orbit $K_{q_2}(q_1)$ is disconnected, say $q_1 = a e_2 \otimes f_2 + b e_3 \otimes f_2 i$, where a, b are distinct positive real numbers. In fact, if $(A, q, B) \in \mathbf{SO}(m-4) \times Q \cdot \mathbf{Sp}(n-1)$ is in the isotropy of q_1, then we must have $B f_2 q^{-1} = f_2$ and $B f_2 i q^{-1} = f_2 i$, which implies that $B f_2 q^{-1} i = B f_2 i q^{-1}$ and then $q^{-1} i = i q^{-1}$, so that $q = \pm 1$ or $q = \pm i$. In particular, no element of the form $(A, j, B) \in \mathbf{SO}(m-4) \times Q \cdot \mathbf{Sp}(n-1)$ is in the isotropy of q_1, which implies that the orbit $K_{q_2}(q_1)$ is disconnected. Since the orbit $K(q_1, q_2)$ is connected, Proposition 3.4 implies that the slice representation at p is not taut. Finally we analyse the case $m = 3$. Then $K = \mathbf{SO}(2) \times \mathbf{Sp}(1)' \cdot \mathbf{Sp}(n-1)$. Consider a regular point $q_2 \in V_2$, say $q_2 = a e_2 \otimes f_1 i + b e_3 \otimes f_1 j$, where a, b are distinct positive real numbers. Then we have that the isotropy K_{q_2} is isomorphic to $\mathbf{Z}_4 \times \mathbf{Sp}(n-1)$. Here $\mathbf{Z}_4$ is generated by $(-1, k)$, where -1 is minus the identity matrix in $\mathbf{SO}(2)$ and $k \in \mathbf{Sp}(1)'$. Consider the restriction of ν_1 to K_{q_2}. We can choose $q_1 \in V_1$ as above such that its isotropy does not contain elements of the form $(-1, k, B)$, for $B \in \mathbf{Sp}(n-1)$. It follows that the orbit $K_{q_2}(q_1)$ is disconnected and the proof follows as above. □

6.7 Lemma *The tensor product ρ of the vector representation of* $\mathbf{SO}(3)$ *on* $\mathbf{R}^3$ *and the spin representation of* $\mathbf{Spin}(9)$ *on* $\mathbf{R}^{16}$ *is not taut.*

Proof. Let $p = v_1 \otimes v_2 \in \mathbf{R}^3 \otimes \mathbf{R}^{16}$. Then the connected component of the isotropy group $K = G_p^0$ is isomorphic to $\mathbf{SO}(2) \times \mathbf{Spin}(7)$. The normal space at p to the orbit $G(p)$ decomposes as $\mathbf{R}p \oplus V_1 \oplus V_2$, where $V_1 = \mathbf{R}^2 \otimes \mathbf{R}^7$, $V_2 = \mathbf{R}^2 \otimes \mathbf{R}^8$ are the representation spaces for the irreducible components ν_1, ν_2 of the slice representation at p. Here ν_1 is the tensor product of the vector representations and ν_2 is the tensor product of the vector representation of $\mathbf{SO}(2)$ and the spin representation of $\mathbf{Spin}(7)$. Now let $q = v_3 \otimes v_4 \in V_2$. Then the connected component of the isotropy group K_q^0 is isomorphic to $\mathbf{G}_2$ and the slice representation at q of the representation of K on $V_1 \oplus V_2$ is a representation of $\mathbf{G}_2$ on $\mathbf{R}^{22}$ which decomposes as a trivial component $\mathbf{R}q$ plus three copies of the 7-dimensional representation of $\mathbf{G}_2$. Next we shall show that the sum $\mu = \mu_1 \oplus \mu_2 \oplus \mu_3$ of three copies of 7-dimensional representation of $H = \mathbf{G}_2$ is not taut. It then follows from Proposition 3.1 that ρ is not taut either.

In fact, let W_i be the representation space of μ_i. We can select $w_i \in W_i$ such that the isotropy groups H_{w_1}, $(H_{w_1})_{w_2}$, $((H_{w_1})_{w_2})_{w_3}$ are respectively $\mathbf{SU}(3)$, $\mathbf{SU}(2)$, $\{1\}$. Since $H_{(w_1,w_2,w_3)} = ((H_{w_1})_{w_2})_{w_3}$, we get that the orbit $H(w_1, w_2, w_3)$ is diffeomorphic to $\mathbf{G}_2$. Now if this orbit is taut, Proposition 3.4 implies that it must have the product homology of the orbits through w_1, w_2 and

w_3, namely, S^6, S^5 and S^3. But it is well known that $\mathbf{G}_2$ has the product homology of S^{11} and S^3 (this also follows from Satz 1 in [19]). Thus, μ is not taut. $\square$

We next recall a reduction principle which will be used below in Lemma 6.11 to prove that a certain representation is not taut. Let $\rho : G \to \mathbf{O}(V)$ be an orthogonal linear action of a compact Lie group G which is not assumed to be connected. Denote by H a fixed principal isotropy subgroup of the G-action on V and let V^H be the subspace of V that is left pointwise fixed by the action of H. Let N be the normalizer of H in G. Then the group N/H acts on V^H with trivial principal isotropy subgroup. Moreover, the following result is known [17, 22, 23, 29, 30, 31]:

6.8 Theorem (Luna-Richardson) *The inclusion $V^H \to V$ induces a stratification preserving homeomorphism between orbit spaces*

$$V^H/N \to V/G. \tag{6.9}$$

The injectivity of the map (6.9) means that $Np = Gp \cap V^H$ for $p \in V^H$. In particular, the H-fixed point set of a G-orbit is a smooth manifold. Observe also that for a regular point $p \in V^H$ the normal space to the principal orbit $M = Gp$ at p is contained in V^H, because the slice representation at p is trivial.

6.10 Lemma *Let $p \in V^H$ be a regular point and consider a normal vector ξ to $M = Gp$ at p. Then the Weingarten operator of M at ξ restricts to the Weingarten operator of $M^H = M \cap V^H$ at ξ, in symbols, $A^M_\xi|_{T_pM^H} = A^{M^H}_\xi$.*

Proof. Let $v \in T_pM^H$. Consider an extension $\tilde{\xi}(t)$ of ξ to a normal vector field along a curve $\gamma(t)$ in M^H with $\dot{\gamma}(0) = v$. Then $A^M_\xi v = -\tilde{\xi}'(0)^T$, where "$T$" denotes the orthogonal projection onto T_pM. Now $\tilde{\xi}(t) \in V^H$, so the derivative $\tilde{\xi}'(0) \in V^H$. The normal component of $\tilde{\xi}'(0)$ is already in V^H, so its component in T_pM is also in V^H. Hence $A^{M^H}_\xi v = -\tilde{\xi}'(0)^T$ which proves the claim. $\square$

6.11 Lemma *We have that the tensor product ρ of the vector representation of* $\mathbf{SO}(3)$ *on* $\mathbf{R}^3$ *and the 7-dimensional representation of* $\mathbf{G}_2$ *is not taut.*

Proof. Note that $G = \mathbf{SO}(3) \times \mathbf{G}_2$ is a closed subgroup of $\hat{G} = \mathbf{SO}(3) \times \mathbf{SO}(7)$ and ρ is the restriction of a representation $\hat{\rho}$ of $\hat{G}$ which is the isotropy representation of a real Grassmann manifold.

We view the Cayley numbers $\mathbf{O} = \mathbf{H} \oplus e\mathbf{H}$, so that $\{1, e, i, j, k, ei, ej, ek\}$ is a real orthonormal basis for $\mathbf{O}$. The Lie group $\mathbf{G}_2$ is the automorphism group of $\mathbf{O}$ with respect to its nonassociative algebra structure. We next describe the space of orbits of G acting by ρ on $V = \mathbf{R}^3 \otimes \mathbf{R}^7$. Take $\{e, i, j, k, ei, ej, ek\}$ as a basis for $\mathbf{R}^7$ and take an orthonormal basis $\{f_1, f_2, f_3\}$ for $\mathbf{R}^3$. Recall that the subspace spanned by $\{i, j, k, ei, ej, ek\}$ has a complex structure given by left multiplication by e. Since all these representations are self-adjoint, it is equivalent to consider ρ acting on $\mathbf{R}^{7*} \otimes \mathbf{R}^3$ which we may identify with the space of 3×7 real matrices. Then the orbit space is described by the matrices of the form

$$\begin{pmatrix} a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c & 0 & d & 0 & 0 \end{pmatrix}, \tag{6.12}$$

subject to the conditions $a \geq b \geq \sqrt{c^2 + d^2}$ and $c \geq 0$.

Next we want to compute a principal isotropy subgroup of G. Let p be an element of the form (6.12) which is regular for both G and $\hat{G}$. The principal isotropy group $\hat{G}_p$ is isomorphic to $\mathbf{Z}_2^2 \times \mathbf{SO}(4)$, and $G_p = \hat{G}_p \cap G$. Now the $\mathbf{Z}_2^2$-factor is contained in G_p, because

$$(\mathrm{diag}(-1,-1,1), \mathrm{diag}(-1,-1,1,-1,1,-1,1))$$

and

$$(\mathrm{diag}(1,-1,-1), \mathrm{diag}(1,-1,-1,1,-1,-1,1))$$

are elements of G that fix p. On the other hand, an element in the $\mathbf{SO}(4)$-factor that fixes p and is in $G = \mathbf{SO}(3) \times \mathbf{G}_2$ is in fact in $\{1\} \times \mathbf{G}_2$ and has to fix e, i, $cj + d(ei) \in \mathbf{R}^7$. Therefore it also fixes $j = \frac{1}{c}[(cj + d(ei)) - d(ei)]$ ($c \neq 0$ because p is regular for G) and hence it is the identity. We conclude that the principal isotropy group $H = G_p$ is exactly $\mathbf{Z}_2^2$.

It is immediate to see that the fixed subspace V^H is

$$\begin{pmatrix} * & 0 & 0 & * & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 & * & 0 \\ 0 & 0 & * & 0 & * & 0 & 0 \end{pmatrix}.$$

Let N be the normalizer of H in G and $\bar{G} = N/H$ the reduced group. The connected component $\bar{G}^0$ must be contained in $\{1\} \times \mathbf{G}_2$. One can use the fact that N equals the normalizer of V^H in G to verify that $\bar{G}^0$ is the 2-torus group consisting of the matrices

$$\begin{pmatrix} \cos t & 0 & 0 & -\sin t & 0 & 0 & 0 \\ 0 & \cos s & 0 & 0 & 0 & -\sin s & 0 \\ 0 & 0 & \cos(s-t) & 0 & -\sin(s-t) & 0 & 0 \\ \sin t & 0 & 0 & \cos t & 0 & 0 & 0 \\ 0 & 0 & \sin(s-t) & 0 & \cos(s-t) & 0 & 0 \\ 0 & \sin s & 0 & 0 & 0 & \cos s & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{6.13}$$

for $s, t \in \mathbf{R}$.

Let $p \in V^H$ be a regular point and let $M = Gp$ be the corresponding principal orbit in V. We next prove that M is not taut by contradiction. In fact, assume M is taut. The orbit $\bar{G}^0 p$ is a connected component of M^H, and it is a substantial 2-torus embedded in the 6-dimensional Euclidean space V^H. Since its second osculating space is at most 5-dimensional, it cannot be taut by Theorem 2.5. It is known that a surface is taut if and only if its lines of curvature are circles. Therefore there is a parallel normal vector field $\xi(t)$ along a curve $\gamma(t)$ in $\bar{G}^0 p \subset M^H$ such that $A^{M^H}_{\xi(t)} \dot\gamma(t) = \lambda(t)\dot\gamma(t)$ and $\lambda(t)$ is not constant. Since M is a principal G-orbit, the normal spaces of M and M^H in V coincide along $\gamma(t)$ implying that $\xi(t)$ is also parallel with respect to M. By Lemma 6.10 we have

$$A^M_{\xi(t)} \dot\gamma(t) = \lambda(t)\dot\gamma(t). \tag{6.14}$$

We do not claim that $\gamma(t)$ is a curvature line of M since the eigenspace corresponding to $\lambda(t)$ might not be one-dimensional. Still the argument in [28], Lemmas 1 and 2 (see also [24]) carries through and shows that (6.14) together with the tautness of M implies that $\lambda(t)$ is constant, which is a contradiction. Hence M is not taut. Thus the representation ρ is not taut either. □

A Table for Dadok's invariant $k(\lambda)$

We next tabulate the value of the Dadok invariant $k(\lambda)$ for the the complex irreducible representations π_λ of the complex simple Lie algebras, where λ is a fundamental highest weight (since $k(\lambda)$ is linear on λ, this already determines the value of $k(\lambda)$ for all λ). The number in parenthesis next to the vertex of the Dynkin diagram which specifies α_i is $k(\lambda_i)$. The last column tells, for each complex simple Lie algebra, whether all representations are self-dual, or else, the dual of π_{λ_i} is given by π_{λ_j}, where the vertices that specify α_i and α_j are images one of the other under the symmetry of the Dynkin diagram.

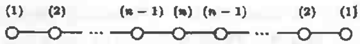
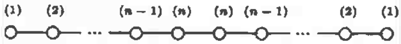
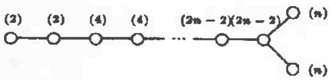
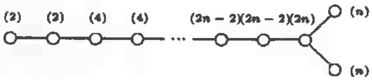
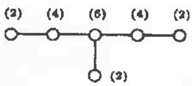
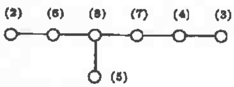
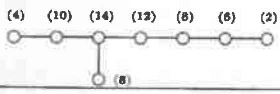
G	$k(\lambda)$	*Conditions*
$\mathbf{A}_{2n-1}$		duality given by symmetry of diagram
$\mathbf{A}_{2n}$		duality given by symmetry of diagram
$\mathbf{B}_{2n-1}$		all self-dual
$\mathbf{B}_{2n}$		all self-dual
$\mathbf{C}_n$		all self-dual
$\mathbf{D}_{2n}$		all self-dual
$\mathbf{D}_{2n+1}$		duality given by symmetry of diagram
$\mathbf{G}_2$		all self-dual
$\mathbf{F}_4$		all self-dual
$\mathbf{E}_6$		duality given by symmetry of diagram
$\mathbf{E}_7$		all self-dual
$\mathbf{E}_8$		all self-dual

Table A.1: The invariant of Dadok.

B Table of candidates to class $\mathcal{O}^2$

In the tables below are compiled all the real irreducible representations of compact connected Lie groups that are not orbit equivalent to the isotropy representation of a symmetric space but that have a chance of being a representation of class $\mathcal{O}^2$. These are just the representations mentioned in Propositions 5.1, 5.2, 5.14, 5.15, 5.16, 5.17 and 5.18; note, however, that the presentation here slightly differs from the presentation in the above mentioned propositions for the sake of convenience.

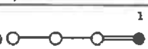
G	ρ	*Conditions*
SO(2) × Spin(9)	$(x \oplus x^{-1}) \otimes$ 	–
U(2) × Sp(n)	$(x \oplus x^{-1}) \otimes$ 	$n \geq 2$
SU(2) × Sp(n)		$n \geq 2$

Table B.1: Taut representations.

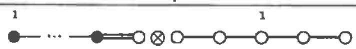
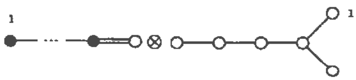
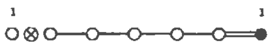
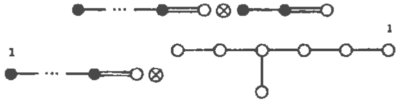
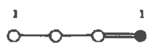
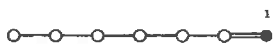
G	ρ	*Conditions*
Sp(n) × SU(6)		$n \geq 2$
Sp(n) × Spin(11)		–
Sp(n) × Spin(12)		$n \geq 2$
Sp(1) × Spin(13)		–
Sp(1) × Sp(2)		–
Sp(n) × Sp(3)		$n \geq 2$
Sp(n) × E_7		$n \geq 2$
Spin(7)		–
Spin(9)		–
Spin(15)		–
Spin(17)		–

Table B.2: Nontaut representations (first group).

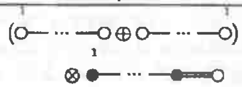
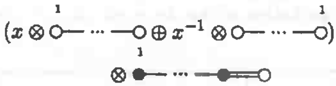
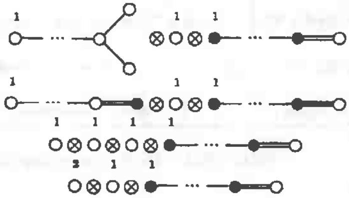
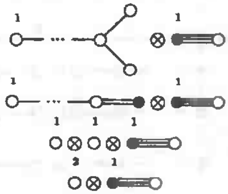
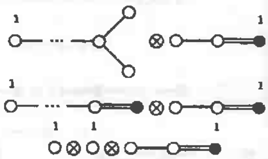
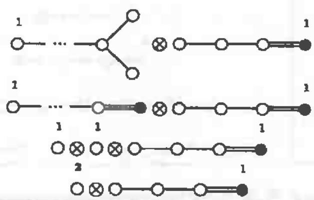
G	ρ	*Conditions*
$\mathbf{SU}(n) \times \mathbf{Sp}(m)$		$n \geq 3,\ m \geq 2$
$\mathbf{U}(n) \times \mathbf{Sp}(m)$		$n \geq 3,\ m \geq 2$
$\mathbf{SO}(m) \times \mathbf{Sp}(1) \cdot \mathbf{Sp}(n)$ $(n \geq 2)$		$m = 2p > 4$ $m = 2p+1 > 3$ $m = 4$ $m = 3$
$\mathbf{SO}(m) \times \mathbf{G}_2$		$m = 2p > 4$ $m = 2p+1 > 3$ $m = 4$ $m = 3$
$\mathbf{SO}(m) \times \mathbf{Spin}(7)$		$m = 2p > 4$ $m = 2p+1 \geq 5$ $m = 4$
$\mathbf{SO}(m) \times \mathbf{Spin}(9)$		$m = 2p > 4$ $m = 2p+1 > 3$ $m = 4$ $m = 3$

Table B.3: Nontaut representations (second group).

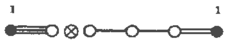
G	ρ	*Conditions*
$\mathbf{G_2} \times \mathbf{G_2}$		–
$\mathbf{G_2} \times \mathbf{Spin(7)}$		–
$\mathbf{G_2} \times \mathbf{Spin(9)}$		–
$\mathbf{Spin(7)} \times \mathbf{Spin(7)}$		–
$\mathbf{Spin(7)} \times \mathbf{Spin(9)}$		–
$\mathbf{Sp(1)} \cdot \mathbf{Sp}(n) \times \mathbf{G_2}$		$n \geq 2$
$\mathbf{Sp(1)} \cdot \mathbf{Sp}(n) \times \mathbf{Spin(7)}$		$n \geq 2$

Table B.4: Nontaut representations (third group).

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