



# Representations of generalized bound path algebras

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Accepted: 18 April 2023

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## Abstract

The concept of generalized path algebras was introduced in Coelho and Liu (Lecture Notes in Pure and Appl. Math. 210, 53–66, 2000). Roughly speaking, these algebras are constructed in a way similar to that of path algebras over a quiver, the difference being that we assign an algebra to each vertex of the quiver and consider paths intercalated with elements from these algebras. Then we use concatenation of paths together with the algebra structure in each vertex to define multiplication. The representations of a generalized path algebra were described in Cobos et al. (Rev Roumaine Math Pures Appl 53(1):25–36, 2008), in terms of the representations of the algebras used in its construction. In this article, we continue our investigation started in Chust and Coelho (Comm Algebra 50(5):2056–2071, 2022) and extend the result mentioned above to describe the representations of the generalized bound path algebras, which are a quotient of a generalized path algebra by an ideal generated by relations. In particular, the representations associated with the projective and injective modules are described.

**Keywords** Generalized path algebras · Generalized bound path algebras · Representations of generalized path algebras · Representations of generalized bound path algebras

**Mathematics Subject Classification** 16G10 · 16G20

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Communicated by Vyacheslav Futorny.

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## 1 Introduction

It is a well-established fact that any finite dimensional basic algebra  $A$  over an algebraically closed field  $k$  can be seen as the quotient of a path algebra, that is,  $A \cong kQ/I$ , where  $Q$  is a finite quiver and  $I$  is an admissible ideal of  $kQ$  (see for instance [1, 2]). In [6], Coelho and Liu studied a generalization of such construction. There, it is assigned an algebra to each vertex of a given finite quiver  $Q$  instead of assigning just the base field. The multiplication in such a construction is given not only by the concatenation of paths over the quiver but also by the multiplication of the algebras associated with the vertices.

More specifically, let  $\Gamma$  denote a finite quiver and  $\mathcal{A} = \{A_i : i \in \Gamma_0\}$  denote a family of basic algebras of finite dimension over an algebraically closed field  $k$  indexed by the set  $\Gamma_0$  of the vertices of  $\Gamma$ . Consider also a set of relations  $I$  on the paths of  $\Gamma$ . In [3], to such a data we have assigned a generalized bound path algebra  $\Lambda = k(\Gamma, \mathcal{A}, I)$  with a natural multiplication (see preliminaries for details).

In [6], where the particular case when  $I = 0$  is considered, the main interest was more of ring-theoretic nature, but clearly, such a construction can be also very useful from the point of view of Representation Theory. In [3], we start our work in this direction for the general case. Observe that any basic algebra  $A$  of finite dimension over  $k$  algebraically closed can be naturally realized as a generalized bound path algebra in two ways. The first, by the well-known description as the quotient of a path algebra. And the other, by taking a quiver with a sole vertex and no arrows and the algebra  $A$  itself assigned to it. Since for most algebras these are the only possibilities, one can wonder which algebras can possibly be described as generalized bound path algebras in a different way from these two above (what we would call a **non-trivial simplification of  $A$** ). Such a description could be useful once one aims to look at properties of a given algebra relatively to those of the smaller ones. We deal with this problem in [3].

Here, following the same strategies of our previous work, the focus will be on the representations of a generalized bound path algebra. When  $I = 0$ , this has been considered in [5] and we shall generalize their results here (Theorem 1). Descriptions of the representations of the projective, injective and simple modules are also given.

This paper is organized as follows. Section 2 below is devoted to the preliminaries needed along the paper. In Sect. 3 we prove the above mentioned theorem which describes the representations of a given generalized bound path algebra. After establishing useful ideas in Sect. 4, Sect. 5 is devoted to the description of the projective, injective and simple modules.

In a forthcoming paper [4], we shall look at some homological relations between the algebras  $A_i$  and the whole algebra  $\Lambda$ .

## 2 Preliminaries

We recall here some basic notions and establish some notations needed along the paper. We indicate the books [1, 2] where details on Representation Theory can be found. For an algebra, we mean an associative and unitary basic algebra of finite dimension

over an algebraically closed field  $k$ . Unless otherwise stated, the modules considered here are right modules.

### 2.1 Quivers and path algebras

A **quiver**  $Q$  is given by  $(Q_0, Q_1, s, e)$  where  $Q_0$  is the set of **vertices**,  $Q_1$  is the set of **arrows** and  $s, e: Q_1 \rightarrow Q_0$  are functions which indicate, for each arrow  $\alpha \in Q_1$ , the **starting vertex**  $s(\alpha) \in Q_0$  of  $\alpha$  and the **ending vertex**  $e(\alpha) \in Q_0$  of  $\alpha$ . Naturally, given a quiver  $Q$  one can consider the **path algebra**  $kQ$  with a  $k$ -basis given by all the paths over  $Q$  and multiplication of elements in this basis defined by concatenation. Even when  $Q$  is finite (that is, when  $Q_0, Q_1$  are finite sets), the corresponding algebra might not be finite dimensional. However, a well-known result established by Gabriel states that given an algebra  $A$ , there exists a finite quiver  $Q$  and a set of relations on the paths of  $Q$  which generates an admissible ideal  $I$  such that  $A \cong kQ/I$ . (See, for example, [1], Section I.2 for details).

Along this paper we assume that quivers are finite.

### 2.2 Generalized path algebras

We shall now recall the definition of a generalized path algebra given in [6].

Let  $\Gamma = (\Gamma_0, \Gamma_1, s, e)$  be a quiver and  $\mathcal{A} = (A_i)_{i \in \Gamma_0}$  be a family of algebras, indexed by  $\Gamma_0$ . An  $\mathcal{A}$ -**path of length**  $n$  over  $\Gamma$  is defined as follows. If  $n = 0$ , it is just an element of  $\bigcup_{i \in \Gamma_0} A_i$ , and, if  $n > 0$ , it is a sequence of the form

$$a_1\beta_1a_2 \dots a_n\beta_na_{n+1}$$

where  $\beta_1 \dots \beta_n$  is an ordinary path over  $\Gamma$ ,  $a_i \in A_{s(\beta_i)}$  if  $i \leq n$ , and  $a_{n+1} \in A_{e(\beta_n)}$ . Denote by  $k[\Gamma, \mathcal{A}]$  the  $k$ -vector space spanned by all  $\mathcal{A}$ -paths over  $\Gamma$ . We shall give it a structure of algebra as follows.

First, consider the quotient vector space  $k(\Gamma, \mathcal{A}) = k[\Gamma, \mathcal{A}]/M$ , where  $M$  is the subspace generated by all elements of the form

$$(a_1\beta_1 \dots \beta_{j-1}(a_j^1 + \dots + a_j^m)\beta_j a_{j+1} \dots \beta_n a_{n+1}) - \sum_{l=1}^m (a_1\beta_1 \dots \beta_{j-1}a_j^l \beta_j \dots \beta_n a_{n+1})$$

or, for  $\lambda \in k$ ,

$$(a_1\beta_1 \dots \beta_{j-1}(\lambda a_j)\beta_j a_{j+1} \dots \beta_n a_{n+1}) - \lambda \cdot (a_1\beta_1 \dots \beta_{j-1}a_j \beta_j a_{j+1} \dots \beta_n a_{n+1}).$$

Now, consider the multiplication in  $k(\Gamma, \mathcal{A})$  induced by those of the  $A_i$ 's and by composition of paths. Namely, it is defined by linearity and the following rule:

$$(a_1\beta_1 \dots \beta_n a_{n+1})(b_1\gamma_1 \dots \gamma_m b_{m+1}) = a_1\beta_1 \dots \beta_n (a_{n+1}b_1)\gamma_1 \dots \gamma_m b_{m+1}$$

if  $e(\beta_n) = s(\gamma_1)$ , and

$$(a_1\beta_1 \dots \beta_n a_{n+1})(b_1\gamma_1 \dots \gamma_m b_{m+1}) = 0$$

otherwise.

With this multiplication, we call  $k(\Gamma, \mathcal{A})$  the **generalized path algebra** of  $\Gamma$  and  $\mathcal{A}$ .

**Remark 1** Clearly, the ordinary path algebras are particular cases of generalized path algebras, simply by taking  $A_i = k$  for every  $i \in \Gamma_0$ .

Note that a generalized path algebra  $k(\Gamma, \mathcal{A})$  is associative. And since we are assuming the quivers to be finite, it also has an identity element, which is equal to  $\sum_{i \in \Gamma_0} 1_{A_i}$ . Finally, it is easy to observe that  $k(\Gamma, \mathcal{A})$  is finite-dimensional over  $k$  if and only if so are the algebras  $A_i$  and if  $\Gamma$  is acyclic.

**Remark 2** As observed in [6], if  $k(\Gamma, \mathcal{A})$  is a generalized path algebra, then it is a tensor algebra: if  $A_{\mathcal{A}} = \prod_{i \in \Gamma_0} A_i$  is the product of the algebras in  $\mathcal{A}$ , then there is an  $(A_{\mathcal{A}} - A_{\mathcal{A}})$ -bimodule  $M_{\mathcal{A}}$  such that  $k(\Gamma, \mathcal{A}) \cong T(A_{\mathcal{A}}, M_{\mathcal{A}})$ .

### 2.3 Generalized bound path algebras (gbp-algebras)

Following [3], we shall extend the definition of generalized path algebras to allow them to have relations. In doing so, these algebras will be called **generalized bound path algebras** or **gbp-algebras** for short. As observed in [3], the idea of taking the quotient of a generalized path algebra by an ideal of relations has already been studied by Li Fang (see [7] for example). However, in [3] and here we deal with a slightly different concept, since in order to prove the results below, we consider an ideal of relations which is in general bigger roughly speaking.

Observe that if  $A_i \in \mathcal{A}$ , then, as explained in Sect. 2.1, there is a quiver  $\Sigma_i$  such that  $A_i \cong k\Sigma_i/\Omega_i$  where  $\Omega_i$  is an admissible ideal of  $k\Sigma_i$ . Let now  $I$  be a finite set of relations over  $\Gamma$  which generates an admissible ideal in  $k\Gamma$ . Consider the ideal  $(\mathcal{A}(I))$  generated by the following subset of  $k(\Gamma, \mathcal{A})$ :

$$\mathcal{A}(I) = \left\{ \sum_{i=1}^t \lambda_i \beta_{i1} \overline{\gamma_{i1}} \beta_{i2} \dots \overline{\gamma_{i(m_i-1)}} \beta_{im_i} : \right. \\ \left. \sum_{i=1}^t \lambda_i \beta_{i1} \dots \beta_{im_i} \text{ is a relation in } I \text{ and } \gamma_{ij} \text{ is a path in } \Sigma_{e(\beta_{ij})} \right\}$$

The quotient  $\frac{k(\Gamma, \mathcal{A})}{(\mathcal{A}(I))}$  is said to be a **generalized bound path algebra (gbp-algebra)**. To simplify the notation, we may also write  $\frac{k(\Gamma, \mathcal{A})}{(\mathcal{A}(I))} = k(\Gamma, \mathcal{A}, I)$ . When the context is clear, we may denote the set  $\mathcal{A}(I)$  simply by  $I$ .

### 2.4 Notations

We use the following notation in this article:  $\Gamma$  denotes an acyclic quiver,  $\mathcal{A} = \{A_i : i \in \Gamma_0\}$  denotes a family of basic algebras of finite dimension over an algebraically closed field  $k$ , and  $I$  is a set of relations in  $\Gamma$  generating an admissible ideal in the path algebra  $k\Gamma$ . We also denote by  $\Lambda = k(\Gamma, \mathcal{A}, I)$  the generalized bound path algebra (gbp-algebra) obtained from these objects. Also,  $A_{\mathcal{A}}$  denotes the product algebra  $\prod_{i \in \Gamma_0} A_i$ . For the purpose of simplifying notation, we also denote the identity element of the algebras  $A_i$  by  $1_i$  instead of  $1_{A_i}$ .

### 3 Representations

The aim of this section is to prove Theorem 1 below, which is an extension of Theorem 2.4 from [5]. As already mentioned above, this result is of key importance here, and sometimes we use it without further clarification.

Based on [5], we start by defining what are generalized representations. However, before this we do a remark about the notation used here:

**Remark 3** Generally speaking, if  $A$  is an algebra and  $M$  is a vector space, an action of  $A$  over  $M$  which turns  $M$  into an  $A$ -module is equivalent to a homomorphism of algebras  $\phi : A \rightarrow \text{End}_k M$ . (This correspondence is given by  $\phi(a)(m) = m.a$  for all  $a \in A$  and  $m \in M$ ). That way, if we understand this correspondence as being canonical, then, at least in the concepts to be treated below, an element  $a$  of  $A$  could denote either the element itself or  $\phi(a)$ , which is the endomorphism given by right translation through  $a: m \mapsto m.a$  for all  $m \in M$ . This shall be done in order to simplify the notations.

**Definition 1** Let  $\Lambda = k(\Gamma, \mathcal{A}, I)$  be a generalized bound path algebra. (a) A **representation** of  $\Lambda$  is given by  $((M_i)_{i \in \Gamma_0}, (M_\alpha)_{\alpha \in \Gamma_1})$  where

- (i) for every  $i \in \Gamma_0$ ,  $M_i$  is an  $A_i$ -module;
- (ii) for every arrow  $\alpha \in \Gamma_1$ ,  $M_\alpha : M_{s(\alpha)} \rightarrow M_{e(\alpha)}$  is a  $k$ -linear transformation.
- (iii) it satisfies any relation  $\gamma$  of  $I$ . That is, if  $\gamma = \sum_{i=1}^t \lambda_i \alpha_{i1} \alpha_{i2} \dots \alpha_{in_i}$  is a relation in  $I$  with  $\lambda_i \in k$  and  $\alpha_{ij} \in \Gamma_1$ , then

$$\sum_{i=1}^t \lambda_i M_{\alpha_{in_i}} \circ \overline{\gamma_{in_i}} \circ \dots \circ M_{\alpha_{i2}} \circ \overline{\gamma_{i2}} \circ M_{\alpha_{i1}} = 0$$

for every choice of paths  $\gamma_{ij}$  over  $\Sigma_{s(\alpha_{ij})}$ , with  $1 \leq i \leq t, 2 \leq j \leq n_i$ .

(b) We say that a representation  $((M_i)_{i \in \Gamma_0}, (M_\alpha)_{\alpha \in \Gamma_1})$  of  $\Lambda$  is **finitely generated** if each of the  $A_i$ -modules  $M_i$  is finitely generated.

(c) Let  $M = ((M_i)_{i \in \Gamma_0}, (M_\alpha)_{\alpha \in \Gamma_1})$  and  $N = ((N_i)_{i \in \Gamma_0}, (N_\alpha)_{\alpha \in \Gamma_1})$  be representations of  $\Lambda$ . A **morphism of representations**  $f : M \rightarrow N$  is given by a tuple  $f = (f_i)_{i \in \Gamma_0}$ , such that, for every  $i \in \Gamma_0$ ,  $f_i : M_i \rightarrow N_i$  is a morphism of  $A_i$ -modules; and such that, for every arrow  $\alpha : i \rightarrow j \in \Gamma_1$ , it holds that  $f_j M_\alpha = N_\alpha f_i$ , that is, the following diagram commutes:

$$\begin{array}{ccc}
 M_i & \xrightarrow{M_\alpha} & M_j \\
 f_i \downarrow & & \downarrow f_j \\
 N_i & \xrightarrow{N_\alpha} & N_j
 \end{array}$$

We denote by  $\text{Rep}_k(\Gamma, \mathcal{A}, I)$  (or  $\text{rep}_k(\Gamma, \mathcal{A}, I)$ , respectively) the category of the representations (or finitely generated representations) of the algebra  $k(\Gamma, \mathcal{A}, I)$ .

The next step will be to establish the promised equivalence between  $k(\Gamma, \mathcal{A}, I)$ -representations and  $\Lambda$ -modules, thus generalizing the well-known result of Gabriel for representations and also Theorem 2.4 from [5], where the equivalence was established only in the case  $I = \emptyset$ . The construction of the functors  $F$  and  $G$  is essentially the same of the original proof, but, for completeness, we will repeat it here.

**Theorem 1** (compare with [5], Theorem 2.4) *There is a  $k$ -linear equivalence*

$$F : \text{Rep}_k(\Gamma, \mathcal{A}, I) \rightarrow \text{Mod } k(\Gamma, \mathcal{A}, I)$$

which restricts to an equivalence

$$F : \text{rep}_k(\Gamma, \mathcal{A}, I) \rightarrow \text{mod } k(\Gamma, \mathcal{A}, I)$$

**Proof** For a given representation  $M = ((M_i)_{i \in \Gamma_0}, (M_\alpha)_{\alpha \in \Gamma_1})$  in  $\text{Rep}_k(\Gamma, \mathcal{A}, I)$ , define

$$F(M) = \bigoplus_{i \in \Gamma_0} M_i$$

which will be an object in  $\text{Mod } k(\Gamma, \mathcal{A}, I)$ .

We have to define the action of  $\Lambda$  over  $F(M)$  in such a way that  $F(M)$  is indeed an object in  $\text{Mod } k(\Gamma, \mathcal{A}, I)$ . This is equivalent to constructing a homomorphism of algebras  $\Phi : \Lambda \rightarrow \text{End } F(M)$ . The idea is to use the universal property of tensor algebras (see [5], Lemma 2.1). Let  $A_{\mathcal{A}}$  and  $M_{\mathcal{A}}$  be as in Remark 2.

First we define a homomorphism of algebras

$$\phi_0 : A_{\mathcal{A}} \rightarrow \text{End}_k F(M)$$

given by

$$\phi_0(a_i)((x_l)_{l \in \Gamma_0}) = (\delta_{li} x_i a_i)_{l \in \Gamma_0}$$

for all  $i \in \Gamma_0$ , for all  $a_i \in A_i$  and all  $(x_l)_{l \in \Gamma_0} \in F(M)$ , where  $\delta_{li}$  is a Kronecker's delta. We also define a morphism of  $(A_{\mathcal{A}} - A_{\mathcal{A}})$ -bimodules

$$\phi_1 : M_{\mathcal{A}} \rightarrow \text{End}_k F(M)$$

as follows: for every  $\mathcal{A}$ -path  $a_i \alpha a_j$  of length 1, where  $\alpha : i \rightarrow j$  is an arrow of  $\Gamma$ ,  $a_i \in A_i, a_j \in A_j$ , and for every tuple  $(x_l)_{l \in \Gamma_0} \in F(M)$ , define

$$\phi_1(a_i \alpha a_j)((x_l)_{l \in \Gamma_0}) = (\delta_{lj} M_\alpha(x_i a_i) a_j)_{l \in \Gamma_0}$$

Now, since  $k(\Gamma, \mathcal{A}) = T(A_{\mathcal{A}}, M_{\mathcal{A}})$ , by the universal property of tensor algebras ([5], Lemma 2.1), there is a homomorphism of algebras

$$\phi : k(\Gamma, \mathcal{A}) \rightarrow \text{End}_k F(M)$$

uniquely determined by the property that  $\phi|_{A_{\mathcal{A}}} = \phi_0$  and  $\phi|_{M_{\mathcal{A}}} = \phi_1$ . This shows that  $F(M)$  is a  $k(\Gamma, \mathcal{A})$ -module. In order to show that  $F(M)$  is a module over  $\Lambda = k(\Gamma, \mathcal{A}, I)$ , it suffices to show that  $\phi(I) = 0$ , because then, due to the Homomorphism Theorem,  $\phi$  induces a homomorphism of algebras  $\Phi : k(\Gamma, \mathcal{A})/I \rightarrow \text{End}_k F(M)$ .

Therefore let us verify that  $\phi(I) = 0$ . Let  $\rho = \sum_{r=1}^t \lambda_r \alpha_{r1} \dots \alpha_{rn_r}$  be a relation in  $I$ , where  $\lambda_r \in k$  and the sequences  $\alpha_{r1} \dots \alpha_{rn_r}$  are paths over  $\Gamma$  starting and ending at the same vertex. And let, for every  $1 \leq r \leq t$  and  $2 \leq j \leq n_r$ ,  $\gamma_{rj}$  be a path over  $\Sigma_S(\alpha_{rj})$ . Then:

$$\begin{aligned} & \phi \left( \sum_{r=1}^t \lambda_r \alpha_{r1} \overline{\gamma_{r2}} \alpha_{r2} \dots \overline{\gamma_{rn_r}} \alpha_{rn_r} \right) \\ &= \sum_{r=1}^t \lambda_r \phi(\alpha_{r1} \overline{\gamma_{r2}} \alpha_{r2} \dots \overline{\gamma_{rn_r}} \alpha_{rn_r}) \\ &= \sum_{r=1}^t \lambda_r \iota_{e(\alpha_{rn_r})} \circ M_{\alpha_{rn_r}} \circ \overline{\gamma_{rn_r}} \circ \dots \circ M_{\alpha_{r2}} \circ \overline{\gamma_{r2}} \circ M_{\alpha_{r1}} \circ \pi_{S(\alpha_{r1})} \\ &= \iota_{e(\alpha_{1n_1})} \circ \left( \sum_{r=1}^t \lambda_r M_{\alpha_{rn_r}} \circ \overline{\gamma_{rn_r}} \circ \dots \circ M_{\alpha_{r2}} \circ \overline{\gamma_{r2}} \circ M_{\alpha_{r1}} \right) \circ \pi_{S(\alpha_{11})} = 0 \end{aligned}$$

where  $\iota$  and  $\pi$  denote respectively canonical inclusions and projections, and the last equality above holds because  $M$  satisfies  $\rho$ . We need to see how  $F$  acts on morphisms. Let  $f = (f_i)_{i \in \Gamma_0} : M \rightarrow N$  be a morphism of representations, where  $M = ((M_i)_{i \in \Gamma_0}, (M_\alpha)_{\alpha \in \Gamma_1})$  and  $N = ((N_i)_{i \in \Gamma_0}, (N_\alpha)_{\alpha \in \Gamma_1})$  are representations satisfying  $I$ . Then each  $f_i : M_i \rightarrow N_i$  is a morphism of  $A_i$ -modules, and thus we may define a linear map

$$F(f) : F(M) = \bigoplus_{i \in \Gamma_0} M_i \rightarrow F(N) = \bigoplus_{j \in \Gamma_0} N_j$$

by establishing that the  $(i, j)$ -th coordinate of  $F(f)$  is  $\delta_{ij} f_j$ . It can be shown that  $F(f)$  is a morphism of  $\Lambda$ -modules and that  $F$  defined as such is indeed a functor.

We now define the quasi-inverse functor of  $F$ :

$$G : \text{Mod } k(\Gamma, \mathcal{A}) \rightarrow \text{Rep}_k(\Gamma, \mathcal{A})$$

Let  $M$  be a module over  $\Lambda$ . We need to define a  $k(\Gamma, \mathcal{A})$ -representation  $G(M) = ((M_i)_{i \in \Gamma_0}, (\phi_\alpha)_{\alpha \in \Gamma_1})$  which satisfies  $I$ .

- For each  $i \in \Gamma_0$ ,  $M_i$  is defined by  $M_i \doteq M \cdot 1_i$ , which is clearly an  $A_i$ -module.
- For each arrow  $\alpha : i \rightarrow j \in \Gamma_1$ , define the  $k$ -linear map  $M_\alpha : M_i \rightarrow M_j$  given by  $\phi_\alpha(m) = m \cdot \alpha$ .

To show that  $G(M)$  thus defined satisfies  $I$ , let  $\rho = \sum_{r=1}^t \lambda_r \alpha_{r1} \dots \alpha_{rn_r}$  be a relation in  $I$ , where  $\lambda_r \in k$  and the sequences  $\alpha_{r1} \dots \alpha_{rn_r}$  are paths over  $\Gamma$  that start and end at the same vertex. Also let, for each  $1 \leq r \leq t$  and  $2 \leq j \leq n_r$ ,  $\gamma_{rj}$  be a path over  $\Sigma_{S(\alpha_{rj})}$ . Then, for  $m \in M_{S(\alpha_{r1})}$ ,

$$\begin{aligned} & \left( \sum_{r=1}^t \lambda_r M_{\alpha_{rn_r}} \circ \overline{\gamma_{rn_r}} \circ \dots \circ M_{\alpha_{r2}} \circ \overline{\gamma_{\alpha_{r2}}} \circ M_{\alpha_{r1}} \right) (m) \\ &= \left( \sum_{r=1}^t \lambda_r M_{\alpha_{rn_r}} \circ \overline{\gamma_{rn_r}} \circ \dots \circ M_{\alpha_{r2}} \circ \overline{\gamma_{r2}} \right) (m\alpha_{r1}) \\ &= \left( \sum_{r=1}^t \lambda_r M_{\alpha_{rn_r}} \circ \overline{\gamma_{rn_r}} \circ \dots \circ M_{\alpha_{r2}} \right) (m\alpha_{r1}\overline{\gamma_{r2}}) \\ &= \dots = \sum_{r=1}^t \lambda_r m\alpha_{r1}\overline{\gamma_{r2}} \dots \overline{\gamma_{rn_r}}\alpha_{rn_r} \\ &= m \left( \sum_{r=1}^t \lambda_r \alpha_{r1}\overline{\gamma_{r2}} \dots \overline{\gamma_{rn_r}}\alpha_{rn_r} \right) = 0 \end{aligned}$$

The last equality above holds because the expression that multiplies  $m$  is equal to 0 in  $\Lambda$ . We have thus shown that  $G(M)$  is an object in  $\text{Rep}_k(\Gamma, \mathcal{A}, I)$ .

Let  $g : M \rightarrow N$  be a morphism in  $\text{Mod } \Lambda$ . We define its image under  $G$ :

$$\begin{aligned} G(g) &= (G(g)_i)_{i \in \Gamma_0} \\ G(g)_i &: M_i \rightarrow N_i, G(g)_i \doteq g|_{M_i} \end{aligned}$$

It is immediately verified that  $G(g)_i$  is well-defined and is a morphism of  $A_i$ -modules for every  $i \in \Gamma_0$ . Let us show that  $G(g)$  is a morphism of representations. Let  $\alpha : i \rightarrow j$  be an arrow in  $\Gamma$ . Then, for every  $m \in M$ ,  $G(g)_j \circ M_\alpha(m \cdot 1_i) = G(g)_j(m\alpha) = g(m\alpha) = g(m)\alpha = G(g)_i(m \cdot 1_i)\alpha = N_\alpha \circ G(g)_i(m \cdot 1_i)$ . So  $G(g)_j \circ M_\alpha = N_\alpha \circ G(g)_i$ , which means that the following diagram commutes:

$$\begin{array}{ccc} M_i & \xrightarrow{M_\alpha} & M_j \\ G(g)_i \downarrow & & \downarrow G(g)_j \\ N_i & \xrightarrow{N_\alpha} & N_j \end{array}$$

Therefore  $G(g)$  is a morphism of representations. It is straightforward to show that  $G$  defined this way is a functor. It is also directly verified that:

- $F$  and  $G$  are quasi-inverse functors and are therefore equivalences.
- $F$  maps finitely generated representations to finitely generated modules, while  $G$  does the opposite. Thus the restrictions of these functors to these subcategories are still quasi-inverse equivalences.

□

**Example 1** In this example we illustrate Theorem 1 above. Let  $A$  be the path algebra given by the quiver



bound by  $\gamma^n = 0$ , where  $n > 1$ . Then consider the gbp-algebra  $\Lambda = k(\Gamma, \mathcal{A}, I)$ , where  $\Gamma$  is the quiver below:

$$1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$$

and where  $\mathcal{A} = \{A_1, A_2, A_3\}$ , with  $A_1 = A_3 = k$ ,  $A_2 = A$ , and  $I = (\alpha\beta)$ . Or simpler,  $\Lambda$  is the gbp-algebra given by

$$k \xrightarrow{\alpha} A \xrightarrow{\beta} k$$

bound by  $\alpha\beta = 0$ . Using the proof of Theorem 1, we are going to calculate the representation associated with the projective  $\Lambda$ -module  $P = 1_{A_1} \cdot \Lambda$ .

We have that  $P_1 = P \cdot 1_{A_1} = 1_{A_1} \cdot \Lambda \cdot 1_{A_1} = (1_{A_1})$  is the  $k$ -vector space spanned by  $1_{A_1}$ . Moreover,  $P_2 = P \cdot 1_{A_2} = 1_{A_1} \cdot \Lambda \cdot 1_{A_2} = (\alpha, \alpha\bar{\gamma}, \dots, \alpha\bar{\gamma}^{n-1})$ , which is a right  $A$ -module easily seen to be isomorphic to the regular  $A$ -module  $A$ . And also  $P_3 = P \cdot 1_{A_3} = 1_{A_1} \cdot \Lambda \cdot 1_{A_3} = 0$  since  $I = (\alpha\beta)$  and thus every  $\mathcal{A}$ -path of the form  $\alpha\bar{\gamma}^i\beta$  for  $i \geq 0$  is identified with 0 in  $\Lambda$ .

Now we have that  $P_\alpha$  is given by right multiplication by  $\alpha$ , so it maps the single element of the basis of  $P_1$ , which is  $1_{A_1}$ , to  $1_{A_1} \cdot \alpha = \alpha$  in  $P_2$ .

If we identify  $P_2 \cong A$  and consider the  $k$ -basis  $\{\bar{1}, \bar{\gamma}, \dots, \bar{\gamma}^{n-1}\}$  for  $A$ , we may conclude that the representation associated with the  $\Lambda$ -module  $P$  is the following:

$$P : \quad k \xrightarrow{[1 \ 0 \ \dots \ 0]^T} A \longrightarrow 0$$

Having obtained the equivalence in Theorem 1 as a tool, we are in conditions to study, over the course of the following sections, the representations associated to simple, projective and injective modules over a gbp-algebra, thus generalizing the well-known description that is done for ordinary path algebras.

From now on, we will assume that the modules are always finitely generated.

### 3.1 Opposite algebra

The aim of this subsection is to obtain some useful lemmas involving opposite algebras, opposite quivers and the duality functor. Again we refer to [2] for the definition of these

concepts. For a quiver  $\Gamma$ , denote by  $\Gamma^{op}$  its opposite quiver (that is, the quiver with the same vertices of  $\Gamma$  and with all its arrows reversed). For a set  $I$  of relations in  $\Gamma$ ,  $I^{op}$  will denote the set of relations in  $\Gamma^{op}$  obtained through inversion of the arrows in  $I$ . Also, if  $\mathcal{A} = \{A_i : i \in \Gamma_0\}$  is a family of algebras, denote by  $\mathcal{A}^{op} = \{A_i^{op} : i \in \Gamma_0\}$  the set where  $A_i^{op}$  is the opposite algebra of  $A_i$ . With these notations, we have the following:

**Proposition 2** *If  $\Lambda = k(\Gamma, \mathcal{A}, I)$  is a gbp-algebra, then  $\Lambda^{op} \cong k(\Gamma^{op}, \mathcal{A}^{op}, I^{op})$ .*

**Proof** As recalled in the preliminaries, the generalized path algebra  $k(\Gamma, \mathcal{A})$  is a quotient of a vector space denoted as  $k[\Gamma, \mathcal{A}]$  by a subspace generated by linearity relations. Let us then use the following auxiliar notation:  $k(\Gamma, \mathcal{A}) \doteq k[\Gamma, \mathcal{A}] / \sim$ . In order to avoid confusion, let us also denote the equivalence class (relatively to  $\sim$ ) of an  $\mathcal{A}$ -path  $x$  by  $[x]$ . With these notations we can define a  $k$ -linear map

$$\bar{\phi} : k[\Gamma, \mathcal{A}] \rightarrow k(\Gamma^{op}, \mathcal{A}^{op})$$

by defining it in the  $k$ -basis of  $k[\Gamma, \mathcal{A}]$ :

$$\bar{\phi}(a_0\beta_1a_1 \dots a_{r-1}\beta_r a_r) \doteq [a_r\beta_r a_{r-1} \dots a_1\beta_1 a_0]$$

for each  $\mathcal{A}$ -path  $a_0\beta_1a_1 \dots a_{r-1}\beta_r a_r$ . Then we must show that  $\sim \subseteq \ker \bar{\phi}$ . Indeed:

$$\begin{aligned} &\bar{\phi}(a_0\beta_1a_1 \dots (a_i^1 + \dots + a_i^s) \dots a_{r-1}\beta_r a_r) - \sum_{j=1}^s a_0\beta_1a_1 \dots a_i^j \dots a_{r-1}\beta_r a_r) = \\ &= \bar{\phi}(a_0\beta_1a_1 \dots (a_i^1 + \dots + a_i^s) \dots a_{r-1}\beta_r a_r) - \sum_{j=1}^s \bar{\phi}(a_0\beta_1a_1 \dots a_i^j \dots a_{r-1}\beta_r a_r) = \\ &= [a_r\beta_r a_{r-1} \dots (a_i^1 + \dots + a_i^s) \dots a_1\beta_1 a_0] - \sum_{j=1}^s [a_r\beta_r a_{r-1} \dots a_i^j \dots a_1\beta_1 a_0] = 0 \end{aligned}$$

and, for  $\lambda \in k$ ,

$$\begin{aligned} &\bar{\phi}(a_0\beta_1a_1 \dots \lambda a_i \dots a_{r-1}\beta_r a_r - \lambda(a_0\beta_1a_1 \dots a_i \dots a_{r-1}\beta_r a_r)) \\ &= \bar{\phi}(a_0\beta_1a_1 \dots \lambda a_i \dots a_{r-1}\beta_r a_r) - \lambda \bar{\phi}(a_0\beta_1a_1 \dots a_i \dots a_{r-1}\beta_r a_r) \\ &= [a_r\beta_r a_{r-1} \dots \lambda a_i \dots a_1\beta_1 a_0] - \lambda [a_r\beta_r a_{r-1} \dots a_i \dots a_1\beta_1 a_0] = 0 \end{aligned}$$

We have just shown that there is a  $k$ -linear map

$$\phi : k(\Gamma, \mathcal{A}) \rightarrow k(\Gamma^{op}, \mathcal{A}^{op})$$

satisfying

$$\phi([a_0\beta_1a_1 \dots a_{r-1}\beta_r a_r]) = [a_r\beta_r a_{r-1} \dots a_1\beta_1 a_0]$$

It is easy to see that  $\phi$  is bijective. To conclude the first part of the statement, it remains to show that  $\phi$  is an anti-homomorphism of algebras. It is easy to see that  $\phi$

preserves the identity element. We will thus show that it antipreserves multiplication. Let  $a = [a_0\beta_1a_1 \dots a_{r-1}\beta_r a_r]$  and  $b = [b_0\gamma_1b_1 \dots b_{s-1}\gamma_s b_s]$  be the classes of two  $\mathcal{A}$ -paths. If  $e(\beta_r) \neq s(\gamma_1)$ , it is straightforward to show that  $\phi(ab) = 0 = \phi(b)\phi(a)$ . So suppose that  $e(\beta_r) = s(\gamma_1)$ . In this case,

$$\begin{aligned} \phi(ab) &= \phi([a_0\beta_1a_1 \dots a_{r-1}\beta_r a_r][b_0\gamma_1b_1 \dots b_{s-1}\gamma_s b_s]) \\ &= \phi([a_0\beta_1a_1 \dots a_{r-1}\beta_r(a_r.b_0)\gamma_1b_1 \dots b_{s-1}\gamma_s b_s]) \\ &= [b_s\gamma_s b_{s-1} \dots b_1\gamma_1(a_r.b_0)\beta_r a_{r-1} \dots a_1\beta_1 a_0] \\ &= [b_s\gamma_s b_{s-1} \dots b_1\gamma_1(b_0.op a_r)\beta_r a_{r-1} \dots a_1\beta_1 a_0] \\ &= [b_s\gamma_s b_{s-1} \dots b_1\gamma_1 b_0][a_r\beta_r a_{r-1} \dots a_1\beta_1 a_0] \\ &= \phi([b_0\gamma_1b_1 \dots b_{s-1}\gamma_s b_s])\phi([a_0\beta_1a_1 \dots a_{r-1}\beta_r a_r]) = \phi(b)\phi(a) \end{aligned}$$

This proves that  $k(\Gamma, \mathcal{A})$  is anti-isomorphic to  $k(\Gamma^{op}, \mathcal{A}^{op})$  via  $\phi$ , which is the same to say that  $k(\Gamma, \mathcal{A})^{op}$  is isomorphic to  $k(\Gamma^{op}, \mathcal{A}^{op})$ . To conclude the proof, we realize that the map  $\phi$  defined above satisfies  $\phi(I) = I^{op}$ , and the statement follows directly.  $\square$

### 3.2 Duality

We now use the results of the previous subsection to dualize the representations of the gbp-algebra  $\Lambda$ . Denote by  $D = \text{Hom}_k(-, k)$  the usual duality functor.

**Proposition 3** *Let  $\Lambda = k(\Gamma, \mathcal{A}, I)$  be a gbp-algebra. If  $((M_i)_{i \in \Gamma_0}, (\phi_\alpha)_{\alpha \in \Gamma_1})$  is the representation of the  $\Lambda$ -module  $M$ , then the representation of the  $\Lambda^{op}$ -module  $DM$  is isomorphic to  $(D(M_i)_{i \in \Gamma_0}, D(\phi_\alpha)_{\alpha \in \Gamma_1})$ .*

**Proof** We need to show that the representations  $((DM)_i)_{i \in \Gamma_0}, ((DM)_\alpha)_{\alpha \in \Gamma_1}$  and  $(D(M_i)_{i \in \Gamma_0}, D(\phi_\alpha)_{\alpha \in \Gamma_1})$  are isomorphic. It is useful to recall how the quasi-inverse equivalences  $F$  e  $G$  discussed in the proof of Theorem 1 were like. Let  $i \in \Gamma_0$ . First of all, note that

$$\begin{aligned} DM &= \text{Hom}_k(M, k), \text{ thus } (DM)_i = 1_i(\text{Hom}_k(M, k)) \\ D(M_i) &= \text{Hom}_k(M_i, k) = \text{Hom}_k(M.1_i, k) \end{aligned}$$

We can define

$$\begin{aligned} f_i : 1_i \text{Hom}_k(M, k) &\rightarrow \text{Hom}_k(M.1_i, k) \\ 1_i \cdot g &\mapsto g|_{M.1_i} \end{aligned}$$

We shall see that  $f_i$  is an isomorphism. It is clear that it is well-defined and  $k$ -linear. To show that  $f_i$  is a morphism of  $A_i^{op}$ -modules, let  $g \in \text{Hom}_k(M, k)$ ,  $a \in A_i^{op}$  and  $x \in M.1_i$ . Then

$$f_i(a \cdot 1_i g)(x) = (a \cdot g)|_{M.1_i}(x) = (a \cdot g)(x) = g(xa) = g(xa1_i) =$$

$$= g|_{M \cdot 1_i}(xa) = f_i(1_i g)(xa) = (a \cdot f_i(1_i g))(x)$$

which implies that  $f_i(a \cdot 1_i g) = a \cdot f_i(1_i g)$ , as required.

Now, to see that  $f_i$  is injective, suppose  $f_i(1_i g) = 0$ . Then  $(1_i g)(x) = 0$  for every  $x \in M \cdot 1_i$  and so  $(1_i \cdot g)(x) = (1_i \cdot g)(x \cdot 1_i) = 0$  for every  $x \in M$ . In particular,  $1_i \cdot g = 0$ , which shows our claim.

It remains to see that  $f_i$  is surjective. Let  $h \in \text{Hom}_k(M \cdot 1_i, k)$ . We know that  $M \cong \bigoplus_{j \in \Gamma_0} M \cdot 1_j$ . We can thus define a  $k$ -linear transformation  $g \in \text{Hom}_k(M, k)$ ,  $g : \bigoplus_{j \in \Gamma_0} M \cdot 1_j \rightarrow k$ ,  $g = (\delta_{ji} h)_{j \in \Gamma_0}$ , where  $\delta_{ji}$  is the Kronecker's delta. Then, if  $x \in M \cdot 1_i$ ,  $f_i(1_i \cdot g)(x) = g|_{M \cdot 1_i}(x) = h(x)$ . Thus  $f_i(1_i \cdot g) = h$ . This concludes the proof that  $f_i$  is an isomorphism of  $A_i$ -modules. The next step is to show the commutativity of the diagram

$$\begin{CD} (DM)_j @>{(DM)_\alpha}>> (DM)_i \\ @V{f_j}VV @VV{f_i}V \\ D(M_j) @>>{D(\phi_\alpha)}> D(M_i) \end{CD}$$

For that, let  $g \in \text{Hom}_k(M, k)$  e  $x \in M$ . Then:

$$\begin{aligned} (f_i \circ (DM)_\alpha)(1_j \cdot g)(x \cdot 1_i) &= f_i((DM)_\alpha(1_j \cdot g))(x \cdot 1_i) = f_i(1_i \alpha g)(x \cdot 1_i) \\ &= (\alpha g)|_{M \cdot 1_i}(x \cdot 1_i) = (\alpha g)(x \cdot 1_i) = g(x\alpha) \\ &= g|_{M \cdot 1_j}(x\alpha 1_j) = g|_{M \cdot 1_j}(x\alpha) = g|_{M \cdot 1_j}(\phi_\alpha(x \cdot 1_i)) \\ &= D(\phi_\alpha)(g|_{M \cdot 1_j})(x \cdot 1_i) = D(\phi_\alpha)(f_j(1_i \cdot g))(x \cdot 1_i) \\ &= (D(\phi_\alpha) \circ f_j)(1_i \cdot g)(x \cdot 1_i) \end{aligned}$$

Hence  $(f_i \circ (DM)_\alpha) = (D(\phi_\alpha) \circ f_j)$ , as required. The fact that  $DM$  satisfies  $I^{op}$  if and only if  $M$  satisfies  $I$  follows easily from the fact that  $D$  is a fully faithful and dense  $k$ -linear functor. □

### 4 Realizing an $A_j$ -module as a $\Lambda$ -module

Let  $i \in \Gamma_0$ , and let  $M$  be a (right)  $A_j$ -module. In this section we shall see three ways of obtaining a  $\Lambda$ -module from  $M$ . The first one is quite natural, while the second one essentially relies on the well-known technique of extension of scalars. By applying the duality functor, we get a third way. It will be interesting to dedicate different notations for each of the three.

### 4.1 The inclusion functors

Given an  $A_i$ -module  $M$ , define the  $\Lambda$ -representation  $\mathcal{I}(M) = ((M_j)_{j \in \Gamma_0}, (\phi_\alpha)_{\alpha \in \Gamma_1})$  given by

$$M_j = \begin{cases} M & \text{if } j = i \\ 0 & \text{if } j \neq i \end{cases} \quad \text{and} \quad \phi_\alpha = 0 \quad \text{for all } \alpha \in \Gamma_1.$$

Clearly, because of Theorem 1,  $\mathcal{I}(M)$  yields a  $\Lambda$ -module, and, since  $\mathcal{I}(M)$  and  $M$  have the same underlying vector space, we may, by abuse of notation, denote  $\mathcal{I}(M) = M$ .

Actually, for every vertex  $i$  we have a functor  $\mathcal{I}_i : \text{mod } A_i \rightarrow \text{mod } \Lambda$  which we shall call **inclusion functor**. (We might even denote it simply by  $\mathcal{I}$  if it is clear which vertex we are talking about). We have just defined its image on objects, and its image on morphisms is defined obviously. It is also easy to see why  $\mathcal{I}$  is called an inclusion functor, because it is covariant and fully faithful.

From now on, unless stated or denoted otherwise, we will always be assuming that we are seeing  $M$  as an  $\Lambda$ -module in this way.

It is not difficult to see that simple  $A_i$ -modules viewed as  $\Lambda$ -modules are also simple. Conversely, any simple  $\Lambda$ -module is of this kind. To see this, we may use a counting argument: for each  $i$ , the set of simple  $A_i$ -modules is in a one-to-one correspondence with any fixed complete set  $E_i$  of primitive idempotent orthogonal elements of  $A_i$ , and it holds that  $E = \cup_{i \in \Gamma_0} E_i$  is a complete set of primitive idempotent orthogonal elements in  $\Lambda$ . So, the description of the simple  $\Lambda$ -modules is easily done.

### 4.2 Cones

We shall now see another way to induce a  $\Lambda$ -module from an  $A_i$ -module  $M$ .

Here again, let  $k(\Gamma, \mathcal{A}) = T(A_{\mathcal{A}}, M_{\mathcal{A}})$  as in Remark 2. Clearly,  $M$  is also an  $A_{\mathcal{A}}$ -module (using the action  $m \cdot (a_j)_j = m \cdot a_i$  for each  $m \in M$  and  $(a_j)_{j \in \Gamma_0} \in A_{\mathcal{A}}$ ).

Since  $\Lambda$  is equal to the quotient  $k(\Gamma, \mathcal{A})/I$ , and  $M_{\mathcal{A}}$  is an  $(A_{\mathcal{A}} - A_{\mathcal{A}})$ -bimodule,  $\Lambda$  is also an  $(A_{\mathcal{A}} - A_{\mathcal{A}})$ -bimodule that contains  $A_{\mathcal{A}}$  as a subalgebra. Therefore it makes sense to consider the extension of scalars of  $M$  to  $\Lambda$ . We shall denote it by  $\mathcal{C}_i(M) = M \otimes_{A_{\mathcal{A}}} \Lambda$ . Just emphasizing, since  $\Lambda$  is a right  $\Lambda$ -module,  $\mathcal{C}_i(M)$  is a right  $\Lambda$ -module too.

**Definition 2**  $\mathcal{C}_i(M)$  is called **cone** over  $M$ .

The reason why we call it a cone is because of the shape that the representation of  $\mathcal{C}_i(M)$  has, as it will be more transparent after the description that will be done later.

The proof of the next proposition is easy and left to the reader:

**Proposition 4** *If  $M$  and  $N$  are  $A_i$ -modules, then  $\mathcal{C}_i(M \oplus N) \cong \mathcal{C}_i(M) \oplus \mathcal{C}_i(N)$ .*

**Remark 4** Since we are assuming  $\Gamma$  to be acyclic, it will be useful to remark that

$$C_i(M) = \left\{ \sum_{\substack{\gamma=\gamma_1 \dots \gamma_r \text{ is a path in } \Gamma \\ s(\gamma_1)=i}} m^\gamma \otimes \overline{\gamma_1 a_{e(\gamma_1)}^\gamma \dots \gamma_r a_{e(\gamma_r)}^\gamma} : m^\gamma \in M, a_{e(\gamma_j)}^\gamma \in A_{e(\gamma_j)} \right\}$$

This equality follows by observing that  $C_i(M) = M \otimes_{A_A} \Lambda = M \cdot 1_i \otimes_{A_A} \Lambda = M \otimes_{A_A} 1_i \cdot \Lambda$ .

The next goal of this subsection is to describe the representation associated to the cone  $C_i(M)$  of an  $A_i$ -module  $M$ .

Let  $((M_j)_{j \in \Gamma_0}, (\phi_\alpha)_{\alpha \in \Gamma_1})$  denote the representation of  $C_i(M)$ . For each  $l \in \Gamma_0$ , let  $\{a_1^l, \dots, a_{\dim_k A_l}^l\}$  denote a  $k$ -basis of  $A_l$ . Also, let  $\{m_1, \dots, m_{\dim_k M}\}$  be a  $k$ -basis of  $M$ .

**Proposition 5** *With the notations above, it holds that  $M_i = M$ , and if  $j \in \Gamma_0$  is different from  $i$ , then  $M_j$  is isomorphic to the free  $A_j$ -module having as basis the set of equivalence classes of the formal sequences of the form*

$$m_p \gamma_1 a_{i_2}^{s(\gamma_2)} \dots a_{i_r}^{s(\gamma_r)} \gamma_r$$

where  $\gamma_1 \dots \gamma_r$  is a path from  $i$  to  $j$ ,  $1 \leq p \leq \dim_k M$  and  $1 \leq i_l \leq \dim_k A_{s(\gamma_l)}$  for every  $1 < l \leq r$ .

Moreover, if  $\alpha : j \rightarrow j'$  is an arrow, then  $\phi_\alpha$  is the only linear transformation that satisfies

$$\phi_\alpha \left( \overline{m_p \gamma_1 a_{i_2}^{s(\gamma_2)} \dots a_{i_r}^{s(\gamma_r)} \gamma_r a_{i_{r+1}}^j} \right) = \overline{m_p \gamma_1 a_{i_2}^{s(\gamma_2)} \dots a_{i_r}^{s(\gamma_r)} \gamma_r a_{i_{r+1}}^j \alpha}$$

**Proof** The key idea here is to recall the equivalence  $G$  constructed in the proof of Theorem 1. By Remark 4, and by the fact that  $\Gamma$  is acyclic,

$$M_i = C_i(M) \cdot 1_i \cong \left\{ \sum_{\gamma:i \rightsquigarrow i} m^\gamma : m^\gamma \in M \right\} = \{m : m \in M\} = M$$

For  $j \neq i$ , we have that

$$M_j = C_i(M) \cdot 1_j = \left\{ \sum_{\gamma=\gamma_1 \dots \gamma_r:i \rightsquigarrow j} m^\gamma \otimes \overline{\gamma_1 a_2^\gamma \gamma_2 \dots a_r^\gamma \gamma_r a_{r+1}^\gamma} : m^\gamma \in M, a_l^\gamma \in A_{s(\gamma_l)} \forall 1 < l \leq r, \text{ and } a_{r+1}^\gamma \in A_j \right\}$$

Since  $\{a_1^l, \dots, a_{\dim_k A_l}^l\}$  is a  $k$ -basis of  $A_l$  and  $\{m_1, \dots, m_{\dim_k M}\}$  is a  $k$ -basis of  $M$ , the above expression equals to

$$\text{span}_k \{ \overline{m_p \otimes \gamma_1 a_{i_2}^{s(\gamma_2)} \dots a_{i_r}^{s(\gamma_r)} \gamma_r a_{r+1}} : \gamma_1 \dots \gamma_r \text{ is a path } i \rightsquigarrow j, \}$$

$$1 \leq p \leq \dim_k M, 1 \leq i_l \leq \dim_k A_{s(\gamma_l)} \quad \forall 1 < l \leq r, \text{ and } a_{r+1} \in A_j \} \quad (1)$$

If one denotes  $\{\theta_1, \dots, \theta_{n_j}\} = \overline{\{m_p \otimes \gamma_1 a_{i_2}^{s(\gamma_2)} \dots a_{i_r}^{s(\gamma_r)} \gamma_r\}}$ , then the expression (1) is equal to

$$\text{span}_k \{\theta_l a : 1 \leq l \leq n_j, a \in A_j\}.$$

An easy calculation shows that it is isomorphic to the free  $A_j$ -module having as basis  $\{\theta_1, \dots, \theta_{n_j}\}$ , as we wanted to prove.

Let  $\alpha : j \rightarrow j'$  be an arrow in  $\Gamma_1$ . Again, by Theorem 1,  $\phi_\alpha : M_j \rightarrow M_{j'}$  is given by

$$\begin{aligned} \phi_\alpha : C_i(M)1_j &\rightarrow C_i(M)1_{j'} \\ m1_j &\mapsto m\alpha \end{aligned}$$

with  $m \in C_i(M)$ . Therefore  $\phi_\alpha$  has the form given in the statement, concluding the proof.  $\square$

**Remark 5** If  $I = 0$ , then it is easier to see how the representation of  $C_i(M)$  looks like: it holds that  $M_i = M$ , and if  $j \neq i$ ,  $M_j \cong A_j^{n_j}$ , where

$$n_j = \sum_{\gamma: i=i_0 \rightarrow i_1 \rightarrow \dots \rightarrow i_{r+1}=j \text{ is a path } i \rightsquigarrow j} (\dim_k M) \cdot (\dim_k A_{i_1}) \cdot \dots \cdot (\dim_k A_{i_r})$$

In particular, if there is no path going from  $i$  to  $j$ ,  $M_j = 0$ .

We finish this subsection with the following result.

**Proposition 6** *The gbp-algebra  $\Lambda$  is projective as a (left)  $A_{\mathcal{A}}$ -module. In particular, given  $i \in \Gamma_0$ , the cone functor  $C_i : \text{mod } A_i \rightarrow \text{mod } \Lambda$  is exact.*

**Proof** Since  $A_{\mathcal{A}} = \prod_{i \in \Gamma_0} A_i$ , in order to prove the first assertion, it is sufficient to prove that  $1_i \Lambda$  is a projective left  $A_i$ -module for every  $i \in \Gamma_0$ . But actually  $1_i \Lambda$  is free as a  $A_i$ -module, by an argument very similar to that in the proof of Proposition 5 above. The second assertion of the statement follows immediately from the first:  $C_i$  is exact because it is equal to the composition of the inclusion functor  $\text{mod } A_i \rightarrow \text{mod } A_{\mathcal{A}}$  with the tensor product  $-\otimes_{A_{\mathcal{A}}} \Lambda : \text{mod } A_{\mathcal{A}} \rightarrow \text{mod } \Lambda$ , with  $\Lambda$  being  $A_{\mathcal{A}}$ -projective.  $\square$

### 4.3 Dual cones

We now dualize the notion of cone.

**Definition 3** Let  $i \in \Gamma_0$ , and let  $M$  be an  $A_i$ -module. Then  $D(M)$  is an  $A_i^{op}$ -module, and therefore the cone  $C_i(DM)$  is a  $\Lambda^{op}$ -module. Finally,  $D(C_i(DM))$  is a  $\Lambda$ -module, which we call **dual cone** of  $M$ . We shall use the notation  $C_i^*(M) \doteq D(C_i(DM))$ .

**Proposition 7** *Given two  $A_i$ -modules  $M$  and  $N$ ,  $C_i^*(M \oplus N) \cong C_i^*(M) \oplus C_i^*(N)$ .*

**Proof** This follows because the duality functor preserves direct sums and because  $C_i$  also preserves direct sums due to Proposition 4. □

**Example 8** Let us give an example to illustrate the differences between the three ways of obtaining a  $\Lambda$ -module from an  $A_i$ -module seen in this section.

Let  $A$  and  $B$  be two finite dimensional algebras over the base field  $k$ . Suppose that  $A$  has dimension 2 over  $k$  and that  $B$  has dimension 3. Consider the gbp-algebra  $\Lambda$  given below:

$$\begin{array}{ccccccc}
 & & & & B & \longrightarrow & A \\
 & & & & \nearrow & & \\
 B & \longrightarrow & k & \xrightarrow{\alpha} & A & \xrightarrow{\beta} & B
 \end{array}$$

bound by  $\alpha\beta = 0$ . Let  $x$  be the vertex of the quiver above to which  $k$  was assigned. If we consider  $k^4$  as a  $\Lambda$ -module via the inclusion functor relative to  $x$ , its representation will be

$$\begin{array}{ccccccc}
 & & & & 0 & \longrightarrow & 0 \\
 & & & & \nearrow & & \\
 0 & \longrightarrow & k^4 & \longrightarrow & 0 & \longrightarrow & 0
 \end{array}$$

By using Proposition 5 above, one concludes that the representation of  $C_x(k^4)$ , which is the cone of  $k^4$ , will be

$$\begin{array}{ccccccc}
 & & & & B^4 & \longrightarrow & A^{12} \\
 & & & & \nearrow & & \\
 0 & \longrightarrow & k^4 & \longrightarrow & A^4 & \longrightarrow & 0
 \end{array}$$

The bottom right vertex needs to be assigned with 0 as a consequence of the existence of the relation  $\alpha\beta = 0$ . Note how the representation of  $C_x(k^4)$  resembles a cone whose vertex is  $x$  and whose basis is the set of vertices which are the end of non-zero paths starting at  $x$ . This is to complement our previous remark explaining why we are calling the functor  $C_x$  a cone. Finally, the dual cone  $C_x^*(k^4)$  of  $k^4$  will be given by

$$\begin{array}{ccccccc}
 & & & & 0 & \longrightarrow & 0 \\
 & & & & \nearrow & & \\
 B^4 & \longrightarrow & k^4 & \longrightarrow & 0 & \longrightarrow & 0
 \end{array}$$

**Remark 6** Proposition 5 allows us to calculate the cones of modules. But thanks to Proposition 3, the same is possible for dual cones: given an  $A_i$ -module  $M$ , we calculate the cone of  $DM$  over  $(\Gamma^{op}, \mathcal{A}^{op}, I^{op})$  and then obtain the dual cone of  $M$  over  $(\Gamma, \mathcal{A}, I)$  using Proposition 3. This proposition tells us that what we need to do is to take the dual of the modules in each vertex and take the transpose linear transformation in each arrow, which, in practical situations, is done by transposing matrices. We shall yield examples of this in Sect. 5.2.

## 5 Projective and injective representations

We shall now apply the results of the previous subsection to describe the indecomposable projective and injective  $\Lambda$ -modules. We remark that [8] contains a description of projective modules over generalized path algebras, although here we manage to extend this to the context of gbp-algebras.

### 5.1 Projective representations

We state the following result, whose proof follows directly from the properties of extension of scalars.

**Proposition 9** *If  $P$  is a projective  $A_i$ -module, then  $C_i(P)$  is a projective  $\Lambda$ -module.*

Now, for each  $i \in \Gamma_0$ , let  $E_i = \{e_{i1}, \dots, e_{is_i}\}$  be a complete set of primitive idempotent and pairwise orthogonal elements in  $A_i$ . Then every indecomposable projective  $A_i$ -module is isomorphic to  $P_i^j \doteq e_{ij}A_i$  for some  $1 \leq j \leq s_i$ . Moreover,  $E = \{\bar{e}_{ij} : i \in \Gamma_0, 1 \leq j \leq s_i\}$  is a complete set of primitive idempotent and pairwise orthogonal elements in  $\Lambda$ . Therefore every indecomposable projective  $\Lambda$ -module is isomorphic to  $P(i, j) \doteq \bar{e}_{ij}\Lambda$  for a certain pair of indexes  $i \in \Gamma_0$  e  $1 \leq j \leq s_i$ .

**Proposition 10** *For each  $i \in \Gamma_0$  and  $1 \leq j \leq s_i$ ,  $P(i, j) = C_i(P_i^j)$ .*

**Proof** Using Remark 4, we have that

$$\begin{aligned} C_i(P_i^j) &= \left\{ \sum_{\substack{\gamma=\gamma_1 \dots \gamma_l \text{ in } \Gamma \\ s(\gamma_1)=i}} m^\gamma \otimes \overline{\gamma_1 a_{e(\gamma_1)}^\gamma \dots \gamma_l a_{e(\gamma_l)}^\gamma} : m^\gamma \in P_i^j, a_{e(\gamma_j)}^\gamma \in A_{e(\gamma_j)} \right\} \\ &= \left\{ \sum_{\substack{\gamma=\gamma_1 \dots \gamma_l \text{ in } \Gamma \\ s(\gamma_1)=i}} \overline{e_{ij} a^\gamma \gamma_1 a_{e(\gamma_1)}^\gamma \dots \gamma_l a_{e(\gamma_l)}^\gamma} : a^\gamma \in A_i, a_{e(\gamma_j)}^\gamma \in A_{e(\gamma_j)} \right\} \\ &= \bar{e}_{ij}\Lambda = P(i, j) \end{aligned}$$

□

Thanks to the last proposition and Proposition 5, we are now able to calculate the representations associated to projective indecomposable modules. The following proposition reflects the particular case of this construction when  $I = 0$ , i.e., when there are no relations:

**Proposition 11** *Suppose  $I = 0$ . Let  $P(i, j) = ((M_i)_{i \in \Gamma_0}, (\phi_\alpha)_{\alpha \in \Gamma_0})$  be the representation associated to  $P(i, j)$ . Then, for  $l \in \Gamma_0$ ,*

- (a) *If  $l = i$ , then  $M_l = M_i = P_i^j$ .*
- (b) *If  $l \neq i$ , denote*

$$n_l = \sum_{\gamma: i=i_0 \rightarrow i_1 \rightarrow \dots \rightarrow i_r=l} (\dim_k P_i^j) \cdot (\dim_k A_{i_1}) \cdot \dots \cdot (\dim_k A_{i_{r-1}})$$

where  $\gamma$  runs through all possible paths  $i \rightsquigarrow l$ .

Then  $M_l \cong (A_l)^{n_l}$  as  $A_l$ -modules. In particular, if there are no paths  $i \rightsquigarrow l$ , then  $M_l = 0$ .

In practical examples, however, difficulties may arise either because the matrices of the  $k$ -linear transformations denoted above as  $\phi_\alpha$  can be too big, or, given their dependence on the choice of a  $k$ -basis of the algebras  $A_i$  or of  $P_i^j$ , there could be some confusion. To avoid that, it is convenient to make use of block matrices. We shall give further details of this in the remark and example below.

**Remark 7** Let  $V$  be a  $k$ -vector space of dimension 1 and fixed basis  $\{v\}$  and let  $A$  be a  $k$ -algebra. Then there is a linear map that shall be treated as canonical from now on: it is defined as  $\mu : V \rightarrow A, \mu(\lambda \cdot v) = \lambda \cdot 1_A$ , where  $\lambda \in k$ . Although the vector space  $V$  may vary, the letter  $\mu$  will always be used for such a map.

**Example 12** Let  $A$  be the path algebra given by the quiver below:

$$1 \longrightarrow 2$$

Then there are two indecomposable projective  $A$ -modules, namely,

$$P_1 : k \xrightarrow{id} k \qquad P_2 : 0 \longrightarrow k$$

Now let  $\Lambda$  be the generalized path algebra given by

$$A \longrightarrow A$$

According to the discussions above, there are exactly 4 indecomposable projective  $\Lambda$ -modules, which are:

$$\begin{array}{ll}
 P(1, 1) : P_1 \xrightarrow{\begin{bmatrix} \mu & 0 \\ 0 & \mu \end{bmatrix}} A^2 & P(1, 2) : P_2 \xrightarrow{[\mu]} A \\
 P(2, 1) : 0 \longrightarrow P_1 & P(2, 2) : 0 \longrightarrow P_2
 \end{array}$$

We can also describe the representations associated to radicals of the projective modules, as expressed in the proposition below:

**Proposition 13** *With the same notations as before, let  $i \in \Gamma_0$  and  $1 \leq j \leq s_i$ . Denote  $P(i, j) = ((M_l)_{l \in \Gamma_0}, (\phi_\alpha)_{\alpha \in \Gamma_1})$ . Then the radical of  $P(i, j)$  is given by the representation  $\text{rad } P(i, j) = ((N_l)_{l \in \Gamma_0}, (\psi_\alpha)_{\alpha \in \Gamma_1})$ , where  $N_i = \text{rad } P_i^j$ ,  $N_l = M_l$  for each  $l \in \Gamma_0$  with  $l \neq i$ , and for each  $\alpha \in \Gamma_1$ ,  $\psi_\alpha = \phi_\alpha|_{M_{s(\alpha)}}$ .*

**Proof** Let  $N = ((N_l)_{l \in \Gamma_0}, (\psi_\alpha)_{\alpha \in \Gamma_1})$ . Note that  $N$  satisfies  $I$  because  $M$  satisfies it. We wish to prove that  $N = \text{rad } P(i, j)$ . Note that, if  $l \neq i$ ,  $N_l = M_l$ , so  $M_l/N_l = 0$ . Moreover,  $M_i = P_i^j$  and  $N_i = \text{rad } P_i^j$ , thus  $M_i/N_i = P_i^j/\text{rad } P_i^j$ . This implies that  $P(i, j)/N$  is isomorphic to the  $A_i$ -module  $P_i^j/\text{rad } P_i^j$  realized as a  $\Lambda$ -module. Since  $P_i^j$  is an indecomposable projective  $A_i$ -module,  $P_i^j/\text{rad } P_i^j$  is a simple  $A_i$ -module, and it is also simple when seen as a  $\Lambda$ -module, as observed in Sect. 4.1. This means that  $P(i, j)/N$  is a simple  $\Lambda$ -module. We have thus proved that  $N$  is a maximal submodule of  $P(i, j)$ , and since  $P(i, j)$  is indecomposable projective, it has a unique maximal submodule, which is  $\text{rad } P(i, j)$ . This concludes the proof that  $N = \text{rad } P(i, j)$ .  $\square$

**Example 14** We continue Example 12 above to apply Proposition 13 and thus obtain the radical of the 4 projective modules seen above. Thus we have:

$$\begin{array}{ll}
 \text{rad } P(1, 1) : \text{rad } P_1 \xrightarrow{\begin{bmatrix} 0 \\ \mu \end{bmatrix}} A^2 & \text{rad } P(1, 2) : 0 \longrightarrow A \\
 \text{rad } P(2, 1) : 0 \longrightarrow \text{rad } P_1 & \text{rad } P(2, 2) : 0 \longrightarrow 0
 \end{array}$$

### 5.2 Injective representations

In this subsection we give a description of the representations associated with indecomposable injective modules. As we shall see, the injective modules will be particular cases of dual cones, in an analogy with the projective modules, which were particular cases of cones, as we saw in Sect. 5.1.

**Proposition 15** For  $i \in \Gamma_0$ , if  $I$  is an injective  $A_i$ -module, then  $C_i^*(I)$  is an injective  $\Lambda$ -module.

**Proof** Since  $I$  is an injective  $A_i$ -module and  $D$  is a duality,  $DI$  is a projective  $A_i^{op}$ -module. Because of Proposition 9,  $C_i(DI)$  is a projective  $\Lambda^{op}$ -module, and again since  $D$  is a duality,  $C_i^*(I) = D(C_i(DI))$  is an injective  $\Lambda$ -module.  $\square$

For each  $i \in \Gamma_0$ , let  $E_i = \{e_{i_1}, \dots, e_{i_{s_i}}\}$  be a complete set of primitive idempotent and pairwise orthogonal elements in  $A_i$ . If  $D : \text{mod } A_i^{op} \rightarrow \text{mod } A_i$  is the duality functor, then it is well-known that a complete set of isomorphism classes of indecomposable injective  $A_i$ -modules is given by  $I_i^1 = D(A_i e_{i_1}), \dots, I_i^{s_i} = D(A_i e_{i_{s_i}})$ .

On the other hand, if  $E = \{\overline{e_{ij}} : i \in \Gamma_0, 1 \leq j \leq s_i\}$ , then  $E$  is a complete set of primitive idempotent and pairwise orthogonal elements in  $\Lambda$ . This means that a complete set of isomorphism classes of indecomposable injective  $\Lambda$ -modules is given by  $\{I(i, j) : i \in \Gamma_0, 1 \leq j \leq s_i\}$ , where  $I(i, j) \doteq D(\Lambda \overline{e_{ij}})$ .

**Proposition 16** With the notations above,  $C_i^*(I_i^j) \cong I(i, j)$ .

**Proof**

$$C_i^*(I_i^j) = D(C_i(D(I_i^j))) = D(C_i(D(D(A_i e_{ij})))) \cong D(C_i(A_i e_{ij})) = D(\Lambda \overline{e_{ij}}) = I(i, j)$$

where the penultimate equality follows from Proposition 10.  $\square$

Proposition 16 gives us a complete description of the indecomposable injective  $\Lambda$ -modules. In order to calculate these modules in practical examples, we need to combine this description with Remark 6 above.

The particular case of when there are no relations is expressed in the following proposition, which is dual to Proposition 11 above:

**Proposition 17** Suppose  $I = 0$ . Let  $I(i, j) = ((M_i)_{i \in \Gamma_0}, (\phi_\alpha)_{\alpha \in \Gamma_0})$  be the representation associated to  $I(i, j)$ . Then, for  $l \in \Gamma_0$ ,

- (a) If  $l = i$ , then  $M_l = M_i = I_i^j$ .
- (b) If  $l \neq i$ , denote

$$n_l = \sum_{\gamma: l=i_0 \rightarrow i_1 \rightarrow \dots \rightarrow i_r=i} (\dim_k A_{i_1}) \cdot \dots \cdot (\dim_k A_{i_{r-1}}) \cdot (\dim_k I_i^j)$$

where  $\gamma$  runs through all possible paths  $l \rightsquigarrow i$ . Then  $M_l \cong (A_l^*)^{n_l}$  as  $A_l$ -modules, where we denote  $A_l^* = D(A_l)$  for brevity. In particular, if there are no paths  $l \rightsquigarrow i$ , then  $M_l = 0$ .

**Example 18** Let  $A$  be the path algebra given by the quiver

$$1 \longleftarrow 2$$

Then there are 2 indecomposable injective  $A$ -modules, namely,

$$I_1 : k \xleftarrow{id} k \qquad I_2 : 0 \xleftarrow{\quad} k$$

Now let  $\Lambda$  be the generalized path algebra given by

$$A \xleftarrow{\quad} A$$

We want to calculate the indecomposable injective  $\Lambda$ -modules. According to the discussions above, we first calculate the indecomposable projective modules over the following generalized path algebra:

$$A^{op} \longrightarrow A^{op}$$

and we note that  $A^{op}$  is the path algebra over the following quiver:

$$1 \longrightarrow 2$$

In our case, this calculation was already done in Example 12. Therefore it remains only to apply Proposition 3. Thus the indecomposable injective  $\Lambda$ -modules are:

$$I(1, 1) : I_1 \xleftarrow{\begin{bmatrix} D(\mu) & 0 \\ 0 & D(\mu) \end{bmatrix}} (A^*)^2 \qquad I(1, 2) : I_2 \xleftarrow{[D(\mu)]} A^*$$

$$I(2, 1) : 0 \xleftarrow{\quad} I_1 \qquad I(2, 2) : 0 \xleftarrow{\quad} I_2$$

**Acknowledgements** The authors gratefully acknowledge financial support by São Paulo Research Foundation (FAPESP), Grants #2018/18123-5, #2020/13925-6 and #2022/02403-4. The second author has also a Grant by CNPq (Pq 312590/2020-2). We also thank the referee for suggesting simplifications to some of the proofs contained here.

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