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INFINITELY GENERATED TILTING MODULES OF FINITE PROJECTIVE DIMENSION

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Abstract

We extend Miyashita's notion of a tilting module of finite projective dimension to infinitely generated modules over an arbitrary ring R and characterize the classes $\mathcal{X} \subset \text{Mod}R$ induced by such tilting modules in terms of the existence of $\mathcal{X}$-preenvelopes. This extends results in [3] and [1]. MSC 16D90 16E30 16G10.

Tilting theory was introduced in the early eighties in the context of finitely generated modules over artin algebras by Brenner and Butler [5] and by Happel and Ringel [14], and since then it has played a central role in the development of the representation theory of artin algebras. While the first papers were dealing with tilting modules of projective dimension at most one, the theory was later extended to tilting modules of finite projective dimension by Miyashita [16]. Auslander and Reiten [3] then investigated the connection between tilting theory and the notion of covariantly finiteness which had been introduced by Auslander and Smalø in [4]. This also shed some new light on the Homological Conjectures, which still are a challenging topic in representation theory.

The aim of this paper is to extend the results of Auslander and Reiten to the setting of infinitely generated modules over arbitrary rings. Hereby we generalize previous results obtained by the first author jointly with Tonolo and Trlifaj [1] for infinitely generated tilting modules of projective dimension at most one.

It should be pointed out that tilting modules of projective dimension at most one over arbitrary rings had already been considered by several authors. The first step in this direction is due to Colby and Fuller [7] who studied the finitely presented case. Later Colpi and Trlifaj [10] dropped the assumption 'finitely presented' and could extend the pioneering works [5] and [14] to this more general situation. In fact, the results in [1] were proven employing Colpi's and Trlifaj's investigations.

Moreover, there was also a theory of covariantly finiteness available for modules over arbitrary rings. Actually, in a parallel development, the corresponding general notions had independently been discovered and studied by Enochs [12] and other authors. We will adopt their terminology, so left and right approximations will be called preenvelopes and precovers.

Let us now describe the contents of this paper in more detail. Section 1 is devoted to some preliminaries. In Section 2 we extend Miyashita's notion of a tilting module of finite projective dimension [16] to infinitely generated modules. In Section 3 and 4 we then relate these tilting modules to the theory of preenvelopes. More precisely, we show that a class of modules $\mathcal{X}$ with suitable closure properties is induced by a tilting module of projective dimension n if and only if every module has a special $\mathcal{X}$-preenvelope and all modules in the Ext-orthogonal class ${}^{\perp}\mathcal{X}$ have projective dimension at most n (Theorem 4.1). Moreover, we study infinitely generated partial tilting modules and see that a module M is tilting if and only if it is a partial tilting module such that the category $M^{\perp}$ is contained in the category Gen$M$ of all M-generated modules (Theorem 4.4).

We will also consider the dual situation. In the artin algebra case, the dual concepts of a cotilting and a partial cotilting module are just obtained by applying the ordinary self-duality. For infinitely generated modules over an arbitrary ring, the situation is more complicated. The case of injective dimension at most one has been studied by Colpi, D'Este, Tonolo and Trlifaj in [8],[9], and has been related to the theory of precovers in [1]. Here we will consider infinitely generated cotilting modules of finite injective dimension and establish the relationship with the theory of precovers by dualizing the statements proven in the tilting case (Theorem 4.2).

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1 Preliminary results

Along this paper R will denote an arbitrary ring and Mod$R$ the category of all right R-modules. The subcategory of the finitely generated right R-modules is denoted by mod$R$.

Let $\mathcal{M} \subset \mathrm{Mod}R$ be a class of modules. Unless otherwise stated, we will always assume that the classes of modules are *closed under direct summands and isomorphic images*. Denote by **Add** $\mathcal{M}$ (respectively, **add** $\mathcal{M}$) the class of modules consisting of all modules isomorphic to direct summands of (finite) direct sums of elements of $\mathcal{M}$. The class consisting of all modules isomorphic to direct summands of products of modules of $\mathcal{M}$ is denoted by **Prod** $\mathcal{M}$. Denote by $\mathcal{M}^{\perp} = \{X_R : \mathrm{Ext}^i_R(M, X) = 0$, for each $M \in \mathcal{M}$ and each $i > 0\}$. Clearly, $\mathcal{M}^{\perp}$ is a **coresolving** subcategory, that is, it is closed under extension and cokernels of monomorphisms and contains all the injective modules. Also, the subcategory ${}^{\perp}\mathcal{M} = \{X_R : \mathrm{Ext}^i_R(X, M) = 0$, for each $M \in \mathcal{M}$ and each $i > 0\}$, is **resolving**, that is, it is closed under extensions and kernels of epimorphisms and contains all the projective modules. If $\mathcal{M}$ contains a unique module M, then we shall denote these subcategories by **Add**M, **add**M, **Prod**M, $M^{\perp}$ and ${}^{\perp}M$, respectively.

Given a module M, we write **Gen**M for the subcategory of all M-generated modules, that is, modules X admitting an epimorphism $M' \longrightarrow X$ with $M' \in$Add M. The category of M-cogenerated modules, that is, modules X admitting a monomorphism $X \longrightarrow M'$ with $M' \in$Prod M, is denoted by **Cogen**M.

Let us now recall some notions concerning preenvelopes and precovers. Let $\mathcal{M} \subset \mathrm{Mod}R$ and $A \in \mathrm{Mod}R$. Then a morphism $f \in \mathrm{Hom}_R(A, X)$, with $X \in \mathcal{M}$, is called an $\mathcal{M}$**-preenvelope** of A provided that the abelian group homomorphism $\mathrm{Hom}_R(f, M)$: $\mathrm{Hom}_R(X, M) \longrightarrow \mathrm{Hom}_R(A, M)$ is surjective for each $M \in \mathcal{M}$. Dually, a morphism $f \in \mathrm{Hom}_R(X, A)$ with $X \in \mathcal{M}$, is called an $\mathcal{M}$**-precover of** A provided that the abelian group homomorphism $\mathrm{Hom}_R(M, f)$: $\mathrm{Hom}_R(M, X) \longrightarrow \mathrm{Hom}_R(M, A)$ is surjective for each $M \in \mathcal{M}$. The class $\mathcal{M}$ is called a **preenvelope class** (respectively, a **precover class**) provided each R-module admits an $\mathcal{M}$-preenvelope (respectively, an $\mathcal{M}$-precover). Preenvelope and precover classes have been characterized by Rada and Saorin in [17]. The following observation will be very useful.

Proposition 1.1 [17] Let M be a module. Then Add$M$ is a precover class and Prod$M$ is a preenvelope class.

Proof: We recall the proof for the reader's convenience. If X_R is a module and $I = \mathrm{Hom}_R(M, X)$, then the codiagonal map $M^{(I)} \to X$ induced by all homomorphisms is an Add$M$-precover. Dually, for $J = \mathrm{Hom}_R(X, M)$ the diagonal map $X \to M^J$ is a Prod$M$-preenvelope. □

Further, an $\mathcal{M}$-precover $f \in \mathrm{Hom}_R(X, A)$ of A is **special** if it is an epimorphism and $\mathrm{Ext}^1_R(M, \mathrm{Ker} f) = 0$ for each $M \in \mathcal{M}$. Dually, an $\mathcal{M}$- preenvelope $f \in \mathrm{Hom}_R(A, X)$ of A is called **special** if it is a monomorphism and $\mathrm{Ext}^1_R(\mathrm{Coker} f, M) = 0$ for each $M \in \mathcal{M}$. We now give a criterion for $\mathrm{Coker} f$ even lying in the Ext-orthogonal class. Observe that the dual version of the following lemma is basically contained in [6, 2.1].

Lemma 1.2 Let $\mathcal{M} \subset \mathrm{Mod}R$ be a class of modules closed under cokernels of monomorphisms and containing all injective modules. If C is a module with $\mathrm{Ext}^1_R(C, M) = 0$ for all $M \in \mathcal{M}$, then $C \in {}^{\perp}\mathcal{M}$.

Proof: We prove by induction on $i \geq 1$ that $\mathrm{Ext}^i_R(C, M) = 0$ for each $M \in \mathcal{M}$. There is nothing to prove for $i = 1$. Consider now a short exact sequence $0 \longrightarrow M \longrightarrow I \longrightarrow M' \longrightarrow 0$, where I is an injective module. By hypothesis, both I and M' belong to $\mathcal{M}$. Applying now the functor $\mathrm{Hom}_R(C, -)$ to it and using the induction hypothesis we get the claim. □

As a consequence, we have the following result.

Corollary 1.3 Let $\mathcal{M} \subset \mathrm{Mod}R$ be a class of modules closed under cokernels of monomorphisms and containing all injective modules. If $f: A \longrightarrow B$ is a special $\mathcal{M}$-preenvelope, then $\mathrm{Coker} f \in {}^{\perp}\mathcal{M}$.

For more information about precovers and preenvelopes we refer to [18].

The following well-known lemma will play a useful role in our investigations.

Lemma 1.4 (Dimension shifting) Let A, Y be R-modules. If $0 \longrightarrow A \longrightarrow B_0 \longrightarrow B_1 \longrightarrow \cdots \longrightarrow B_n \longrightarrow 0$ is a long exact sequence with $B_i \in Y^{\perp}$ for $0 \leq i < n$, then $\mathrm{Ext}^i_R(Y, B_n) \cong \mathrm{Ext}^{i+n}_R(Y, A)$ for each $i > 0$. In particular, if $n = \mathrm{pd}_R Y$, then $B_n \in Y^{\perp}$ as well.

2 Tilting and cotilting modules

In this section we shall start our investigation on the relations between tilting and cotilting modules and the notions of preenvelopes and precovers. We first give the definition of (generalized) tilting modules in the context we are working with.

Definition. A module M_R is called a **tilting module** provided

(T1) $\mathrm{pd}_R M < \infty$;

(T2) $\mathrm{Ext}^i_R(M, M^{(I)}) = 0$ for each $i > 0$ and all sets I;

(T3) There exists a long exact sequence $0 \longrightarrow R_R \longrightarrow M_0 \longrightarrow \cdots \longrightarrow M_r \longrightarrow 0$ with $M_i \in \mathrm{Add}M$ for each $0 \leq i \leq r$.

Also, M_R is called a **partial tilting module** if it satisfies conditions (T1) and (T2) above.

This definition generalizes the classical ones for artin algebras given by Happel and Ringel [14] and Miyashita [16], as well as the one given by Colpi and Trlifaj [10] for infinitely generated tilting modules of projective dimension at most one. In fact, for any finitely generated module M over an artin algebra Λ, we have that the functors $\mathrm{Ext}^i_\Lambda(M, \ )$, $i \geq 0$, commute with coproducts, hence any $M \in \mathrm{mod}\Lambda$ with $\mathrm{Ext}^i_\Lambda(M, M) = 0$ for each $i > 0$ satisfies even (T2).

Example 2.1 (a) Let Λ be an artin algebra. There is an easy way of constructing infinitely generated tilting Λ-modules from finitely generated ones. Indeed, if M is a tilting module in $\mathrm{mod}\Lambda$ of $\mathrm{pd}_R M = n$ and I is any set, then $M^{(I)}$ is a tilting module in $\mathrm{Mod}\Lambda$ of $\mathrm{pd}_R M = n$.
(b) Let R be a right noetherian ring of finite global dimension n, e. g. $R = K[x_1, \ldots, x_n]$ the polynomial ring in n indeterminates over a field K. Since R is right noetherian, there is an injective module M such that the class of all injective modules coincides with $\mathrm{Add}M$ (see [17, 3.6]). Then M is a (in general not finitely generated) tilting module of $\mathrm{pd}_R M = n$. In fact, any injective resolution of a module of projective dimension n must contain some injective of projective dimension n, which shows $\mathrm{pd}_R M = n$. Moreover, for any set I, the module $M^{(I)}$ is again injective, and therefore $\mathrm{Ext}^i_R(M, M^{(I)}) = 0$ for each $i > 0$. Finally, any injective resolution of R will give a long exact sequence as in (T3). □

We turn now to the dual definition of cotilting modules.

Definition. A module M_R is called a **cotilting module** provided

(C1) $\mathrm{id}_R M < \infty$;

(C2) $\mathrm{Ext}^i_R(M^I, M) = 0$ for each $i > 0$ and all sets I;

(C3) There exists an injective cogenerator Q and a long exact sequence $0 \longrightarrow M_n \longrightarrow \cdots \longrightarrow M_0 \longrightarrow Q \longrightarrow 0$, with $M_i \in \mathrm{Prod}M$ for each $0 \leq i \leq r$.

Also, M_R is called a **partial cotilting module** if it satisfies conditions (C1) and (C2) above.

Also this definition generalizes the classical one for artin algebras, as well as the one given by Colpi, D'Este and Tonolo [8] for infinitely generated cotilting modules of injective dimension at most one (see [1, 2.3]). In fact, since any finitely generated module M over an artin algebra Λ is product-complete in the sense of [15], for any set I we have that M^I is a direct summand of some $M^{(J)}$, hence any $M \in \mathrm{mod}\Lambda$ with $\mathrm{Ext}^i_\Lambda(M, M) = 0$ for each $i > 0$ satisfies even (C2).

The next lemmata will be very important in our investigations. If $\mathcal{M} \subset \mathrm{Mod}R$, then $\check{\mathcal{M}}$ will denote the subcategory of all modules X_R such that there exists a long exact sequence $0 \longrightarrow X \longrightarrow M_0 \longrightarrow M_1 \longrightarrow \cdots \longrightarrow M_r \longrightarrow 0$ with $r \geq 0$ and $M_i \in \mathcal{M}$ for $i = 0, \cdots, r$. Further, $\widehat{\mathcal{M}}$ will denote the dual subcategory of all modules X_R admitting a long exact sequence $0 \longrightarrow M_r \longrightarrow \cdots \longrightarrow M_1 \longrightarrow M_0 \longrightarrow X \longrightarrow 0$ with $r \geq 0$ and $M_i \in \mathcal{M}$ for $i = 0, \cdots, r$.

Lemma 2.2 Let $M_R \in \mathrm{Mod}R$.

(a) Denote $\mathcal{X} = M^\perp$. If $\mathrm{pd}_R M = n$, then $\check{\mathcal{X}} = \mathrm{Mod}R$, and all $Y \in {}^\perp\mathcal{X}$ have $\mathrm{pd}_R Y \leq n$.

(b) Denote $\mathcal{X} = {}^\perp M$. If $\mathrm{id}_R M = n$, then $\widehat{\mathcal{X}} = \mathrm{Mod}R$, and all $Y \in \mathcal{X}^\perp$ have $\mathrm{id}_R Y \leq n$.

Proof: (a) Let $A_R \in \mathrm{Mod}R$ and consider a long exact sequence

$$0 \longrightarrow A_R \longrightarrow I_0 \longrightarrow \cdots \longrightarrow I_{n-1} \longrightarrow Q_n \longrightarrow 0$$

with I_j injective modules, hence in $\mathcal{X}$. Since $\mathrm{pd}_R M = n$, we infer by Lemma 1.4 that $Q_n \in \mathcal{X}$ and so $A_R \in \check{\mathcal{X}}$. Let now $Y \in {}^\perp\mathcal{X}$. Clearly, $I_j, Q_n \in Y^\perp$. Hence, by Lemma 1.4 again, $\mathrm{Ext}^{i+n}_R(Y, A) = 0$, for each $i > 0$. Since this is true for each $A \in \mathrm{Mod}R$, we have that $\mathrm{pd}Y \leq n$. The proof of (b) is dual. □

Let $M \in \mathrm{Mod}R$. Then M is called **finendo** if it is finitely generated over its endomorphism ring $\mathrm{End}(M_R)$. Observe that M is finendo if and only if there exists an $\mathrm{Add}M$-preenvelope of R_R (see [1](1.2)). Also, following [1], $M \in \mathrm{Mod}R$ is called **cofinendo** if there exists an injective cogenerator Q, a cardinal γ and a map $f: M^\gamma \longrightarrow Q$ such that for any cardinal α, all maps $M^\alpha \longrightarrow Q$ factor through f or, equivalently, there exists a $\mathrm{Prod}M$-precover of an injective cogenerator of $\mathrm{Mod}R$.

Lemma 2.3 Let $M_R \in \text{Mod}R$.

(a) If M_R satisfies conditions (T2) and (T3), then M is finendo and $M^{\perp} \subset \text{Gen}M$.

(b) If M_R satisfies conditions (C2) and (C3), then M is cofinendo and ${}^{\perp}M \subset \text{Cogen}M$.

Proof: (a) Denote $\mathcal{X} = M^{\perp}$ and consider the following sequence given by condition (T3)

$$0 \longrightarrow R_R \xrightarrow{f_0} M_0 \xrightarrow{f_1} M_1 \cdots \xrightarrow{f_n} M_n \longrightarrow 0$$

with $M_i \in \text{Add}M$ for each $0 \leq i \leq n$. Clearly, we have that $M_i \in {}^{\perp}\mathcal{X}$. Since ${}^{\perp}\mathcal{X}$ is resolving, we infer that $K_i = \text{Ker} f_i \in {}^{\perp}\mathcal{X}$ for each i. In particular, $\text{Ext}^1_R(K_2, X) = 0$ for all $X \in \mathcal{X}$. So, $f_0: R \longrightarrow M_0$ is an $\mathcal{X}$-preenvelope, and from [1](1.1), it follows that $\mathcal{X} \subset \text{Gen}M_0 \subset \text{Gen}M$. Moreover, since $M_0 \in \text{Add}M$, and since $\text{Add}M \subset \mathcal{X}$ by (T2), we infer that $f_0: R \longrightarrow M_0$ is also an $\text{Add}M$-preenvelope, which means by [1](1.2) that M is finendo. The proof of (b) is dual. □

Lemma 2.4 Let $M_R \in \text{Mod}R$.

(a) Suppose M_R satisfies condition (T2) and denote $\mathcal{X} = M^{\perp}$. If $\mathcal{X} \subset \text{Gen}M$, then

  (i) for each $X \in \mathcal{X}$, there exists a short exact sequence $0 \longrightarrow K \longrightarrow M' \longrightarrow X \longrightarrow 0$, with $M' \in \text{Add}M$ and $K \in \mathcal{X}$.

  (ii) every map $A \longrightarrow X$ with $A \in {}^{\perp}\mathcal{X}$ and $X \in \mathcal{X}$ factors through $\text{Add}M$. In particular, we then have $\text{Add}M = \mathcal{X} \cap {}^{\perp}\mathcal{X}$.

(b) Suppose M_R satisfies condition (C2) and denote $\mathcal{X} = {}^{\perp}M$. If $\mathcal{X} \subset \text{Cogen}M$, then

  (i) for each $X \in \mathcal{X}$, there exists a short exact sequence $0 \longrightarrow X \longrightarrow M' \longrightarrow L \longrightarrow 0$, with $M' \in \text{Prod}M$ and $L \in \mathcal{X}$.

  (ii) every map $X \longrightarrow C$ with $C \in \mathcal{X}^{\perp}$ and $X \in \mathcal{X}$ factors through $\text{Prod}M$. In particular, we then have $\text{Prod}M = \mathcal{X} \cap \mathcal{X}^{\perp}$.

Proof: (a) (i) Let $X \in \mathcal{X}$ and consider an $\text{Add}M$-precover $g: M' \longrightarrow X$ (see Proposition 1.1). Clearly, g is an epimorphism because $X \in \text{Gen}M$. We claim now that K =Ker$g$ belongs to $\mathcal{X}$. Indeed, first observe that $\text{Ext}^1_R(M,K) = 0$ because $\text{Hom}_R(M,g)$ is an epimorphism and $\text{Ext}^1_R(M,M') = 0$ (by (T2)). For $i \geq 1$ consider the exact sequence

$$\text{Ext}^i_R(M,X) \longrightarrow \text{Ext}^{i+1}_R(M,K) \longrightarrow \text{Ext}^{i+1}_R(M,M')$$

Since $X \in \mathcal{X}$, $\text{Ext}^i_R(M,X) = 0$, and from (T2) it follows that $\text{Ext}^{i+1}_R(M,M') = 0$, giving the required claim.
(ii) Let now $f: A \longrightarrow X$ with $A \in {}^\perp\mathcal{X}$ and $X \in \mathcal{X}$ and consider a short exact sequence $0 \longrightarrow K \longrightarrow M' \xrightarrow{g} X \longrightarrow 0$ with $M' \in \text{Add}M$ as in (i). Since $\text{Ext}^1_R(A,K) = 0$, we have that f factors through g as required. To show that $\text{Add}M = \mathcal{X} \cap {}^\perp\mathcal{X}$, observe initially that for $A \in \mathcal{X} \cap {}^\perp\mathcal{X}$ its identity map id_A factors through $\text{Add}M$, and so $A \in \text{Add}M$. The other inclusion follows directly from condition (T2). The proof of (b) is dual. □

3 $M^\perp$-preenvelopes and ${}^\perp M$-precovers

As a further step towards our main result, we show in this section that the Ext-orthogonal class corresponding to a tilting (respectively, cotilting) module yields special preenvelopes (respectively, special precovers). In the tilting case, this is an immediate consequence of the following general result which was recently proven by Eklof and Trlifaj. We should remark that the result is stated in [11] for the category $\{X_R \mid \text{Ext}^1_R(M,X) = 0\}$, but, as pointed out to us by J. Trlifaj, it can easily be extended to $M^\perp$ by using the dual version of Lemma 1.4.

Theorem 3.1 [11, Theorem 10] Let $M \in \text{Mod}R$. Then every right R-module has a special $M^\perp$-preenvelope.

So far, it is still open whether the statement dual to Theorem 3.1 is true in full generality. But for a cotilting module M, we can obtain the existence of special ${}^\perp M$-precovers from the following result due to Auslander and Buchweitz.

Theorem 3.2 [2, 1.1, 2.2, 2.3, 2.5] Let $\mathcal{X} \subset \text{Mod}R$ be closed under extensions and $\omega \subset \mathcal{X}$. Suppose there exists, for each $X \in \mathcal{X}$, a short exact sequence $0 \longrightarrow X \longrightarrow M' \longrightarrow L \longrightarrow 0$ with $L \in \mathcal{X}$ and $M' \in \omega$. Then

(i) For each $X \in \widehat{\mathcal{X}}$, there exist short exact sequences

$$0 \longrightarrow K_X \longrightarrow M_X \xrightarrow{g_X} X \longrightarrow 0 \quad \text{with } K_X \in \widehat{\omega}, M_X \in \mathcal{X} \text{ and}$$

$$0 \longrightarrow X \xrightarrow{f_X} B_X \longrightarrow C_X \longrightarrow 0 \quad \text{with } B_X \in \hat{\omega}, C_X \in \mathcal{X}.$$

(ii) If $\omega \subset \mathcal{X}^\perp$, then $\hat{\omega} \subset \mathcal{X}^\perp$ and f_X is an $\hat{\omega}$-preenvelope, g_X is an $\mathcal{X}$-precover.

Proposition 3.3 If $M \in \text{Mod}R$ is a cotilting module, then every right R-module has a special ${}^\perp M$-precover.

Proof: Denote $\omega = \text{Prod}M$ and $\mathcal{X} = {}^\perp M$. It follows from Lemma 2.3 and 2.4 that $\omega = \mathcal{X} \cap \mathcal{X}^\perp$, and that for each $X \in \mathcal{X}$, there exists a short exact sequence $0 \longrightarrow X \longrightarrow M' \longrightarrow L \longrightarrow 0$, with $M' \in \omega$ and $L \in \mathcal{X}$. So, we are in the conditions of Theorem 3.2. Observe also that $\widehat{\mathcal{X}} = \text{Mod}R$ by Lemma 2.2. Hence, for each right R-module X, there exists a short exact sequence $0 \longrightarrow K_X \longrightarrow M_X \xrightarrow{g_X} X \longrightarrow 0 \quad$ with $K_X \in \hat{\omega}, M_X \in \mathcal{X}$. Since $\omega \subset \mathcal{X}^\perp$, we have that g_X is an $\mathcal{X}$- precover with kernel $K_X \in \hat{\omega} \subset \mathcal{X}^\perp$, as required. □

4 Main results

Let Λ be an artin algebra and $\mathcal{X} \subset \text{mod}\Lambda$ a coresolving subcategory. Auslander and Reiten have shown in [3] that $\mathcal{X}$ is of the form $M^\perp \cap \text{mod}\Lambda$ for some finitely generated tilting module M if and only if $\mathcal{X}$ is covariantly finite and $\check{\mathcal{X}} \cap \text{mod}\Lambda = \text{mod}\Lambda$. Recall that $\mathcal{X}$ is covariantly finite if and only if every finitely generated Λ-module A has an $\mathcal{X}$-preenvelope $f : A \to X$. Since the modules involved are all of finite length, it is well known that we can always find a left minimal version for f, that is, we can assume f to have the property that each endomorphism $h : X \to X$ with $hf = f$ is an isomorphism. In this case, Wakamatsu's Lemma (see [3, 1.3]) tells us that $\text{Ext}^1_R(\text{Coker}\, f, X) = 0$ for each $X \in \mathcal{X}$. Moreover, since $\mathcal{X}$ contains all injectives, f has to be a monomorphism. Thus f is a special $\mathcal{X}$-preenvelope. Observe further that by [3, 5.3 and 3.9] the condition $\check{\mathcal{X}} \cap \text{mod}\Lambda = \text{mod}\Lambda$ is equivalent to the existence of a common bound n for the projective dimensions of the finitely generated modules in ${}^\perp\mathcal{X}$. So, we can reformulate Auslander's and Reiten's result as follows. The subcategory $\mathcal{X}$ is of the form $M^\perp \cap \text{mod}\Lambda$ for some finitely generated tilting module M if and only if every finitely generated Λ-module has a special $\mathcal{X}$-preenvelope and there is an integer n such that all finitely generated $Y \in {}^\perp \mathcal{X}$ have $\text{pd}_\Lambda Y \leq n$. We now show the corresponding result for infinitely generated modules.

Theorem 4.1 Let $\mathcal{X} \subset \text{Mod}R$ be a class of modules closed under cokernels of monomorphisms and such that $\mathcal{X} \cap^{\perp} \mathcal{X}$ is closed under coproducts. The following statements are equivalent:

(a) There exists a tilting module M with $\text{pd}_R M \leq n$ such that $\mathcal{X} = M^{\perp}$;

(b) Every right R-module has a special $\mathcal{X}$-preenvelope and all $Y \in^{\perp} \mathcal{X}$ have $\text{pd}_R Y \leq n$.

Proof: (a) $\Rightarrow$ (b) By Proposition 3.1, we have that, for each $A_R \in \text{Mod}R$, there exists a short exact sequence $0 \longrightarrow A \xrightarrow{f} B \longrightarrow C \longrightarrow 0$, where f is a special $\mathcal{X}$-preenvelope. By Lemma 2.2 it follows that all $Y \in^{\perp} \mathcal{X}$ have $\text{pd}_R Y \leq n$.
(b) $\Rightarrow$ (a) The proof will be done in several steps.
Claim 1. For each $A_R \in \text{Mod}R$, there exists a long exact sequence

$$0 \longrightarrow A \longrightarrow N_0 \longrightarrow N_1 \longrightarrow \cdots \longrightarrow N_n \longrightarrow 0$$

with $N_i \in \mathcal{X}$, and all cokernels in ${}^{\perp}\mathcal{X}$.
Let $A_R \in \text{Mod}R$. By hypothesis, there exists a short exact sequence $0 \longrightarrow A \xrightarrow{f_0} N_0 \longrightarrow C_1 \longrightarrow 0$, where f_0 is a special $\mathcal{X}$-preenvelope. Observe that $\mathcal{X}$ contains all the injective modules (indeed, an injective module has itself a special $\mathcal{X}$-preenvelope which must split). So, by Corollary 1.3, we have that $C_1 \in^{\perp} \mathcal{X}$. Iterating this procedure, one gets for A_R a sequence

$$0 \longrightarrow A \xrightarrow{f_0} N_0 \xrightarrow{f_1} N_1 \longrightarrow \cdots \xrightarrow{f_n} N_n \longrightarrow C_{n+1} \longrightarrow 0$$

with $N_i \in \mathcal{X}$ and $C_i = \text{Coker} f_{i-1} \in^{\perp} \mathcal{X}$. Therefore, $N_0, \cdots, N_{n-1} \in C_{n+1}^{\perp}$. By Lemma 1.4, $\text{Ext}^1_R(C_{n+1}, C_n) \cong \text{Ext}^{n+1}_R(C_{n+1}, A)$, which vanishes because $\text{pd}_R C_{n+1} \leq n$. Hence $C_n \in \mathcal{X}$, and

$$0 \longrightarrow A \xrightarrow{f_0} N_0 \xrightarrow{f_1} N_1 \longrightarrow \cdots \xrightarrow{f_{n-1}} N_{n-1} \longrightarrow C_n \longrightarrow 0$$

is a sequence as required.
Claim 2. There exists a long exact sequence

$$0 \longrightarrow R_R \longrightarrow B_0 \longrightarrow B_1 \longrightarrow \cdots \longrightarrow B_n \longrightarrow 0$$

with $B_i \in \mathcal{X} \cap^{\perp} \mathcal{X}$.
It is enough to take a sequence like in Claim 1 for $A = R_R$. Since R and all cokernels are in ${}^{\perp}\mathcal{X}$, and ${}^{\perp}\mathcal{X}$ is extension closed, we infer that $B_i \in^{\perp} \mathcal{X}$, which shows Claim 2.

We shall now show that the module $M = \bigoplus_{i=0}^{n} B_i$ is tilting. Clearly, $M \in \mathcal{X} \cap^{\perp} \mathcal{X}$ and so, by hypothesis, $\mathrm{pd}_R M \leq n$. Also, $\mathrm{Ext}_R^i(M, M^{(I)}) = 0$ for each $i > 0$ and any set I because $\mathcal{X} \cap^{\perp} \mathcal{X}$ is closed under coproducts and $M \in \mathcal{X} \cap^{\perp} \mathcal{X}$. The remaining condition, (T3), is given by Claim 2 above.
Observe also that $\mathcal{X} \subset M^{\perp}$.
It remains to show that $M^{\perp} \subset \mathcal{X}$. We shall first show that $\mathcal{X} \cap^{\perp} \mathcal{X} = \mathrm{Add}M$. The inclusion $\mathrm{Add}M \subset \mathcal{X} \cap^{\perp} \mathcal{X}$ follows by construction and the fact that $\mathcal{X} \cap^{\perp} \mathcal{X}$ is closed under coproducts. Let now $X \in \mathcal{X} \cap^{\perp} \mathcal{X}$. Iterating Lemma 2.4, we obtain a long exact sequence

$$0 \longrightarrow K_n \longrightarrow M_n \xrightarrow{f_n} M_{n-1} \longrightarrow \cdots \xrightarrow{f_1} M_0 \xrightarrow{f_0} X \longrightarrow 0$$

with $M_i \in \mathrm{Add}M$, $K_i = \mathrm{Ker} f_i \in M^{\perp}$, for $i = 0, \cdots, n$. Since $X \in^{\perp} \mathcal{X}$, by hypothesis, we have $\mathrm{pd}_R X \leq n$. Hence, by the dual version of Lemma 1.4, we have

$$\mathrm{Ext}_R^1(K_{n-1}, K_n) \cong \mathrm{Ext}_R^{n+1}(X, K_n) = 0$$

and so $K_n \in \mathrm{Add}M \subset \mathcal{X}$. We also obtain that $K_0 \in \mathcal{X}$ because $\mathcal{X}$ is closed under cokernels of monomorphisms. Since now $\mathrm{Ext}_R^1(X, K_0) = 0$, we infer that $X \in \mathrm{Add}M$ as required.
Finally, let $A_R \in M^{\perp}$. By Claim 1, there exists a long exact sequence

$$0 \longrightarrow A \xrightarrow{f_0} N_0 \xrightarrow{f_1} N_1 \longrightarrow \cdots \xrightarrow{f_n} N_n \longrightarrow 0$$

with $N_i \in \mathcal{X}$, and all cokernels in ${}^{\perp}\mathcal{X}$, hence $N_n \in \mathcal{X} \cap^{\perp} \mathcal{X} = \mathrm{Add}M$, by the above. Observe that $C_i = \mathrm{Coker} f_{i-1} \in M^{\perp}$ because $A \in M^{\perp}$, $N_i \in \mathcal{X} \subset M^{\perp}$ and $M^{\perp}$ is coresolving. An easy induction argument now gives that f_0 is a split monomorphism and thus $A \in \mathcal{X}$. □

Using Proposition 3.3, we can obtain the dual version of the above result. We leave the details of the proof for the reader.

Theorem 4.2 Let $\mathcal{X} \subset \mathrm{Mod}R$ be a class of modules closed under kernels of epimorphisms and such that $\mathcal{X} \cap \mathcal{X}^{\perp}$ is closed under products. The following statements are equivalent:

(a) There exists a cotilting module M with $\mathrm{id}_R M \leq n$ such that $\mathcal{X} =^{\perp} M$;

(b) Every right R-module has a special $\mathcal{X}$-precover and all $Y \in \mathcal{X}^{\perp}$ have $\mathrm{id}_R Y \leq n$.

We shall now analyse some consequences of the above results. We first show that Theorem 4.1 generalizes the corresponding result proven in [1] for tilting modules of projective dimension at most one. This also yields a new proof for that result, since the arguments employed in [1] made use of Colpi's and Trlifaj's investigations [10]. Recall that a class $\mathcal{X} \subset \operatorname{Mod}R$ is called a **pretorsion class** provided it is closed under direct sums and factors.

Corollary 4.3 ([1, 2.1]) Let $\mathcal{X} \subset \operatorname{Mod}R$ be a pretorsion class. The following are equivalent:

(i) There exists a tilting module M of projective dimension at most 1 such that $\mathcal{X} = M^{\perp}$.

(ii) Every right R-module has a special $\mathcal{X}$-preenvelope.

Proof: The pretorsion class $\mathcal{X}$ obviously satisfies the closure properties in the assumption of Theorem 4.1. So, we obtain the implication (i)⇒(ii), and in order to prove (ii)⇒(i), we have only to show that every module $Y \in {}^{\perp}\mathcal{X}$ has projective dimension at most one. Let A_R be a module and $0 \longrightarrow A \stackrel{f}{\longrightarrow} B \longrightarrow C \longrightarrow 0$ an exact sequence where f is a special $\mathcal{X}$-preenvelope. Since $\mathcal{X}$ is closed under factors, $C \in \mathcal{X}$. For any $i > 0$ we then have $\operatorname{Ext}^i_R(Y,C) = \operatorname{Ext}^{i+1}_R(Y,B) = 0$, hence $\operatorname{Ext}^{i+1}_R(Y,A) = 0$, and the claim is proven. □

Dually, we obtain [1, 2.5] as a consequence of Theorem 4.2.

Colpi and Trlifaj have shown in [10, 1.3] that a module M is a tilting module of projective dimension at most one if and only if $M^{\perp} = \operatorname{Gen} M$. Observe that the latter is equivalent to M being a partial tilting module of projective dimension at most one such that $M^{\perp} \subset \operatorname{Gen} M$. In fact, $\operatorname{pd}_R M \leq 1$ implies that $M^{\perp}$ is closed under factors, hence from (T2) we deduce $\operatorname{Gen} M \subset M^{\perp}$.

We are interested in a corresponding result for tilting modules of higher projective dimension. Over an artin algebra Λ, it was already proven by Happel [13, §3] that $M \in \operatorname{mod}\Lambda$ is a tilting module if and only if it is a partial tilting module such that $M^{\perp} \cap \operatorname{mod}\Lambda \subset \operatorname{Gen} M$. We are now going to see that Happel's characterization extends to arbitrary modules.

Theorem 4.4 Let $M \in \operatorname{Mod}R$, and denote $\mathcal{X} = M^{\perp}$. The following statements are equivalent:

(i) M is a tilting module;

(ii) M is a partial tilting module such that $\mathcal{X} \subset \text{Gen}M$.

(iii) M is a partial tilting module, $\mathcal{X} \subset \text{Gen}M$, and all $A \in {}^{\perp}\mathcal{X}$ have an Add$M$-preenvelope.

Proof: The implication (i) $\Rightarrow$ (ii) follows immediately from Lemma 2.3.
(ii) $\Rightarrow$ (iii) By Theorem 3.1, every $A \in {}^{\perp}\mathcal{X}$ has an $\mathcal{X}$-preenvelope $f: A \longrightarrow B$. By Lemma 2.4, f factors through a map $f': A \longrightarrow B'$ where $B' \in \text{Add}M$, and since $\text{Add}M \subset \mathcal{X}$ by (T2), f' must be an Add$M$-preenvelope.
(iii) $\Rightarrow$ (i) By hypothesis, every $A \in {}^{\perp}\mathcal{X}$ has an Add$M$-preenvelope $f: A \longrightarrow B$. From Lemma 2.4 we infer that all homomorphisms $A \to X$ where $X \in \mathcal{X}$ factor through Add$M$ and therefore through f. In particular, this applies to any monomorphism $A \longrightarrow I$ with I injective, showing that f is a monomorphism. We claim that $C = \text{Coker}f \in {}^{\perp}\mathcal{X}$. In fact, for any $X \in \mathcal{X}$ we have that $\text{Hom}(f, X)$ is an epimorphism and $\text{Ext}^1_R(B, X) = 0$, which implies $\text{Ext}^1_R(C, X) = 0$, hence $C \in {}^{\perp}\mathcal{X}$ by Lemma 1.2. Let us now take $A = R$ and $n = \text{pd}M$. Iterating the above construction, we get an exact sequence

$$0 \longrightarrow R \longrightarrow B_0 \longrightarrow B_1 \longrightarrow \cdots \longrightarrow B_{n-1} \longrightarrow C_n \longrightarrow 0$$

with $B_i \in \text{Add}M$, and all cokernels in ${}^{\perp}\mathcal{X}$. By Lemma 1.4, we have that $\text{Ext}^i_R(M, C_n) \cong \text{Ext}^{i+n}_R(M, R) = 0$ for each $i > 0$, hence $C_n \in \mathcal{X} \cap {}^{\perp}\mathcal{X}$. Since, by Lemma 2.4, $\text{Add}M = \mathcal{X} \cap {}^{\perp}\mathcal{X}$, the above sequence gives the required one in condition (T3), and the result is proven. □

For the cotilting case, note that the proof of Proposition 3.3 also works if M is a partial cotilting module such that ${}^{\perp}M \subset \text{Cogen}M$. So, we can dualize the above result and obtain that a module M is cotilting if and only if it is a partial cotilting module such that ${}^{\perp}M \subset \text{Cogen}M$.

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