Journal of Dynamics and Differential Equations
https://doi.org/10.1007/s10884-023-10248-4

Homogenization for Nonlocal Evolution Problems with Three Different Smooth Kernels

Monia Capanna[1] · Jean C. Nakasato[2] · Marcone C. Pereira[3] · Julio D. Rossi[1]

Received: 18 August 2021 / Revised: 28 December 2022 / Accepted: 9 January 2023

Abstract
In this paper we consider the homogenization of the evolution problem associated with a jump process that involves three different smooth kernels that govern the jumps to/from different parts of the domain. We assume that the spacial domain is divided into a sequence of two subdomains $A_n \cup B_n$ and we have three different smooth kernels, one that controls the jumps from A_n to A_n, a second one that controls the jumps from B_n to B_n and the third one that governs the interactions between A_n and B_n. Assuming that $\chi_{A_n}(x) \to X(x)$ weakly in L^∞ (and then $\chi_{B_n}(x) \to 1 - X(x)$ weakly in L^∞) as $n \to \infty$ and that the initial condition is given by a density u_0 in L^2 we show that there is an homogenized limit system in which the three kernels and the limit function X appear. When the initial condition is a delta at one point, $\delta_{\bar{x}}$ (this corresponds to the process that starts at $\bar{x}$) we show that there is convergence along subsequences such that $\bar{x} \in A_{n_j}$ or $\bar{x} \in B_{n_j}$ for every n_j large enough. We also provide a probabilistic interpretation of this evolution equation in terms of a stochastic process that describes the movement of a particle that jumps in Ω according to the three different kernels and show that the underlying process converges in distribution to a limit process associated with the limit equation. We focus our analysis in Neumann type boundary conditions and briefly describe at the end how to deal with Dirichlet boundary conditions.

✉ Julio D. Rossi
jrossi@dm.uba.ar

Monia Capanna
moniacapanna@gmail.com

Jean C. Nakasato
nakasato@ime.usp.br
http://www.ime.usp.br/∼nakasato

Marcone C. Pereira
marcone@ime.usp.br
http://www.ime.usp.br/∼marcone

[1] CONICET and Departamento de Matemática, FCEyN, Universidad de Buenos Aires, Ciudad Universitaria, Pabellon I, (1428), Buenos Aires, Argentina

[2] Dpto. de Matemática, ICMC, Universidade de São Paulo, Avenida Trabalhador São-Carlense, 400, São Carlos, SP, Brazil

[3] Dpto. de Matemática Aplicada, IME, Universidade de São Paulo, Rua do Matão 1010, São Paulo, SP, Brazil

Published online: 14 February 2023

Keywords Heterogeneous media · Homogenization · Nonlocal equations · Neumann problem · Dirichlet problem

Mathematics Subject Classification 45K05 · 35B27 · 35B40

1 Introduction

Our main goal in this paper is to study the homogenization that occurs when one deals with nonlocal evolution problems with different non-singular kernels that act in different domains. This paper is a natural continuation of [7] where the stationary case was studied.

Consider a partition of the ambient space $\overline{\Omega}$ (a bounded domain in $\mathbb{R}^N$) into two subdomains A, B, and consider a nonlocal problem in which we have three different smooth kernels. One (that we call J) that measures the probability of jumping from A to A ($J(x, y)$ is the probability that a particle that is at $x \in A$ moves to $y \in A$), another one (G) that is involved in jumps from B to B and a third one (R) that gives the interactions between A and B. Remark that the involved kernels can be of convolution type, that is, we could have for instance, $J(x, y) = J(x - y)$ (this special form of the kernels is often used in applications). However, we only use in our arguments that the kernels $V = J$, G and R are non-singular functions which satisfy the following hypotheses that will be assumed from now on

$$
(\mathbf{H}) \qquad \begin{aligned} & V \in \mathcal{C}(\mathbb{R}^N \times \mathbb{R}^N, \mathbb{R}) \text{ is non-negative with } V(x,x) > 0, \text{ symmetric}, \\ & \quad V(x,y) = V(y,x) \text{ for every } x, y \in \mathbb{R}^N, \text{ and} \\ & \int_{\mathbb{R}^N} V(x,y)\,dy = 1. \end{aligned}
$$

We take a sequence of partitions A_n, B_n of the fixed ambient space $\overline{\Omega}$ such that $\overline{\Omega} = A_n \cup B_n$, $A_n \cap B_n = \emptyset$, B_n is open, has a Lipchitz boundary (consequently $|\partial B_n \cap \Omega| = |\partial A_n \cap \Omega| = 0$) and

$$
\begin{aligned} & \bullet\ \chi_{A_n}(x) \rightharpoonup X(x), \qquad \text{weakly in } L^\infty(\overline{\Omega}), \\ & \bullet\ \chi_{B_n}(x) \rightharpoonup 1 - X(x), \qquad \text{weakly in } L^\infty(\overline{\Omega}), \\ & \quad \text{with } 0 < X(x) < 1. \end{aligned} \tag{1.1}
$$

Notice that the assumption $0 < X(x) < 1$ implies that for every set E with positive measure inside Ω we have $E \cap A_n \neq \emptyset$ and $E \cap B_n \neq \emptyset$ for n large enough. This says that eventually we find both A_n and B_n everywhere in Ω. We use this assumption in our arguments to obtain results for the correctors, see (2.5) in Sect. 2.

Associated to this sequence of partitions we consider the diffusion process that we describe next. We want to analyze the evolution of a particle that moves in $\overline{\Omega}$. To do that we introduce three families $\{E_k^1\}_{k\in\mathbb{N}}$, $\{E_k^2\}_{k\in\mathbb{N}}$ and $\{E_k^3\}_{k\in\mathbb{N}}$ of independent random variables with exponential distribution of parameter $\frac{1}{3}$. Define

$$
\Upsilon_k := \min_{i\in\{1,2,3\}} \left\{E_k^i\right\}, \qquad \forall k \in \mathbb{N}.
$$

The set $\{\Upsilon_k\}_{k\in\mathbb{N}}$ is a family of independent random variables distributed as an exponential of parameter 1. Fixing $\tau_0 = 0$, we define recursively the random times

$$
\tau_k = \tau_{k-1} + \Upsilon_k, \qquad \forall k \in \mathbb{N}.
$$

We denote by $Y_n(t)$ the position of the particle at time t. The evolution of the particle is described as follows: At the times $\{\tau_k\}$ the particle chooses a site $y \in \overline{\Omega}$ according to the kernels J, R or G. The jumps from a site in A_n to another site in A_n are ruled by J, the jumps between A_n and B_n (or vice versa) are ruled by R, the jumps from a site in B_n to a site in B_n are ruled by G. More precisely, if $\Upsilon_k = E_k^1$ the particle chooses a site $y \in \overline{\Omega}$ according to $J(Y_n(\tau_{k-1}), y)$ and it jumps on it only if $Y_n(\tau_{k-1}) \in A_n$ and $y \in A_n$ otherwise the particle remains in its current position. If $\Upsilon_k = E_k^2$ the particle chooses a site $y \in \overline{\Omega}$ according to the kernel $R(Y_n(\tau_{k-1}), y)$ and it jumps on it only if $Y_n(\tau_{k-1}) \in A_n$ and $y \in B_n$ (or if $Y_n(\tau_{k-1}) \in B_n$ and $y \in A_n$). Finally, if $\Upsilon_k = E_k^3$ the particle chooses a site $y \in \overline{\Omega}$ according to $G(Y_n(\tau_{k-1}), y)$ and it jumps on it only if $Y_n(\tau_{k-1}) \in B_n$ and $y \in B_n$.

The process $Y_n(t)$ is a Markov process whose generator L_n is defined on functions $f : \overline{\Omega} \to \mathbb{R}$ with $f|_{A_n} \in C(A_n)$ and $f|_{B_n} \in C(B_n)$ as

$$
\begin{aligned}
L_n f(x) = {} & \chi_{A_n}(x) \int_\Omega \chi_{A_n}(y) J(x, y)(f(y) - f(x))\, dy \\
& + \chi_{B_n}(x) \int_\Omega \chi_{B_n}(y) G(x, y)(f(y) - f(x))\, dy \\
& + \chi_{A_n}(x) \int_\Omega \chi_{B_n}(y) R(x, y)(f(y) - f(x))\, dy \\
& + \chi_{B_n}(x) \int_\Omega \chi_{A_n}(y) R(x, y)(f(y) - f(x))\, dy.
\end{aligned}
$$

With an initial distribution of the position of the particle at time $t = 0$, u_0, the associated evolution problem (whose solution is the density of the process Y_n, see Corollary 2.5) reads as

$$
\begin{cases}
\dfrac{\partial u_n}{\partial t}(t, x) = L_n u_n(t, x), & t > 0,\ x \in \overline{\Omega}, \\
u_n(0, x) = u_0(x), & x \in \overline{\Omega}.
\end{cases}
\tag{1.2}
$$

Notice that we have an evolution equation of Neumann type since the particle remains inside $\overline{\Omega}$ for every positive time (there are no particles entering or leaving the domain). For this evolution the total mass is preserved in time, that is,

$$
\int_\Omega u_n(t, x)\, dx = \int_\Omega u_0(x)\, dx, \qquad \forall t > 0,
$$

as is expected for a Neumann problem. We will focus in this case, but at the end of this paper we will briefly comment on Dirichlet type problems (in this case the particle is allowed to jump outside Ω and is killed when doing so).

Our goal is to take the limit, as $n \to +\infty$, both in the processes $Y_n(t)$ and in the associated densities $u_n(t, x)$. To this end we need to look at the process $Y_n(t)$ as a couple $(Y_n(t), I_n(t))$. In our notation $I_n(t)$ contains explicitly the information over the set (A_n or B_n) in which $Y_n(t)$ is located. More precisely, $I_n(t) = 1$ (or 2) if the particle is in A_n (or in B_n respectively) at time t.

First, we assume that the initial position $Y_n(0)$ is described in terms of a given distribution $u_0 \in L^2(\Omega)$. We suppose that

$$
P(Y_n(0) \in E) = \int_E u_0(z)\, dz,
$$

for every measurable set $E \subseteq \overline{\Omega}$.

Theorem 1.1 *Let the initial condition be given by a distribution $u_0 \in L^2(\Omega)$. Assume (1.1) and fix $T > 0$. We have that, as $n \to \infty$,*

$$\begin{aligned} &u_n(t,x) \rightharpoonup u(t,x), \quad && \text{weakly in } L^2((0,T)\times\Omega),\\ &\chi_{A_n}(x)u_n(t,x) \rightharpoonup a(t,x), \quad && \text{weakly in } L^2((0,T)\times\Omega),\\ &\chi_{B_n}(x)u_n(t,x) \rightharpoonup b(t,x), \quad && \text{weakly in } L^2((0,T)\times\Omega). \end{aligned} \tag{1.3}$$

These limits verify

$$u(t,x) = a(t,x) + b(t,x)$$

and are characterized by the fact that (a, b) is the unique solution to the following system,

$$\begin{cases} \dfrac{\partial a}{\partial t}(t,x) = \displaystyle\int_\Omega J(x,y)\left(X(x)a(t,y) - X(y)a(t,x)\right) dy \\ \qquad\qquad + \displaystyle\int_\Omega R(x,y)\left(X(x)b(t,y) - (1-X(y))\,a(t,x)\right) dy & t>0,\ x\in\Omega, \\ \dfrac{\partial b}{\partial t}(t,x) = \displaystyle\int_\Omega G(x,y)\left[(1-X(x))\,b(t,y) - (1-X(y))\,b(t,y)\right] dy \\ \qquad\qquad + \displaystyle\int_\Omega R(x,y)\left[(1-X(x))\,a(t,y) - X(y)b(t,x)\right] dy\,dy & t>0,\ x\in\Omega, \\ a(0,x) = X(x)\,u_0(x), \quad b(0,x) = (1-X(x))\,u_0(x) & x\in\overline{\Omega}. \end{cases} \tag{1.4}$$

Remark that in the limit we obtain a system rather than a single equation. However, as we show here, there is uniqueness for solutions to the limit system and hence this characterizes the limit $u(t,x) = a(t,x) + b(t,x)$.

Before stating the next theorem that describes the limit distribution of our stocastic process, we introduce some notation. Given a metric space $\mathcal{X}$, for $T > 0$, we denote by $D([0,T],\mathcal{X})$ the space of all trajectories cadlag defined in $[0,T]$ and taking values in $\mathcal{X}$. We consider $D([0,T],\mathcal{X})$ endowed with the Skorohod topology (see Chapter 3 of [4] for more details). Our process $(Y_n(t), I_n(t))_{t\in[0,T]}$ is in $D\left([0,T],\overline{\Omega}\right) \times D([0,T],\{1,2\})$ which we consider endowed with the product topology.

Theorem 1.2 *The sequence of processes converges in distribution*

$$(Y_n(t), I_n(t)) \xrightarrow[n\to+\infty]{D} (Y(t), I(t)) \tag{1.5}$$

in $D\left([0,T],\overline{\Omega}\right) \times D([0,T],\{1,2\})$, where the distribution of the limit $(Y(t), I(t))$ is characterized by having as probability densities $a(t,x)$ and $b(t,x)$ defined in (1.3), that is,

$$P\Big(Y(t)\in E, I(t)=1\Big) = \int_E a(t,z)\,dz \quad \text{and} \quad P\Big(Y(t)\in E, I(t)=2\Big) = \int_E b(t,z)\,dz,$$

for every measurable set $E \subseteq \overline{\Omega}$.

In the following theorem we finally study the asymptotic behaviour of u_n, as $t \to \infty$, proving exponential convergence to the unique stationary distribution.

Theorem 1.3 *Let the initial condition be given by a probability density $u_0 \in L^2(\Omega)$. There exist two constants $A > 0$ (depending only on the domain and the kernels) and $C > 0$ (that depends on the initial condition), such that*

$$\left\| u_n(t,\cdot) - \frac{1}{|\Omega|} \right\|^2_{L^2(\Omega)} \le C e^{-At}.$$

Notice that using Theorem 1.3 (that involve a uniform exponential decay), Theorem 1.1 can be extended to hold in $L^2((0,\infty)\times\Omega)$.

Corollary 1.4 *Let $u_0 \in L^2(\Omega)$, then the conclusions of Theorem 1.1 hold with convergences taking place in $L^2((0,\infty)\times\Omega)$.*

Now, we fix a point $\bar{x} \in \Omega$ and analyze the case in which the initial position is given by $Y_n(0) = \bar{x}$, that is, we assume that

$$P\left(Y_n\left(0\right) \in E\right) = \delta_{\bar{x}}(E) = \begin{cases} 1 & \bar{x} \in E, \\ 0 & \bar{x} \notin E, \end{cases}$$

for every measurable set $E \subseteq \overline{\Omega}$.

In this case there is no convergence of the whole sequence a_n, b_n, but only convergence along subsequences. This can be expected from the fact that the initial condition for a_n (the initial condition fir b_n is similar) satisfies

$$\int_\Omega \varphi(x)a_n(0,x)dx = \int_\Omega \varphi(x)\chi_{A_n}(x)\delta_{\bar{x}}(dx) = \begin{cases} \varphi(\bar{x}) & \bar{x} \in A_n, \\ 0 & \bar{x} \notin A_n, \end{cases}$$

that only converges along subsequences with $\bar{x} \in A_{n_j}$ or $\bar{x} \notin A_{n_j}$ for every n_j.

Call $\left(\nu_t^n\right)_t$ the law of $(Y_n(t))_t$. By Dynkin's formula we know that for every G continuous

$$\begin{cases} \displaystyle\frac{d}{dt}\int_\Omega G(x)\nu_t^n(dx) = \int_\Omega L_n G(x)\nu_t^n(dx), \\ \displaystyle\int_\Omega G(x)\nu_0^n(dx) = G(\bar{x}), \end{cases}$$

where L_n is the generator of our process as described before. Since the involved kernels are smooth, this evolution problem does not have a regularizing effect and therefore we expect that the measure $\delta_{\bar{x}}$, that is the initial condition, remains for positive times. Hence, we write ν_t^n as an absolutely continuous part plus a time-dependent multiple of $\delta_{\bar{x}}$, that is,

$$\nu_t^n(dx) := z_n(t,x)dx + \sigma_n(t)\delta_{\bar{x}}(dx). \tag{1.6}$$

Now, we assume that

$$\chi_{A_n}(\bar{x})\int_\Omega \chi_{A_n}(y)J(\bar{x},y)dy + \chi_{B_n}(\bar{x})\int_\Omega \chi_{B_n}(y)G(\bar{x},y)dy + \chi_{A_n}(\bar{x})\int_\Omega \chi_{B_n}(y)R(\bar{x},y)dy$$
$$+ \chi_{B_n}(\bar{x})\int_\Omega \chi_{A_n}(y)R(\bar{x},y)dy = 1.$$

This condition says that the particle jumps with full probability (that is, the probability of staying at the same location when the exponential clock rings is zero). In fact, assume, for example, that $\bar{x} \in A_n$, then

$$\int_\Omega \chi_{A_n}(y)J(\bar{x},y)dy$$

is the probability to jump to a new position in A_n and

$$\int_\Omega \chi_{B_n}(y)R(\bar{x},y)dy$$

gives the probability to jump to B_n. Then,

$$\int_\Omega \chi_{A_n}(y)J(\bar{x},y)dy + \int_\Omega \chi_{B_n}(y)R(\bar{x},y)dy$$

is the probability to jump to a new position in Ω (and we have that it is equal to 1 since the particle is obliged to jump). A similar analysis can be done when $\bar{x} \in B_n$.

Under this condition, from the expression of L_n we obtain that the time-dependent multiple of $\delta_{\bar{x}}$ is exponentially decreasing in time (independent on n) (see Sect. 3)

$$\sigma_n(t) = e^{-t}, \quad \forall n \in \mathbb{N}.$$

This fact can be interpreted as follows: when the particle jumps for the first time the probability density passes from being a delta at times $s < \tau_1$ to an absolutely continuous measure (recall that the kernels are smooth) for times greater $s > \tau_1$ and this first jump τ_1 is distributed as an exponential of parameter 1.

On the other hand, we have an equation for z_n,

$$\begin{aligned} \frac{\partial z_n}{\partial t}(t,x) &= \chi_{A_n}(x) \int_\Omega \chi_{A_n}(y) J(x,y) \left(z_n(t,y) - z_n(t,x)\right) dy + e^{-t} \chi_{A_n}(\bar{x}) \chi_{A_n}(x) J(\bar{x},x) \\ &+ \chi_{B_n}(x) \int_\Omega \chi_{B_n}(y) G(x,y) \left(z_n(t,y) - z_n(t,x)\right) dy + e^{-t} \chi_{B_n}(\bar{x}) \chi_{B_n}(x) G(\bar{x},x) \\ &+ \chi_{A_n}(x) \int_\Omega \chi_{B_n}(y) R(x,y) \left(z_n(t,y) - z_n(t,x)\right) dy + e^{-t} \chi_{A_n}(\bar{x}) \chi_{B_n}(x) R(\bar{x},x) \\ &+ \chi_{B_n}(x) \int_\Omega \chi_{A_n}(y) R(x,y) \left(z_n(t,y) - z_n(t,x)\right) dy + e^{-t} \chi_{B_n}(\bar{x}) \chi_{A_n}(x) R(\bar{x},x), \end{aligned} \tag{1.7}$$

with initial condition $z_n(0,x) = 0$.

Notice that $z_n(t,x)$ is a function in $L^2(\Omega)$ for every $t > 0$. Therefore, the solution to our evolution problem with initial condition $\delta_{\bar{x}}$ is given by an absolutely continuos (with respect to the Lebesgue measure) part, z_n, and a singular part, $e^{-t}\delta_{\bar{x}}$ (in this singular part the delta measure remains but decays exponentially fast in time).

Now, we want to look at the limit as $n \to \infty$. Since the singular part of the solution, $e^{-t}\delta_{\bar{x}}$, is independent of n we have to look for the behaviour of z_n as $n \to \infty$ (here as we already mentioned we can only show convergence along subsequences).

Theorem 1.5 *Given $(z_n)_{n\in\mathbb{N}}$ there is a subsequence z_{n_k} that is weakly convergent in $L^2((0,T)\times\Omega)$. Moreover, it holds that*

$$\chi_{A_{n_k}}(x) z_{n_k}(t,x) \rightharpoonup a_k(t,x),$$
$$\chi_{B_{n_k}}(x) z_{n_k}(t,x) \rightharpoonup b_k(t,x),$$

weakly in $L^2((0,T)\times\Omega)$, where $(a_k(t,x), b_k(t,x))$ is a solution to

$$\begin{cases} \dfrac{\partial a}{\partial t}(t,x) = \displaystyle\int_\Omega J(x,y) \left(X(x) a(t,y) - X(y) a(t,x)\right) dy & \\ \quad + \displaystyle\int_\Omega R(x,y) \left(X(x) b(t,y) - (1 - X(y))\, a(t,x)\right) dy + e^{-t} J(\bar{x},x) X(x), & t > 0,\ x \in \Omega, \\ \dfrac{\partial b}{\partial t}(t,x) = \displaystyle\int_\Omega G(x,y) \left((1 - X(x))\, b(t,y) - (1 - X(y))\, b(t,x)\right) dy & \\ \quad + \displaystyle\int_\Omega R(x,y) \left((1 - X(x))\, a(t,y) - X(y) b(t,x)\right) dy\, dy + e^{-t} R(\bar{x},x) \left(1 - X(x)\right), & t > 0,\ x \in \Omega, \\ a(0,x) = 0, \quad b(0,x) = 0, & x \in \overline{\Omega}, \end{cases} \tag{1.8}$$

or to

$$
\begin{cases}
\dfrac{\partial a}{\partial t}(t,x) = \displaystyle\int_{\Omega} J(x,y)\left(X(x)a(t,y) - X(y)a(t,x)\right) dy \\
\quad + \displaystyle\int_{\Omega} R(x,y)\left(X(x)b(t,y) - (1-X(y))\,a(t,x)\right) dy + e^{-t} R(\bar{x},x)X(x), & t>0,\ x\in\Omega, \\
\dfrac{\partial b}{\partial t}(t,x) = \displaystyle\int_{\Omega} G(x,y)\left((1-X(x))\,b(t,y) - (1-X(y))\,b(t,x)\right) dy \\
\quad + \displaystyle\int_{\Omega} R(x,y)\left((1-X(x))\,a(t,y) - X(y)b(t,x)\right) dy\, dy + e^{-t} G(\bar{x},x)\,(1-X(x)), & t>0,\ x\in\Omega, \\
a(0,x)=0, \quad b(0,x)=0, & x\in\overline{\Omega}.
\end{cases}
\tag{1.9}
$$

The first system, (1.8), occurs when the convergent subsequence is such that $\bar{x} \in A_{n_k}$ for every n_k; while the second one, (1.9), appears when $\bar{x} \in B_{n_k}$ for every n_k.

Notice that the two possible limit systems are similar but different since in (1.8) we have exponential terms like $e^{-t}J(\bar{x},x)X(x)$ and $e^{-t}R(\bar{x},x)\,(1-X(x))$ while in (1.9) the terms $e^{-t}R(\bar{x},x)X(x)$ and $e^{-t}G(\bar{x},x)\,(1-X(x))$ appear. Also remark that both systems (1.8) and (1.9) are similar to (1.4) except by the fact that the exponential terms do not appear in (1.4) (c.f. Theorem 1.1). In addition, we have that also the limit $u(t,x) = a(t,x) + b(t,x)$ is different in the two previously mentioned cases (notice that in the first system the term $e^{-t}G(\bar{x},x)\,(1-X(x))$ that involves the kernel G does not appear; while in the second one the term $e^{-t}J(\bar{x},x)X(x)$ is missing).

We finally analyze the asymptotic behaviour of z_n, as $t \to \infty$, proving exponential convergence to the unique stationary distribution.

Theorem 1.6 *There exist $A > 0$ and $C > 0$, such that*

$$
\left\| z_n(t,\cdot) - \frac{1}{|\Omega|} \right\|^2_{L^2(\Omega)} \leq Ce^{-At},
$$

for t large enough.

As a consequence of this theorem we obtain the asymptotic behaviour for $\left(\nu_t^n\right)_t$ the law of our process $(Y_n(t))_t$ starting at $\delta_{\bar{x}}$. For every continuous function g, there exist $\widehat{A} > 0$ and $\widehat{C} > 0$ (independent of g and $\bar{x}$) such that

$$
\begin{aligned}
&\left| \int_{\Omega} g(x)\nu_t^n(dx) - \frac{1}{|\Omega|}\int_{\Omega} g(x)\,dx \right| \\
&\quad = \left| \int_{\Omega} g(x) z_n(t,x)\,dx + e^{-t}g(\bar{x}) - \frac{1}{|\Omega|}\int_{\Omega} g(x)\,dx \right| \leq \|g\|_{L^\infty(\Omega)} \widehat{C} e^{-\widehat{A}t}.
\end{aligned}
$$

That is, we have that $\left(\nu_t^n\right)$ converges exponentially in the sense of measures to the unique stationary distribution as $t \to +\infty$.

Now, let us end the introduction with a brief description of previous results and comments on the ideas and difficulties involved in our proofs.

Nonlocal equations with smooth kernels like the ones considered here has been widely studied and used in the literature as models in different applied scenarios, see for example, [1, 2, 9, 10, 13–15, 18, 20]. Here we have a model in which the jumping probabilities depend on three different kernels J, G and R that act in different parts of the domain (thus, our model problem can be seen as a coupling between two nonlocal equations that occur in the sets A and B). For other couplings (even considering local equations and nonlocal ones) we refer to [8, 13–16, 20, 21, 23].

For references concerning convergence of evolution processes in the fiwld of probability theory we refer to [11, 19, 29] and references therein.

As a motivation for this study we mention that it is closely related to a homogenization procedure. In fact, when the two sets are given by a chessboard (the union of the white squares is A and the black squares is B) of side ε then the involved operator reads as

$$\begin{aligned} L_\varepsilon f(x) = a\left(\frac{x}{\varepsilon}\right)\int_\Omega a\left(\frac{y}{\varepsilon}\right) J(x,y)(f(y)-f(x))\,dy \\ + b\left(\frac{x}{\varepsilon}\right)\int_\Omega b\left(\frac{y}{\varepsilon}\right) G(x,y)(f(y)-f(x))\,dy \\ + a\left(\frac{x}{\varepsilon}\right)\int_\Omega b\left(\frac{y}{\varepsilon}\right) R(x,y)(f(y)-f(x))\,dy \\ + b\left(\frac{x}{\varepsilon}\right)\int_\Omega a\left(\frac{y}{\varepsilon}\right) R(x,y)(f(y)-f(x))\,dy. \end{aligned}$$

Here a and b are the 1−periodic functions given by

$$a(x_1,x_2) = \begin{cases} 1 & [0,1/2]\times[0,1/2]\cup[1/2,1]\times[1/2,1], \\ 0 & (0,1/2)\times(1/2,1)\cup(1/2,1)\times(0,1/2), \end{cases}$$

and

$$b(x_1,x_2) = 1 - a(x_1,x_2)$$

extended periodically to the whole $\mathbb{R}^2$. Notice that in a chessboard with squares of side $\varepsilon/2$ we have that the white part χ_A can be written as $a(x/\varepsilon)$ (similarly, the black part is given by $b(x/\varepsilon)$). Therefore, in the particular case of a chessboard configuration of the sets we have a homogenization problem with periodic coefficients. Remark that in this case we have that (1.1) holds with $X \equiv 1/2$, since in this case we have

$$\lim_{\varepsilon\to 0}\int_{A_\varepsilon} f(x)\,dx = \lim_{\varepsilon\to 0}\int_\Omega a\left(\frac{x}{\varepsilon}\right) f(x)\,dx = \frac{1}{2}\int_\Omega f(x)\,dx$$

for every continuous function f. Moreover, we emphasize that we can tackle a far more general setting in which we only ask for weak convergence of the characteristic functions of the involved sets A_n, B_n, recall (1.1). For example, take a the characteristic function of a measurable set $D \subset [0,1]\times[0,1]$ (with $0 < |D| < 1$) extended periodically and consider L_ε as before. Our results also apply here (now we have that (1.1) holds with $X \equiv |D|$).

Homogenization for PDEs is by now a classical subject that originated in the study of the behaviour of the solutions to elliptic and parabolic local equations with highly oscillatory coefficients (periodic homogenization). We refer to [3, 12, 28] as general references for the subject. For other kinds of homogenization for pure nonlocal problems with one kernel we refer to [24–26]. For homogenization results for nonlocal equations with a singular kernel (like the one that appears in the fractional Laplacian) we refer to [6, 27, 30] and references therein. We emphasize that the previously mentioned references deal with homogenization in the coefficients involved in the equation. For random homogenization of an obstacle problem we refer to [5]. For mixing local and nonlocal processes we refer to [8]. Here we deal with an homogenization problem that is different in nature with the ones treated in the previously mentioned references as we homogenize mixing three different jump operators with smooth kernels.

We observe that our results can be generalized to cover the case of N−subdomains, $(A_1)_n \cup \ldots \cup (A_N)_n = \Omega$. In this case there are kernels J_i that govern the jumps inside each of the A_i and also kernels $G_{i,j}$ that are the ones that encode the jumps from A_i to A_j. In this case the operator reads as

$$L_n f(x) = \sum_{i=1}^{N} \chi_{(A_i)_n}(x) \int_{\Omega} \chi_{(A_i)_n}(y)\, J_i(x, y)\, (f(y) - f(x))\, dy$$
$$+ \sum_{j=1}^{N} \chi_{(A_i)n}(x) \int_{\Omega} \chi_{(A_j)_n}(y)\, G_{i,j}(x, y)\, (f(y) - f(x))\, dy.$$

Similar results also hold in this case. Assuming

$$\chi_{(A_i)_n}(x) \rightharpoonup X_i(x), \qquad \text{weakly in } L^{\infty}(\overline{\Omega}),$$

for every $i = 1, \ldots, N$, we get that

$$\chi_{(A_i)_n}(x) u_n(t, x) \to a_i(t, x)$$

where the a_i satisfy a limit system of equations.

Finally, let us describe the main ingredients that appear in the proofs. First, we show weak convergence along subsequences of u_n, $\chi_{A_n} u_n$ and $\chi_{B_n} u_n$ (these convergences comes from a uniform bound in L^2). Next, we find the system that these limits verify. This part of the proof is delicate since we have to pass to the limit in the week form of the equation that involves terms like $\chi_{B_n}(x)\chi_{A_n}(y)u_n(t, y)J(x, y)$ and we only have weak convergence of χ_{B_n} and $\chi_{A_n} u_n$. Here we need to rely on the continuity of J and use the fact that the product $\chi_{B_n}(x)\chi_{A_n}(y)u_n(t, y)J(x, y)$ involves two different variables, x and y. Finally, we show uniqueness of the limit by proving uniqueness of solutions to the limit system.

To show the convergence of the process we first prove that $Y_n(t)$ has a probability density $u_n(t, x)$ which is the unique solution to system (1.2). Next, we prove that the laws of the processes $(Y_n(t), I_n(t))_{t\in[0,T]}$ form a tight sequence and finally we characterize the limit as the limit process.

When the initial condition is $u_0 = \delta_{\bar{x}}$ we use analogous arguments, but in this case we need extra care since, due to the lack of regularizing effect, we have a term of the form $e^{-t}\delta_{\bar{x}}$ in the solution (see formula (1.6)). This creates the extra exponential terms in the equation satisfied by z_n, (1.7). We remark again that here there is only convergence along subsequences for which the point at which the process starts, $\bar{x}$, satisfies that $\bar{x} \in A_{n_k}$ or $\bar{x} \in A_{n_k}$ for every n_k. Notice that these two possible limits along subsequences are different (the limits are solutions to two different systems), so in general, the full limit does not exists. Also remark that the measure ν_t^n has the decomposition (1.6) with a fixed Dirac measure since the process Y_n is purely a jump process without drift (otherwise the drift should be considered to obtain the location of the Dirac measure).

The paper is organized as follows: in Sect. 2 we analyze the case in which $u_0 \in L^2(\Omega)$ by proving convergence of the densities and convergence of the processes as $n \to \infty$, and also analyzing the asymptotic behaviour of the densities as $t \to \infty$; in Sect. 3 we discuss the convergence via subsequences when $u_0 = \delta_{\bar{x}}$ and the asymptotic behaviour of z_n. Finally in Sect. 4 we include a brief description of the same problem with Dirichlet boundary conditions.

2 Initial Conditions $u_0 \in L^2(\Omega)$

2.1 Convergence of the Densities

This subsection is dedicated to the proof of Theorem 1.1. We start by showing the following lemma which guarantees that the sequence u_n is uniformly bounded in the L^2 norm.

Lemma 2.1 *Let u_n be the solution of (1.2). Then there exists a constant C (independent of n) such that*

$$\|u_n\|_{L^\infty(0,T:L^2(\Omega))} \leq C.$$

Proof To prove the uniform bound we just multiply by u_n both sides of (1.2) and integrate in Ω and in $[0, T]$ to obtain

$$\begin{aligned}
&\frac{1}{2}\int_\Omega (u_n)^2(T,x)dx - \frac{1}{2}\int_\Omega (u_0)^2(x)dx = \int_0^T \int_\Omega L_n u_n(t,x) u_n(t,x) dxdt \\
&= \int_0^T \int_\Omega \chi_{A_n}(x) \int_\Omega \chi_{A_n}(y) J(x,y)(u_n(t,y) - u_n(t,x)) dy u_n(t,x) dxdt \\
&\quad + \int_0^T \int_\Omega \chi_{A_n}(x) \int_\Omega \chi_{B_n}(y) R(x,y)(u_n(t,y) - u_n(t,x)) dy u_n(t,x) dxdt \\
&\quad + \int_0^T \int_\Omega \chi_{B_n}(x) \int_\Omega \chi_{B_n}(y) G(x,y)(u_n(t,y) - u_n(t,x)) dy u_n(t,x) dxdt \\
&\quad + \int_0^T \int_\Omega \chi_{B_n}(x) \int_\Omega \chi_{A_n}(y) R(x,y)(u_n(t,y) - u_n(t,x)) dy u_n(t,x) dxdt \\
&\leq -\int_0^T \int_{\Omega\times\Omega} (1 - \chi_{B_n}(x)\chi_{B_n} \\
&(y))(J(x,y) + G(x,y) + 2R(x,y))(u_n(t,y) - u_n(t,x))^2 dxdydt \leq 0,
\end{aligned}$$

and hence the L^2-norm of the solution is decreasing in time and the result follows. □

Now, consider

$$a_n(t,x) := \chi_{A_n}(x) u_n(t,x) \qquad \text{and} \qquad b_n(t,x) := \chi_{B_n}(x) u_n(t,x).$$

We are ready to proceed with the proof of Theorem 1.1.

Proof of Theorem 1.1 By Lemma 2.1 we can extract a weakly in $L^2((0,T)\times\Omega)$ convergent subsequence of $a_n(t,x)$ and $b_n(t,x)$ that for simplicity of notation we index again with n. We call a and b their respective weak limits.

Take a smooth function ϕ such that $\phi(T,\cdot) \equiv 0$ and consider Eq. (1.2). Multiply both sides by $\chi_{B_n}(x)\phi(t,x)$ and then integrate respect to the variables x and t. Since by construction $\phi(T,\cdot) \equiv 0$, integrating by parts we obtain

$$\begin{aligned}
&-\int_0^T \int_\Omega \frac{\partial\phi}{\partial t}(t,x) b_n(t,x)\, dxdt - \int_\Omega \chi_{B_n}(x) u_0(x)\phi(0,x)\, dx \\
&= \int_0^T \int_\Omega \int_\Omega \chi_{B_n}(x)\chi_{B_n}(y) G(x,y)(u_n(t,y) - u_n(t,x))\phi(t,x) dydxdt \\
&\quad + \int_0^T \int_\Omega \int_\Omega \chi_{B_n}(x)\chi_{A_n}(y) R(x,y)(u_n(t,y) - u_n(t,x))\phi(t,x) dydxdt \\
&= \int_0^T \int_\Omega \int_\Omega G(x,y)\chi_{B_n}(x) b_n(t,y)\phi(t,x)\, dydxdt
\end{aligned}$$

$$
\begin{aligned}
&-\int_0^T \int_\Omega \int_\Omega G(x,y)\,\chi_{B_n}(y)\,b_n(t,x)\,\phi(t,x)\;dydxdt\\
&+\int_0^T \int_\Omega \int_\Omega R(x,y)\,\chi_{B_n}(x)\,a_n(t,y)\,\phi(t,x)\;dydxdt\\
&-\int_0^T \int_\Omega \int_\Omega R(x,y)\,\chi_{A_n}(y)\,b_n(t,x)\,\phi(t,x)\;dydxdt.
\end{aligned}
$$

Since

$$
\chi_{A_n}(\cdot) \xrightarrow[n\to+\infty]{} X(\cdot) \qquad \text{and} \qquad \chi_{B_n}(\cdot) \xrightarrow[n\to+\infty]{} 1-X(\cdot)
$$

weakly in $L^2(\Omega\times(0,T))$, we obtain the following limits

$$
\begin{aligned}
&\int_0^T \int_\Omega \frac{\partial\phi}{\partial t}(t,x)\,b_n(t,x)\;dxdt \xrightarrow[n\to+\infty]{} \int_0^T \int_\Omega \frac{\partial\phi}{\partial t}(t,x)\,b(t,x)\;dxdt,\\
&\int_\Omega \chi_{B_n}(x)\,u_0(x)\,\phi(0,x)\;dx \xrightarrow[n\to+\infty]{} \int_\Omega (1-X(x))\,u_0(x)\phi(0,x)\;dx.
\end{aligned}
\tag{2.1}
$$

Now, as we assumed that $G(x,y)$ is continuous, we have that

$$
h_n(y)=\int_\Omega G(x,y)\chi_{B_n}(x)\phi(t,x)\;dx \xrightarrow[n\to+\infty]{} \int_\Omega G(x,y)\,(1-X(x))\,\phi(t,x)\;dx \tag{2.2}
$$

uniformly in y. Therefore, we get

$$
\begin{aligned}
&\int_0^T \int_\Omega \int_\Omega G(x,y)\,\chi_{B_n}(x)\,b_n(t,y)\,\phi(t,x)\;dydxdt\\
&\qquad \xrightarrow[n\to+\infty]{} \int_0^T \int_\Omega \int_\Omega G(x,y)\,(1-X(x))\,b(t,y)\,\phi(t,x)\,dydxdt,
\end{aligned}
\tag{2.3}
$$

and, arguing similarly,

$$
\begin{aligned}
&\int_0^T \int_\Omega \int_\Omega G(x,y)\,\chi_{B_n}(y)\,b_n(t,x)\,\phi(t,x)\;dydxdt\\
&\qquad \xrightarrow[n\to+\infty]{} \int_0^T \int_\Omega \int_\Omega G(x,y)\,(1-X(y))\,b(t,x)\,\phi(t,x)\;dydxdt,\\
&\int_0^T \int_\Omega \int_\Omega R(x,y)\,\chi_{B_n}(x)\,a_n(t,y)\,\phi(t,x)\;dydxdt\\
&\qquad \xrightarrow[n\to+\infty]{} \int_0^T \int_\Omega \int_\Omega R(x,y)\,(1-X(x))\,a(t,y)\,\phi(t,x)\;dydxdt,\\
&\int_0^T \int_\Omega \int_\Omega R(x,y)\,\chi_{B_n}(y)\,b_n(t,x)\,\phi(t,x)\;dydxdt,\\
&\qquad \xrightarrow[n\to+\infty]{} \int_0^T \int_\Omega \int_\Omega R(x,y)\,X(y)\,b(t,x)\,\phi(t,x)\;dydxdt.
\end{aligned}
\tag{2.4}
$$

Collecting all these limits we conclude that

$$
\begin{aligned}
&-\int_0^T \int_\Omega \frac{\partial\phi}{\partial t}(t,x)\,b(t,x)\,dxdt-\int_\Omega (1-X(x))\,u_0(x)\,\phi(0,x)\;dx\\
&\quad=\int_0^T \int_\Omega \int_\Omega G(x,y)\,(1-X(x))\,b(t,y)\,\phi(t,x)\;dydxdt
\end{aligned}
$$

$$
\begin{aligned}
&- \int_0^T \int_\Omega \int_\Omega G(x,y)(1-X(y))\, b(t,x)\, \phi(t,x)\; dydxdt \\
&+ \int_0^T \int_\Omega \int_\Omega R(x,y)(1-X(x))\, a(t,y)\, \phi(t,x)\; dydxdt \\
&- \int_0^T \int_\Omega \int_\Omega R(x,y)\, X(y)\, b(t,x)\, \phi(t,x)\; dydxdt.
\end{aligned}
$$

Since this holds for every ϕ, we conclude that $b(t,x)$ is a solution to

$$
\begin{cases}
\dfrac{\partial b}{\partial t}(t,x) = \displaystyle\int_\Omega G(x,y)\,[(1-X(x))\, b(t,y) - (1-X(y))\, b(t,x)]\, dy \\
\quad + \int_\Omega R(x,y)\,[(1-X(x))\, a(t,y) - X(y) b(t,x)]\, dy \\
b(x,0) = (1-X(x))\, u_0(x)\,.
\end{cases}
$$

In a similar way we get the equation for $a(x,t)$ and this concludes the proof of Theorem 1.1. □

Next, let us prove a corrector result. We proceed as in [7, 26] setting the corrector as

$$
\omega_n(t,x) = \frac{\chi_{A_n}(x) a(t,x)}{X(x)} + \frac{\chi_{B_n}(x) b(t,x)}{1-X(x)} \quad (t,x) \in [0,T]\times\Omega. \tag{2.5}
$$

Corollary 2.2 *Under the conditions of Theorem 1.1, we have for each $t \in [0,T]$ that*

$$
\|u_n(t,\cdot) - \omega_n(t,\cdot)\|_{L^2(\Omega)} \to 0 \quad \text{as } n \to +\infty.
$$

Proof First, we observe that we can use the variation of constants formula to write the solutions of (1.2) as

$$
u_n(t,x) = e^{-m_n(x)t} u_0(x) + \int_0^t e^{-m_n(x)(t-s)} \int_\Omega H_n(x,y) u_n(s,y) dyds
$$

where

$$
\begin{aligned}
m_n(x) = {} & \chi_{A_n}(x) \int_\Omega \big(\chi_{A_n}(y) J(x,y) + \chi_{B_n}(y) R(x,y)\big)\, dy \\
& + \chi_{B_n}(x) \int_\Omega \big(\chi_{B_n}(y) G(x,y) + \chi_{A_n}(y) R(x,y)\big)\, dy \\
& \text{and} \quad H_n(x,y) = \chi_{A_n}(x) \big(\chi_{A_n}(y) J(x,y) + \chi_{B_n}(y) R(x,y)\big) \\
& + \chi_{B_n}(x) \big(\chi_{B_n}(y) G(x,y) + \chi_{A_n}(y) R(x,y)\big)\,.
\end{aligned}
$$

Also, we have that

$$
\|u_n(t,\cdot) - \omega_n(t,\cdot)\|^2_{L^2(\Omega)} = \|u_n(t,\cdot)\|^2_{L^2(\Omega)} - 2\int_\Omega u_n(t,x)\, \omega_n(t,x)\, dx + \|\omega_n(t,\cdot)\|^2_{L^2(\Omega)}. \tag{2.7}
$$

We will obtain the result passing to the limit in each term of (2.7). Now, in order to do that, we need to study the sequence of functions $\{e^{-m_n(\cdot)t}\}_{n\in\mathbb{N}} \subset L^\infty(\Omega)$ and $\{\int_\Omega H_n(\cdot,y) u_n(t,y)\, dy\}_{n\in\mathbb{N}} \subset L^\infty(\Omega)$ as $n \to +\infty$.

First, let us consider $\{e^{-m_n(x)t}\}_{n\in\mathbb{N}}$. For all $\varphi \in L^1(\Omega)$ one has

$$
\int_\Omega \varphi(x) e^{-m_n(x)t} dx
$$

$$= \int_\Omega \varphi(x) e^{-t\left(\chi_{A_n}(x)\int_\Omega\{\chi_{A_n}(y)J(x,y)+\chi_{B_n}(y)R(x,y)\}dy\right)} e^{-t\left(\chi_{B_n}(x)\int_\Omega\{\chi_{B_n}(y)G(x,y)+\chi_{A_n}(y)R(x,y)\}dy\right)} dx$$
$$= \int_\Omega \chi_{A_n}(x)\,\varphi(x)\,e^{-t\left(\int_\Omega\{\chi_{A_n}(y)J(x,y)+\chi_{B_n}(y)R(x,y)\}dy\right)} dx$$
$$+ \int_\Omega \chi_{B_n}(x)\,\varphi(x)\,e^{-t\left(\int_\Omega\{\chi_{B_n}(y)G(x,y)+\chi_{A_n}(y)R(x,y)\}dy\right)} dx$$
$$= \int_\Omega \chi_{A_n}(x)\,\varphi(x)\,e^{-tm_n^1(x)}dx + \int_\Omega \chi_{B_n}(x)\,\varphi(x)\,e^{-tm_n^2(x)}dx$$

where

$$m_n^1(x) = \int_\Omega \left\{\chi_{A_n}(y)J(x,y) + \chi_{B_n}(y)R(x,y)\right\}dy \quad \text{and}$$
$$m_n^2(x) = \int_\Omega \left\{\chi_{B_n}(y)G(x,y) + \chi_{A_n}(y)R(x,y)\right\}dy.$$

Notice that, for each $x \in \Omega$, we have

$$\begin{aligned} &m_n^1(x) \to m^1(x) := \int_\Omega \{X(y)J(x,y) + (1 - X(y))R(x,y)\}\,dy \\ &\text{and} \\ &m_n^2(x) \to m^2(x) := \int_\Omega \{(1 - X(y))G(x,y) + X(y)R(x,y)\}\,dy \end{aligned} \tag{2.8}$$

as $n \to \infty$. Hence, from [26, Proposition 2.1] we obtain that

$$e^{-tm_n^1(x)} \to e^{-tm^1(x)} \quad \text{and} \quad e^{-tm_n^2(x))} \to e^{-tm^2(x)} \tag{2.9}$$

strongly in $L^\infty(\Omega)$ for each $t \in [0, T]$. Here we recall that we assumed that the involved kernels are continuous and therefore m_n^1 and m_n^2 are equicontinuous.

In particular, for all $\varphi \in L^1(\Omega)$, one has

$$\int_\Omega \varphi(x)e^{-m_n(x)t}dx \to \int_\Omega X(x)\,\varphi(x)\,e^{-t\left(\int_\Omega\{X(y)J(x,y)+(1-X(y))R(x,y)\}dy\right)}dx$$
$$+ \int_\Omega (1 - X(x))\,\varphi(x)\,e^{-t\left(\int_\Omega\{(1-X(y))G(x,y)+X(y)R(x,y)\}dy\right)}dx.$$

On the other hand, we can write

$$\int_\Omega H_n(x,y)u_n(t,y)\,dy = \chi_{A_n}(x)\Phi_n^1(t,x) + \chi_{B_n}(x)\Phi_n^2(t,x)$$

where

$$\Phi_n^1(t,x) = \int_\Omega \left\{\chi_{A_n}(y)J(x,y) + \chi_{B_n}(y)R(x,y)\right\}u_n(t,y)\,dy$$
$$\text{and}$$
$$\Phi_n^2(t,x) = \int_\Omega \left\{\chi_{B_n}(y)G(x,y) + \chi_{A_n}(y)R(x,y)\right\}u_n(t,y)\,dy.$$

We can argue as in (2.9) to obtain

$$\Phi_n^1(t,x) \to \Phi^1(t,x) := \int_\Omega \{a(t,y)\,J(x,y) + b(t,y)\,R(x,y)\}\,dy$$
$$\text{and} \tag{2.10}$$
$$\Phi_n^2(t,x) \to \Phi^2(t,x) := \int_\Omega \{b(t,y)\,G(x,y) + a(t,y)\,R(x,y)\}\,dy$$

strongly in $L^\infty(\Omega)$ for each $t \in [0,T]$.

Now, let us pass to the limit in $\|u_n(t,\cdot)\|_{L^2(\Omega)}$. Due to (2.6), we get from (2.8) and (2.10) that

$$\begin{aligned}\int_\Omega u_n^2(t,x)\,dx &= \int_\Omega u_n(t,x)\left(e^{-m_n(x)t}\,u_0(x) + \int_0^t e^{-m_n(x)(t-s)}\int_\Omega H_n(x,y)\,u_n(s,y)\,dyds\right)dx\\ &= \int_\Omega \chi_{A_n}(x)\,u_n(t,x)\,e^{-tm_n^1(x)}u_0(x)\,dx + \int_\Omega \chi_{B_n}(x)\,u_n(t,x)\,e^{-tm_n^2(x)}u_0(x)\,dx\\ &+ \int_0^t\int_\Omega \chi_{A_n}(x)\,u_n(t,x)\,e^{-tm_n^1(x)}\,\Phi_n^1(s,x)\,dxds\\ &+ \int_0^t\int_\Omega \chi_{B_n}(x)\,u_n(t,x)\,e^{-tm_n^2(x)}\,\Phi_n^2(s,x)\,dxds.\end{aligned}$$

Consequently, it follows from Theorem 1.1, (2.9) and (2.10) that

$$\begin{aligned}\|u_n(t,\cdot)\|^2_{L^2(\Omega)} &\to \int_\Omega a(t,x)\,e^{-tm^1(x)}u_0(x)\,dx + \int_\Omega b(t,x)\,e^{-tm^2(x)}u_0(x)\,dx\\ &+ \int_0^t\int_\Omega a(t,x)\,e^{-tm^1(x)}\,\Phi^1(s,x)\,dxds + \int_0^t\int_\Omega b(t,x)\,e^{-tm_n^2(x)}\,\Phi^2(s,x)\,dxds\\ &= \int_\Omega \frac{a(t,x)}{X(x)}\left[(Xu_0)(x)e^{-m^1(x)t} + \int_0^t e^{-m^1(x)(t-s)}\,\{X(x)J(x,y) + (1-X(x))R(x,y)\}\,dyds\right]dx\\ &+ \int_\Omega \frac{b(t,x)}{1-X(x)}\left[((1-X)u_0)(x)e^{-m^2(x)t} + \int_0^t e^{-m^2(x)(t-s)}\,\{(1-X)(x)G(x,y)\right.\\ &\left.+X(x)R(x,y)\Big\}\,dyds\right]dx\\ &= \int_\Omega \left\{\frac{a^2(t,x)}{X(x)} + \frac{b^2(t,x)}{1-X(x)}\right\}dx,\end{aligned}$$

since we have that

$$a(t,x) = (Xu_0)(x)e^{-m^1(x)t} + \int_0^t e^{-m^1(x)(t-s)}\,\{X(x)J(x,y) + (1-X(x))R(x,y)\}\,dyds$$
$$\text{and}$$
$$b(t,x) = ((1-X)u_0)(x)e^{-m^2(x)t} + \int_0^t e^{-m^2(x)(t-s)}\,\{(1-X)(x)G(x,y) + X(x)R(x,y)\}\,dyds.$$

Finally, let us pass to the limit in the other terms of (2.7). One can see that

$$\begin{aligned}\int_\Omega u_n(t,x)\,\omega_n(t,x)\,dx &= \int_\Omega \left\{\chi_{A_n}(x)u_n(t,x)\frac{a(t,x)}{X(x)}\right.\\ &\left.+\chi_{B_n}(x)u_n(t,x)\frac{b(t,x)}{1-X(x)}\right\}dx \to \int_\Omega \left\{\frac{a^2(t,x)}{X(x)} + \frac{b^2(t,x)}{1-X(x)}\right\}dx\end{aligned}$$

and

$$\int_\Omega \omega_n^2(t,x)\,dx = \int_\Omega \left\{\chi_{A_n}(x)\frac{a^2(t,x)}{X^2(x)} + \chi_{B_n}(x)\frac{b^2(t,x)}{(1-X(x))^2}\right\}dx \to \int_\Omega \left\{\frac{a^2(t,x)}{X(x)}\right.$$

$$+\frac{b^2(t,x)}{1-X(x)}\Bigg\}dx, \qquad \text{as } n \to +\infty.$$

Hence, we can conclude that

$$\lim_{n\to+\infty} \|u_n(t,\cdot)-\omega_n(t,\cdot)\|^2_{L^2(\Omega)} = 0,$$

proving the result. □

2.2 Convergence of the Stochastic Process

In this subsection we prove Theorem 1.2. Our first goal is to show that $Y_n(t)$ has a probability density $u_n(t,x)$ which is the unique solution to system (1.2). To this end we will prove uniqueness of weak solutions to (1.2).

Lemma 2.3 *Let $u_0 \in L^2(\Omega)$. There exists a unique solution $u_n(t,x) \in L^\infty\big([0,T]; L^2(\Omega)\big)$ to the system*

$$\begin{cases} \dfrac{\partial u_n}{\partial t}(t,x) = L_n u_n(t,x), & t>0,\ x\in\Omega, \\ u_n(0,x) = u_0(x), & x \in \Omega. \end{cases} \tag{2.11}$$

Proof Fix $t_0 \in [0,T]$ and consider the space $L^\infty\big([0,T]; L^2(\Omega)\big)$ endowed with the norm $\|\cdot\|_{2,\infty}$ defined as

$$\|f\|_{2,\infty} = \sup_{t\in[0,t_0]} \int_\Omega [f(t,x)]^2 dx$$

and $L^\infty(\Omega \times [0,t_0])$ with the norm

$$\|f\|_{\infty,\infty} := \sup\Big\{ |f(s,x)|,\ x\in\Omega, s\in[0,t] \Big\}.$$

We let Φ be the map defined on $u \in L^\infty\big([0,T]; L^2(\Omega)\big)$ as

$$\begin{aligned}\Phi(u(t,x)) = u_0(x) &+ \int_0^t \chi_{A_n}(x)\left[\int_\Omega \chi_{A_n}(y)\, J(x,y)\,(u(s,y)-u(s,x))\,dy\right.\\ &\left. + \int_\Omega \chi_{B_n}(y)\, R(x,y)\,(u(s,y)-u(s,x))\,dy\right]ds\\ &+ \int_0^t \chi_{B_n}(x)\left[\int_\Omega \chi_{B_n}(y)\, G(x,y)\,(u(s,y)-u(s,x))\,dy\right.\\ &\left. + \int_\Omega \chi_{A_n}(y)\, R(x,y)\,(u(s,y)-u(s,x))\,dy\right]ds.\end{aligned}$$

For every $u, v \in L^\infty\big([0,T]; L^2(\Omega)\big)$ it holds that

$$\begin{aligned}&|\Phi(u(t,x)) - \Phi(v(t,x))|\\ &\quad\le \Big(\|J\|_\infty + 2\|R\|_\infty + \|G\|_\infty\Big)\int_0^t\int_\Omega (|u(s,y)-v(s,y)| + |u(s,x)-v(s,x)|)\,dyds.\end{aligned} \tag{2.12}$$

Calling $\sqrt{C} = \|J\|_\infty + 2\|R\|_\infty + \|G\|_\infty$ we get that

$$\begin{aligned}
&\int_\Omega |\Phi(u(t,x)) - \Phi(v(t,x))|^2 dx \\
&\quad \le C \int_\Omega \left(\int_0^t \int_\Omega (|u(s,y) - v(s,y)| + |u(s,x) - v(s,x)|) \, dyds \right)^2 dx \\
&\quad \le 2C \int_\Omega \left(\int_0^t \int_\Omega |u(s,y) - v(s,y)| \, dyds \right)^2 dx \\
&\qquad + 2C \int_\Omega \left(\int_0^t \int_\Omega |u(s,x) - v(s,x)| \, dyds \right)^2 dx.
\end{aligned}$$

By Cauchy-Schwartz's inequality we obtain

$$\begin{aligned}
\left(\int_0^t \int_\Omega |u(s,y) - v(s,y)| \, dyds \right)^2 &\le t \int_0^t \left(\int_\Omega |u(s,y) - v(s,y)| \, dy \right)^2 ds \\
&\le t\,|\Omega| \int_0^t \int_\Omega |u(s,y) - v(s,y)|^2 \, dyds \\
&\le t^2\,|\Omega|\,\|u - v\|_{2,\infty},
\end{aligned}$$

and, analogously,

$$\left(\int_0^t \int_\Omega |u(s,x) - v(s,x)| \, dyds \right)^2 \le t\,|\Omega|^2 \int_0^t |u(s,x) - v(s,x)|^2 \, ds.$$

Consequently, by (2.12), we get

$$\begin{aligned}
\int_\Omega |\Phi(u(t,x)) - \Phi(v(t,x))|^2 dx \le 2Ct^2\,|\Omega|^2\,\|u - v\|_{2,\infty} \\
+ 2Ct\,|\Omega|^2 \int_\Omega \int_0^t |u(s,x) - v(s,x)|^2 \, dsdx \\
\le 4Ct^2\,|\Omega|^2\,\|u - v\|_{2,\infty}.
\end{aligned}$$

Therefore by choosing $t_0 < \frac{1}{2|\Omega|\sqrt{C}}$ we get that the map Φ is a contraction in $L^\infty\left([0,T]; L^2(\Omega)\right)$. By the Banach fixed-point Theorem we can deduce that there exists a unique solution of system (2.11) in $L^\infty\left([0,T]; L^2(\Omega)\right)$. We can iterate the previous argument in order to show existence and uniqueness globally in $L^\infty\left([0,T]; L^2(\Omega)\right)$. □

Lemma 2.4 *Let $u_0 \in L^2(\Omega)$. There exists a unique measure ν_t solution to*

$$\begin{cases} \dfrac{\partial}{\partial t} \displaystyle\int_{\overline{\Omega}} f(x)\nu_t(dx) = \int_{\overline{\Omega}} L_n f(x)\nu_t(dx), & t \in [0,T],\ x \in \overline{\Omega}, \\ \displaystyle\int_{\overline{\Omega}} f(x)\nu_0(dx) = \int_{\overline{\Omega}} f(x)u_0(x)dx, & x \in \overline{\Omega}, \end{cases} \tag{2.13}$$

for every $f \in C(A_n) \cap C(B_n)$.

Such a solution is given by

$$\nu_t^n(dx) = u_n(t,x)dx,$$

where $u_n(t,x)$ is the unique solution to (1.2).

Proof The existence of a solution to (2.13) follows just by taking

$$\nu_t^n(dx) = u_n(t,x)dx$$

where $u_n(t,x)$ is a solution of system (1.2) whose existence is guaranteed by Lemma 2.3 (here we are using that L_n is self-adjoint due to the symmetry of the kernels). Next, we prove the uniqueness. Suppose that there exist two trajectories of measures $\nu_t^n(dx)$ and $\tilde{\nu}_t^n(dx)$ such that (2.13) holds. Call

$$\omega_t^n(dx) = \nu_t^n(dx) - \tilde{\nu}_t^n(dx).$$

The evolution of $\omega_t^n(dx)$ satisfies Eq. (2.13) with initial condition

$$\int_{\overline{\Omega}} f(x)\omega_0^n(dx) = 0.$$

Therefore, for all $t \in [0,T]$, it holds that

$$\begin{aligned}\int_{\overline{\Omega}} f(x)\omega_t^n(dx) = &\int_0^t \left\{\int_{\overline{\Omega}} \left[\chi_{A_n}(x)\int_{\Omega}\chi_{A_n}(y)\,J(x,y)\,(f(y)-f(x))\,dy\right.\right.\\ &\left.\left.+\chi_{B_n}(x)\int_{\Omega}\chi_{B_n}(y)\,G(x,y)\,(f(y)-f(x))\,dy\right]\omega_s^n(dx)\right\}ds\\ &+\int_0^t \left\{\int_{\overline{\Omega}} \left[\chi_{A_n}(x)\int_{\Omega}\chi_{B_n}(y)\,R(x,y)\,(f(y)-f(x))\,dy\right.\right.\\ &\left.\left.+\chi_{B_n}(x)\int_{\Omega}\chi_{A_n}(y)\,R(x,y)\,(f(y)-f(x))\,dy\right]\omega_s^n(dx)\right\}ds.\end{aligned}$$

In what follows, for all measures μ on $\overline{\Omega}$, we denote by

$$\|\mu\|_{\mathrm{TV}} := \sup_{g\in C(\overline{\Omega}):\|g\|_\infty\le 1}\int_{\overline{\Omega}} g(x)\mu(dx),$$

the dual norm of μ (total variation). It holds that

$$\left\|\omega_t^n(dx)\right\|_{\mathrm{TV}} \le (\|J\|_\infty + 2\,\|R\|_\infty + \|G\|_\infty)\,(|\Omega|+1)\int_0^t \left\|\omega_s^n(dx)\right\|_{\mathrm{TV}} ds.$$

By Gronwall's inequality we can conclude that $\omega_t^n(dx)$ coincides with the null measure and therefore $\nu_t^n(dx) = \tilde{\nu}_t^n(dx)$. The uniqueness of the solution to system (2.13) follows. □

By this uniqueness result and Lemma A.1.5.1 of [22], we get that the process $Y_n(t)$ has a density. This is the content of the following corollary.

Corollary 2.5 *Let $u_0 \in L^2(\Omega)$. The process $Y_n(t)$ has a density that is characterized as the unique solution $u_n(t,x)$ to*

$$\begin{cases}\dfrac{\partial u_n}{\partial t}(t,x) = L_n u_n(t,x), & t>0,\ x\in\Omega,\\ u_n(0,x) = u_0(x), & x\in\overline{\Omega}.\end{cases}$$

Consider now the coupled process $(Y_n(t), I_n(t)) \in D\left([0,T],\overline{\Omega}\right) \times D([0,T],\{1,2\})$, where

$$I_n(t) = \begin{cases}1 & \text{if } Y_n(t)\in A_n,\\ 2 & \text{if } Y_n(t)\in B_n.\end{cases}$$

The pair $(Y_n(t), I_n(t))$ is a Markov process whose generator $\mathcal{L}_n$ is defined on functions $f \in T_n$, with

$$T_n := \left\{ f : \overline{\Omega} \times \{1, 2\} \to \mathbb{R} : f(\cdot, 1) \in C(A_n) \text{ and } f(\cdot, 2) \in C(B_n) \right\},$$

as

$$\mathcal{L}_n f(x,i) = \begin{cases} \chi_{A_n}(x) \displaystyle\int_\Omega \chi_{A_n}(y) J(x,y) (f(y,1) - f(x,1))\, dy \\ + \chi_{A_n}(x) \int_\Omega \chi_{B_n}(y) R(x,y)(f(y,2) - f(x,1))\, dy & \text{if } i = 1, \\ \chi_{B_n}(x) \displaystyle\int_\Omega \chi_{B_n}(y) G(x,y)(f(y,2) - f(x,2))\, dy \\ + \chi_{B_n}(x) \int_\Omega \chi_{A_n}(y) R(x,y)(f(y,1) - f(x,2))\, dy & \text{if } i = 2. \end{cases} \tag{2.14}$$

By Lemma A.1.5.1 of [22] we know that, for every bounded function $f \in \overline{\Omega} \times \{1, 2\} \to \mathbb{R}$,

$$M_n^f(t) = f(Y_n(t), I_n(t)) - f(Y_n(0), I_n(0)) - \int_0^t \mathcal{L}_n f(Y_n(s), I_n(s))\, ds \tag{2.15}$$

and

$$N_n^f(t) = \left(M_n^f(t)\right)^2 - \int_0^t \left(\mathcal{L}_n (f(Y_n(s), I_n(s)))^2 - 2f(Y_n(s), I_n(s)) \mathcal{L}_n f(Y_n(s) I_n(s))\right) ds \tag{2.16}$$

are martingales with respect to the natural filtration generated by the process.

Let $P_n \in \mathcal{M}_1\left(D\left([0,T], \overline{\Omega}\right) \times D([0,T], \{1,2\})\right)$ be the law of the process $(Y_n(t), I_n(t))_{t\in[0,T]}$; in our notation $\mathcal{M}_1(X)$ denotes the space of probability measures on a metric space X. The next lemma guarantees the tightness of the sequence $(P_n)_{n\in\mathbb{N}}$.

Lemma 2.6 *The sequence of probability measures $(P_n)_{n\in\mathbb{N}}$ is tight.*

Proof Let P_n^1 and P_n^2 be the two marginals of P_n. Since $D\left([0,T], \overline{\Omega}\right) \times D([0,T], \{1,2\})$ is endowed with the product topology, in order to conclude, it is enough to show that the marginals P_n^1 and P_n^2 are tight.

We start by proving that the sequence P_n^1 is tight. By Theorem 1.3 and Proposition 1.6 of Chapter 4 in [22], it is sufficient to show that the following conditions hold:

1. For every $t \in [0, T]$ and $\epsilon > 0$ there exists a compact set $K(t, \epsilon) \subseteq \overline{\Omega}$ such that
$$\sup_n P_n^1\Big(Y(\cdot) : Y(t) \notin K(t, \epsilon)\Big) \le \epsilon,$$
2. For every $\epsilon > 0$, we have that
$$\lim_{\zeta\to 0} \limsup_{n\to+\infty} \sup_{\tau\in\Lambda_T, \theta\le\zeta} P_n^1\Big(Y(\cdot) : |Y(\tau+\theta) - Y(\tau)| > \epsilon\Big) = 0,$$
where Λ_T is the family of all stopping times bounded by T.

The first condition is satisfied since $\overline{\Omega}$ is a compact space. To prove the second condition, fix $\tau \in \Lambda_T$, $\epsilon > 0$ and observe that, considering the function $g(x,i) = x$ in (2.15), we get that

$$M_n^g(t) = Y_n(t) - Y_n(0) - \int_0^t \mathcal{L}_n g(Y_n(s), I_n(s))\, ds.$$

Therefore,

$$|Y_n(\tau+\theta) - Y_n(\tau)| \le \left| \int_\tau^{\tau+\theta} \mathcal{L}_n g(Y_n(s), I_n(s))\, ds \right| + \left| M_n^g(\tau+\theta) - M_n^g(\tau) \right|. \tag{2.17}$$

Since

$$\begin{aligned}\mathcal{L}_n g(x,i) &= \chi_{A_n}(x)\chi_1(i)\left[\int_\Omega \chi_{A_n}(y) J(x,y)(y-x)\,dy + \int_\Omega \chi_{B_n}(y) R(x,y)(y-x)\,dy\right] \\ &\quad + \chi_{B_n}(x)\chi_2(i)\left[\int_\Omega \chi_{B_n}(y) G(x,y)(y-x)\,dy + \int_\Omega \chi_{A_n}(y) R(x,y)(y-x)\,dy\right],\end{aligned}$$

we have that

$$\left| \int_\tau^{\tau+\theta} \mathcal{L}_n g(Y_n(s), I_n(s))\, ds \right| \le C_1\theta, \tag{2.18}$$

where $C_1 := C_1(\|J\|_\infty, \|G\|_\infty, \|R\|_\infty, |\Omega|)$ is a constant depending on $\|J\|_\infty, \|G\|_\infty,$ $\|R\|_\infty$ and $|\Omega|$. Moreover, by (2.16) we get that

$$\begin{aligned}&\mathbb{E}\left(\left(M_n^g(\tau+\theta)\right)^2 - \left(M_n^g(\tau)\right)^2\right) \\ &\quad = \mathbb{E}\left(\int_\tau^{\tau+\theta} \left(\mathcal{L}_n(g(Y_n(s), I_n(s)))^2 - 2g(Y_n(s), I_n(s))\,\mathcal{L}_n g(Y_n(s) I_n(s))\right) ds\right).\end{aligned}$$

Since

$$\begin{aligned}&\mathcal{L}_n(g(x,i))^2 - 2g(x,i)\,\mathcal{L}_n g(x,i) \\ &\quad = \chi_{A_n}(x)\chi_1(i)\left[\int_\Omega \chi_{A_n}(y) J(x,y)\left(y^2-x^2\right)dy + \int_\Omega \chi_{B_n}(y) R(x,y)\left(y^2-x^2\right)dy\right] \\ &\quad + \chi_{B_n}(x)\chi_2(i)\left[\int_\Omega \chi_{B_n}(y) G(x,y)\left(y^2-x^2\right)dy + \int_\Omega \chi_{A_n}(y) R(x,y)\left(y^2-x^2\right)dy\right] \\ &\quad - 2x\chi_{A_n}(x)\chi_1(i)\left[\int_\Omega \chi_{A_n}(y) J(x,y)(y-x)\,dy + \int_\Omega \chi_{B_n}(y) R(x,y)(y-x)\,dy\right] \\ &\quad - 2x\chi_{B_n}(x)\chi_2(i)\left[\int_\Omega \chi_{B_n}(y) G(x,y)(y-x)\,dy + \int_\Omega \chi_{A_n}(y) R(x,y)(y-x)\,dy\right].\end{aligned}$$

we obtain that

$$\mathbb{E}\left(\left(M_n^g(\tau+\theta)\right)^2 - \left(M_n^g(\tau)\right)^2\right) \le C_1\theta,$$

Therefore, by Markov's inequality

$$\mathbb{P}\left(\left|M_n^g(\tau+\theta) - M_n^g(\tau)\right| > \epsilon\right) \le \frac{\mathbb{E}\left(\left(M_n^g(\tau+\theta)\right)^2 - \left(M_n^g(\tau)\right)^2\right)}{\epsilon^2} \le \frac{C\theta}{\epsilon^2}, \tag{2.19}$$

for all $\epsilon > 0$. The bounds (2.17), (2.18) and (2.19) allow to conclude the second condition that guarantees the tightness of the sequence P_n^1.

We proceed now in a similar way to prove the tightness of the sequence P_n^2. As before it is enough to show that

1. For every $t \in [0, T]$ and every $\epsilon > 0$ there exists a compact set $K(t, \epsilon) \subseteq \{1, 2\}$ such that

$$\sup_n P_n^2\Big(I(\cdot) : I(t) \notin K(t, \epsilon)\Big) \leq \epsilon,$$

2. For every $\epsilon > 0$ it holds that

$$\lim_{\zeta \to 0} \limsup_{n \to +\infty} \sup_{\tau \in L_T, \theta \leq \zeta} P_n^2\Big(I(\cdot) : |I(\tau + \theta) - I(\tau)| > \epsilon\Big) = 0.$$

The first condition is trivially satisfied taking $K(t, \epsilon) = \{1, 2\}$. Hence, we need to prove the second condition. Considering the function $h(x, i) = i$ in (2.15), we get that

$$M_n^h(t) = I_n(t) - I_n(0) - \int_0^t \mathcal{L}_n h(Y_n(s), I_n(s))\, ds.$$

Therefore

$$|I_n(\tau + \theta) - I_n(\tau)| \leq \left| \int_\tau^{\tau+\theta} \mathcal{L}_n h(Y_n(s), I_n(s))\, ds \right| + \left| M_n^h(\tau + \theta) - M_n^h(\tau) \right|.$$

Since

$$\mathcal{L}_n h(x, i) = \chi_{A_n}(x)\, \chi_1(i) \int_\Omega \chi_{B_n}(y)\, R(x, y)\, dy - \chi_{B_n}(x)\, \chi_2(i) \int_\Omega \chi_{A_n}(y)\, R(x, y)\, dy,$$

we have that

$$\left| \int_\tau^{\tau+\theta} \mathcal{L}_n h(Y_n(s), I_n(s))\, ds \right| \leq C_2 \theta, \tag{2.20}$$

where $C_2 = C_2(\|R\|_\infty, |\Omega|)$ is a constant depending on $\|R\|_\infty$ and $|\Omega|$. Moreover, by (2.16), we get that

$$\mathbb{E}\left(\left(M_n^h(\tau + \theta)\right)^2 - \left(M_n^h(\tau)\right)^2 \right)$$
$$= \mathbb{E}\left(\int_\tau^{\tau+\theta} \left(\mathcal{L}_n (h(Y_n(s), I_n(s)))^2 - 2h(Y_n(s), I_n(s))\, \mathcal{L}_n h(Y_n(s)\, I_n(s))\right) ds \right) \leq C_2 \theta.$$

The last inequality follows from the fact that

$$\mathcal{L}_n (h(x, i))^2 - 2h(x, i)\, \mathcal{L}_n f_2(x, i) = (3 - 2i)\, \chi_{A_n}(x)\, \chi_1(i) \int_\Omega \chi_{B_n}(y)\, R(x, y)\, dy$$
$$+ (-3 + 2i)\, \chi_{B_n}(x)\, \chi_2(i) \int_\Omega \chi_{A_n}(y)\, R(x, y)\, dy.$$

Finally, by Markov's inequality we get that

$$\mathbb{P}\left(\left| M_n^h(\tau + \theta) - M_n^h(\tau) \right| > \epsilon \right) \leq \frac{\mathbb{E}\left(\left(M_n^h(\tau + \theta)\right)^2 - \left(M_n^h(\tau)\right)^2 \right)}{\epsilon^2} \leq \frac{C_2 \theta}{\epsilon^2}, \tag{2.21}$$

for all $\epsilon > 0$. Bounds (2.17), (2.20) and (2.21) allow to conclude the second condition that guarantees the tightness of the sequence P_n^2. □

Lemma 2.6 guarantees that the sequence of processes $(Y_n(t), I_n(t))_{t \in [0,T]}$ converges in distribution along subsequences. In the following theorem we prove that all subsequences converge to the same limit and we characterize the generator of the limit process.

Theorem 2.7 *The sequence* $(Y_n(t), I_n(t))$ *converges*

$$(Y_n(t), I_n(t)) \xrightarrow[n\to+\infty]{D} (Y(t), I(t))$$

in $D\left([0,T],\overline{\Omega}\right) \times D([0,T],\{1,2\})$. *The limit* $(Y(t), I(t))$ *is a Markov process whose generator* $\widetilde{\mathcal{L}}$ *is defined on functions* $f \in C\left(\overline{\Omega} \times \{1,2\}\right)$ *as*

$$\begin{aligned}\widetilde{\mathcal{L}}f(x,i) = \chi_1(i)&\left\{\int_\Omega X(y)J(x,y)\left(f(y,1)-f(x,1)\right)dy\right.\\&\left.+\int_\Omega (1-X(y))R(x,y)\left(f(y,2)-f(x,1)\right)dy\right\}\\&+\chi_2(i)\left\{\int_\Omega (1-X(y))G(x,y)\left(f(y,2)-f(x,2)\right)dy\right.\\&\left.+\int_\Omega X(y)R(x,y)\left(f(y,1)-f(x,2)\right)dy\right\}.\end{aligned}$$

Proof Lemma 2.6 implies that any subsequence of P_n has a convergent sub-subsequence; it remains then to characterize all the limit points of the sequence P_n. Let $\tilde{P}$ be a limit point and P_{n_k} be a subsequence converging to $\tilde{P}$. To prove the theorem it is enough to show that $\tilde{P}$ concentrates its mass on a process $(Y(\cdot), I(\cdot))$ such that,

$$f(Y(t),I(t)) - f(Y(0),I(0)) - \int_0^t \widetilde{\mathcal{L}}f(Y(s),I(s))\,ds$$

is a martingale, for every $f \in C\left(\overline{\Omega}\times\{1,2\},\mathbb{R}\right)$ and for every $t \in [0,T]$. This implies convergence of the entire sequence P_n and characterizes the limit $\tilde{P}$ as the law of the Markov process with generator $\widetilde{\mathcal{L}}$, we refer the reader to Chapter 4 in [17] for a deeper discussion of the issue. Therefore, to conclude the proof we need to show that,

$$\mathbb{E}^{\tilde{P}}\left[g\left((Y(s),I(s)),0\le s\le t_0\right)\left(f(Y(t),I(t))-f(Y(t_0),I(t_0))-\int_{t_0}^t \widetilde{\mathcal{L}}f(Y(s),I(s))\,ds\right)\right]=0, \tag{2.22}$$

for every bounded continuous function $g : D\left([0,T],\overline{\Omega}\right)\times D([0,T],\{1,2\}) \to \mathbb{R}$, for every $f \in C\left(\overline{\Omega}\times\{1,2\}\right)$ and for every $0 \le t_0 < t \le T$.

By the tightness proved in Lemma 2.6 we know that

$$\begin{aligned}&\mathbb{E}^{\tilde{P}}\left[g\left((Y(s),I(s)),0\le s\le t_0\right)\left(f(Y(t),I(t))-f(Y(t_0),I(t_0))-\int_{t_0}^t \tilde{\mathcal{L}}f(Y(s),I(s))\,ds\right)\right]\\&=\lim_{n_k\to+\infty}\mathbb{E}^{P^{n_k}}\left(g\left((Y(s),I(s)),0\le s\le t_0\right)\left(f(Y(t),I(t))-f(Y(t_0),I(t_0))-\int_{t_0}^t \tilde{\mathcal{L}}f(Y(s),I(s))\,ds\right)\right).\end{aligned}$$

By the triangular inequality, we have

$$\begin{aligned}&\left|\mathbb{E}^{P^{n_k}}\left(g\left((Y(s),I(s)),0\le s\le t_0\right)\left(f(Y(t),I(t))-f(Y(t_0),I(t_0))-\int_{t_0}^t \tilde{\mathcal{L}}f(Y(s),I(s))\,ds\right)\right)\right|\\&\le\left|\mathbb{E}^{P_{n_k}}\left(g\left((Y(s),I(s)),0\le s\le t_0\right)\left(f(Y(t),I(t))-f(Y(t_0),I(t_0))-\int_{t_0}^t \mathcal{L}_{n_k}f(Y(s),I(s))\,ds\right)\right)\right|\\&\quad+\left|\mathbb{E}^{P_{n_k}}\left(g\left((Y(s),I(s)),0\le s\le t_0\right)\int_{t_0}^t\left(\mathcal{L}_{n_k}f(Y(s),I(s))-\tilde{\mathcal{L}}f(Y(s),I(s))\right)ds\right)\right|.\end{aligned} \tag{2.23}$$

Let us analyze the first term in the right hand side of (2.23). By (2.15), we have that

$$f\left(Y\left(t\right),I\left(t\right)\right)-f\left(Y\left(t_0\right),I\left(t_0\right)\right)-\int_{t_0}^{t}\mathcal{L}_{n_k}f\left(Y\left(s\right),I\left(s\right)\right)ds$$

is a martingale. Therefore,

$$\mathbb{E}^{P_{n_k}}\left(g\left(\left(Y\left(s\right),I\left(s\right)\right),0\leq s\leq t_0\right)\left(f\left(Y\left(t\right),I\left(t\right)\right)-f\left(Y\left(t_0\right),I\left(t_0\right)\right)-\int_{t_0}^{t}\mathcal{L}_{n_k}f\left(Y\left(s\right),I\left(s\right)\right)ds\right)\right)=0.$$

Hence, to conclude (2.22) we just need to show that

$$\lim_{n_k\to\infty}\left|\mathbb{E}^{P_{n_k}}\left(g\left(\left(Y\left(s\right),I\left(s\right)\right),0\leq s\leq t_0\right)\int_{t_0}^{t}\left(\mathcal{L}_{n_k}f\left(Y\left(s\right),I\left(s\right)\right)-\widetilde{\mathcal{L}}f\left(Y\left(s\right),I\left(s\right)\right)\right)ds\right)\right|=0.$$

Since

$$\begin{aligned}&\left|\mathbb{E}^{P_{n_k}}\left(g\left(\left(Y\left(s\right),I\left(s\right)\right),0\leq s\leq t_0\right)\int_{t_0}^{t}\left(\mathcal{L}_{n_k}f\left(Y\left(s\right),I\left(s\right)\right)-\widetilde{\mathcal{L}}f\left(Y\left(s\right),I\left(s\right)\right)\right)ds\right)\right|\\&\quad\leq\|g\|_\infty\,\mathbb{E}^{P_{n_k}}\left(\int_{t_0}^{t}\left|\mathcal{L}_{n_k}f\left(Y\left(s\right),I\left(s\right)\right)-\widetilde{\mathcal{L}}f\left(Y\left(s\right),I\left(s\right)\right)\right|ds\right)\end{aligned}$$

it is enough to prove that

$$\lim_{n_k\to\infty}\mathbb{E}^{P_{n_k}}\left(\int_{t_0}^{t}\left|\mathcal{L}_{n_k}f\left(Y\left(s\right),I\left(s\right)\right)-\widetilde{\mathcal{L}}f\left(Y\left(s\right),I\left(s\right)\right)\right|ds\right)=0. \tag{2.24}$$

Denoting by $D:=D\left([0,T],\overline{\Omega}\right)\times D\left([0,T],\{1,2\}\right)$ and using Fubini's theorem we get

$$\begin{aligned}&\mathbb{E}^{P_{n_k}}\left(\int_{t_0}^{t}\left|\mathcal{L}_{n_k}f\left(Y\left(s\right),I\left(s\right)\right)-\widetilde{\mathcal{L}}f\left(Y\left(s\right),I\left(s\right)\right)\right|ds\right)\\&\quad=\int_{t_0}^{t}\int_D\left|\mathcal{L}_{n_k}f\left(Y\left(s\right),I\left(s\right)\right)-\widetilde{\mathcal{L}}f\left(Y\left(s\right),I\left(s\right)\right)\right|dP_{n_k}\left(Y,I\right)ds.\end{aligned}$$

Observe that $I_n(s)=1+\chi_{B_n}\left(Y_n(s)\right)$ and $\chi_{A_n}\left(Y_n(s)\right)=\chi_1\left(I_n(s)\right)$. Then, recalling that $u_n(x,s)$ is the probability density of the process $Y_n(s)$, we get

$$\begin{aligned}&\int_{t_0}^{t}\int_D\left|\mathcal{L}_{n_k}f\left(Y\left(s\right),I\left(s\right)\right)-\widetilde{\mathcal{L}}f\left(Y\left(s\right),I\left(s\right)\right)\right|dP_{n_k}\left(Y,I\right)ds\\&\quad=\int_{t_0}^{t}\int_\Omega\left|\mathcal{L}_{n_k}f\left(x,1+\chi_{B_{n_k}}\left(x\right)\right)-\widetilde{\mathcal{L}}f\left(x,1+\chi_{B_{n_k}}\left(x\right)\right)\right|u_{n_k}\left(x,s\right)dxds\\&\quad\leq\int_{t_0}^{t}\int_\Omega\left|\Upsilon^1_{n_k}\left(x\right)+\Upsilon^2_{n_k}\left(x\right)+\Upsilon^3_{n_k}\left(x\right)+\Upsilon^4_{n_k}\left(x\right)\right|u_{n_k}\left(x,s\right)dxds,\end{aligned}$$

where

$$\begin{aligned}\Upsilon^1_{n_k}\left(x\right)&=\chi_{A_{n_k}}\left(x\right)\int_\Omega\chi_{A_n}\left(y\right)J\left(x,y\right)\left(f\left(y,1\right)-f\left(x,1\right)\right)dy\\&\quad-\chi_{A_{n_k}}\left(x\right)\int_\Omega X(y)J\left(x,y\right)\left(f\left(y,1\right)-f\left(x,1\right)\right)dy,\\\Upsilon^2_{n_k}\left(x\right)&=\chi_{A_{n_k}}\left(x\right)\int_\Omega\chi_{B_{n_k}}\left(y\right)R\left(x,y\right)\left(f\left(y,2\right)-f\left(x,1\right)\right)dy\\&\quad-\chi_{A_{n_k}}\left(x\right)\int_\Omega\left(1-X(y)\right)R\left(x,y\right)\left(f\left(y,2\right)-f\left(x,1\right)\right)dy,\end{aligned}$$

$$\Upsilon^3_{n_k}(x) = \chi_{B_{n_k}}(x) \int_\Omega \chi_{B_{n_k}}(y)\, G(x,y)\,(f(y,2) - f(x,2))\,dy$$
$$- \chi_{B_{n_k}}(x) \int_\Omega (1 - X(y))\, G(x,y)\,(f(y,2) - f(x,2))\,dy,$$
$$\Upsilon^4_{n_k}(x) = \chi_{B_{n_k}}(x) \int_\Omega \chi_{A_{n_k}}(y)\, R(x,y)\,(f(y,1) - f(x,2))\,dy$$
$$- \chi_{B_{n_k}}(x) \int_\Omega X(y) R(x,y)\,(f(y,1) - f(x,2))\,dy.$$

Therefore, (2.24) is proved once we show that

$$\lim_{n_k\to\infty} \int_{t_0}^t \int_\Omega \left|\Upsilon^i_{n_k}(x)\right| u_{n_k}(s,x) ds = 0, \qquad \forall i \in \{1,2,3,4\}. \tag{2.25}$$

Since $u_n(s,x)\chi_{A_n}(x) = a_n(s,x)$, we get

$$\begin{aligned}
&\int_{t_0}^t \int_\Omega \left|\Upsilon^1_{n_k}(x)\right| u_{n_k}(s,x)\,dxds \\
&= \int_{t_0}^t \int_\Omega \left| \int_\Omega \chi_{A_n}(y)\, J(x,y)\,(f(y,1) - f(x,1))\,dy \right. \\
&\quad \left. - (x) \int_\Omega X(y) J(x,y)\,(f(y,1) - f(x,1))\,dy \right| a_{n_k}(s,x)\,dxds \\
&\le \int_{t_0}^t \int_\Omega \left| \int_\Omega \left(\chi_{A_{n_k}}(y) - X(y)\right) J(x,y)\, f(y,1)\,dy \right| a_{n_k}(s,x)\,dxds \\
&\quad + \int_{t_0}^t \int_\Omega \left| \int_\Omega \left(\chi_{A_{n_k}}(y) - X(y)\right) J(x,y)\,dy \right| |f(x,1)|\, a_{n_k}(s,x)\,dxds.
\end{aligned} \tag{2.26}$$

By the continuity of J and the fact that $\chi_{A_n}(x) \rightharpoonup X(x)$ weakly in $L^\infty(\overline{\Omega})$ (see (1.1)) we get

$$\sup_{x\in\Omega} \left| \int_\Omega \left\{\chi_{A_n}(y) - X(y)\right\} f(y,1) J(x,y)\,dy \right| \xrightarrow[n\to+\infty]{} 0, \tag{2.27}$$

and

$$\sup_{x\in\Omega} \left| \int_\Omega \left\{\chi_{A_n}(y) - X(y)\right\} J(x,y)\,dy \right| \xrightarrow[n\to+\infty]{} 0. \tag{2.28}$$

Recall that $a_{n_k}(s,x) \rightharpoonup a(s,x)$ weakly in $L^2((0,T)\times\Omega)$ (see Theorem 1.1). By (2.27) and (2.28), we obtain that the right hand side of (2.26) converges to 0 as $n \to +\infty$. Arguing as before we can conclude that

$$\int_{t_0}^t \int_\Omega \left|\Upsilon^i_{n_k}(x)\right| u_{n_k}(s,x)\,dxds \xrightarrow[n_k\to\infty]{} 0, \qquad \forall i \in \{2,3,4\}.$$

This concludes the proof of (2.25). □

We can prove now the last statement of Theorem 1.2, i.e., that the distribution of the limit process $(Y(t), I(t))$ is characterized by the densities $a(t,x)$ and $b(t,x)$.

First of all, observe that, for every measurable $E \subseteq \overline{\Omega}$,

$$\begin{aligned}
\mathbb{P}\left((Y_n(0), I_n(0)) \in E \times \{1\}\right) &= \mathbb{P}\left(Y_n(0) \in E \cap A_n\right) \\
&= \int_{E\cap A_n} u_0(x) dx
\end{aligned}$$

$$= \int_E u_0(x)\chi_{A_n}(x)dx \xrightarrow[n\to\infty]{} \int_E u_0(x)X(x)\,dx.$$

Therefore, by the tightness result proved in Lemma 2.6, we can write

$$\mathbb{P}\left((Y(0), I(0)) \in E \times \{1\}\right) = \int_E u_0(x)X(x)\,dx. \tag{2.29}$$

Analogously, we get that

$$\mathbb{P}\left((Y(0), I(0)) \in E \times \{2\}\right) = \int_E u_0(x)\left(1 - X(x)\right)dx. \tag{2.30}$$

Let $\mu(dx, i) = (\mu_t(dx, i))_{t\in[0,T]}$ be the law of the limit process $(Y(t), I(t))_{t\in[0,T]}$. We can decompose

$$\mu_t(dx, i) = \chi_1(i)\mu_t(dx, 1) + \chi_2(i)\mu_t(dx, 2)$$

where, by (2.29) and (2.30), $(\mu_t(dx, 1))_{t\in[0,T]}$ and $(\mu_t(dx, 2))_{t\in[0,T]}$ are such that

$$\mu_0(dx, 1) = u_0(x)X(x)\,dx \quad \text{and} \quad \mu_0(dx, 2) = u_0(x)\left(1 - X(x)\right)dx. \tag{2.31}$$

Since $\widetilde{\mathcal{L}}$ is the generator of the process $(Y(t), I(t))$ (see Theorem 2.7), by Lemma A.5.1 of [22] we can conclude that

$$\frac{\partial}{\partial t}\sum_{i=1}^{2}\int_{\overline{\Omega}} f(x,i)\mu_t(dx,i) = \sum_{i=1}^{2}\int_{\overline{\Omega}} \widetilde{\mathcal{L}}f(x,i)\mu_t(dx,i),$$

for all bounded $f : \overline{\Omega} \times \{1, 2\} \to \mathbb{R}$. Therefore, fixing $g \in C\left(\overline{\Omega}\right)$ and choosing $f(x, i) = g(x)\chi_1(i)$, we get

$$\begin{aligned}\frac{\partial}{\partial t}\int_{\overline{\Omega}} g(x)\mu_t(dx, 1) &= \int_{\overline{\Omega}}\int_{\overline{\Omega}} X(y)J(x, y)\left(g(y) - g(x)\right)dy\ \mu_t(dx, 1)\\ &\quad - \int_{\overline{\Omega}}\int_{\overline{\Omega}} \left(1 - X(y)\right)R(x, y)\,g(x)\,dy\ \mu_t(dx, 1)\\ &\quad + \int_{\overline{\Omega}}\int_{\overline{\Omega}} X(y)R(x, y)\,g(y)\,dy\ \mu_t(dx, 2).\end{aligned} \tag{2.32}$$

Choosing $f(x, i) = g(x)\chi_2(i)$ we get

$$\begin{aligned}\frac{\partial}{\partial t}\int_{\overline{\Omega}} g(x)\mu_t(dx, 2) &= \int_{\overline{\Omega}}\int_{\overline{\Omega}} \left(1 - X(y)\right)R(x, y)\,g(y)\,dy\ \mu_t(dx, 1)\\ &\quad + \int_{\overline{\Omega}}\int_{\overline{\Omega}} \left(1 - X(y)\right)G(x, y)\left(g(y) - g(x)\right)dy\ \mu_t(dx, 2)\\ &\quad - \int_{\overline{\Omega}}\int_{\overline{\Omega}} X(y)\,R(x, y)\,g(x)\,dy\ \mu_t(dx, 2).\end{aligned} \tag{2.33}$$

We analyse now the right hand side of (2.32). Since J is symmetric, by a change of variables, we can write

$$\begin{aligned}\int_{\overline{\Omega}}\int_{\overline{\Omega}} X(y)J(x, y)\left(g(y) - g(x)\right)dy\ \mu_t(dx, 1) &= \int_{\overline{\Omega}}\int_{\overline{\Omega}} X(x)J(x, y)g(x)dx\ \mu_t(dy, 1)\\ &\quad - \int_{\overline{\Omega}}\int_{\overline{\Omega}} X(y)J(x, y)g(x)dy\ \mu_t(dx, 1).\end{aligned} \tag{2.34}$$

Moreover, it holds that

$$\int_{\overline{\Omega}}\int_{\overline{\Omega}} X(y)\, R(x,y)\, g(y)\, dy\ \mu_t(dx,2) = \int_{\overline{\Omega}}\int_{\overline{\Omega}} X(x) R(x,y)\, g(x)\, dx\ \mu_t(dy,2). \tag{2.35}$$

Replacing (2.34) and (2.35) in the right hand side of (2.32) we obtain

$$\begin{aligned} &\frac{\partial}{\partial t}\int_{\overline{\Omega}} g(x)\mu_t(dx,1) \\ &\quad= \int_{\overline{\Omega}}\int_{\overline{\Omega}} g(x)\, J(x,y)X(x)dx\ \mu_t(dy,1) - \int_{\overline{\Omega}}\int_{\overline{\Omega}} g(x)J(x,y)X(y)dy\ \mu_t(dx,1) \\ &\qquad - \int_{\overline{\Omega}}\int_{\overline{\Omega}} g(x)R(x,y)\,(1-X(y))\, dy\ \mu_t(dx,1) + \int_{\overline{\Omega}}\int_{\overline{\Omega}} g(x)\, R(x,y)\, X(x)dx\ \mu_t(dy,2). \end{aligned} \tag{2.36}$$

As before, via a change of variable in the right hand side of (2.33), we can write

$$\begin{aligned} &\frac{\partial}{\partial t}\int_{\overline{\Omega}} g(x)\mu_t(dx,2) \\ &\quad= \int_{\overline{\Omega}}\int_{\overline{\Omega}} g(x)\,(1-X(x))\, R(x,y)\, dx\ \mu_t(dy,1) + \int_{\overline{\Omega}}\int_{\overline{\Omega}} g(x)\,(1-X(x))\, G(x,y)\, dx\ \mu_t(dy,2) \\ &\qquad - \int_{\overline{\Omega}}\int_{\overline{\Omega}} g(x)\,(1-X(y))\, G(x,y)\, dy\ \mu_t(dx,2) - \int_{\overline{\Omega}}\int_{\overline{\Omega}} g(x)\, X(y)R(x,y)\, dy\ \mu_t(dx,2). \end{aligned} \tag{2.37}$$

Moreover, by (2.31) we get that

$$\begin{aligned} &\int_{\overline{\Omega}} g(x)\mu_0(dx,1) = \int_{\overline{\Omega}} g(x)u_0(x)X(x)dx \quad \text{and} \\ &\qquad\qquad \int_{\overline{\Omega}} g(x)\mu_0(dx,2) = \int_{\overline{\Omega}} g(x)u_0(x)\,(1-X(x))\, dx. \end{aligned} \tag{2.38}$$

By Lemma 2.8 below we know that there exists a unique pair of trajectories of measures $(\mu_t(dx,1), \mu_t(dx,2))_{t\in[0,T]}$ which, for every $g \in C\left(\overline{\Omega}\right)$, satisfies (2.36), (2.37) and (2.30). Such pair is given by

$$\mu_t(dx,1) = a(t,x)dx \qquad \text{and} \qquad \mu_t(dx,2) = b(t,x)dx,$$

where the couple $(a(t,x), b(t,x))$ is the unique solution to system (1.4). This concludes the proof of Theorem 1.2.

Lemma 2.8 *There exists a unique pair $(\mu_t(dx,1), \mu_t(dx,2))$ such that, for every $g \in C\left(\overline{\Omega}\right)$, (2.36), (2.37) and (2.38) are satisfied. Such solution is given by*

$$(\mu_t(dx,1), \mu_t(dx,2)) = (a(t,x)dx, b(t,x)dx),$$

where the pair $(a(t,x), b(t,x))$ is the unique solution to system (1.4).

Proof The fact that the pair $(a(t,x)dx, b(t,x)dx)$ is a solution to the system (2.36)–(2.37)–(2.38) is a consequence of the fact that, by Theorem 1.1, $(a(t,x), b(t,x))$ is a solution to the system (1.4).

We prove now the uniqueness. Suppose that there exist two pairs $(\nu_t(dx,1), \nu_t(dx,2))$ and $(\tilde{\nu}_t(dx,1), \tilde{\nu}_t(dx,2))$ for which system (2.36)–(2.37)–(2.38) is satisfied. Let

$$\omega_t(dx,1) := \nu_t(dx,1) - \tilde{\nu}_t(dx,1) \qquad \text{and} \qquad \omega_t(dx,2) := \nu_t(dx,2) - \tilde{\nu}_t(dx,2).$$

Therefore, we know that, for all $g \in C(\overline{\Omega})$,

$$\begin{aligned}\int_{\overline{\Omega}} g(x)\omega_t(dx,1) = & \int_0^t \int_{\overline{\Omega}} \int_{\overline{\Omega}} g\,(x)\; J(x,y)X(x)dx\;\omega_s\,(dy,1)\,ds \\ & - \int_0^t \int_{\overline{\Omega}} \int_{\overline{\Omega}} g(x)J(x,y)X(y)dy\;\omega_s\,(dx,1)\,ds \\ & - \int_0^t \int_{\overline{\Omega}} \int_{\overline{\Omega}} g(x)R(x,y)\,(1-X(y))\,dy\;\omega_s\,(dx,1)\,ds \\ & + \int_0^t \int_{\overline{\Omega}} \int_{\overline{\Omega}} g\,(x)\;R\,(x,y)\;X(x)dx\;\omega_s(dy,2)ds \end{aligned} \tag{2.39}$$

and

$$\begin{aligned}\int_{\overline{\Omega}} g(x)\omega_t(dx,2) = & \int_0^t \int_{\overline{\Omega}} \int_{\overline{\Omega}} g(x)\,(1-X\,(x))\;R\,(x,y)\,dx\omega_s\,(dy,1)\,ds \\ & + \textstyle\int_0^t \int_{\overline{\Omega}} \int_{\overline{\Omega}} g\,(x)\,(1-X\,(x))\;G\,(x,y)\,dx\omega_s(dy,2)ds \\ & - \int_0^t \int_{\overline{\Omega}} \int_{\overline{\Omega}} g\,(x)\,(1-X\,(y))\;G\,(x,y)\,dy\;\omega_s(dx,2)ds \\ & - \textstyle\int_0^t \int_{\overline{\Omega}} \int_{\overline{\Omega}} g\,(x)\;X(y)R\,(x,y)\,dy\;\omega_s(dx,2)ds, \end{aligned} \tag{2.40}$$

with initial conditions

$$\int_{\overline{\Omega}} g(x)\omega_0(dx,1) = \int_{\overline{\Omega}} g(x)\omega_0(dx,2) = 0. \tag{2.41}$$

Recalling that we denote by $\|\cdot\|_{\mathrm{TV}}$ the dual norm (total variation), from (2.39) and (2.41) we get

$$\begin{aligned}\|\omega_t(dx,1)\|_{\mathrm{TV}} = & \sup_{g\in C(\overline{\Omega}):\|g\|_\infty\le 1} \Bigg\{ \int_0^t \int_{\overline{\Omega}} \int_{\overline{\Omega}} g\,(x)\;J(x,y)X(x)dx\;\omega_s\,(dy,1)\,ds \\ & - \int_0^t \int_{\overline{\Omega}} \int_{\overline{\Omega}} g(x)J(x,y)X(y)dy\;\omega_s\,(dx,1)\,ds \\ & - \int_0^t \int_{\overline{\Omega}} \int_{\overline{\Omega}} g(x)R(x,y)\,(1-X(y))\,dy\;\omega_s\,(dx,1)\,ds \\ & + \int_0^t \int_{\overline{\Omega}} \int_{\overline{\Omega}} g\,(x)\;R\,(x,y)\;X(x)dx\;\omega_s(dy,2)ds \Bigg\} \\ & \le C_1 \int_0^t \left(\|\omega_s(dx,1)\|_{\mathrm{TV}} + \|\omega_s(dx,2)\|_{\mathrm{TV}}\right) ds, \end{aligned}$$

where $C_1 = C_1\left(\|J\|_\infty, \|R\|_\infty, |\Omega|\right)$ is a constant depending on $\|J\|_\infty$, $\|R\|_\infty$ and $|\Omega|$. Analogously, by (2.40) and (2.41), we obtain

$$\|\omega_t(dx,2)\|_{\mathrm{TV}} \le C_2 \int_0^t \left(\|\omega_s(dx,1)\|_{\mathrm{TV}} + \|\omega_s(dx,2)\|_{\mathrm{TV}}\right) ds,$$

where $C_2 = C_2\left(\|G\|_\infty, \|R\|_\infty, |\Omega|\right)$ is a constant depending on $\|G\|_\infty$, $\|R\|_\infty$ and $|\Omega|$. Therefore by Gronwall's inequality we conclude that

$$\|\omega_t(dx,1)\|_{\mathrm{TV}} + \|\omega_t(dx,2)\|_{\mathrm{TV}} = 0.$$

Therefore, $\omega_t(dx,1)$ and $\omega_t(dx,2)$ coincide with the null measure on $\overline{\Omega}$ and consequently $\nu_t(dx,i) = \tilde{\nu}_t(dx,i)$, for $i \in \{1,2\}$. This concludes the proof. □

Remark 2.9 It holds that

$$(Y_n(t))_{t\in[0,T]} \xrightarrow[n\to\infty]{D} (Y(t))_{t\in[0,T]},$$

where $Y(t)$ has probability density

$$u(t,x) = a(t,x) + b(t,x).$$

Indeed, the convergence in distribution to the process $(Y(t))_{t\in[0,T]}$ is a consequence of (1.5). Moreover, since $Y(t)$ is the marginal in the first variable of $(Y(t), I(t))$, we can write

$$\begin{aligned}\mathbb{P}(Y(t) \in E) &= \sum_{i=1}^{2} \mathbb{P}(Y(t) \in E, I(t) = i) = \sum_{i=1}^{2} \int_E \mu_t(dx, i) \\ &= \int_E (a(t,x) + b(t,x))\, dx = \int_E u(t,x) dx,\end{aligned}$$

for every measurable set $E \subseteq \overline{\Omega}$.

2.3 Asymptotic Behavior of $u_n(t,x)$.

This subsection contains the proof of the fact that $u_n(t,x)$ converges exponentially fast to $\frac{1}{|\Omega|}$.

Proof of Theorem 1.3 Let u_n be the solution to (1.2) and define

$$w_n(t,x) := u_n(t,x) - \frac{1}{|\Omega|}.$$

Observe that w_n is a solution to

$$\begin{cases} \dfrac{\partial w_n}{\partial t}(t,x) = L_n w_n(t,x), \ \ t > 0, \ x \in \overline{\Omega}, \\ w_n(0,x) = u_0(x) - \dfrac{1}{|\Omega|}, \ \ x \in \overline{\Omega}. \end{cases} \tag{2.42}$$

Therefore we have that

$$\int_\Omega w_n(0,x)\, dx = 0.$$

Since our equation preserves the total mass, we have

$$\int_\Omega w_n(t,x)\, dx = 0, \qquad \forall t \geq 0.$$

Now, multiply by w_n both sides in (2.42) and integrate in Ω to obtain

$$\begin{aligned}\frac{\partial}{\partial t}\frac{1}{2}\int_\Omega |w_n(t,x)|^2\, dx &= \int_\Omega L_n w_n(t,x) w_n(t,x)\, dx \\ &= -2E_n(w_n),\end{aligned} \tag{2.43}$$

with

$$\begin{aligned}E_n(w) = \frac{1}{4}\int_{A_n}\int_{A_n} J(x,y)(w(y)-w(x))^2 dy\, dx + \frac{1}{4}\int_{B_n}\int_{B_n} G(x,y)(w(y)-w(x))^2 dy\, dx \\ + \frac{1}{2}\int_{A_n}\int_{B_n} R(x,y)(w(y)-w(x))^2 dy\, dx.\end{aligned} \tag{2.44}$$

By Lemma 4.1 in [7] we know that there exists a positive constant c (independent of n) such that

$$E_n(w_n) \geq c \int_\Omega w_n^2(t,x)\,dx.$$

Therefore by (2.43) we obtain

$$\frac{\partial}{\partial t}\int_\Omega |w_n(t,x)|^2\,dx \leq -2c\int_\Omega w_n^2(t,x)\,dx.$$

Then, by Gronwall's inequality, we conclude that there is a constant c independent of n such that

$$\|w_n^2(t,\cdot)\|^2_{L^2(\Omega)} \leq \|w_0\|^2_{L^2(\Omega)} e^{-2ct}$$

and this concludes the proof of Theorem 1.3. □

Using Theorem 1.3 we can prove the following proposition.

Proposition 2.10 *For every $f \in C(\overline{\Omega})$ it holds that*

$$\mathbb{E}\Big(\int_0^\infty f(Y_n(t))\,dt\Big) \xrightarrow[n\to\infty]{} \mathbb{E}\Big(\int_0^\infty f(Y(t))\,dt\Big).$$

Proof Since $u_n(t,x)$ and $u(t,x)$ are the probability densities of $Y_n(t)$ and $Y(t)$ respectively it is enough to show that

$$\left|\int_0^\infty\int_\Omega f(y)u_n(t,y)dydt - \int_0^\infty\int_\Omega f(y)u(t,y)dydt\right| \xrightarrow[n\to\infty]{} 0. \tag{2.45}$$

For every fixed $T > 0$ it holds that

$$\begin{aligned}&\left|\int_0^\infty\int_\Omega f(y)u_n(t,y)dydt - \int_0^\infty\int_\Omega f(y)u(t,y)dydt\right|\\ &\leq \left|\int_0^T\int_\Omega f(y)u_n(t,y)dydt - \int_0^T\int_\Omega f(y)u(t,y)dydt\right|\\ &\quad+\left|\int_T^\infty\int_\Omega f(y)u_n(t,y)dydt - \int_T^\infty\int_\Omega f(y)u(t,y)dydt\right|.\end{aligned}$$

By Theorem 1.1 we know that $u_n(t,x) \rightharpoonup u(t,x)$ in $L^2((0,T)\times\Omega)$ and therefore

$$\left|\int_0^T\int_\Omega f(y)u_n(t,y)dydt - \int_0^T\int_\Omega f(y)u(t,y)dydt\right| \xrightarrow[n\to\infty]{} 0.$$

Thus to conclude (2.45) it is enough to show that

$$\lim_{T\to\infty}\lim_{n\to\infty}\left|\int_T^\infty\int_\Omega f(y)u_n(t,y)dydt - \int_T^\infty\int_\Omega f(y)u(t,y)dydt\right| = 0. \tag{2.46}$$

To this end, we observe that

$$\begin{aligned}
&\left|\int_T^\infty \int_\Omega f(y)u_n(t,y)dydt - \int_T^\infty \int_\Omega f(y)u(t,y)dydt\right| \\
&\quad \le \int_T^\infty \left|\int_\Omega f(y)\left(u_n(t,y)dy - u(y,t)\right)dy\right| dt \\
&\quad \le \int_T^\infty \left|\int_\Omega f(y)\left(u_n(t,y) - \frac{1}{|\Omega|}\right)dy\right| + \int_t^\infty \left|\int_\Omega f(y)\left(\frac{1}{|\Omega|} - u(t,y)\right)dy\right| \\
&\quad \le \|f\|^2_{L^2(\Omega)}\left(\int_T^\infty \left\|u_n(t,\cdot) - \frac{1}{|\Omega|}\right\|^2_{L^2(\Omega)} dt + \int_T^\infty \left\|u(t,\cdot) - \frac{1}{|\Omega|}\right\|^2_{L^2(\Omega)} dt\right).
\end{aligned} \tag{2.47}$$

Let C and A be the constants that appear in Theorem 1.3, we have that

$$\left\|u_n(t,\cdot) - \frac{1}{|\Omega|}\right\|^2_{L^2(\Omega)} \le Ce^{-At}, \tag{2.48}$$

and therefore

$$\int_T^\infty \left\|u_n(t,\cdot) - \frac{1}{|\Omega|}\right\|^2_{L^2(\Omega)} dt \le \int_T^\infty Ce^{-At}dt \xrightarrow[T\to\infty]{} 0. \tag{2.49}$$

Now, using that u_n is uniformly bounded in $L^\infty([0,T]; L^2(\Omega))$ we can assert that

$$u_n(\cdot,\cdot) - \frac{1}{|\Omega|} \rightharpoonup u(\cdot,\cdot) - \frac{1}{|\Omega|} \qquad \text{weakly in } L^2([0,T]\times\Omega)$$

as $n \to \infty$. Then taking the limit as $n \to \infty$ in (2.48) we get

$$\int_T^N \left\|u(t,\cdot) - \frac{1}{|\Omega|}\right\|^2_{L^2(\Omega)} dt \le \liminf_{n\to\infty} \int_T^N \int_\Omega \left|u_n(t,x) - \frac{1}{|\Omega|}\right|^2 dx\, dt \le \int_T^N Ce^{-At}dt \tag{2.50}$$

and therefore we conclude that

$$\int_T^\infty \left\|u(t,\cdot) - \frac{1}{|\Omega|}\right\|^2_{L^2(\Omega)} dt \xrightarrow[T\to\infty]{} 0.$$

From (2.47), (2.49) and (2.50) we conclude (2.46). □

Now we are ready to prove Corollary 1.4 that is an immediate consequence of our previous results Theorem 1.1 and Theorem 1.3.

Proof From Theorem 1.1 (using a diagonal procedure) we have that there exists a function u such that for every N

$$u_n(t,x) \rightharpoonup u(t,x), \qquad \text{weakly in } L^2((0,N)\times\Omega).$$

Now, from Theorem 1.3 we get that

$$\left\|u(t,\cdot) - \frac{1}{|\Omega|}\right\|^2_{L^2(\Omega)} \le Ce^{-At}.$$

Then, we get that the convergence $u_n(t,x) \rightharpoonup u(t,x)$ takes place in $L^2((0,\infty)\times\Omega)$. In fact, for a given function $w \in L^2((0,\infty)\times\Omega)$ it holds that

$$\begin{aligned}&\left|\int_0^\infty \int_\Omega (u_n-u)(t,x)w(t,x)dxdt\right| \\ &\quad \le \left|\int_0^N \int_\Omega (u_n-u)(t,x)w(t,x)dxdt\right| + 2\|w\|_{L^2((0,\infty)\times\Omega)} \int_N^\infty Ce^{-At}dt,\end{aligned}$$

from where the weak convergence in $L^2((0,\infty)\times\Omega)$ follows (first choose N large in order to make the last term small and then use that $u_n(t,x) \rightharpoonup u(t,x)$ in $L^2((0,N)\times\Omega)$ to conclude.

We argue similarly to obtain that $\chi_{A_n}(x)u_n(t,x) \rightharpoonup a(t,x)$ and $\chi_{B_n}(x)u_n(t,x) \rightharpoonup b(t,x)$ weakly in $L^2((0,\infty)\times\Omega)$.

The proof of the fact that the limits are characterized as the solution to (1.4) is exactly as before. □

3 Initial Conditions $u_0 = \delta_{\bar{x}}$

In this section, we analyze now the case in which $Y_n(0) = \bar{x} \in \Omega$.

Let P_n be the law of the process $(Y_n(t), I_n(t))_{t\ge 0}$ and call $(\nu_n(t))_t$ the law of $(Y_n(t))_t$ that is the first marginal of P_n.

By Dynkin's formula we know that, for every $g \in C(A_n)\cap C(B_n)$, it holds that

$$\begin{cases} \dfrac{d}{dt}\displaystyle\int_\Omega g(x)\nu_t^n(dx) = \int_\Omega L_n g(x)\nu_t^n(dx) \\ \displaystyle\int_\Omega g(x)\nu_0^n(dx) = g(\bar{x}), \end{cases}$$

where L_n is the generator defined in (2.14). Since the evolution problem does not have a regularizing effect we expect that the initial measure $\delta_{\bar{x}}$ remains as time evolves, hence we write

$$\nu_t^n(dx) := z_n(t,x)dx + \sigma_n(t)\delta_{\bar{x}}(dx).$$

By the expression of L_n we obtain

$$\begin{aligned}&\frac{d}{dt}\int_\Omega g(x)z_n(t,x)dx + \frac{d}{dt}\sigma_n(t)g(\bar{x}) \\ &\quad = \int_\Omega \chi_{A_n}(x)\int_\Omega \chi_{A_n}(y)J(x,y)\left(g(y)-g(x)\right)z_n(t,x)dx \\ &\qquad + \sigma_n(t)\chi_{A_n}(\bar{x})\int_\Omega \chi_{A_n}(y)J(\bar{x},y)\left(g(y)-g(\bar{x})\right)dy \\ &\qquad + \int_\Omega \chi_{B_n}(x)\int_\Omega \chi_{B_n}(y)G(x,y)\left(g(y)-g(x)\right)z_n(t,x)dx \\ &\qquad + \sigma_n(t)\chi_{B_n}(\bar{x})\int_\Omega \chi_{B_n}(y)G(\bar{x},y)\left(g(y)-g(\bar{x})\right)dy \\ &\qquad + \int_\Omega \chi_{A_n}(x)\int_\Omega \chi_{B_n}(y)R(x,y)\left(g(y)-g(x)\right)z_n(t,x)dx \\ &\qquad + \sigma_n(t)\chi_{A_n}(\bar{x})\int_\Omega \chi_{B_n}(y)R(\bar{x},y)\left(g(y)-g(\bar{x})\right)dy\end{aligned}$$

$$+\int_\Omega \chi_{B_n}(x)\int_\Omega \chi_{A_n}(y)R(x,y)\left(g(y)-g(x)\right)z_n(t,x)dx$$
$$+\sigma_n(t)\chi_{B_n}(\bar{x})\int_\Omega \chi_{A_n}(y)R(\bar{x},y)\left(g(y)-g(\bar{x})\right)dy.$$

Therefore, we get that

$$\frac{d}{dt}\sigma_n(t)=-\sigma_n(t)\Bigg(\chi_{A_n}(\bar{x})\int_\Omega \chi_{A_n}(y)J(\bar{x},y)dy+\chi_{B_n}(\bar{x})\int_\Omega \chi_{B_n}(y)G(\bar{x},y)dy$$
$$+\chi_{A_n}(\bar{x})\int_\Omega \chi_{B_n}(y)R(\bar{x},y)dy+\chi_{B_n}(\bar{x})\int_\Omega \chi_{A_n}(y)R(\bar{x},y)dy\Bigg),$$

with initial datum $\sigma_n(0)=1$. Now, recall that we assumed that

$$\chi_{A_n}(\bar{x})\int_\Omega \chi_{A_n}(y)J(\bar{x},y)dy+\chi_{B_n}(\bar{x})\int_\Omega \chi_{B_n}(y)G(\bar{x},y)dy$$
$$+\chi_{A_n}(\bar{x})\int_\Omega \chi_{B_n}(y)R(\bar{x},y)dy+\chi_{B_n}(\bar{x})\int_\Omega \chi_{A_n}(y)R(\bar{x},y)dy=1,$$

for every $\bar{x}\in\Omega$. This condition has a clear probabilistic interpretation. It says that the particle has to jump with full probability (that is, the probability of staying at the same location when the exponential clock rings is zero).

Therefore, we conclude that

$$\sigma_n(t)=e^{-t},\quad \forall n\in\mathbb{N}.$$

On the other hand, we get

$$\frac{d}{dt}\int_\Omega g(x)z_n(t,x)dx=\int_\Omega \chi_{A_n}(x)\int_\Omega \chi_{A_n}(y)J(x,y)\left(g(y)-g(x)\right)z_n(t,x)dx$$
$$+\sigma_n(t)\chi_{A_n}(\bar{x})\int_\Omega \chi_{A_n}(y)J(\bar{x},y)g(y)dy$$
$$+\int_\Omega \chi_{B_n}(x)\int_\Omega \chi_{B_n}(y)G(x,y)\left(g(y)-g(x)\right)z_n(t,x)dx$$
$$+\sigma_n(t)\chi_{B_n}(\bar{x})\int_\Omega \chi_{B_n}(y)G(\bar{x},y)g(y)dy$$
$$+\int_\Omega \chi_{A_n}(x)\int_\Omega \chi_{B_n}(y)R(x,y)\left(g(y)-g(x)\right)z_n(t,x)dx$$
$$+\sigma_n(t)\chi_{A_n}(\bar{x})\int_\Omega \chi_{B_n}(y)R(\bar{x},y)g(y)dy$$
$$+\int_\Omega \chi_{B_n}(x)\int_\Omega \chi_{A_n}(y)R(x,y)\left(g(y)-g(x)\right)z_n(t,x)dx$$
$$+\sigma_n(t)\chi_{B_n}(\bar{x})\int_\Omega \chi_{A_n}(y)R(\bar{x},y)g(y)dy,$$

which implies the following equation for z_n,

$$\frac{d}{dt}z_n(t,x)=\chi_{A_n}(x)\int_\Omega \chi_{A_n}(y)J(x,y)\left(z_n(t,y)-z_n(t,x)\right)dy+\sigma_n(t)\chi_{A_n}(\bar{x})\chi_{A_n}(x)J(\bar{x},x)$$
$$+\chi_{B_n}(x)\int_\Omega \chi_{B_n}(y)G(x,y)\left(z_n(t,y)-z_n(t,x)\right)dy+\sigma_n(t)\chi_{B_n}(\bar{x})\chi_{B_n}(x)G(\bar{x},x)$$

$$+\chi_{A_n}(x)\int_\Omega \chi_{B_n}(y)R(x,y)\left(z_n(t,y)-z_n(t,x)\right)dy+\sigma_n(t)\chi_{A_n}(\bar{x})\chi_{B_n}(x)R(\bar{x},x)$$
$$+\chi_{B_n}(x)\int_\Omega \chi_{A_n}(y)R(x,y)\left(z_n(t,y)-z_n(t,x)\right)dy+\sigma_n(t)\chi_{B_n}(\bar{x})\chi_{A_n}(x)R(\bar{x},x), \quad (3.1)$$

with initial condition $z_n(0,x)=0$.

3.1 Convergence Along Subsequences

We devote this subsection to the proof of Theorem 4.1.

Proof of Theorem 4.1 The sequence z_n converges weakly in $L^2([0,T]\times\Omega)$ along subsequences as it is bounded the L^2-norm. The same holds for $\chi_{A_n}(x)z_n(t,x)$ and $\chi_{B_n}(x)z_n(t,x)$. This fact can be easily obtained working as in Lemma 2.1.

Take $\chi_{A_{n_k}}(x)z_{n_k}(t,x)$ and $\chi_{B_{n_k}}(x)z_{n_k}(t,x)$ two convergent subsequences. We have to distinguish between two cases:

Case 1. There exists a sub-subsequence $z_{n_{k_j}}$ such that

$$\chi_{A_{n_{k_j}}}(\bar{x})=1, \qquad \forall n_{k_j}.$$

We call $a_k(t,x)$ and $b_k(t,x)$ the weak limits of $\chi_{A_{n_{k_j}}}(x)z_{n_{k_j}}(t,x)$ and $\chi_{B_{n_{k_j}}}(x)z_{n_{k_j}}(t,x)$ respectively. Observe that $a_k(t,x)$ and $b_k(t,x)$ coincide with the weak limits of $\chi_{A_{n_k}}(x)z_{n_k}(t,x)$ and $\chi_{B_{n_k}}(x)z_{n_k}(t,x)$, respectively, as we know that the two sequences converge along subsequences.

Take now a smooth function ϕ such that $\phi(T,\cdot)\equiv 0$ and consider equation (3.1). Multiply both sides by $\chi_{B_n}(x)\,\phi(t,x)$ and then integrate respect to the variables x and t. Since by construction $\phi(T,\cdot)\equiv 0$, integrating by parts we obtain

$$\begin{aligned}
&-\int_0^T\int_\Omega \frac{\partial\phi}{\partial t}(t,x)b_{n_{k_j}}(t,x)\,dxdt\\
&=\int_0^T\int_\Omega\int_\Omega \chi_{B_{n_{k_j}}}(x)\,\chi_{B_{n_{k_j}}}(y)\,G(x,y)\left(z_{n_{k_j}}(t,y)-z_{n_{k_j}}(t,x)\right)\phi(t,x)\,dydxdt\\
&\quad+\int_0^T\int_\Omega\int_\Omega \chi_{B_{n_{k_j}}}(x)\,\chi_{A_{n_{k_j}}}(y)\,R(x,y)\left(z_{n_{k_j}}(t,y)-z_{n_{k_j}}(t,x)\right)\phi(t,x)\,dydxdt\\
&\quad+\int_0^T\int_\Omega e^{-t}R(\bar{x},x)\chi_{B_{n_{k_j}}}(x)\phi(t,x)\,dxdt.
\end{aligned}$$

We can analyze the terms in the left hand side and the first two terms in the right hand side of the previous equality exactly as we did in the proof of Theorem 1.1 (see (2.1), (2.2), (2.3) and (2.4)). Moreover, we have that

$$\int_0^T\int_\Omega e^{-t}G(\bar{x},x)\chi_{B_{n_{k_j}}}(x)\phi(t,x)\,dxdt \xrightarrow[n\to\infty]{} \int_0^T\int_\Omega e^{-t}G(\bar{x},x)\left(1-X(x)\right)\phi(t,x)\,dxdt.$$

Therefore, we get

$$\begin{aligned}
\frac{\partial b_j}{\partial t}(t,x)=&\int_\Omega G(x,y)\left((1-X(x))\,b_j(t,y)-(1-X(y))\,b_j(t,x)\right)dy\\
&+\int_\Omega R(x,y)\left((1-X(x))\,a_j(t,y)-X(y)b_j(t,x)\right)dy+e^{-t}R(\bar{x},x)\left(1-X(x)\right),
\end{aligned}$$

and, in a similar way, we obtain

$$\frac{\partial a_j}{\partial t}(t,x) = \int_\Omega J(x,y)\big(X(x)a_j(t,y) - X(y)a_j(t,x)\big)\,dy$$
$$+\int_\Omega R(x,y)\big(X(x)b_j(t,y) - (1-X(y))\,a_j(t,x)\big)\,dy + e^{-t}J(\bar{x},x)X(x).$$

Hence, the limit is a solution to (1.8).

Case 2. There exists a sub-subsequence n_{k_j} such that

$$\chi_{B_{n_{k_j}}}(\bar{x}) = 1, \qquad \forall n_{k_j}.$$

For this case, arguing as we did before, it is possible to prove that the limits $a_k(t,x)$ and $b_k(t,x)$ satisfy system (1.9). □

For what concerns the stochastic process $(Y_n(t), I_n(t))$ we can prove an analogous result to Theorem 2.7 even in the case in which the process starts with $\delta_{\bar{x}}$, but now we are able to characterize the measure of the limit process $(Y(t), I(t))$ only when $\bar{x} \in A_n$ or $\bar{x} \in B_n$ for every n (since in this case we have convergence of the densities u_n). The details of this characterization can be done as in Sect. 2 and are left to the reader. Remark that the convergence of the measure holds only along subsequences.

3.2 Asymptotic Behavior of $z_n(t,x)$

In this subsection we look for the asymptotic behaviour as $t \to +\infty$ of $z_n(t,x)$.

Proof of Theorem 1.6 Let

$$w_n(t,x) := z_n(t,x) - \frac{1}{|\Omega|}(1-e^{-t})$$

and note that

$$\int_\Omega w_n(t,x) = 0.$$

To conclude the proof it is enough to show that there exists $C > 0$ and $A > 0$ such that, for t large enough,

$$\|w_n(t,\cdot)\|^2_{L^2(\Omega)} \leq Ce^{-At}. \tag{3.2}$$

Recall the definition of $E_n(w)$ given in (2.44). Following the same strategy we used to prove Theorem 1.3 we get that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega w_n^2(t,x)dx = -2E_n(w_n) + e^{-t}\int_\Omega c(x,\bar{x},n)w_n(t,x)dx,$$

with

$$c(x,\bar{x},n) := \chi_{A_n}(\bar{x})\chi_{A_n}(x)J(\bar{x},x) + \chi_{B_n}(\bar{x})\chi_{B_n}(x)G(\bar{x},x)$$
$$+\chi_{A_n}(\bar{x})\chi_{B_n}(x)R(\bar{x},x) + \chi_{B_n}(\bar{x})\chi_{A_n}(x)R(\bar{x},x).$$

By Lemma 4.1 in [7] it holds that there exists a constant c_1 (independent of n) such that $E_n(w_n) \geq 2c_1\,\|w_n(t,\cdot)\|^2_{L^2(\Omega)}$ and therefore, by Cauchy-Schwartz's inequality, we get

$$\frac{d}{dt}\|w_n(t,\cdot)\|^2_{L^2(\Omega)} \leq -4c_1\,\|w_n(t,\cdot)\|^2_{L^2(\Omega)} + 2e^{-t}\int_\Omega c(x,\bar{x},n)\,|w_n(t,x)|\,dx$$

$$\leq -4c_1 \left\| w_n(t,\cdot) \right\|_{L^2(\Omega)}^2 + 2e^{-t} \left(c_2 \left\| w_n(t,\cdot) \right\|_{L^2(\Omega)}^2 \right),$$

where $c_2 = (\|J\|_\infty + 2\|R\|_\infty + \|G\|_\infty)\,|\Omega|$. Fix $\bar{t} >> 1$ such that $-4c_1 + 2c_2e^{-\bar{t}} \leq -2c_1$, then for every $t \geq \bar{t}$ we get

$$\frac{d}{dt} \left\| w_n(t,\cdot) \right\|_{L^2(\Omega)}^2 \leq -4c_1 \left\| w_n(t,\cdot) \right\|_{L^2(\Omega)}^2 + 2c_2e^{-t}. \tag{3.3}$$

By (3.3) we conclude that,

$$\left\| w_n(t,\cdot) \right\|_{L^2(\Omega)}^2 \leq e^{-4c_1 t} \left\| w_n(\bar{t},\cdot) \right\|_{L^2(\Omega)}^2 + \frac{2c_2}{4c_1 - 1} e^{-t}$$

and therefore we can conclude (3.2). □

4 The Dirichlet Case

In this final section we analyze the Dirichlet problem in which we take a sequence of partitions A_n, B_n of the entire space $\mathbb{R}^N$ such that $\mathbb{R}^N = A_n \cup B_n$, $A_n \cap B_n = \emptyset$ and

$$\begin{aligned} &\bullet\ \chi_{A_n}(x) \rightharpoonup X(x), \qquad \text{weakly in } L^\infty(\mathbb{R}^N),\\ &\bullet\ \chi_{B_n}(x) \rightharpoonup 1 - X(x), \qquad \text{weakly in } L^\infty(\mathbb{R}^N),\\ &\quad \text{with } 0 < X(x) < 1. \end{aligned} \tag{4.1}$$

As for the Neumann case, at the times $\{\tau_k\}$ a particle that is at $x \in \Omega$ chooses a new site y, but now $y \in \mathbb{R}^N$, according to the kernels J, R or G. The jumps from a site in A_n to another site in A_n are ruled by J, the jumps between A_n and B_n (or vice versa) are ruled by R and the jumps from a site in B_n to a site in B_n are ruled by G. Hence, the movement of the particle obeys the same rules as before, but now the particle is allowed to jump outside $\overline{\Omega}$, and, as soon as this happens, the particle is killed and disappears from the system. In this new model we denote by $Z_n(t)$ the position of the particle that is alive in $\overline{\Omega}$ and we suppose (as we did before, but this time in the whole $\mathbb{R}^N$) that we have probability kernels in our equations, that is,

$$\int_{\mathbb{R}^N} J(x,y)dy = 1, \quad \int_{\mathbb{R}^N} R(x,y)dy = 1, \quad \int_{\mathbb{R}^N} G(x,y)dy = 1, \qquad \forall x \in \overline{\Omega}.$$

The process $(Z_n(t))_{t\geq 0}$ is a Markov process whose generator L_n is defined on functions $f \in C(A_n) \cap C(B_n)$ such that $\operatorname{supp} f \subseteq \overline{\Omega}$ as

$$\begin{aligned} L_n f(x) = {} & \chi_{\{A_n \cap \overline{\Omega}\}}(x) \int_{\mathbb{R}^N} \chi_{A_n}(y)\, J(x,y)\,(f(y) - f(x))\,dy \\ & + \chi_{\{B_n \cap \overline{\Omega}\}}(x) \int_{\mathbb{R}^N} \chi_{B_n}(y)\, G(x,y)\,(f(y) - f(x))\,dy \\ & + \chi_{\{A_n \cap \overline{\Omega}\}}(x) \int_{\mathbb{R}^N} \chi_{B_n}(y)\, R(x,y)\,(f(y) - f(x))\,dy \\ & + \chi_{\{B_n \cap \overline{\Omega}\}}(x) \int_{\mathbb{R}^N} \chi_{A_n}(y)\, R(x,y)\,(f(y) - f(x))\,dy. \end{aligned}$$

Again the initial position $Z_n(0)$ is described in terms of a given distribution u_0 in $\overline{\Omega}$. We suppose that

$$P\left(Z_n(0) \in E\right) = \int_E u_0(z)\,dz,$$

for every measurable set $E \subseteq \overline{\Omega}$.

The associated evolution problem reads as

$$\begin{cases} \dfrac{\partial u_n}{\partial t}(t,x) = L_n u_n(t,x), & t > 0,\ x \in \overline{\Omega}, \\ u_n(t,x) = 0, & t \geq 0,\ x \in \overline{\Omega}^c, \\ u_n(0,x) = u_0(x), & x \in \Omega. \end{cases} \tag{4.2}$$

As before we are interested in taking the limit, as $n \to +\infty$, both in the processes $Z_n(t)$ and in the associated densities $u_n(t,x)$. To this end we need to look at the process $Z_n(t)$ as a couple $(Z_n(t), I_n(t))$. In our notation $I_n(t)$ contains explicitly the information over the set (A_n or B_n) in which $Z_n(t)$ is located. More precisely, $I_n(t) = 1$ (or 2) if the particle is in A_n (or in B_n respectively) at time t.

The following theorem holds.

Theorem 4.1 *Assume (4.1) and fix $T > 0$. We have that, as $n \to \infty$,*

$$\begin{aligned} &u_n(t,x) \rightharpoonup u(t,x), && \text{weakly in } L^2((0,T)\times\mathbb{R}^N), \\ &\chi_{A_n}(x) u_n(t,x) \rightharpoonup a(t,x), && \text{weakly in } L^2((0,T)\times\mathbb{R}^N), \\ &\chi_{B_n}(x) u_n(t,x) \rightharpoonup b(t,x), && \text{weakly in } L^2((0,T)\times\mathbb{R}^N). \end{aligned}$$

These limits verify

$$u(t,x) = a(t,x) + b(t,x)$$

and are characterized by the fact that (a, b) is the unique solution to the following system,

$$\begin{cases} \dfrac{\partial a}{\partial t}(t,x) = \displaystyle\int_{\mathbb{R}^N} J(x,y)\left(X(x)a(t,x) - X(y)a(t,x)\right) dy \\ \qquad\qquad + \displaystyle\int_{\mathbb{R}^N} R(x,y)\left(X(x)b(t,y) - (1-X(y))\,a(t,x)\right) dy & t > 0,\ x \in \Omega, \\ \dfrac{\partial b}{\partial t}(t,x) = \displaystyle\int_{\mathbb{R}^N} G(x,y)\left[(1-X(x))\,b(t,y) - (1-X(y))\,b(t,x)\right] dy \\ \qquad\qquad + \displaystyle\int_{\mathbb{R}^N} R(x,y)\left[(1-X(x))\,a(t,y) - X(y)b(t,x)\right] dy\, dy & t > 0,\ x \in \Omega, \\ a(t,x) = b(t,x) = 0, & t \geq 0,\ x \in \overline{\Omega}^c, \\ a(0,x) = X(x)\,u_0(x), \quad b(0,x) = (1-X(x))\,u_0(x) & x \in \Omega. \end{cases}$$

Moreover, it holds that the sequence of processes converges in distribution

$$(Z_n(t), I_n(t)) \xrightarrow[n\to+\infty]{D} (Z(t), I(t))$$

in $D\left([0,T], \mathbb{R}^N\right) \times D([0,T], \{1,2\})$, where the distribution of the limit $(Z(t), I(t))$ is characterized by having as probability densities $a(t,x)$ and $b(t,x)$, that is,

$$P\Big(Z(t) \in E, I(t) = 1\Big) = \int_E a(t,z)\,dz \quad \text{and} \quad P\Big(Z(t) \in E, I(t) = 2\Big) = \int_E b(t,z)\,dz,$$

for every measurable set $E \subseteq \mathbb{R}^N$.

The proof of the previous theorem follows exactly the same strategy that we used to prove Theorem 1.1 as it still holds that

$$\sup_{y\in\mathbb{R}^N} \left| \int_\Omega V(x,y)\chi_{A_n}(x)\phi(t,x)dx - \int_\Omega V(x,y)X(x)\phi(t,x)dx \right| \xrightarrow[n\to\infty]{} 0,$$

and

$$\sup_{x\in\overline{\Omega}}\left|\int_{\mathbb{R}^N}\left\{\chi_{A_n}(y)-X(y)\right\}V(x,y)dy\right|\xrightarrow[n\to\infty]{}0,$$

for every $V\in\{J,G,R\}$ and $\phi:\mathbb{R}^N\times[0,T]\to\mathbb{R}$ smooth.

The fact that the limits verify the Dirichlet condition

$$a(t,x)=b(t,x)=0$$

for all $x\in\overline{\Omega}^c$ and $t\geq 0$ is a consequence of the second condition in (4.2).

Acknowledgements J.C.N. supported by CAPES - INCTmat grant 465591/2014-0 (Brazil). M.C.P. partially supported by CNPq grant 303253/2017-7 (Brazil). M.C. and J.D.R. partially supported by CONICET grant PIP GI No 11220150100036CO (Argentina), PICT-2018-03183 (Argentina) and UBACyT grant 20020160100155BA (Argentina).

Data availability There is no data associated with this manuscript.

Declarations

Conflicts of interest The authors declare that there is no conflict of interest.

References

1. Andreu, F., Mazón, J.M., Rossi, J.D., Toledo, J.: Nonlocal Diffusion Problems. Mathematical Surveys and Monographs, vol. 165. American Mathematical Society; Real Sociedad Matemática Española, Providence, RI; Madrid (2010)
2. Bates, P., Chmaj, A.: An integrodifferential model for phase transitions: stationary solutions in higher dimensions. J. Stat. Phys. **95**(5–6), 1119–1139 (1999)
3. Bensoussan, A., Lions, J.L., Papanicolaou, G.: Asymptotic Analysis of Periodic Structures. North-Holland, Amsterdam (1978)
4. Billingsley, P.: Convergence of Probability Measures. Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics. John Wiley & Sons Inc., New York (1968)
5. Caffarelli, L.A., Mellet, A.: Random homogenization of fractional obstacle problems. Netw. Heterog. Media **3**(3), 523–554 (2008)
6. Cazeaux, P., Grandmont, C.: Homogenization of a multiscale viscoelastic model with nonlocal damping, application to the human lungs. Math. Models Methods Appl. Sci. **25**(6), 1125–1177 (2015)
7. Capanna, M., Nakasato, J.C., Pereira, M.C., Rossi, J.D.: Homogenization for nonlocal problems with smooth kernels. Discrete Contin. Dyn. Syst. **41**(6), 2777–2808 (2021)
8. Capanna M., Rossi, J.D.: Mixing local and nonlocal evolution equations. Preprint arXiv:2003.03407v1 (2020) (**to appear in Mediterr. J. Math.**)
9. Carrillo, C., Fife, P.: Spatial effects in discrete generation population models. J. Math. Biol. **50**(2), 161–188 (2005)
10. Chasseigne, E., Chaves, M., Rossi, J.D.: Asymptotic behavior for nonlocal diffusion equations. J. Math. Pures Appl. (9) **86**(3), 271–291 (2006)
11. Chen, Z., Wang, B.: Existence, exponential mixing and convergence of periodic measures of fractional stochastic delay reaction–diffusion equations on $\mathbb{R}^n$. J. Differ. Equ. **336**, 505–564 (2022)
12. Cioranescu, D., Donato, P.: An Introduction to Homogenization. Oxford University Press, New York (1999)
13. D'Elia, M., Du, Q., Gunzburger, M., Lehoucq, R.: Nonlocal convection–diffusion problems on bounded domains and finite-range jump processes. Comput. Methods Appl. Math. **17**(4), 707–722 (2017)
14. D'Elia, M., Perego, M., Bochev, P., Littlewood, D.: A coupling strategy for nonlocal and local diffusion models with mixed volume constraints and boundary conditions. Comput. Math. Appl. **71**(11), 2218–2230 (2016)
15. D'Elia, M., Ridzal, D., Peterson, K.J., Bochev, P., Shashkov, M.: Optimization-based mesh correction with volume and convexity constraints. J. Comput. Phys. **313**, 455–477 (2016)

16. Du, Q., Li, X.H., Lu, J., Tian, X.: A quasi-nonlocal coupling method for nonlocal and local diffusion models. SIAM J. Numer. Anal. **56**(3), 1386–1404 (2018)
17. Ethier, S.N., Kurtz, T.G.: Markov Processes, Characterization and Convergence. Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics. John Wiley & Sons Inc., New York (1986)
18. Fife, P.: Some nonclassical trends in parabolic and parabolic-like evolutions. In: Kirkilionis, M., Krömker, S., Rannacher, R., Tomi, F. (eds.) Trends in Nonlinear Analysis, pp. 153–191. Springer, Berlin (2003)
19. Fu, X., Magal, P.: Asymptotic behavior of a nonlocal advection system with two populations. J. Dyn. Differ. Equ. **34**(3), 2035–2077 (2022)
20. Gal, C.G., Warma, M.: Nonlocal transmission problems with fractional diffusion and boundary conditions on non-smooth interfaces. Commun. Partial Differ. Equ. **42**(4), 579–625 (2017)
21. Gárriz, A., Quirós, F., Rossi, J.D.: Coupling local and nonlocal evolution equations. Calc. Var. Partial Differ. Equ. **59**(4), 1–24 (2020). (**article 112**)
22. Kipnis, C., Landim, C.: Scaling Limits of Interacting Particle Systems. Grundlehren der mathematischen Wissenschaften. Springer, Berlin (1999)
23. Kriventsov, D.: Regularity for a local-nonlocal transmission problem. Arch. Ration. Mech. Anal. **217**, 1103–1195 (2015)
24. Pereira, M.C., Rossi, J.D.: An obstacle problem for nonlocal equations in perforated domains. Potential Anal. **48**(3), 361–373 (2018)
25. Pereira, M.C., Rossi, J.D.: Nonlocal problems in perforated domains. R. Soc. Edinb. Proc. A **150**(1), 305–340 (2020)
26. Pereira, M.C.: Nonlocal evolution equations in perforated domains. Math. Methods Appl. Sci. **41**, 6368–6377 (2018)
27. Schwab, R.W.: Periodic homogenization for nonlinear integro-differential equations. SIAM J. Math. Anal. **42**(6), 2652–2680 (2010)
28. Tartar, L.: The General Theory of Homogenization. A Personalized Introduction. Lecture Notes of the Unione Matematica Italiana. Springer-Verlag, Berlin (2009)
29. Videla, L., Rebolledo, R.: Evolving systems of stochastic differential equations. J. Theor. Probab. **35**(3), 1662–1705 (2022)
30. Waurick, M.: Homogenization in fractional elasticity. SIAM J. Math. Anal. **46**(2), 1551–1576 (2014)