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ON PAIRS OF FOLIATIONS DEFINED BY VECTOR FIELDS IN THE PLANE

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Abstract. We obtain a smooth and analytic local classification of pairs of foliations of the plane assuming that one of the foliations is defined by a nonsingular vector field and the other foliation is defined by a singular vector field.

1. Introduction. Given a vector field v on the plane we denote by F_v the foliation defined by its phase curves (or orbits). This foliation is singular at equilibria (i.e. the zeroes) of v. In this paper we study the local smooth or analytic classification of pairs (F_ξ, F_v) around the origin, where ξ is a nonvanishing vector field and $v(0) = 0$. The results of this paper reduce this classification to that of singularities of single foliations F_v. The advantage of this reduction is to make it possible to apply the well-known results on local orbital classification of singularities of vector fields on R^2 (see [1, 4, 9]).

The classification under homeomorphisms of pairs such as (F_ξ, F_v) arrises in the the study of bifurcations of vector fields on manifold with boundary [12]. In the setting above, the boundary is represented by the leaves of F_ξ, depending on a transveral real parameter. The change of such parameter produces the unfolding due to a singularity (of v) exiting the manifold. The extension of this study to three dimensional manifold with boundary has been carried out in [11].

Another situation leading to the above mentioned pairs appears in the study of constrained differential systems of the form $A(x)\dot{x} = v(x)$, near impasse points at which $A(x)$ is a singular endomorphism and $v(x)$ belongs to its image. The case of smooth equivalence has been treated in [13] and that of topological equivalence in [10].

The classification of pair of foliations is also a problem of control theory, see, for example, [7].

Theorem 1 of this paper gives a simple genericity condition (G1), expressed in algebraic terms, under which the reduction of the classification of pairs to that of singularities of vector fields is possible. Proposition 1 gives a geometric interpretation of (G1) which implies an important Corollary 1. Theorem 2 completes Theorem 1. These results imply a number of normal forms for pairs (F_ξ, F_v) (Section 3), and a reduction theorem for pairs consisting of a foliation of the plane and

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a curve, generalizing some results of the works [5] and [6] (Corollary 2). Our results show that under the condition (G1) all invariants of a pair of foliations (F_ξ, F_v) are determined by the pair (γ, F_v), where γ is the phase curve of the vector field ξ passing through the singular point of v (Corollary 3). The proofs are contained in Section 4.

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2. Statements of the main results.

All results of this section hold for germs in either C^∞ or real analytic category. In what follows Ω is a fixed nondegenerate volume form on R^2. By $[.,.]$ we denote the Lie bracket of vector fields.

Theorem 1. *Let* ξ_1, ξ_2 *and* v *be vector fields on* R^2 *such that* $v(0) = 0$ *and*

$$\Omega(\xi_1, [\xi_1, v])(0) \;\cdot\; \Omega(\xi_2, [\xi_2, v])(0) > 0. \tag{G1}$$

Then there exists a local diffeomorphism preserving the foliation F_v *and sending the foliation* F_{ξ_1} *to the foliation* F_{ξ_2}.

To give a geometric interpretation of the condition (G1) introduce the following notations. By $P(T_0R^2)$ we denote the projectivization of the tangent plane T_0R^2, and by E the set of points of $P(T_0R^2)$ corresponding to the eigenspaces of the linearization of v at the origin.

Proposition 1. *1. The condition*

$$\Omega(\xi_1, [\xi_1, v])(0) \;\cdot\; \Omega(\xi_2, [\xi_2, v])(0) \neq 0 \tag{G2}$$

holds if and only if $\xi_1(0) \neq 0$, $\xi_2(0) \neq 0$ *and neither* $\xi_1(0)$ *nor* $\xi_2(0)$ *is an eigenvector of the linearization of* v.

2. The condition (G1) holds if and only if $\xi_1(0) \neq 0$, $\xi_2(0) \neq 0$ *and the two points* l_1, l_2 *of* $P(T_0R^2)$ *corresponding to the lines* $span\{\xi_1(0)\}$ *and* $span\{\xi_2(0)\}$ *belong a single connected component of the set* $P(T_0R^2) - E$.

If the eigenvalues of v are not real then E is empty, and if the eigenvalues are equal then either E consists of one point and then the set $P(T_0R^2) - E$ is connected, or (if the linear approximation of v is diagonalizable) $E = P(T_0R^2)$. Therefore we obtain the following corollary.

Corollary 1. *The condition (G2) implies (G1) unless the vector field* v *has real distinct eigenvalues at the origin. If the eigenvalues are not real then the condition (G1) holds for any vector fields* ξ_1 *and* ξ_2 *nonvanishing at the origin.*

In the case of real distinct eigenvalues the set $P(T_0R^2) - E$ contains two connected components and by Proposition 1 the inequality

$$\Omega(\xi_1, [\xi_1, v])(0) \;\cdot\; \Omega(\xi_2, [\xi_2, v])(0) < 0 \tag{G3}$$

is possible *only* in this case. Then the existence of a diffeomorphism preserving the foliation F_v and sending the foliation F_{ξ_1} to the foliation F_{ξ_2} depends on the existence of "orientation reversing" symmetries of the foliation F_v, in the sense defined below.

Definition 1. A local diffeomorphism Ψ will be called an orientation reversing symmetry of the foliation F_v if $\Psi_* v = Qv$ for some function Q such that $Q(0) det(\Psi'(0)) < 0$.

For example, the foliation defined by the vector field $\dot{x} = Nx + y^N, \dot{y} = y$ has an orientation reversing symmetry $(x, y) \to (x, -y)$ if N is even number and a simple analysis shows that there are no C^∞ orientation reversing symmetries if N is odd.

To formulate our second theorem we need the following natural definitions.

Definition 2. A singularity at 0 of a local vector field v is called algebraicly isolated if the ideal generated by the components of v contain a power of the maximal ideal in the ring of germs at 0 of smooth or analytic functions.

Definition 3. The pair of foliations (F_{ξ_1}, F_{v_1}) is said to be equivalent to the pair (F_{ξ_2}, F_{v_2}) if there exists a diffeomorphism sending F_{ξ_1} to F_{ξ_2} and F_{v_1} to F_{v_2}.

Theorem 2. *Assume that $v(0) = 0$ and the origin is an algebraicly isolated singularity of v. Assume that (G3) holds for some vector fields ξ_1 and ξ_2. Then the pair of foliations (F_{ξ_1}, F_v) is equivalent to the pair (F_{ξ_2}, F_v) if and only if the foliation F_v admits an orientation reversing symmetry. The condition that the origin be an algebraically isolated singularity of v is not essential when replacing "if and only if" by "if".*

The given results can be directly applied to the classification of pairs of foliations of the plane. In fact, using results on the orbital classification of vector fields on the plane (see [1, 4, 9]) and distinguishing the normal forms of vector fields with real distinct eigenvalues such that the corresponding foliations admit an orientation reversing symmetry, we obtain a list of normal forms for pairs of foliations (F_ξ, F_v), see Section 3.

Another corollary concerns the classification of pairs consisting of a foliation of R^2 and a curve passing through a singular point of this foliation. In what follows by a curve we mean a smooth or analytic 1-dimensional submanifold.

Corollary 2. *Let v be a vector field on R^2 with algebraically isolated singularity at the origin and let $\gamma_1 = \{f_1 = 0\}$ and $\gamma_2 = \{f_2 = 0\}$ be curves passing through the origin, $df_1(0) \neq 0$, $df_2(0) \neq 0$. Assume that neither $T_0\gamma_1$ nor $T_0\gamma_2$ is an eigenspace of the linearization of v. Then the pair (γ_1, F_v) is equivalent to the pair (γ_2, F_v) (i.e., there exists a diffeomorphism preserving the foliation defined by v and sending the curve γ_1 to the curve γ_2) if and only if either*
(a) the 2-forms

$$\Omega_1 = df_1 \wedge d(v(f_1)), \quad \Omega_2 = df_2 \wedge d(v(f_2))$$

define the same orientation of R^2, or
(b) the foliation F_v admits an orientation reversing symmetry.

The condition (a) always holds if the eigenvalues of the linearization of v are not real or equal.

This result generalizes some results of the works [5] and [6]. Finally, we obtain that under the condition (G1) the classification of pairs (F_ξ, F_v) is equivalent to the classification of pairs (γ, F_v), where γ is the phase curve of the vector field ξ passing through the singular point of the vector field v. Namely, the following corollary holds.

Corollary 3. *Let v_1 and v_2 be vector fields on R^2 with algebraically isolated singularity at the origin, and let ξ_1 and ξ_2 be nonvanishing vector fields such that $\xi_1(0)$ is not an eigenvector of the linearization of v_1 and $\xi_2(0)$ is not an eigenvector of the linearization of v_2. Let γ_i be the phase curve of ξ_i passing through the origin, $i = 1, 2$. Then the pair (F_{ξ_1}, F_{v_1}) is equivalent to the pair (F_{ξ_2}, F_{v_2}) if and only if the pair (γ_1, F_{v_1}) is equivalent to the pair (γ_2, F_{v_2}).*

3. Normal forms. To apply the results of Section 2 to the classification of pairs (F_ξ, F_v) and to the classification of pairs consisting of a foliation F_v and a curve, we distinguish normal forms of vector fields with real distinct eigenvalues such that the corresponding foliations admit an orientation reversing symmetry. We also restrict ourselves to normal forms in the C^∞ category and avoid certain degeneracies of infinite codimension, namely
(a) the origin is not algebraically isolated singular point of v;
(b) the eigenvalues of v are pure imaginary and v is formally orbitally equivalent to its linear approximation (this is so if and only if all Lyapunov focal numbers are equal to zero, see [1]).

It follows from the well known results on the orbital classification of vector fields on the plane (see [1, 4, 9]) that if at least one of the eigenvalues of v is not zero and neither (a) nor (b) holds then v is C^∞ orbitally equivalent to one of the normal forms
V_1: $\dot{x} = x + \lambda y,\ \dot{y} = -\lambda x + y$, where $\lambda \neq 0$;
V_2: $\dot{x} = \lambda x,\quad \dot{y} = y$;
V_3: $\dot{x} = x + y,\quad \dot{y} = y$;
V_4: $\dot{x} = Nx + y^N,\quad \dot{y} = y$, where $N \in \{2, 3, ...\}$;
V_5: $\dot{x} = x(-p/q \pm z^r + az^{2r}),\quad \dot{y} = y$, where $a \in R$, $z = x^q y^p$, $p, q, r \in \{1, 2, ...\}$, p/q is an irreducible fraction, or $p = 0, q = 1$;
V_6: $\dot{x} = y + xA(z),\quad \dot{y} = -x + yA(z)$, where $z = x^2 + y^2$, $A(z) = \pm z^r + az^{2r}$, $r \in \{1, 2, ...\}$, $a \in R$.

If the eigenvalues of v are equal to zero, but $j_0^1 v \neq 0$, then v is formally (on the level of formal series) orbitally equivalent to the normal form
V_7: $\dot{x} = y,\quad \dot{y} = f(x) + yg(x)$, where $f(0) = f'(0) = g(0) = 0$.

The normal forms V_2, V_4 and V_5 correspond to vector fields with real distinct eigenvalues. Note now that the foliation defined by V_2 admits an orientation reversing symmetry $(x, y) \to (x, -y)$ and the same diffeomorphism is an orientation reversing symmetry of the foliation defined by V_4 and V_5 provided that N in V_4 is even and one of the numbers p, q in V_5 is even. One can show that the inverse is also true: if N in V_4 is odd (respectively both p and q in V_5 are odd) then the foliation defined by V_4 (respectively V_5) does not have orientation reversing symmetries. These facts and the results of Section 1 imply the following corollary.

Corollary 4. *Let ξ and v be vector fields on R^2 such that $v(0) = 0$, $\xi(0) \neq 0$ and $\xi(0)$ is not an eigenvector of the linearization of v. If at least one of the eigenvalues of v is not equal to zero and none of the degenerations (a), (b) of infinite codimension holds then the pair of foliations defined by ξ and v can be reduced by a C^∞ diffeomorphism to a pair defined by vector fields U and V, where $V \in \{V_1, V_2, V_3, V_4, V_5, V_6\}$ and*
(α) $U = \frac{\partial}{\partial y}$ if $V \in \{V_1, V_3, V_6\}$,
(β) $U = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$ if $V = V_2$, or $V = V_4$ and N is even, or $V = V_5$ and one of the numbers p, q is even,
(γ) $U \in \{U_+, U_-\}$, $U_\pm = \frac{\partial}{\partial x} \pm \frac{\partial}{\partial y}$ if $V = V_4$ and N is odd or $V = V_5$ and p and q are odd. In these cases the pair of foliations defined by vector fields (U_-, V) is not equivalent to the pair of foliations defined by the vector fields (U_+, V).

If the eigenvalues of v are equal to zero and $j_0^1 v \neq 0$ then there the pair of foliations defined by ξ and v is formally equivalent to a pair defined by the vector field $\frac{\partial}{\partial y}$ and a vector field of the form V_7.

By Corollary 3, a similar result holds for pairs consisting of a foliation F_v, $v(0) =$

0, and a curve γ containing the origin and such that $T_0\gamma$ is not an eigenspace of the linearization of v: one has to change, throughout Corollary 4, the pair (F_ξ, F_v) by the pair (γ, F_v) and U by the phase curve of U passing through the origin.

3. Proofs of the main results.

Proof of Proposition 1. To prove the first statement it suffices to note that if ξ and v are nonvanishing vector fields then the condition $\Omega(\xi(0), [\xi, v](0)) = 0$ means that $[\xi, v](0) = \lambda\xi(0)$ for some $\lambda \in R$. In local coordinates this means that $Aa = \lambda a$, where A is the matrix of the linear approximation of v at the origin, $a = (a_1, a_2)^t$, $\xi(0) = a_1\frac{\partial}{\partial x} + a_2\frac{\partial}{\partial y}$, i.e., $\xi(0)$ is an eigenvector of the linear approximation of v.

The proof of the second statemet is based on the following observation: the points l_1 and l_2 belong to one connected component of $P(T_0R^2) - E$ if and only if either for $\delta = 1$ or for $\delta = -1$ there exists a smooth path $\nu_t \in T_0R^2, t \in [0, 1]$ from $\xi_1(0)$ to $\delta\xi_2(0)$ $(\nu_0 = \xi_1(0), \nu_1 = \xi_2(0))$ such that $\nu_t \neq 0$ and ν_t is not an eigenvector of the linearization of v for all $t \in [0, 1]$. Assuming that this is so, take any family of vector fields X_t such that $X_t(0) = \nu_t$, $X_0 = \xi_1, X_1 = \delta\xi_2$. Let $f(t) = \Omega(X_t, [X_t, v])(0)$. By the first statement, $f(t) \neq 0$ for all $t \in [0, 1]$. Consequently, the numbers $f(0) = \Omega(\xi_1, [\xi_1, v])(0)$ and $f(1) = \Omega(\delta\xi_2, [\delta\xi_2, v])(0) = \delta^2\Omega(\xi_2, [\xi_2, v])(0)$ have the same sign and (G1) holds.

Assume now that the points l_1 and l_2 belong to different connected components of $P(T_0R^2) - E$. Then there exists a smooth path $\nu_t \in T_0R^2, t \in [0, 1]$ from $\xi_1(0)$ to $\xi_2(0)$ such that $\nu_t \neq 0$ for all $t \in [0, 1]$ and ν_t is an eigenvector of the linearization of v for exactly one $t \in (0, 1)$. The function $f(t)$ constructed as above has exactly one zero in the segment $[0, 1]$, therefore $f(0)f(1) < 0$, i.e., (G1) does not hold. Therefore, Proposition 1 is proved.

Proof of Theorem 1. This theorem will be proved using the homotopy method (see, for example, [2], [9]). This method reduces the nonlinear problems of equivalence of objects of local analysis (vector fields, mappings, etc.) to the solution of infinitesimal linear equations with a parameter varying on the segment $[0, 1]$.

In the proof of Proposition 1 we observed that under condition (G1) there exists a path X_t of vector fields such that $X_0 = \xi_1$, $X_1 = \pm\xi_2$ and $\Omega(X_t, [X_t, v])(0) \neq 0$ for all $t \in [0, 1]$, i.e., the vector fields X_t and $[X_t, v]$ are linearly independent at the origin. It is clear that the path can be chosen to depend analytically on t. We will show that there exists a family of diffeomorphisms Φ_t preserving the foliation defined by v, and, for each $t \in [0, 1]$, sending the foliation defined by X_t to that defined by $X_0 = \xi_1$. Obviously, Theorem 1 is a consequence of this statement.

To prove the existence of such a family Φ_t, consider the following system of infinitesimal equations with respect to a family of vector fields Z_t, and two families of functions α_t and β_t:

$$[Z_t, X_t] + \alpha_t X_t = -\frac{dX_t}{dt}, \quad [Z_t, v] + \beta_t v = 0. \tag{1}$$

Assume that this system has a solution Z_t, α_t, β_t such that $Z_t(0) = 0$ for all $t \in [0, 1]$. Then the family Φ_t can be defined by a system of ordinary differential equations $\frac{d\Phi_t}{dt} = Z_t(\Phi_t)$ with the initial condition $\Phi_0 = id$. It is easy to see that Φ_t is a local diffeomorphism for each t. To check that Φ_t sends the foliation defined by X_t to the foliation defined by X_0, preserving the foliation defined by v, introduce the mappings $t \to A(t) = (\Phi_t)_*X_t$ and $t \to B(t) = (\Phi_t)_*v$, and differentiate them

$$A'(t) = (\Phi_t)_*([Z_t, X_t] + \frac{dX_t}{dt}), \quad B'(t) = (\Phi_t)_*[Z_t, v].$$

Using (1) we obtain that

$$A'(t) = \tilde{\alpha}_t A(t), \quad B'(t) = \tilde{\beta}_t B(t),$$

where $\tilde{\alpha}_t = -\alpha_t(\Phi_t)$, $\tilde{\beta}_t = -\beta_t(\Phi_t)$. It follows that $A(t) = A(0)H_t$ and $B(t) = B(0)Q_t$, where H_t and Q_t are families of nonvanishing functions. Since $A(0) = X_0$ and $B(0) = v$ we obtain $(\Phi_t)_* X_t = H_t X_0$ and $(\Phi_t)_* v = Q_t v$ which means that Φ_t sends the foliation defined by X_t to the foliation defined by X_0 preserving the foliation defined by v.

We have reduced Theorem 1 to the solvability of the system (1) with the constrain $Z_t(0) = 0$. We will seek for Z_t in the form $Z_t = h_t v$ introducing instead of the unknown family of vector fields Z_t the family of functions h_t. Since $v(0) = 0$ then the condition $Z_t(0) = 0$ holds for any choice of h_t. The second equation in the system (1) takes the form $(\beta_t - v(h_t))v = 0$, therefore to solve (1) it suffices to solve only the first equation with respect to h_t and α_t (the solution of the system will be h_t, α_t and $\beta_t = v(h_t)$). Notice that the choice of Z_t in the form $h_t v$ means that for any t the diffeomorphisms Φ_t moves any point along the phase curve of v, which is the simplest(and in many cases the unique) way to preserve the foliation defined by v).

After the substitution $Z_t = h_t v$ the first equation in system (1) takes the form

$$h_t[v, X_t] - X_t(h_t)v + \alpha_t X_t = -\frac{dX_t}{dt}. \tag{2}$$

and Theorem 1 will be proved if we show that this equation has a solution h_t, α_t. At this point we use the linear independence of X_t and $[v, X_t]$ at the origin. This condition implies that any family of vector fields parametrized by $t \in [0,1]$ is a linear combination of X_t and v with functional coefficients depending analytically on t ((locally, in a sufficient small neighbourhood of the origin). Let

$$v = A_t X_t + B_t[v, X_t], \quad -\frac{dX_t}{dt} = C_t X_t + D_t[v, X_t]. \tag{3}$$

Then (2) can be rewritten as a system

$$-A_t X_t(h_t) + \alpha_t = C_t$$

$$h_t - B_t X_t(h_t) = D_t. \tag{4}$$

To solve this system (with respect to h_t and α_t) it suffices to solve the equation (6) with respect to the family of functions h_t. Note that since $v(0) = 0$ then (3) implies that $A_t(0) = B_t(0) = 0$ for all t, therefore this equation is singular. Nevertheless, we will prove that it is solvable for any family D_t.

To achieve this end, we will find a certain relation between the vector field X_t and the function B_t. The first relation in (3) implies $[v, X_t] = [A_t X_t, X_t] + [B_t[v, X_t], X_t]$. Since $B_t(0) = 0$ it follows that $[v, X_t](0) = -X_t(A_t)X_t(0) - X_t(B_t)[v, X_t](0)$. Since the vectors $X_t(0)$ and $[v, X_t](0)$ are linearly independent, we obtain that

$$X_t(B_t)(0) = -1. \tag{5}$$

This relation plays an important role in the solvability of the equation (4). Namely, since $X_t(0) \neq 0$ and t varies on the compact interval $[0,1]$, there exists a family of diffeomorphism (parametrized by t) reducing X_t to $\frac{\partial}{\partial x}$. This family of diffeomorphisms brings B_t to some family $\tilde{B}_t(x,y)$, and by (5) we have $\frac{\partial \tilde{B}_t}{\partial x}(0) = -1$. Then $\tilde{B}_t(x,y), y$ is a system of coordinates. Changing $\tilde{B}_t(x,y)$ by x we transform the vector field $\frac{\partial}{\partial x}$ to a vector field of the form $C_t(x,y)\frac{\partial}{\partial x}$. Now (5) implies that $C_t(0,0) = -1$. These transformations allow us to reduce the equation (4) to the form $h_t - C_t(x,y)x\frac{\partial h_t}{\partial x} = D_t$. Dividing by $(-C_t)$, we get a simple equation of the form

$$f_t(x,y)h_t(x,y) + x\frac{\partial h_t}{\partial x} = g_t(x,y), \tag{6}$$

where $f_t(x,y)$ and $g_t(x,y)$ are given families of functions, $f(0,0) = 1$, and h_t is the unknown family. To prove Theorem 1 we have to show that (6) has a local solution (in either C^∞ or analytic category).

Though (6) can be treated as a linear ordinary differential equation with parameters t and y and can be integrated, the proof of existence of local analytic and especially C^∞ solution h_t (with f_t and g_t belonging to the same category) requires certain technical prepartion. At first we will simplify (6) reducing $f_t(x,y)$ to a family depending on y only. Namely, introducing instead of h_t another unknown family $\tilde{h}_t$ such that $h_t = \tilde{h}_t exp(\nu_t(x,y))$, where ν_t will be chosen below, we transform (6) to an equation of the same form with $f_t(x,y)$ replaced by $f_t(x,y)+x\frac{\partial \nu_t}{\partial x}$ and $g_t(x,y)$ replaced by $g_t(x,y)exp(-\nu_t(x,y))$. Therefore the equation (6) can be reduced to the form

$$F_t(y)h_t(x,y) + x\frac{\partial h_t}{\partial x} = G_t(x,y), \tag{6'}$$

where F_t satisfies (important) condition $F_t(0) = 1$. Now the solvability of (6') in the analytic category follows without difficulty:

$$h_t(x,y) = \sum_{i=0}^{\infty} \frac{\partial^i G_t}{\partial x^i}(0,y)\frac{x^i}{i!(i+1)}, \tag{7}$$

which is, obviously, a convergent series (provided $G_t(x,y)$ is analytic).

In the C^∞ category the arguments are more difficult. The existence of a solution h_t in the form (7) (which , in C^∞ category, is nothing more than a formal power series with respect to x) allows us to reduce (6') to the same equations with $G_t(x,y)$ replaced by $\tilde{G}_t(x,y)$ such that any partial derivative of $\tilde{G}_t(x,y)$ vanishes at any point of the y-axis (such a function is called flat along the y-axis). To achieve this reduction we take a C^∞ function $H_t(x,y)$ whose Taylor series with respect to x coincides with (7) (such a function exists due to the Borel theorem) and replace in (6') h_t by the new unknown $\tilde{h}_t$ such that $h_t = H_t + \tilde{h}_t$. Now, following Belitskii's method (see [3]) we can present a C^∞ solution in the form

$$h_t(x,y) = \int_{-\infty}^{0} e^{F_t(y)s}G_t(e^s x, y)ds. \tag{8}$$

It is easy to check that if (8) is a C^∞ function then it is a solution of (6'), and the fact that (8) is a C^∞ function follows from the flatness of G_t on the y-axes: for all negative s and any positive r the norm of $G_t(e^s x, y)$ in a fixed neighbourhood of 0 can be estimated from above by $C_r e^{sr}$, where C_r is a constant, and the same is true for any partial derivative of G_t. Theorem 1 is proved.

Proof of Theorem 2. Assume that the vector fields v, ξ_1 and ξ_2 satisfy (G3), $v(0) = 0$ and Ψ is an orientation reversing symmetry of the foliation F_v: $v = Q\Psi_* v$, where $Q(0)det(\Psi')(0) < 0$. Let $\tilde{\xi}_1 = \Psi_* \xi_1$. Since $v(0) = 0$ it follows that $[\tilde{\xi}_1, v](0) = Q(0)(\Psi_*[\xi_1, v])(0)$ and then $\Omega(\tilde{\xi}_1(0), [\tilde{\xi}_1, v](0)) = Q(0)(\Psi^*\Omega)(\xi_1(0), [\xi_1, v](0))$. This relation and the condition (G3) imply that (G3) holds when replacing ξ_1 by $\tilde{\xi}_1$. By Theorem 1 there exists a diffeomorphism Φ preserving the foliation F_v and sending $F_{\tilde{\xi}_1}$ to F_{ξ_2}. Then the diffeomorphism $\Phi\Psi$ also preserves F_v and sends $F_{\tilde{\xi}_1}$ to F_{ξ_2}.

Now let us show that if (G3) holds and the pair (F_{ξ_1}, F_v) is equivalent to the pair (F_{ξ_2}, F_v) then F_v admits an orientation reversing symmetry provided that 0 is an algebraically isolated singular point of v. The latter condition will be used as follows: it implies that any diffeomorphism preserving the foliation F_v sends v to Qv, where Q is a function (without condition of algebraic isolateness this is, in general, not true). In fact, if the origin is an algebraically isolated singularity of v then by the results of [8] v has the following division property: if $\tilde{v}$ is another vector field such that $v \wedge \tilde{v} \equiv 0$ (i.e., the vectors $v(x)$ and $\tilde{v}(x)$ span $T_x R^2$ at no point $x \in R^2$) then $\tilde{v} = Qv$ for some function Q. Note now that if v and $\Phi_* v$ define the same foliation then $v \wedge \Phi_* v \equiv 0$ and by the mentioned result $\Phi_* v = Qv$.

It follows that if the pair (F_{ξ_1}, F_v) is equivalent to the pair (F_{ξ_2}, F_v) via a diffeomorphism Φ then there exist functions Q and H such that $v = Q\Phi_* v$ and $\xi_2 = H\Phi_*\xi_1$. Then it is easy to compute that

$$\Omega(\xi_2(0), [\xi_2, v](0)) = H_2^2(0)Q(0)(\Phi^*\Omega)(\xi_1(0), [\xi_1, v](0)),$$

and the relation (G3) implies that $Q(0)det(\Phi')(0) < 0$, i.e., Φ is an orientation reversing symmetry of the foliation F_v. Theorem 2 is proved.

Proof of Corollaries 2 and 3. By Proposition 1, the validity of the condition (G1) depends only on the phase curves of ξ_1 and ξ_2 passing through the origin, therefore to prove Corollaries 2 and 3 it suffices to show that the condition (G1) is equivalent to the condition (a) in Corollary 2 (we assume that ξ_1 and ξ_2 do not vanish and their phase curves γ_1 and γ_2 passing through the origin are given by equations $f_1 = 0$ and $f_2 = 0$ respectively).

At first let us notice that the condition $(df \wedge d(v(f)))(0) \neq 0$ is equivalent to the condition that $T_0\gamma$ is not an eigenspace of the linearization of v. To show this, one can take local coordinates x, y in which $f = x$ and the linear approximation of v is $\dot{x} = a_{11}x + a_{12}y,\ \dot{y} = a_{21}x + a_{22}y$, then the condition $(df \wedge d(v(f)))(0) = 0$ is equivalent to the condition $a_{12} = 0$ which holds if and only if γ is tangent to one of the eigenspaces of the linearization of v.

Now repeating the proof of Proposition 1 we can show that the condition (a) in Corollary 2 holds if and only if the points $T_0\gamma_1$ and $T_0\gamma_2$ in the projective line $P(T_0R^2)$ belong to the same connected component of the set $P(T_0R^2) - E$, where E consists of points corresponding to the eigenspaces of the linearization of v. By Proposition 1 this condition is equivalent to (G1). Therefore, the corollaries are proved.

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