UNIVERSIDADE DE SÃO PAULO
Instituto de Ciências Matemáticas e de Computação

ISSN 0103-2569

A NEW LIFETIME DISTRIBUTION

GLADYS D. C. BARRIGA
FRANCISCO LOUZADA-NETO
VICENTE G. CANCHO

Nº 311

RELATÓRIOS TÉCNICOS

São Carlos – SP
Dez./2007

A New Lifetime Distribution

Gladys D.C. Barriga * Francisco Louzada-Neto †
Des-UFSCAR *Des-UFSCAR*

Vicente G. Cancho‡
ICMC-USP

Abstract

A new lifetime distribution capable of modeling a bathtub-shaped, unimodal, increasing hazard rate function is proposed. The model has tree parameters is easy tractable for applications in reliability study and survival analysis.

Key Words: *Weibull Distribution; Bathtub-shaped failure rate function; Lifetime data.*

1 Introduction

The Weibull distribution, is commonly used for analyzing of lifetime data. This distribution family can accommodate constant, increasing and decreasing failure rates. However, the Weibull distribution does not provide a reasonable parametric fit for some practical applications where the underlying hazard rates may be bathtub or unimodal shapes. In order to achieve these behaviors from a single distribution several models were introduced, as the generalized gamma distributions proposed by Stacy (1962), the generalized F distributions proposed by Prentice (1975), the exponential-power family by Smith and Bain (1975), the IDB distribution proposed by Hjort (1980). In recent years, some new classes of models have been proposed based on modifications of the Weibull distribution. For instance, Mudlholkar and Srivastava (1993) proposed a new class of models generalizing the Weibull distributions given by the introduction of an additional shape parameter. This extended family provides models for a large number of survival or reliability problems, which includes unimodal, bathtub an other classes of monotonotonic failure rate functions. Another modifications of the Weibull distribution with bathtub-shaped failure rates were proposed for Xei *et al.* (2002), Lai *et al.* (2003) and Bebbington *et al.* (2007). A compreensive review of such models is presented in Pham and Lai (2007).

*Address: Des, Universidade Federal de São-Carlos, Brazil. E-mail: gladyscacsire@yahoo.com.br

†Address: DEs, Universidade Federal de São-Carlos, Brazil. E-mail: dfln@power.com.br

‡Address: ICMC, Universidade de São paulo, São Carlos, São Paulo, Brazil. E-mail: garibay@icmc.usp.br

In this paper, we propose a new distribution family which can accommodate Increasing, Decreasing, Unimodal and Bathtub-shaped hazard functions, hereafter called IDUB distribution. As we show later, this model family has the Weibull distribution as a special case obtained asymptoticaly and hence it can be considering as a Weibull extension.

The remainder of this paper is organized as follows. In Section 2 we present IDUB model that accommodates all five major hazard shapes: constant, increasing, decreasing, bathtub and unimodal failure rates. In Section 3 the total time on test (TTT) transform procedure is used as a tool to identify the hazard behavior of the proposed distribution. In Section 4 a inferencial procedure based in maximum likelihood is presented. In Section 5 we presents the results of an analysis with two real data set

2 The IDUB distribution

The random variable T nonnegative has a IDUB distribution if its probability density is given by

$$f(t) = \frac{\beta\theta t^{\beta-1}}{\alpha^{\beta}}\exp(1+(\frac{t}{\alpha})^{\beta}-\exp((\frac{t}{\alpha})^{\beta}))(1-\exp(1-\exp((\frac{t}{\alpha})^{\beta})))^{\theta-1},\ t>0, \qquad (1)$$

where $\alpha > 0$ is a scale parameter and $\beta >$ and $\theta > 0$ are form parameters. The distribution (1) is mainly related to the exponential-power model proposed by Smith and Bain (1975) with the additional form parameter. For $\theta = 1$ the IDUB distribution becomes the Smith and Bain distribution. Some probability density functions the IDUB distribution are shown in Figure 1.

The survival function of IDUB model is given by

$$S(t) = 1 - \left[1-\exp\left(1-\exp\left(\left(\frac{t}{\alpha}\right)^{\beta}\right)\right)\right]^{\theta}, \qquad (2)$$

and hazard function is given by

$$h(t) = \frac{\beta\theta t^{\beta-1}\exp(1+(\frac{t}{\alpha})^{\beta}-\exp((\frac{t}{\alpha})^{\beta}))(1-\exp(1-\exp((\frac{t}{\alpha})^{\beta})))^{\theta-1}}{\alpha^{\beta}\left(1-\left[1-\exp\left(1-\exp\left((\frac{t}{\alpha})^{\beta}\right)\right)\right]^{\theta}\right)}. \qquad (3)$$

The kth moment for IDUB distribution is given by

$$\mu'_k = E(T^k) = \int_0^1 Q(u)^k du, \qquad (4)$$

where $Q(u) = F^{-1}(u) = \alpha[\log(1-\log(1-u^{1/\theta}))]^{1/\beta}$ and $F(.)$ is the distribution function. For the moment calculation a numerical integration procedure is needed , since the integral in (4) do not have a closed form.

The flexibility of the IDUB distribution for fitting survival data can be visualized by the different forms that its hazard function (3) can assume. For instance, (i) if $\theta \geq 1$

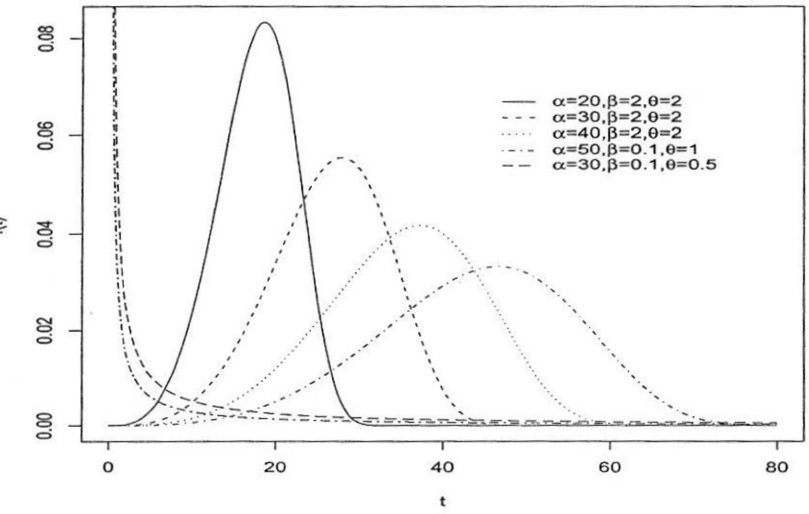

Figure 1: The probability density function (1) of the IDUB distribution, for several values of parameters.

and $\beta\theta \leq 1$ or $\beta < 1$ and $\theta < 1$, we have a monotone decreasing failure rate function; (ii) if $\theta < 1$ and $\beta\theta \geq 1$ or $\theta > 1$ and $\beta > 1$, we have a monotone increasing failure rate function; (iii) if $\theta = 1$ and $\beta < 1$ or $\theta < 1$ and $\beta\theta \leq 1$, we have a bathtub-shaped for the failure rate function; (iv) if $\beta < 1$ and $\beta\theta > 1$, we have a unimodal failure rate function.

The proposed model is related to Weibull distribution in a interesting way. The Weibull distribution can be seen as an asymptotic case of the IDUB distribution. When the scale parameter α becomes very large or approach infinity we have that

$$1 - \exp\left((\frac{t}{\alpha})^{\beta}\right) \approx 1 - [1 + (t/\alpha)^{\beta} + o(t^{\beta})] \approx -(\frac{t}{\alpha})^{\beta}. \tag{5}$$

In this case, we have that

$$S(t) = 1 - \left[1 - \exp\left(-(\frac{t}{\alpha})^{\beta}\right)\right]^{\theta}, \tag{6}$$

which is the survival function of an exponentiated-Weibull distribution with scale parameter $\alpha > 0$ and shape parameters $\beta > 0$ and $\theta > 0$. That is, in the case α approach infinity, the IDUB distribution becomes an exponentiated-Weibull distribution, implicating that the distribution family includes the Weibull distribution when $\theta = 1$ and the exponential distribution when $\theta = 1$ and $\beta = 1$.

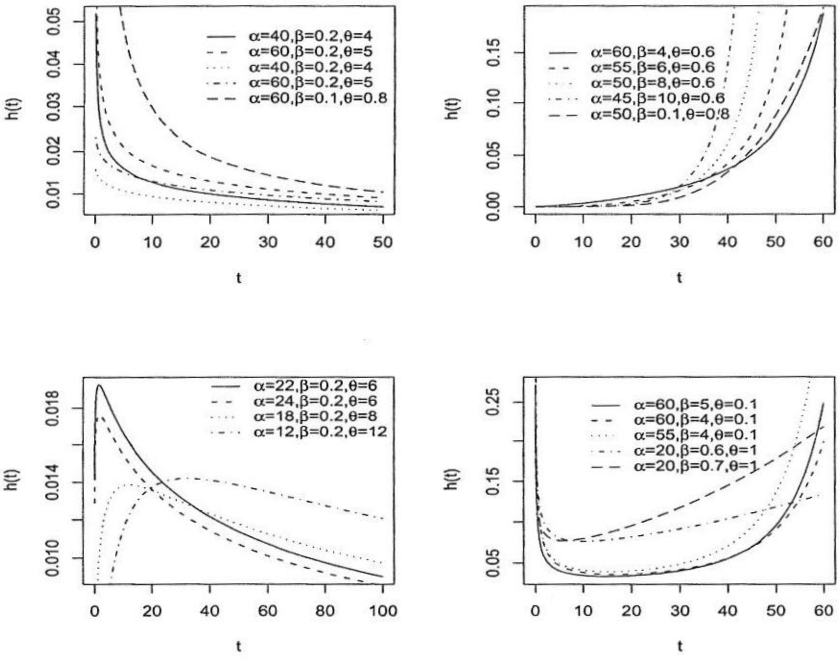

Figure 2: Some specials cases for the failure rate function of IDUB distribution (3).

3 Scaled TTT transformes

In order to identify the type of failure rate of the lifetime data, many approach have been proposed (see, Glaser, 1980). In this study, a graphical method based on the total time on test (TTT) transformed introduced by Barlow and Campo (1975) will be used to illustrate the variety hazard rate shapes. It has been shown that the hazard function of $F(t)$ increases (decreases) if the scaled TTT-transform, $\phi_F(u) = H_F^{-1}(u)/H_F^{-1}(1)$, where $H_F^{-1}(u) = \int_0^{F^{-1}(u)}(1-F(t))dt$, $0 \leq u \leq 1$, is concave (convex). In addition, for a distribution with bathtub (unimodal) failure rate the TTT-transform is first convex (concave) and then concave (convex). The empirical version of TTT-transform (Aarset,1987) is given by $G(r/n) = [(\sum_{i=1}^{r} Y_{i:n}) - (n-r)Y_{r:n}]/(\sum_{i=1}^{r} Y_{i:n})$, where $r = 1, \ldots, n$ and $Y_{i:n}$ represent the order statistics of the sample.

The TTT-transform illustrates the variety of hazard rate functions include in the IDUB distribution family. In the order to obtain $\phi_F(u)$, the scaled TTT-transform, the definite integrals are obtain numerically by using the *integrate* function of the R software. Figure 3, gives the scaled TTT-tranforms of the IDBU distributions for diferent parameter values.

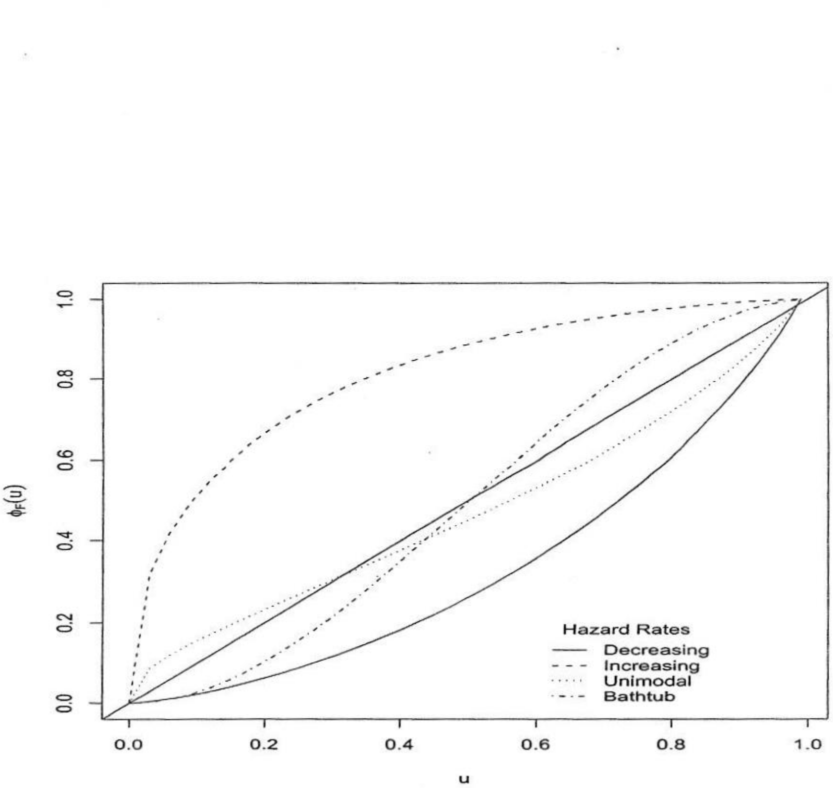

Figure 3: Scaled TTT-Transforms of the IDBU distributions.

4 Maximum likelihood estimation

We assume that the lifetime are independently distributed, and also independent from the censoring mechanism. Considering lifetime data, we observe $t_i = \min(T_i^0, C_i)$, where T_i^0 is the lifetime for the ith individual with distribution given by (1) and C_i is the censoring time for the ith individual, $i = 1, \ldots, n$. In this case the log-likelihood function of α, β and θ is given by

$$\begin{aligned}\ell(\alpha,\beta,\theta) &= r\log(\theta\beta) - r\beta\log(\alpha) + (\beta-1)\sum_{i\in F}\log(t_i) + r - \sum_{i\in F}(\frac{t_i}{\alpha})^{\beta} + \\ & \sum_{i\in F}\exp\left((\frac{t_i}{\alpha})^{\beta}\right) + (\theta-1)\sum_{i\in F}\log(g(t_i;\alpha,\beta)) + \sum_{i\in C}\log(1-g(t_i;\alpha,\beta)^{\theta})\end{aligned} \quad (7)$$

where F denotes the set of non censured observations (failures), C denotes of set censored observations, r is the non censured observation number and $g(t_i;\alpha,\beta) = 1 - \exp\left(1-\exp\left((\frac{t_i}{\alpha})^{\beta}\right)\right)$.

The maximum likelihood estimates $\boldsymbol{\theta} = (\alpha,\beta,\theta)$ can be obtained by maximizing log-likelihood function $\ell(\boldsymbol{\theta})$ which is equivalent to solve the following nonlinear equation system

$$\frac{\partial \ell(\boldsymbol{\theta})}{\partial \alpha} = -\frac{\beta}{\alpha}\sum_{i\in F}\left[1-(\frac{t_i}{\alpha})^{\beta}(1-e^{(\frac{t_i}{\alpha})^{\beta}})\right] - \frac{(\theta-1)\beta}{\alpha}\sum_{i\in F}\left[\frac{\exp(1+(\frac{t_i}{\alpha})^{\beta}-e^{(\frac{t_i}{\alpha})^{\beta}})}{g(t_i;\alpha,\beta)}\right]$$
$$+\frac{\beta\theta}{\alpha}\sum_{i\in C}\frac{g(t_i;\alpha,\beta)^{\theta-1}\exp\left(1+(\frac{t_i}{\alpha})^{\beta}-e^{(\frac{t_i}{\alpha})^{\beta}}\right)}{1-g(t_i;\alpha,\beta)^{\theta}} = 0; \quad (8)$$

$$\frac{\partial \ell(\boldsymbol{\theta})}{\partial \beta} = \frac{r}{\beta} - \sum_{i\in F}(\frac{t_i}{\alpha})^{\beta}\log(\frac{t_i}{\alpha})\left[1-e^{(\frac{t_i}{\alpha})^{\beta}}\right] + \sum_{i\in F}\left[\frac{(\frac{t_i}{\alpha})^{\beta}\log(\frac{t_i}{\alpha})\exp(1+(\frac{t_i}{\alpha})^{\beta}-e^{(\frac{t_i}{\alpha})^{\beta}})}{g(t_i;\alpha,\beta)}\right]$$
$$-r\log(\alpha) - \theta\sum_{i\in C}\frac{g(t_i;\alpha,\beta)^{\theta-1}(\frac{t_i}{\alpha})^{\beta}\log(\frac{t_i}{\alpha})\exp\left(1+(\frac{t_i}{\alpha})^{\beta}-e^{(\frac{t_i}{\alpha})^{\beta}}\right)}{1-g(t_i;\alpha,\beta)^{\theta}} = 0; \quad (9)$$

$$\frac{\partial \ell(\boldsymbol{\theta})}{\partial \beta} = \frac{r}{\theta} + \sum_{i\in F}\log(g(t_i;\alpha,\beta)) - \sum_{i\in C}\frac{g(t_i;\alpha,\beta)^{\theta}\log(g(t_i;\alpha,\beta))}{1-g(t_i;\alpha,\beta)^{\theta}} = 0. \quad (10)$$

Under, suitable regularity conditions, it can be shown that (Cox and Hinkley, 1974)

$$\mathbf{R}(\boldsymbol{\theta})(\hat{\boldsymbol{\theta}}-\boldsymbol{\theta})\xrightarrow{D}N(\mathbf{0},\mathbf{I}),$$

as $n \to \infty$, where $\mathbf{I}$ denotes the unity matrix and $\mathbf{R}(\boldsymbol{\theta})$ denotes Choleski decomposition of the observed information matrix $\mathcal{I}_{obs}(\boldsymbol{\theta})$, that is, $\mathbf{R}(\boldsymbol{\theta})^{\top}\mathbf{R}(\boldsymbol{\theta}) = \mathcal{I}_{obs}(\boldsymbol{\theta})$. Thus, the approximate distribution of , $\hat{\boldsymbol{\theta}}$ in large samples is a multivariate normal distribution with mean vector $\boldsymbol{\theta}$ and covariance matrix $\mathcal{I}_{obs}^{-1}(\boldsymbol{\theta})$, which can be estimated by $\left\{-\frac{\partial \ell(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}^{\top}}\right\}^{-1}$ evaluated at $\boldsymbol{\theta} = \hat{\boldsymbol{\theta}}$. Second derivative computation can be obtained numerically.

5 Applications

5.1 Example 1: Unimodal failure rate illustration

In this section, we consider a reanalysis of a data set presented in Efron (1980) on survival times of 51 patients, of a head and neck cancer clinical trial. The data from Efron (1988) are reproduced in the Table 1. This data set has been analyzed by Mudholkar, *et al.* (1996) considering a generalized Weibull distribution.

In the fit of the IDUB model, the survival times were transformed to months. After maximizing the log-likelihood (7), the maximum likelihood estimates and their respective standard erros (in parenthesis) are given by $\hat{\alpha} = 0.06017$ (0.2554), $\hat{\beta} = 0.1073$ (0.0556) and $\hat{\theta} = 58.108$ (97.157). The plots of the fitted scaled TTT-transform with estimates are shown in Figure (4). The corresponding fitted failure rate function and survival function of the IDUB distribution with the parameters is given in Figure 5.

Table 1: Survival times in days, from a clinical trial considered by Efron (1980).

7	34	42	63	64	74+	83	84	91	108	112	129	133
133	139	140	140	146	149	154	157	160	160	165	173	176
185+	218	225	241	248	273	277	279+	297	319+	415	417	420
440	523	523+	583	594	1101	1116+	1146	1226+	1349+	1412+	1417	

Note: + indicates censoring

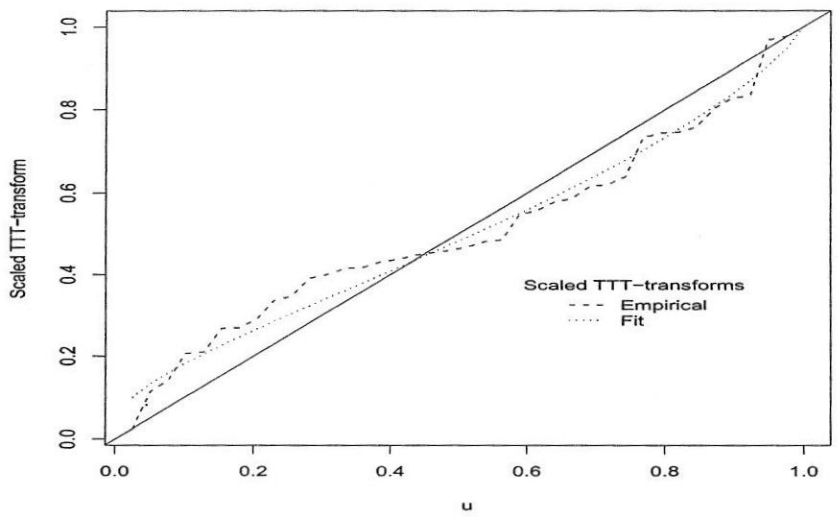

Figure 4: Empirical and Fitted Scaled TTT-Transforms for the Efron data.

For a simple model choice illustration, we consider a comparison between the IDUB distribution fitting against the exponentiated-Weibull distribution fitting (Mudholkar, *et al.*, 1996) with survival function given in (6). The comparison is made by inspection of the Akaike's information criterion (AIC, $-2\ell(\widehat{\boldsymbol{\theta}}) + 2p$) where p is the number of parameters in the model. The maximum likelihood estimates of parameter of exponentiated-Weibull model for the Efron date set are given by $\hat{\alpha} = 0.16309$, $\hat{\beta} = 0.30168$ and $\hat{\theta} = 17.485$. The AIC for exponentiated-Weibull model is equals to 311.61 , while the AIC for IDUB model is equals to 311.46, which is a positive evidence in favour of the IDUB model.

5.2 Example 2: Bathtub-shaped failure rate illustration

As a second example, we consider the data set presented in Aarset (1987). The data describe lifetimes of 50 industrial devices put on life test at time zero. The data from Aarset (1987) are reproduced in Table 2.

After maximizing the log-likelihood (7), we obtained the maximum likelihood esti-

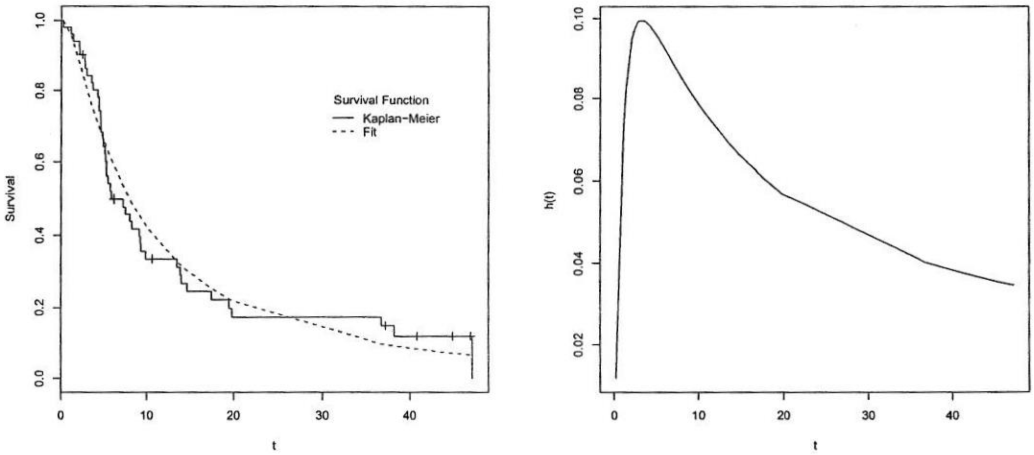

Figure 5: Fitted failure rate function and survival function for the data Efron.

Table 2: Lifetimes for the 50 devices.

0.1	0.2	1	1	1	1	1	2	3	6	7	11	12	18
18	18	18	18	21	32	36	40	45	46	47	50	55	60
63	63	67	67	67	67	72	75	79	82	82	83	84	84
84	85	85	85	85	85	86	86						

mates and their respective standard erros (in parenthesis) given by $\hat{\alpha} = 90.131$ (6.511), $\hat{\beta} = 2.0658$ (0.2737) and $\hat{\theta} = 0.33583$ (0.0647). The corresponding hazard plot of the IDUB distribution with the estimate parameters is given in Figure 6.

Further analysis of this data set was performed in Mudlholkar and Shivastava (1993) using the exponentiated-Weibull model with survival function given in (6). Parameter estimation by the maximum likelihood approach under this model leads to the following estimates, $\hat{\alpha} = 91.023$, $\hat{\beta} = 4.69$ and $\hat{\theta} = 0.146$. The corresponding AIC for the exponentiated-Weibull model is equals to 464.228, while the AIC for the IDUB model is equals to 452.575. The results leads to a positive evidence in favour of the IDUB model.

6 Concluding remarks

In this paper a simple new model for lifetime data is provided and discussed. The new family distribution is flexible and can accomodate increasing, decreasing, unimodal and bath-tube hazard functions. Parameter estimation is easily implemented by considering maximum likelihood approach, while asymptotic theory can be used, at least in principle, for inferential purposes.

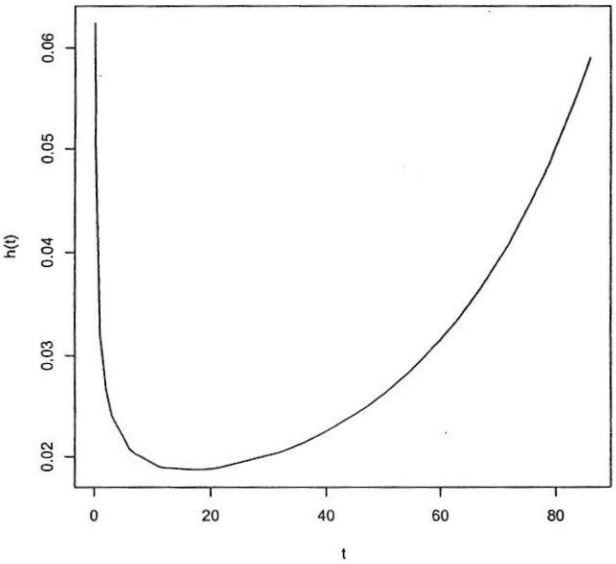

Figure 6: Fitted hazard function to the Aarset data.

Acknowledgement: Francisco Louzada-Neto research is granted by CNPq-Brasil.

References

[1] Aarset, M.V.(1987). How to identify bathtub hazard rate. *IEEE Transactions on Reliability,* v.36, 106-108.

[2] Barlow, R.E.and Campo, R.(1975). Total time on test processes and applications to failure data analysis. In Barlow, Fussell, Singpurwalla, editors. Realiability abd Fault Tree Analysis. SIAM, New York,.235-272.

[3] Bebbington, M., Lai C.D. and Zitikis, R. (2007). A flexible Weibull extension. *Realiability Engineering and System Safety,* v.92, 719-726.

[4] Cox, D.R. and Hinkley, D.V. (1974). *Theorical Statistics.* London: Chapman and Hall.

[5] Efron, B. (1988). Logistics regression, survival analysis and the Kaplan-Meier curve. *Journal of the Americam Statistical Association,* v.83, 414-425.

[6] Glaser, R.E. (198). Bathtub shaped and related failure rate characterizations. *Journal of the American Statistical Association,* v.75, 667-672.

[7] Hjort, U. A realiability distributions with increasing, decreasing, constant and bathtub failure rates. *Technometrics,* v.22, 99-107.

[8] Lai,C.D., Xei, M. and Murthy, D.N.P. (2003). Modified Weibull model, *IEEE Transactions on Reliability,* v.56, 454-458.

[9] Lawless, J.F.(1982) *Statistical models and methods for lifetime data,* New York: John Wiley & Sons.

[10] Mudholkar, G.S., Srivastava, D.K. and Kollia, G.D. (1996). A generalization of the Weibull distribution with application to the analysis of survival data. . *Journal of the Americam Statistical Association,* v.91, 1575-1583.

[11] Mudholkar, G.S.and Srivastava, D.K.(1993). The exponentiated Weibull family for analyzing bathub failure-rate. *IEEE Transactions on Reliability,* v.42, 299-302.

[12] Pham, H. and Lai, C.D. (2007). On recent generalizations of the Weibull distribution. *IEEE Transactions on Reliability,* v.56, 454-458.

[13] Prentice, R.L. (1975) Discrimination among some parametric models. *Biomentrics,* Washimgton, v.61, 539-544.

[14] Smith, R.M. and Bain, L.J. (1975). An exponential power life-testing distribution, *Communications Statistics,* v.4, 469-481.

[15] Stacy, E.W. (1962) A generalization of the gamma distributions. *The Annals of Mathematical Statistics,* Hayward, v.33, 1178-1192.

[16] Xei, M. and Lai, C.D.(1995) Realibility analysis using additive Weibull model with bathtub-shaped failure rate function. *Realiability Engineering and System Safety,* Ireland. v. 52 87-93.

[17] Xei, M.; Tang, Y.; Goh, T.N. (2002). A modified Weibull extension with bathtub-shaped failure rate function. *Realiability Engineering and System Safety,* v.76, 279-285.