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Contact and equivalence of submanifolds of homogenous spaces

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CONTACT AND EQUIVALENCE OF SUBMANIFOLDS OF HOMOGENEOUS SPACES

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§1 - INTRODUCTION - Let M be a homogeneous space of a Lie group G; for an element $g \in G$, let $Lg: M \to M$ denote the diffeomorphism defined by G. Given two submanifolds S_1, S_2 of M, of same dimension p, it is a classical problem of differential geometry to find conditions on S_1 and S_2 for the existence of $g \in G$ such that $L_g(S_1) = S_2$. When this happens, S_1 and S_2 are said to be G-equivalent.

Given two points $a_1 \in S_1$ and $a_2 \in S_2$, we say that S_1 has G-contact of order k with S_2 at the points a_1 and a_2 if there exists $g \in G$ such that $L_g(a_1) = a_2$ and $L_g(S_1)$ has contact of order k with S_2 at the point a_2. The equivalence problem of submanifolds of a homogeneous space was extensively treated by E. Cartan [1], by his method of the moving frame. One of the underlying ideas in Cartan's method is that for each homogeneous space M there exists an integer k, depending on p, such that if there exists a diffeomorphism $\phi: S_1 \longrightarrow S_2$ having the property that S_1 has G-contact of order k with S_2 at all points $x \in S$, and $\phi(x) \in S_2$, then S_1 and S_2 are locally equivalent. Let $a_0 \in M$ be a fixed point and H the isotropy group of a_0. Using Cartan's method of the moving frame, G.R.Jensen [3] proved the existence of k, assuming the existence of local sections in the space of orbits of the action of H in the Grassmann manifold $G^p(T_{a_0}M)$ and also assuming regularity conditions on S_1 and S_2.

In the method of the moving frame, one defines imbeddings σ_1. $S_1 \longrightarrow G$ and σ_2: $S_2 \longrightarrow G$ such that, for $g \in G$, $L_g(S_1) = S_2$ if and only if $\sigma_2(S_2)$ is the image of $\sigma_1(S_1)$ by the left translation defined by g. This reduces the equivalence problem to the case where $M = G$ and G acts on G by left translations. In this case, the problem is easely solved by means of Frobenius theorem. Ultimatly, the method of the moving frame reduces the necessary integration to solve the problem to the integration of a differential system of order 1. In this paper we prove the existence of the integer k and consequently solve the equivalence problem by integrating directly a differential system of order k of finite type. This makes easier to state the regularity conditions which the manifolds S_1 and S_2 have to satisfy; at the same time the regularity conditions become geometrically more meaningfull.

Our regularity conditions bear only on contact elements of S_1 and S_2 of two consecutive orders ℓ and $\ell-1$ whereas in the method of the moving frame the regularity conditions bear on all orders from 1 up to k. Moreover, the order of regularity for which the theorem of equivalence 3.3 applies is not fixed for all submanifolds of M. If a submanifold does not satisfy the regularity conditions at a given order it may satisfy these conditions at higher order. This allows to prove equivalence theorems which can not be immediatly derived by the method of the moving frame. For instance, theorem 3.3 can be applied to curves γ in the Euclidean space $\mathbb{R}^3$, in the neighborhood of a point $a \in \gamma$ where the curvature $\rho(a)$ and all derivatives

$$\frac{d\rho}{ds} a) \ldots \frac{d^{\ell-2}\rho}{ds^{\ell-2}}(a)$$

vanishe up to some order $\ell-2$ but

$$\frac{d^{\ell-1}\rho}{ds^{\ell-1}}(a) \text{ and } \frac{d^{\ell}\rho}{ds^{\ell}}(a)$$

are different from zero, s being the are lenght (see §4).

Usually, equivalence theorems are stated imposing that the submanifolds S_1 and S_2 have same invariants at corresponding points. This type of theorem can be derived from theorem 3.3 by taking a complete set of invariant functions for the orbits of G in the manifold $C^{k,p}(M)$ of elements of contact of order k in theorem 3.3. Let us remark that the condition

It is natural to ask how generic are the regularity conditions in theorem 3.3. Let us remark that the condition

$$h_1^k(a) = 0 \text{ (see §3)}$$

depends only on the contact element $S_{1a_1}^{k+1}$ of S_1 at the point $a_1 \in S_1$. Hence, this question will be answered (see Corollary 3.4) if one proves the existence of an integer k such that the set of points $X \in C^{k+1,p}(M)$ for which the isotropy group $G^{k+1}(X)$ of G is discreete and $h^{k+1}(X) = 0$ is dense and open in $C^{k+1,p}(M)$. It seems reasonable to conjecture the existence of this integer for all homogeneous spaces. In the case $p = 1$, it follows from a theorem of I.Kupka [6] that the set of points of $C^{k,1}(M)$ for which the isotropy is discreete is dense and open in $C^{k,1}(M)$, for all sufficiently heigh k.

The equivalence problem can be posed for two imersions

$f,h: S \longrightarrow M$ where S is any differentiable manifold. f and h are equivalent if there exists $g \in G$ such that $h = L_g \circ f$. This fixed parametrization equivalence problem has been solved be means of a differential system of finite type of higher order by J. A.Verderesi [9], see also [4,10].

In §2 we state a generalization of Frobenius theorem to differential systems defined by contact elements of higher order. This theorem will be our main tool in the proof of theorem 3.3. §3 is devoted to the proof of the equivalence theorem 3.3. In §4 we give a necessary and sufficient condition for a submanifold $S \subset M$ to be an open set of an orbit of a Lie sub-group L of G. This theorem can be generalized to characterize the submanifolds S of M which are locally invariant by the action of a Lie sub-group L of G and which are fibered by the orbits of L which meet S. We end the paper with some simple remarks about curves in $\mathbb{R}^3$.

§2 - HIGHER ORDER FROBENIUS THEOREM

All differentiable manifolds and maps will be considered to be of class C^∞. If M is a differentiable manifold, we shall denote by $C^{k,p}M$ the differentiable manifold of contact elements of order k and dimension p of M [2]; $\pi^k_{k'}: C^{k,p}M \longrightarrow C^{k',p}M$, $k' \leq k$, will denote the canonical projection. If $k = 0$, the manifold $C^{0,p}M$ is identifyed with M. If $k' = 0$, we shall use the notation $\pi^k: C^{k,p}M \longrightarrow M$ instead of π^k_0. The fiber of $C^{k,p}M$ over a point $x \in M$ will be denoted by $C^{k,p}_x M$.

Let $S \subset M$ be a submanifold of dimension p. We shall denote by

$S_x^k \in C^{k,p}M$ the contact element of S at the point $x \in S$ and by S^k the image of the imbbeding $x \in S \longrightarrow S_x^k \in C^{k,p}M$. Given two submanifolds $S_1, S_2 \subset M$ of same dimension and a point $x \in S_1 \cap S_2$, by definition, S_1 and S_2 have contact order k at x if $S_{1x}^k = S_{2x}^k$.

A differential system of order k and dimension p defined over M is, by definition, a submanifold $\Omega \subset C^{k,p}M$. An integral manifold of Ω is a submanifold $S \subset M$ of dimension p such that $S_x^k \in \Omega$ for every $x \in S$.

An important notion associated to differential systems is the notion of prolongation. Let $\lambda^{k+1}: C^{k+1,p}M \longrightarrow C^{1,p}(C^{k,p}M)$ be the map defined as follows: If $X \in C^{k+1,p}M$ and $S \subset M$ is a submanifold such that $X = C_a^{k+1}S$, $a \in S$, then, $\lambda^{k+1}(X) = (S^k)_{X'}^1$, where $X' = \pi_k^{k+1}(X) \in S^k$. It is easy to verify that λ^{k+1} is an imbedding of $C^{k+1,p}M$ into $C^{1,p}(C^{k,p}M)$. Clearly, there is also a natural imbedding of $C^{1,p}\Omega$ into $C^{1,p}(C^{k,1}M)$; we shall identify $C^{1,p}\Omega$ with its image in $C^{1,p}(C^{k,1}M)$. The first prolongation of Ω is then defined to be the set $p\Omega = C^{k+1,p}\Omega \cap C^{1,p}\Omega$ [8]. The following generalization of Frobenius theorem is just a geometric formulation of the existence and uniqueness theorem of solutions of differential systems of finite type [7].

THEOREM 2.1 - Let $\Omega \subset C^{k,p}M$, $k \geq 1$, be a differential system such that: 1) $\pi_{k-1}^k: \Omega \longrightarrow C^{k-1,p}M$ is an imersion. 2) The projection $\pi_k^{k+1}: p\Omega \longrightarrow \Omega$ is surjective. Then, for all $X_0 \in \Omega$, there exists a solution of Ω defined in a neighborhood of $x_0 = \pi^k(X_0) \in M$. Moreover, if S_1 and S_2 are two such solutions, there exists $V \subset S_1 \cap S_2$ which is an open neighborhood of x_0 in both S_1 and S_2.

A differential system satisfying conditions 1) and 2) is called completely integrable. A proof of theorem 2.1 will appear elsewhere.

§3 - k-ADMISSIBLE SUBMANIFOLDS

Let G be a Lie group and let M be a homogeneous space of G. The action of G on M extends naturally to an action of G on $C^{k,p}M$. If $X\in C^{k,p}M$, $X = S^k_x$, $x\in N$, and $g\in G$ then, by definition, $g\cdot X = (g\cdot S)^k_{gx}$ where $g\cdot S = L_g(S)$ and $g\cdot x = L_g(x)$, L_g being the diffeomorphism of M induced by g.

Given two submanifolds S_1 and S_2 of M, we say that they are G-equivalent if there exists $g\in G$ such that $g\cdot S_1 = S_2$; we say that they are locally G-equivalent at the points $a_1\in S_1$ and $a_2\in S_2$ if there are open neighborhoods V_1 and V_2 of a_1 and a_2 in S_1 and S_2 which are G-equivalent. Given an element $g\in G$ and a point $a_1\in S_1$, g makes contact of order $k\geq 0$ between S_1 and S_2 at the point $x\in S_1$ if $gx\in S_2$ and $g\cdot S_1$ and S_2 have contact of order k at the point $g\cdot x$, or equivalently $g\cdot S^k_{1x} = S^k_{2gx}$. Clearly, if $g\cdot S_1 = S_2$, then g makes contact of any order $k\geq 0$ between S_1 and S_2 at any point $x\in S$.

DEFINITION 3.1 - The submanifold S is k-admissible if there exists a submanifold U of $C^{k,p}(M)$ and a neighborhood A of the identity e in G such that:

1) For all $g\in A$ and $x\in S$, $g\cdot S^k_x \in U$.
2) U is a completely integrable differential system of order k.

THEOREM 3.1 - Given two submanifolds S_1 and S_2 of dimension p of M, assume that

1) S_2 is k-admissible for some $k\geq 1$.

2) There exists a continous map $\phi: S_1 \longrightarrow G$ such that for all $x\in S_1$ $\phi(x)\cdot x\in S_2$ and $\phi(x)\cdot S^k_{1x} = S^k_{2\phi(x)\cdot x}$.

Then, for any point $a_1\in S_1$, S_1 and S_2 are locally equivalent at the points a_1 and $a_2 = \phi(a_1)\cdot a_1$.

PROOF - Choose U and A as in definition 3.1, with respect to S_2. Since the map $x\in S_1 \longrightarrow \phi(a_1)\phi(x)^{-1}\in G$ is continuous, there exists an open neighborhood V of a_1 such that $\phi(a_1)\phi(x)^{-1}\in A$ for all $x\in V$. Hence,

$$\phi(a_1)\cdot S^k_{1x} = \phi(a_1)\cdot\phi(x)^{-1}(\phi(x)\cdot S^k_{1x}) = (\phi(a)\phi(x)^{-1})\cdot S^k_{2\phi(x)x}\in U.$$

Therefore, $\phi(a_1)\cdot S_1$ is an integral manifold of U and

$$(\phi(a_1)\cdot S_1)^k_{a_2} = S^k_{2a_2}.$$

Since S_2 is also an integral manifold of U, it follows from theorem 2.1 that there exists a set V_2 which is is open neighborhood of a_2 in $\phi(a_1)\cdot S_1$ and in S_2. Then, the neighborhoods $V_1 = \phi(a_1)^{-1}\cdot V_2\subset S_1$ and V_2 are equivalent.

We shall now give sufficient conditions for a submanifold $S\subset M$ to be locally k-admissible. For $x\in S$, let $g^k(x) \subset G$ be the dimension of $G^k(x)$. Let T^k_x be the tangent space of S^k at the point S^k_a and let $T_xO^k_x$ be the tangent space at the point S^k_x of the orbit O^k_x, of the point $S^k_x\in C^{k,p}_M$. Put

$$h^k(x) = \dim(T^k_x\cap T_xO^k_x).$$

Clearly, $g^{k-1}(x) \leq g^k(x)$ and $h^{k-1}(x) \leq h^k(x)$ for all $k\geq 1$.

DEFINITION 3.2 - A point $a\in S$ is k-regular, $k\geq 1$, if the integers

$d^k(x) = g^k(x) + h^k(x)$, $d^{k-1}(x) = g^{k-1}(x) + h^{k-1}(x)$ are constant and $d^k(x) = d^{k-1}(x)$ for x in a neighborhood of a.

THEOREM 3.2 - Let $a \in S$ be a k-regular point. Then, there exists a neighborhood V of a in S which is k-admissible.

PROOF - Let ψ^k: $G \times S^k \longrightarrow C^{k,p}(M)$ be the map defined by

$$\psi^k(g, S_x^k) = g \cdot S_x^k.$$

Denote by $L_g^{k,p}$: $C^{k,p}(M) \longrightarrow C^{k,p}(M)$ the diffeomorphism defined by g and let id^k: $S^k \longrightarrow S^k$ be the identity map. Then

$$\psi^k \circ (L_g \times id^k) = L_g^{k,p} \circ \psi^k.$$

Since $L_g \times id^k$ and $L_g^{k,p}$ are diffeomorphisms, it follows that the rank of ψ^k at the point $(g, S_x^k) \in G \times S^k$ is equal to the rank of ψ^k at the point $(e, S_x^k) \in G \times S^k$, where $e \in G$ is the neutral element. Clearly, the rank of ψ^k at the point (e, S_x^k) is equal to

$$\dim G - g^k(x) + p - h^k(x) = \dim G + p - d^k(x).$$

Hence, the rank of ψ^k at the point (e, S_x^k) does not depend on x for x in a neighborhood of a. Therefore, the rank of ψ^k is constant in a neighborhood of (e, S_a^k) in $G \times S^k$. Let $A \times V^k$ be an open neighborhood of (e, S_a^k) in $G \times S^k$ which is mapped by ψ^k onto a submanifold U of $C^{k,p}(M)$ and put $V = \pi^k(V^k)$. We shall show that U satisfies both conditions of definition 3.1 with respect to the neighborhood V and the neighborhood A of e. By construction of U, condition 1) holds true. To prove condition 2), consider the following commutative diagram:

$$\begin{array}{ccc} A \times V^k & \xrightarrow{\quad \psi^k \quad} & U \\ \Big\downarrow {\scriptstyle id \times \pi_{k-1}^k} & & \Big\downarrow {\scriptstyle \pi_{k-1}^k} \\ a \times V^{k-1} & \xrightarrow{\quad \psi^{k-1} \quad} & C^{k-1,p}(M) \end{array}$$

Denote by ψ_*^k the map induced by ψ^k on tangent vectors. The kernel of ψ_*^k at the point $(g,S_a^k)\in A\times V^k$, $x\in V$, has dimension $d^k(x)=g^k(x)+h^k(x)$. Similarly the kernel of ψ_*^{k-1} at the point (g,S_x^{k-1}) has dimension $d^{k-1}(x)$. Since $(id\times\pi_{k-1}^k)_*$ is an isomorphism, it follows that $(\pi_{k-1}^k|U)_*$ is injective at the point $\psi^k(g,S_x^k)$ if and only if $d^{k-1}(x)=d^k(x)$. Hence, U is a completely integrable differential system.

As a consequence from theorems 3.1 and 3.2 we have the following

THEOREM 3.3 - Let $S_1,S_2\subset M$ be two submanifolds of same dimension p, and let $a_1\in S_1$ and $a_2\in S_2$ be two points. Assume that a_2 is k-regular and that there exists a continuous map $\phi:S_1\longrightarrow G$ such that $\phi(a_1)=a_2$, $\phi(x)\cdot x\in S_2$ and $\phi(x)\cdot S_{1x}^k=S_{2\phi(x)}^k$. for all $x\in S_1$. Then, S_1 and S_2 are locally G-equivalent at the points a_1 and a_2.

Remark that, under the hypothesis of theorem 3.3,it follows from the theorem that the point $a_1\in S_1$ is also k-regular. Remark also that the map $x\in S_1\longrightarrow\phi(x)\cdot x\in S_2$ is not necessarely a diffeomorphism; in fact, this map may even be constant equal to a_2.

The following corollary is an important special case of theorem 3.3. Assume that for a point a of a submanifold $S\subset M$, $h^k(a)=0$, that is, the orbit of S_a^k in $C^{k,p}(M)$ cuts S^k transversely at the point S_a^k. Assume also that $g^k(a)=0$ that is, the isotropy group of S_a^k is discreete. Then, the same conditions clearly hold in a neighborhood of a in S. Since

$$h^{k+1}(a)\leq h^k(a)\quad\text{and}\quad g^{k+1}(a)\leq g^k(a),$$

it follows that a is k+1-regular. Hence, keeping the notations as in theorem 3.3, we have the following corollary.

COROLLARY 3.4 - Let $a_2 \in S_2$ be a point and let $k \geq 1$ be an integer such that $g^k(a_2) = h^k(a_2) = 0$. Assume that there exists a continuous map $\phi: S_1 \longrightarrow G$ and a point $a_1 \in S_1$ such that

$$\phi(a_1)\cdot a_1 = a_2, \ \phi(x)\cdot x \in S_2$$

and $\phi(x)\cdot S^{k+1}_{1x} = S^{k+1}_{2\phi(x)\cdot x}$ for all $x \in S_1$. Then S_1 and S_2 are locally equivalent at the points $a_1 \in S_1$ and $a_2 \in S_2$.

Assuming stronger regularity conditions on the contact elements of S_2, theorem 3.3 can be reformulated in the following way.

THEOREM 3.5. Let S_1, $S_2 \subset M$ be two submanifolds of same dimension p and let $a_1 \in S_1$ and $S_2 \in S_2$ be two points. Let $W^k(S_2)$ be the set of all contact elements $g\cdot S^k_{2x}$ for all $x \in S_2$ and $g \in G$. Assume that there exists $k \geq 1$ such that $W^k(S_2)$ is an imbedded submanifold of $C^{k,p}(M)$ (i.e. the topology of $W^k(S_w)$ is the induced topology) and $a_2 \in S_2$ is k-regular. Assume also that there exists a map $\phi: S_1 \longrightarrow S_2$ such that $\phi(a_1) = a_2$ and $S^k_{2\phi(x)}$ and S^k_{1x} are in the same orbit of G for all $x \in S_1$. Then S_1 and S_2 are locally G-equivalent at the points a_1 and a_2.

PROOF. As in the proof of theorem 3.2 let $\psi^k: G \times S^k_2 \longrightarrow W^k(S_2)$ be the map $\psi^k(g, S^k_{2x}) = g\cdot S^k_{2x}$. Since $W^k(S_2)$ is an imbedded submanifold ψ^k is a differentiable map. Repeating the argument in the proof of theorem 3.2, one can show that there exists an open neighborhood U of $S^k_{2a_2}$ in $W^k(S_2)$ which is a completely integrable differential system. Let $g \in G$ be such that $g\cdot S^k_{1a_1} = S^k_{2a_2}$. Since $g\cdot S^k_1 \subset$ $\subset W^k(S_2)$ and the topology of $W^k(S_2)$ is the induced topology, one can choose an open neighborhood V of a_1 in S_1 such that $g\cdot V^k$ is a submanifolf of U. Then $g\cdot V$ is an integral submanifold of U. Since there exists a neighborhood of a_2 in S_2 which is also an integral submanifold of U it follows that $g\cdot V$ and S_2 coincide in a neighborhood of a_2.

COROLLARY 3.6. Assume $W^k(S_2)$ is an imbedded submanifold and that $g^k(a_2) = h^k(a_2) = 0$. Assume also that there exists a map $\phi : S_1 \longrightarrow S_2$ such that $S^{k+1}_{2\phi(2)}$ and S^{k+1}_{1x} are in the same orbit of G for $x \in S$, and $\phi(a_1) = a_2$. Then S_1 and S_2 are locally G-equivalent at the points a_1 and a_2.

§4 - HOMOGENEOUS SUBMANIFOLDS OF M

In this § we shall characterize the submanifolds of M which are open sets of an orbit of a Lie subgroup L of G. If $S \subset M$ is an open set of an orbit of L then, $h^k(x) = p$ and $g^k(x)$ is constant for $x \in S$ and for all $k \geq 0$; moreover, for sufficiently high k, every point of S is k-regular. The following theorem is the converse to the above statement.

THEOREM 4.1 - Let S be a connected submanifold of dimension p of M. Assume that there exists $k \geq 1$ such that $h^k(x) = p$ and every point $x \in S$ is k-regular. Then, S is an open set of an orbit of a connected Lie sub-group L of G.

PROOF - Given a point $a \in S$, we are going to show the existence of a neighborhood of S^k_a in S^k which is contained in the G-orbit of S^k_a in $C^{k,p}M$. Keeping the notations as in theorem 3.2, the set $A\cdot S^k_a = \{g\cdot S^k_a | g \in A\}$ is open in the orbit of S^k_a. From the hypothesis $h^k(x) = p$ it follows that $A\cdot S^k_a$ and U have same dimension; hence, $A\cdot S^k_a$ is open in U. Since we can assume that V^k is a submanifold of U with the induced topology, it follows that $A\cdot S^k_a \cap V^k$ is an open neighborhood of S^k_a in S^k contained in the orbit of S^k_a. Since this holds for every $a \in S$ and S^k is connected, S^k is contained in the orbit of S_a^k.

Consider the map $\lambda : g \in G \longrightarrow g \cdot S_a^k \in C^{k,p}M$. and let $N = \lambda^{-1}(S_a^k)$. N is a submanifold of G and $e \in N$. Let Σ be the invariant distribution which associates to each point $g \in G$ the sub-space $\Sigma(g) = (L_g)_*(T_e N) \subset T_g G$. Then, N is an integral manifold of Σ. To prove this, let $g \in N$ and let $b = g \cdot a \in S$. Theorem 3.3 implies the existence of a neighborhood V_1 of a and the existence of a neighborhood V_2 of b such that $L_g(V_1) = V_2$. Choose a neighborhood W of e in N such that $g \cdot S_a^k \in V_1$ for all $g \in W$. For $g_1 \in W$ we have, $(gg_1) \cdot S_a^k = g \cdot (g_1 S_a^k) \in L_g(V_1) = V_2$. Hence, $gg_1 \in N$ and therefore $g \cdot W$ is a neighborhood of g in N. Consequently, $T_g N = (L_g)_*(T_e N) = \Sigma(g)$. Since the image of N by left translations of G are also integral manifolds of Σ, Σ is completely integrable. Let L be the maximal integral manifold of Σ which goes trough $e \in G$. L is a connected sub-group of G and the L-orbit of S_a^k in $C^{k,p}(M)$ has same dimension as S^k. Hence, S^k is an open set of this orbit. It follows that S is contained in the L-orbit of a in M. Clearly, the sub-group of L which leaves a fixed coincides with the sub-group of L which leaves S_a^k fixed. Hence, S is open in the L-orbit of a in M.

§5 - REGULAR POINTS OF CURVES IN $\mathbb{R}^3$

We shall characterize, up to order 3, the regular points of a curve γ in the Euclidean space R^3, the group G being the group of rigid motions. Let $a \in \gamma$ be a point of γ, since $g^0(a) = 3$ and $g^1(a) = 1$, x is not regular of order 1. For a to be regular of order 2 it is necessary that $g^1(a) = g^2(a) = 1$; the condition $g^2(a) = 1$ is equivalent to the vanishing of the curvature at the point a. Hence, it is necessary that the curvature vanishes in a neighborhood of a. Conversely, if the curvature vanishes in an open neighborhood V of a, then, for $x \in V$,

$$g^2(x) = g^1(x) = h^2(x) = h^1(x) = 1$$

and a is 2-regular. Therefore, a is 2-regular if and only if there exists a neighborhood of a in γ which is contained in a straight line. Let $\rho, s: \gamma \to \mathbb{R}$ be respectively the curvature and the are lenght of γ measured from a (i.e. $s(a) = 0$). $g^2(a) = 0$ if and only if $\rho(a) \neq 0$ and $h^2(a) = 0$ if and only if the derivative $\frac{d\rho}{ds}(a) \neq 0$. Hence, if $\rho(a) \neq 0$ and $\frac{d\rho}{ds}(a) \neq 0$ then,

$$g^2(x) = g^3(x) = h^2(x)\ 2\ h^3(x) = 0$$

in a neighborhood of a and a is 3-regular. If $\rho(a) \neq 0$ and $\frac{d\rho}{ds}(a) = 0$ then $h^2(a) = 1$. Hence, for a to be 3-regular it is necessary that $\frac{d\rho}{ds}(x) = 0$ in a neigborhood of a. Conversely, if $\rho(a) \neq 0$ and if ρ is constant in a neighborhood V of a then, $g^3(x) = g^2(x) = 0$ and $h^3(x) = h^2(x) = 1$ for $x \in V$. Consequently, if $\rho(a) \neq 0$ and $\frac{d\rho}{ds}(a) = 0$ then, a is 3-regular if and only if ρ is constant in a neighborhood of a.

It is well know that the classical theorem of congruence of curves in $\mathbb{R}^3$ may not hold in a neighborhood of a point where the curvature vanishes. The following refinement of the classical theorem is an easy consequence of theorem 3.3. Assume that $\rho(a) = 0$ for a point $a \in \gamma$. Then, the derivative $\frac{d\rho}{ds}$ may not exist but the right derivative

$$\left(\frac{d\rho}{ds}\right)^+(a) = \lim_{s \to 0^+} \frac{\rho(s)}{s}$$

always exists. Assume that

$$\left(\frac{d\rho}{ds}\right)^{+}(a) \quad \text{and} \quad \lim_{s\to 0^+}\frac{d^2\rho}{ds^2}$$

do not vanish at a. Then, $g^3(x) = g^4(x) = h^3(x) = h^4(x) = 0$ in a neighborhood of a in γ and a is a regular point of order 4. From the hypothesis $\left(\frac{d\rho}{ds}\right)^{+}(a) \neq 0$ it follows that $\rho(x) \neq o$ for x in a neighborhood V of a and $x \neq a$; hence the torsion $\tau(x)$ is defined for $x \in V$ and $x \neq a$. Let $\bar{\gamma} \subset \mathbb{R}^3$ be a second curve satisfying the same conditions at the point $\bar{a} \in \bar{\gamma}$ and denote by $\bar{\rho}$ $\bar{\tau}$ and $\bar{s}$ the curvature, the torsion and the arc lenght of $\bar{\gamma}$. Let $f\colon \gamma \to \bar{\gamma}$ be a diffeomorphism such that $f(a) = \bar{a}$, $\gamma = \bar{\gamma}\circ f$, $\tau = \bar{\tau}\circ f$ and $s = \bar{s}\circ f$. Then, in a neighborhood of a, f is the restriction to γ of a rigid motion in $\mathbb{R}^3$. This theorem can also be deduced from the usual theorem of congruence of curves in $\mathbb{R}^3$; we state it here to give an example of a situation which is covered by theorem 3.3 whereas the method of the moving frame can not be directly applied.

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