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δ-KOSZUL ALGEBRAS

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Abstract

Let $A = A_0 \oplus A_1 \oplus A_1 \oplus \cdots$ be a graded K-algebra such that A_0 is a finite product of copies of the field K, A is generated in degrees 0 and 1, and $\dim_K A_1 < \infty$. We study those graded algebras A with the property that A_0, viewed as a graded A-module, has a graded projective resolution, $\cdots \to P^t \to \cdots \to P^1 \to P^0 \to A_0 \to 0$, such that each P^i can be generated in a single degree. The paper describes necessary and sufficient conditions for the Ext-algebra of A, $\bigoplus_{n\geq 0} \mathrm{Ext}^n_A(A_0, A_0)$, to be finitely generated. We also investigate classes of modules over such algebras and Veronese subrings of the Ext-algebra.

1 Introduction

Throughout this paper, K will denote a field. Let $A = A_0 \oplus A_1 \oplus A_2 \oplus \cdots$ be a graded K-algebra such that A_0 is isomorphic to a finite product of copies of K, $\prod_{i=1}^n K$, A is generated in degrees 0 and 1, and $\dim_K A_0 < \infty$. Since A is generated in degrees 0 and 1, it follows that A is isomorphic to a quotient of the tensor algebra $T_{A_0}(A_1) = A_0 \oplus A_1 \oplus (A_1 \otimes_{A_0} A_1) \oplus (\otimes^3_{A_0} A_1) \oplus \cdots$. There is a quiver Q such that $T_{A_0}(A_1)$ is isomorphic to the path algebra KQ [5]. Thus, $A \simeq KQ/I$ for some ideal

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I in KQ. Since A is graded, I can be generated by length homogeneous elements. Conversely, if A is isomorphic to KQ/I for some quiver Q and ideal I, with I generated by length homogeneous elements, then giving A the grading induced from the length grading on KQ, A_0 is isomorphic to a finite product of copies of K, A is generated in degrees 0 and 1, and $\dim_K A_0 < \infty$.

Viewing A_0 as a graded A-module supported in degree 0, let

$$\mathcal{P} : \cdots \to P^t \xrightarrow{d^t} P^{t-1} \to \cdots \to P^1 \xrightarrow{d^1} P^0 \to A_0 \to 0$$

be a graded resolution. Let $\mathbf{r}$ denote the graded Jacobson radical of A, namely, $\mathbf{r} = A_1 \oplus A_2 \oplus \cdots$. The resolution $\mathcal{P}$ is *minimal* if, for each $t \geq 1$, $d^t(P^t) \subset \mathbf{r}P^{t-1}$. The assumptions about the structure of A imply that minimal graded projective resolutions exist for graded A-modules. We study the class of graded rings A with the property that, for each $i \geq 0$, P^i can be generated in exactly one degree, $\delta(i)$. Included in this class are the Koszul algebras, where $\delta(i) = i$ [7, 3]. Also included in this class are the D-Koszul algebras introduced by R. Berger [4] and later studied by Green, Marcos, Martínez, and Zhang [6]. For D-Koszul algebras, there is some d such that

$$\delta(n) = \begin{cases} \frac{n}{2}\, d, & \text{for } n \text{ even}, \\ \frac{(n-1)}{2}\, d + 1, & \text{for } n \text{ odd}. \end{cases}$$

Note that a Koszul algebra is a D-Koszul algebra for $d = 2$.

We will be interested in the Ext-algebra, $E(A) = \bigoplus_{n\geq 0} \operatorname{Ext}^n_A(A_0, A_0)$, of a graded algebra A. The multiplicative structure of $E(A)$ is given by the Yoneda product. From the definition of $E(A)$, we see that $E(A)$ is a graded algebra. One of the fundamental results concerning Koszul algebras, is that a graded algebra A is a Koszul algebra if and only if $E(A)$ can be generated in degrees 0 and 1 [7]. For D-Koszul algebras, we have the following result. If $A = KQ/I$ is a graded algebra with I generated by homogenous elements of degree d, then A is a D-Koszul algebra if and only if $E(A)$ can be generated in degrees $0, 1$, and 2.

The main goal of this paper is to study graded algebras A of the form KQ/I with the following two conditions:

1) If $\mathcal{P} : \cdots \to P^t \xrightarrow{d^t} \cdots \xrightarrow{d^2} P^1 \xrightarrow{d^1} P^0 \to A_0 \to 0$ is a minimal graded resolution then, for each i, P^i can be generated in exactly one degree, $\delta(i)$.

2) The Ext-algebra, $E(A) = \bigoplus_{n\geq 0} \operatorname{Ext}^n_A(A_0, A_0)$, is finitely generated.

We call such algebras *δ-Koszul algebras.* The main result of the paper is Theorem 3.6 which provides necessary and sufficient conditions on δ for A to be a δ-Koszul algebra. At the end of this section we provide an example of a graded algebra that satisfies condition 1) but not condition 2).

The paper begins with the study of a special class of bigraded algebras in Section 2. When A is a graded algebra, then $E(A)$ is naturally bigraded. That is, each $\text{Ext}^n(A_0, A_0)$ is a direct sum of graded pieces. If A satisfies condition 1) above then, for each $n \geq 0$, $Ext^n(A_0, A_0)$, as a graded module, is supported in degree $\delta(n)$. This motivates the study of bigraded algebras, $B = \bigoplus_{i,j=1}^{n} B_{i,j}$ such that $B_{i,j} = 0$ if $j \neq \delta(i)$, for some function $\delta : \mathbf{N} \to \mathbf{N}$, where $\mathbf{N} = \{0, 1, 2, \ldots\}$. Such bigraded algebras are called *δ-determined bigraded algebras.* We introduce a new condition, which we call condition SG, on δ-determined bigraded algebras; namely whenever $\delta(i) + \delta(j) = \delta(i+j)$, $B_{i,\delta(i)} \cdot B_{j,\delta(j)} = B_{i+j,\delta(i+j)}$. The main result of Section 2 is Theorem 2.2 which provides necessary and sufficient conditions on δ, for a δ-determined bigraded algebra satifying condition SG to be finitely generated.

In Section 3 we return to our study of graded algebras satisfying condition 1). If A is such an algebra, we show that, as a bigraded algebra, $E(A)$ satisfies condition SG. This leads to Theorem 3.6 which gives necessary and sufficient conditions on δ for an algebra satisfying condition (1) to satisfy condition (2). In Section 4, we study a special class of δ-Koszul algebras and their modules which include Koszul algebras and D-Koszul algebras. We show that a subalgebra C of $E(A)$ can be regraded so that C is a Koszul algebra in this regrading. As a consequence of this result, if A is a Koszul algebra and α is a nonnegative integer, then $C = \bigoplus_{n \geq 0} \text{Ext}_A^{n\alpha}(A_0, A_0)$ is a Koszul algebra.

Section 5 investigates a class of algebras which satisfy condition (1) and there is some n such that $\delta(kn) = k\delta(n)$ for all $k \geq 1$. Let Λ be such an algebra. We show that the Veronese subring $E(\Lambda)^{[n]} = \bigoplus_{k \geq 0} \text{Ext}_\Lambda^{kn}(\Lambda_0, \Lambda_0)$ is a Koszul algebra. We also investigate Backelin's rate of such algebras.

The paper ends with the consideration of algebras satisfying condition 1) with different restrictions on δ. We obtain subalgebras of $E(A)$ which, after regrading, are Koszul algebras.

We end this section with an example of a graded algebra A which satisfies condition 1) but $E(A)$ is not finitely generated. Let $K\langle x, y\rangle$ denote the free associative algebra on two noncommuting indeterminates x and y and write $I = (a, b, c)$ for the ideal generated by a, b, c. We note that $K\langle x, y\rangle$ is isomorphic to the path algebra with quiver consisting of one vertex and two loops. Let $I = (x^2y,\ y^2x)$ and $A = K\langle x, y\rangle / I$. We note that A is a monomial algebra and the ring structure of

$E(A)$ is determined in [10]. If $\mathcal{P} : \cdots \to P^t \xrightarrow{d^t} \cdots \xrightarrow{d^2} P^1 \xrightarrow{d^1} P^0 \to A_0 \to 0$ is a minimal graded resolution then P^n is isomorphic to $A \oplus A$ for $n \geq 1$, P^0 is generated in degree 0, and P^n is generated in $2n-1$ for $n \geq 1$. It is not hard to show that $\mathrm{Ext}^r_A(A_0, A_0) \cdot \mathrm{Ext}^s_A(A_0, A_0) = 0$ for all $r, s \geq 1$ (see [10, 6]). It follows that $E(A)$ is not finitely generated.

2 Graded algebras and δ-determined bigraded algebras

Let $B = \bigoplus_{i,j\geq 0}^{\infty} B_{i,j}$ be a bigraded algebra; that is, if $b_{i,j} \in B_{i,j}$ $b_{s,t} \in B_{s,t}$ then $b_{i,j} \cdot b_{s,t} \in B_{i+s,j+t}$. Associated to a bigraded algebra B is a graded algebra $A(B) = A(B)_0 \oplus A(B)_1 \oplus \cdots$ where $A(B)_n = \bigoplus_{j=0}^{n} B_{n,j}$. We assume throughout this section that if B is a bigraded K-algebra then, for each n, $\dim_K(A(B)_n) < \infty$.

If B is a bigraded K-algebra and $\delta : \mathbf{N} \to \mathbf{N}$, we say B is *δ-determined from m* if, for $i \geq m$, $B_{i,j} = 0$ whenever $j \neq \delta(i)$. If $m = 0$ we simply say that B is *δ-determined* and if we do not need to explicitly mention m we will say that B is *eventually δ-determined*. Clearly, if B is δ-determined then $A(B) = \bigoplus_{n\geq 0}^{\infty} B_{n,\delta(n)}$.

We note that if $A = A_0 \oplus A_1 \oplus A_2 \oplus \cdots$ is a graded K-algebra then the Ext-algebra $E(A)$ inherits a natural bigrading. Furthermore, if A satisfies condition 1) of Section 1, then $E(A)$ is δ-determined. This observation motiviates studying δ-determined algebras.

In this paper a graded algebra A is called *strongly graded* if, for each i and j, if $A_iA_j \neq 0$ then $A_iA_j = A_{i+j}$. We say A is *strongly graded from m* if, for all $i, j \geq m$, if $A_iA_j \neq 0$ then $A_iA_j = A_{i+j}$. We say A is *eventually strongly graded* if it is strongly graded from m for some m.

We begin with an elementary result about graded algebras whose proof we include for completeness.

Proposition 2.1 *Let $A = A_0 \oplus A_1 \oplus \cdots$ be an eventually strongly graded algebra with $\dim_K A_i < \infty$. The following statements are equivalent.*

i) A is finitely generated.

ii) There is an integer t, such that, for all $k > t$, there exists i, $0 < i < k$, such that $A_k = A_iA_{k-i}$.

Proof. We are assuming that $\dim_K(A_i) < \infty$ for all $i \geq 0$. Thus, $A_0 \oplus A_1 \oplus \cdots \oplus A_t$ is a finite dimensional vector space. It is clear that if ii) holds, A can be generated by a K-basis of $A_0 \oplus A_1 \oplus \cdots \oplus A_t$. Thus, ii) implies i).

We now show i) implies ii). Assume i) holds and suppose that A is generated by $A_0 \oplus \cdots A_s$ for some integer s. Suppose that A is strongly graded from m. Without loss of generality, we may assume that $s \geq m$.

Let $t = 3s$. Suppose that $a \in A_k$ with $a \neq 0$ and $k > t$. Then we may write a as a sum of terms of the form $a_{i_1} \cdots a_{i_r}$ where each $a_{i_j} \in A_{u_{i_j}}$, each $0 < u_{i_j} \leq s$, and $a_{i_1} \cdots a_{i_r} \neq 0$ is an element in A_k. Consider one of these terms, say $a_1 \cdots a_r$ with $a_i \in A_{u_i}$ and $0 < u_i \leq s$. We take j such that $u_1 + \cdots + u_{j-1}$ is less than s but $u_1 + \cdots + u_j$ is greater than s. Hence, if d is the degree of $a_{i_1} \cdots a_{i_j}$, then $s > d$ and $d < k - s$. Now $a_{j_1} \cdots a_r$ is degree $k - d \geq s$ and $s \geq m$. Since $a_1 \cdots a_j, a_{j+1} \cdots a_r$ and $a_1 \cdots a_r$ are all nonzero, we conclude that $A_d A_{k-d} = A_k$ and the result follows.

□

A condition similar to being strongly graded for a δ-determined algebra B is the following.

Condition SG: For all $i, j \geq 0$ such that $\delta(i) + \delta(j) = \delta(i+j)$, $B_{i,\delta(i)} B_{j,\delta(j)} = B_{i+j,\delta(i+j)}$.

The importance of δ-determimed bigraded algebras satisfying condition SG will be shown in the next section. We will prove that if a graded algebra A satisfies condition 1) of the previous section, then $E(A)$ is a δ-determined K-algebra which satisfies condition SG.

Note that if B satisfies condition SG, then $A(B)$ is strongly graded, but, in general, the converse fails. Furthermore note that if B is a δ-determined bigraded algebra and if $\delta(i) + \delta(j) \neq \delta(i+j)$ then $B_{i,\delta(i)} \cdot B_{j,\delta(j)} = 0$.

The next result is the main result of this section which classifies when $A(B)$ is finitely generated for a bigraded algebra that satisfies condition SG.

Theorem 2.2 *Let $B = \bigoplus_{i,j\geq 0}^{\infty} B_{i,j}$ be a δ-determined bigraded algebra such that $dim_K B_{i,\delta(i)} < \infty$. Assume that B satisfies condition SG. Then the following statements are equivalent.*

i) B is finitely generated.

ii) $A(B)$ is finitely generated.

iii) There is an integer t such that, for all $k > t$, there exits i, $0 < i < t$ such that $\delta(k) = \delta(i) + \delta(k-i)$.

Proof. The equivalence of i) and ii) is clear since we are assuming that $\dim_K(A(B)_i) < \infty$ for all $i \geq 0$.

By Proposition 2.1, ii) is equivalent to

ii′) there is an integer t, such that, for all $k > t$, there exists i, $0 < i < k$, such that $A(B)_k = A(B)_i A(B)_{k-i}$.

Since we are assuming that B satisfies condition SG, it follows that B is δ-determined and that, for each $i \geq 0$, $A(B) = B_{i,\delta(i)}$. Thus, if $A(B)_i A(B)_j \neq 0$ then $\delta(i) + \delta(j) = \delta(i+j)$. But by condition SG, it follows that $A(B)_i A(B)_j \neq 0$ implies $A(B)_i A(B)_j = A(B)_{i+j}$. From these remarks, it is easy to see that ii′) and iii) are equivalent. This completes the proof. □

The following example shows that the above result is not valid if one replaces that B satisfies condition SG with B eventually satisfies condition SG. Let $\delta(i) = i$ for $i = 0$ and 1, and $\delta(i) = i+1$ for $i > 1$. Let $B = \bigoplus_{i \geq 0} B_{i,\delta(i)}$, where $B_{i,\delta(i)} = K$ for all i. For $i, j \geq 1$, we require $B_i \cdot B_j = 0$. Then B is δ-determined. We note that B satisfies condition SG from 2 vacuously. But for $t = 2$, if $k > 1$, $\delta(k) = \delta(1) + \delta(k-1)$. Hence condition iii) of Theorem 2.2 holds but B is not finitely generated.

We end this section with another result about finite generation of graded algebras.

Proposition 2.3 *Let $A = A_0 \oplus A_1 \oplus \cdots$ be a graded K-algebra with each A_i finite dimensional over K. Let d be a positive integer and consider the subalgebra $C = A_0 \oplus A_d \oplus A_{d+1} \oplus A_{d+2} \oplus \cdots$ of A. Then A is finitely generated if and only if C is finitely generated.*

Proof. Since each A_i is finite dimensional, it is clear that if C is finitely generated then so is A.

Assume that A is finitely generated by homogeneous elements $a_1, \ldots, a_t$ such that, for $1 \leq i \leq t$, $\deg(a_i) \leq m$. We may assume that $m > d$. We show that C can be generated by elements from $A_0 \oplus A_d \oplus \cdots A_{2m+d}$. Since $A_0 \oplus A_d \oplus A_{d+1} \cdots A_{2m+d}$ is finite dimensional, the result will follow.

Let x be homogeneous element in A of degree $t > 2m+d$. We will show it is a sum products of elements of smaller degree in C. Since we are assuming that A can be finitely generated by $a_1, \ldots, a_t$, we may write x as a sum of terms of the form $kb_1 \cdots b_r$ where $k \in K$ and each $b_i \in \{a_1, \ldots, a_t\}$. It suffice to show that each term $b_1 \cdots b_r$ can be written as a product elements from $A_d \oplus A_{d+1} \cdots A_{2m+d}$. Choose j such that

$b_1 \cdots b_{j-1} < m$ and $b_1 \cdots b_j \geq m$. It follows that $m \geq b_1 \cdots b_j < 2m-1$. Thus the degree of $b_{j+1} \cdots b_r$ is $t - \deg(b_1 \cdots b_j) > t-(2m-1) \geq 2m+d-(2m-1) = d+1$. Thus, both $b_1 \cdots b_j$ and $b_{j+1} \cdots b_r$ are in C and $b_1 \cdots b_j \in A_0 \oplus A_d \oplus A_{2m+d}$. If $b_{j+1} \cdots b_r \in A_0 \oplus A_d \oplus A_{2m+d}$ we are done. If not, $b_{j+1} \cdots b_r$ has degree $> 2m+d$ and we use the argument above replacing x by $b_{j+1} \cdots b_r$. Eventually we get x as a product of elements of $A_0 \oplus A_d \oplus A_{2m+d}$ and the proof is complete. □

3 δ-Koszul algebras

In this section we introduce a class of algebras which generalize both Koszul and D-Koszul algebras. Let $A = A_0 \oplus A_1 \oplus \cdots$ be a graded K-algebra such that $A_0 = \prod_{i=1}^n K$, A can be generated in degrees 0 and 1, and $\dim_K A_1 < \infty$. As mentioned in the introduction, A is isomorphic to KQ/I, where Q is a finite quiver, I is a length homogeneous ideal in the path algebra KQ contained in square of the ideal generated by the arrows of Q. The grading of A is induced from the length grading of KQ. We keep these assumptions on A throughout this section.

Let $\mathcal{P} : \cdots \to P^n \to \cdots \to P^1 \to P^0 \to A_0 \to 0$ be a minimal graded projective resolution of A_0 (viewed as a graded A-module living in degree 0). We say that A is *δ-resolution determined* if there is a function $\delta : \mathbb{N} \to \mathbb{N}$ such that, for each $n \geq 0$, P^n can be generated in degree $\delta(n)$. This is the same as condition 1) of the introduction. We note that A is a Koszul algebra if and only if A is δ-resolution determined for δ given by $\delta(n) = n$. Furthermore, A is a D-Koszul algebra if and only if A is δ-resolution determined for δ given by

$$\delta(n) = \begin{cases} \frac{n}{2}d, & \text{for } n \text{ even,} \\ \frac{(n-1)}{2}d+1, & \text{for } n \text{ odd.} \end{cases}$$

In the general graded case, the grading on A induces a bigrading on the Ext-algebra $E(A) = \bigoplus_{n\geq 0} \mathrm{Ext}_A^n(A_0, A_0)$ as follows. We note that $\mathrm{Ext}_A^n(A_0, A_0) = \mathrm{Hom}_A(P^n, A_0)$. Now P^n can be written as $\oplus_j P^n[j]$ where $P^n[j]$ is a graded projective A-module generated in degree j. Then $E(A)_{i,j} = \mathrm{Hom}_A(P^i[j], A_0)$.

Viewed as a bigraded algebra, $E(A)$ is δ-determined if and only if A is δ-resolution determined.

We begin with some elementary observations. Suppose, as usual, that $\cdots \to P^n \to \cdots \to P^1 \to P^0 \to A_0 \to 0$ is a minimal graded projective resolution of A_0. Then we always have that P^0 can be generated in degree 0 and P^1 can be generated in degree 1. It is easy to show (see [7]) that P^2 is generated in degree

d if and only if I can be generated by homogeneous elements of degree d. There are δ-resolution determined algebras with $\delta(2)$ a positive integer greater than 2; for example D-Koszul algebras [6]. On the other hand, there are restrictions on δ; for example the generalized triangle inequality given in the next proposition.

Proposition 3.1 *Let A be a δ-resolution determined algebra and suppose that $n = n_1 + n_2 + \cdots + n_k$ is a partition of n. Then $\delta(n) \geq \delta(n_1) + \delta(n_2) + \cdots + \delta(n_k)$.*

Proof. We use Corollary 3.3 of [6] which states that if M is a graded module generated in degree 0 which has a graded resolution $\cdots \to Q^1 \to Q^0 \to M \to 0$ with Q^n generated in exactly one degree t, then $t \geq \delta(n)$.

Observe that $\Omega^n(A_0) = \Omega^{n_1}(\Omega^{n-n_1}(A_0))$. We also have that $\Omega^{n-n_1}(A_0)$ is generated in degree $\delta(n-n_1)$. Using the result above and also the fact that $\Omega^{n-n_1}(A_0)$ shifted $-\delta(n-n_1)$ is generated in degree 0, we get that $\Omega^n(A_0)$ is generated in degree bigger or equal $\delta(n_1) + \delta(n-n_1)$. Hence $\delta(n) \geq \delta(n_1) + \delta(n-n_1)$, and the result follows by induction. □

This result has some interesting consequences.

Corollary 3.2 *Keeping the hypothesis of Proposition 3.1, δ is strictly increasing.*

Proof. By Proposition 3.1, $\delta(n+1) \geq \delta(n) + \delta(1) = \delta(n) + 1$. □

Corollary 3.3 *Keeping the hypothesis of Proposition 3.1, if $d = \delta(2)$, then, for $n \geq 0$,*

$$\delta(n) \geq \begin{cases} \frac{n}{2}\, d, & \text{for } n \text{ even}, \\ \frac{(n-1)}{2}\, d + 1, & \text{for } n \text{ odd}. \end{cases}$$

Proof. The result follows by an easy induction argument from Proposition 3.1. □

We now state an important result from [6] which will imply that if A is a δ-resolution determined algebra, then $E(A)$ satisfies condition SG.

Proposition 3.4 *[6] Let A be a graded algebra and $\cdots \to P^2 \to P^1 \to A_0 \to 0$ a minimal graded resolution of A_0. Suppose that P^i is finitely generated with generators in degree d_i, for $i = \alpha, \beta, \alpha+\beta$. Assume that*

$$d_{\alpha+\beta} = d_\alpha + d_\beta.$$

Then the Yoneda map $Ext_A^\alpha(A_0, A_0) \otimes_K Ext_A^\beta(A_0, A_0) \to Ext_A^{\alpha+\beta}(A_0, A_0)$ is surjective.

Thus,

$$Ext_A^{\alpha+\beta}(A_0, A_0) = Ext_A^{\alpha}(A_0, A_0) \cdot Ext_A^{\beta}(A_0, A_0)$$
$$= Ext_A^{\beta}(A_0, A_0) \cdot Ext_A^{\alpha}(A_0, A_0).$$

□

As an immediate application of the proposition, we have the following result.

Proposition 3.5 *If A is a δ-resolution determined graded algebra then $E(A)$ is a δ-determined bigraded algebra that satisfies condition SG.*

Proof. We have seen that if A is a δ-resolution determined graded algebra then $E(A)$ is a δ-determined bigraded algebra. Proposition 3.4 directly implies that $E(A)$ satisfies condition SG. □

Recalling that both Koszul algebras and D-Koszul algebras have Ext-algebras that are finitely generated, we are led to the following definition. We say a δ-resolution determined algebra A is a *δ-Koszul algebra* if $E(A) = \bigoplus_{n\geq 0} \text{Ext}_A^n(A_0, A_0)$ is finitely generated.

The following example shows that not all δ-resolution determined algebras are δ-Koszul. Let $A = K\langle x, y, z, w\rangle/(xyz, zwx)$. Then A is a monomial algebra and the minimal graded projective resolution of $A_0 = K$ is given in [9]. It is immediate to check that $\delta(0) = 0$, $\delta(1) = 1$, and, for $n \geq 2$, $\delta(n) = 2n - 1$. The multiplicative structure of the Ext-algebra of a monomial algebra is given in [10]. Using the bigrading, one sees that $\text{Ext}_A^i(A_0, A_0)\text{Ext}_A^j(A_0, A_0) = 0$ for $i, j \geq 1$. It follows that A is a δ-resolution determined algebra but $E(A)$ is not finitely generated. We note that the algebra given in the example at the end of Section 1 also has the property that A is δ-resolution determined and $E(A)$ is not finitely generated.

The next result is the main result of this section.

Theorem 3.6 *Let $A = A_0 \oplus A_1 \oplus \cdots = KQ/I$ be a graded algebra where I is an ideal generated by length homogeneous elements in KQ and the grading is induced from the length grading in KQ. Assume that A is δ-resolution determined. Then A is a δ-Koszul algebra if and only if there is some positive integer t, such that, if $k > t$, then there exists i, with $0 < i < k$, such that $\delta(i) + \delta(k - i) = \delta(k)$.*

Proof. By Proposition 3.5 that $E(A)$ is a δ-determined bigraded algebra that satisfies condition SG. The result follows from Theorem 2.2. $\square$

Note that if A is a Koszul algebra then, for each k and each $i < k$ then $\delta(k) = \delta(i) + \delta(k-i)$. On the other hand, if A is a D-Koszul algebra with $D > 2$ then, for each i, k and $0 < i < k$, $\delta(k) = \delta(i) + \delta(k-i)$ if and only if i or $k - i$ is even. This shows that, in general, if there is some positive integer t, such that, if $k > t$, then there exists i, with $0 < i < k$, such that $\delta(i) + \delta(k-i) = \delta(k)$ one must be careful how i is chosen.

4 δ-Koszul modules and applications

We keep the same assumptions on a graded K-algebra A as in the previous sections. In this section, we study δ-Koszul algebras and δ-Koszul modules. If A is a δ-Koszul algebra and M is a graded A-module, we say M is a *δ-Koszul module* if there is a graded projective resolution of M, $\cdots \to Q^n \to \cdots \to Q^1 \to Q^0 \to M \to 0$ such that, for each $n \geq 0$, Q^n can be generated in degree $\delta(n)$.

In [6], the authors prove that if A is a D-Koszul algebra, $D \geq 2$, then, after regrading, $E^{ev}(A) = \bigoplus_{n \geq 0} \operatorname{Ext}_A^{2n}(A_0, A_0)$ is a Koszul algebra. They also prove that if M is D-Koszul module, then $\bigoplus_{n \geq 0} \operatorname{Ext}_A^{2n}(M, A_0)$ is a Koszul $E^{ev}(A)$-module. In this section, we extend and generalize these results.

If $A = A_0 \oplus A_1 \oplus A_2 \oplus \cdots$ then we let $\mathbf{r} = A_1 \oplus A_2 \oplus \cdots$. Given our assumptions on A, $\mathbf{r}$ is the graded Jacobson radical of A. If M is a graded A-module, we let $\Omega^n(M)$ denote the n^{th}-syzygy of M of a minimal graded projective resolution of M. If $M = \bigoplus_{n \in \mathbf{Z}} M_n$ is a graded A-module, $M[k]$ is the graded module $\bigoplus_{n \in \mathbf{Z}} N_n$ where $N_n = M_{n-k}$.

From the remarks at the end of the previous section, we see that for a D-Koszul algebra, if n even, $\delta(k+n) = \delta(k) + \delta(n)$ for all $k \in \mathbf{N}$. The next proposition gives a generalization of a known result, for the even part of the Ext-algebra,[6].

Proposition 4.1 *Let A be a δ-Koszul algebra and assume there is an integer n such that $\delta(k+n) = \delta(k) + \delta(n)$ for all $k \in \mathbf{N}$. Let M be a δ-Koszul module. Then $\Omega^n(M)[-\delta(n)]$ and $\Omega^{n-1}(\mathbf{r}M)[-\delta(n)]$ are δ-Koszul modules.*

Proof. Since M is a δ-Koszul module, $\Omega^n(M)[-\delta(n)]$ is generated in degree 0. If $\cdots \to Q^1 \to Q^0 \to M \to 0$ is a minimal graded projective resolution of M then $\cdots \to Q^{n+1}[-\delta(n)] \to Q^n[-\delta(n)] \to \Omega^n(M)[-\delta(n)] \to 0$ is a minimal

graded projective resolution of $\Omega^n(M)[-\delta(n)]$ and $Q^{n+k}[-\delta(n)]$ is generated in degree
$\delta(n+k) - \delta(n) = \delta(n) + \delta(k) - \delta(n) = \delta(k)$. Hence we have that $\Omega^n(M)[-\delta(n)]$ is δ-koszul.

Applying Proposition 5.1 of [6] we get an exact sequence

$$(*) \qquad 0 \to \Omega^t(M) \to \Omega^t(M/\mathbf{r}M) \to \Omega^{t-1}(\mathbf{r}M) \to 0,$$

where each module is generated in degree $\delta(t)$. Thus $\Omega^{n-1}(\mathbf{r}M)$ is generated in degree $\delta(n)$. Moreover, $\Omega^t(\Omega^{n-1}(\mathbf{r}M))[-\delta(n)]$ is generated in degree $\delta(t+n) = \delta(t) + \delta(n)$. The proof now follows exactly like the proposition 5.2 of [6]. $\square$

If $C = C_0 \oplus C_1 \oplus \cdots$ is a graded K-algebra, we let $C^{[n]}$ denote the graded K-algebra $C_0 \oplus C_n \oplus C_{2n} \oplus C_{3n} \oplus \cdots$ regraded by viewing C_{kn} as the homogeneous part of $C^{[n]}$ of degree k. $C^{[n]}$ is called the n^{th}-*Veronese subring* of C. If $X = \bigoplus_{i\in\mathbf{Z}} X_i$ is graded C-module, we let $X^{[n]} = \oplus_{i\in\mathbf{Z}} Y_i$ where $Y_i = X_{in}$. Then $X^{[n]}$ is a graded $C^{[n]}$-module.

If A is a graded algebra and M is a graded A-module, we let $\mathcal{E}(M) = \bigoplus_{n\geq 0} \operatorname{Ext}_A^n(M, A_0)$. We now prove the main result of the section which extends Theorem 6.2 of [6]. The result is similar in flavor to Backelin and Froberg's result [1] that the n^{th}-Veronese subring of a Koszul algebra is Koszul except that the algebra in question, $E(A)$ is not in general generated in degrees $0, 1$ as required in Backelin's work.

Theorem 4.2 *Let A be a δ-koszul algebra and assume there is an integer n such that $\delta(n+k) = \delta(n) + \delta(k)$ for all $k \in \mathbf{N}$. Let M be a δ-Koszul module. Then $\mathcal{E}(M)^{[n]} = \bigoplus_{k\in\mathbf{N}} Ext^{nk}(M, A_0)$ (regraded) is a Koszul module, over the algebra $E(A)^{[n]} = \bigoplus_{k\geq 0} Ext_A^{nk}(A_0, A_0)$, regraded. In particular, $E(A)^{[n]}$ is a Koszul algebra, after regrading.*

Proof. From the short exact sequence $(*)$, for each t, we obtain a short exact sequence

$$0 \to \operatorname{Ext}^{t-1}(\mathbf{r}M, A_0) \to \operatorname{Ext}^t(M/\mathbf{r}M, A_0) \to \operatorname{Ext}^t(M, A_0) \to 0.$$

Thus, we have an epimorphism of $E(A)^{[n]}$-modules, $\mathcal{E}(M/\mathbf{r}M)^{[n]} \to \mathcal{E}(M)^{[n]} \to 0$ whose kernel is

$$\bigoplus_{k\geq 1} \operatorname{Ext}_A^{nk-1}(\mathbf{r}M, A_0) = \coprod_{k\geq 0} \operatorname{Ext}^{nk-n}(\Omega^{n-1}(\mathbf{r}M), A_0) = \mathcal{E}(\Omega^{n-1}(\mathbf{r}M)[-\delta(n)])^{[n]}.$$

By Proposition 4.1, $\Omega^{n-1}(\mathbf{r}M)[-\delta(n)]$ is a δ-Koszul module. Using the hypothesis that $\delta(n+k) = \delta(n) + \delta(k)$ for all $k \in \mathbf{N}$ and induction, we conclude that $\mathcal{E}(M)^{[n]}$ is a $E(A)^{[n]}$-Koszul module. Taking $M = A_0$, we see that,after regrading, $E(A)^{[n]}$ is a Koszul algebra. □

We apply the theorem to a Koszul algebra and obtain another proof of a reult of Backelin and Fröberg [1].

Corollary 4.3 *Let A be a Koszul algebra. Then $A^{[n]}$ and $E(A)^{[n]}$ are both Koszul algebras.*

Proof. For a Koszul algebra, $\delta(n) = n$ and hence, for all $k, n \in \mathbf{N}$, $\delta(n+k) = \delta(n) + \delta(k)$. By Theorem 4.2, since A_0 is a δ-Koszul A-module, $E(A)^{[n]}$ is a Koszul algebra. Since A is isomorphic to $E(E(A))$ and $E(A)$ is a Koszul algebra, we see that $A^{[n]}$ is a Koszul algebra. □

5 Veronese subrings

In this section we fix a positive integer n and a δ-resolution determined algebra Λ. We say that Λ satisfies the *n^{th}-Veronese property*, or the *n-V property*, if $\delta(nk) = k\delta(n)$ for all $k \geq 0$. A Λ-module M is called an *n-V module* if $\Omega^{nk}(M)$ is generated in degree $k\delta(n) = \delta(kn)$ for all $k \geq 0$.

Observe that a Koszul algebra satisfies the 1-V property and a D-Koszul algebra satisfies the 2-V property. In general, if Λ satisfies property n-V then Λ is δ-Koszul and, after regrading, the n^{th}-Veronese subring of $E(\Lambda)$, $E(\Lambda)^{[n]}$, is Koszul.

Proposition 5.1 *Let Λ be a δ-resolution determined algebra which satisfies the n-V property. Let M be a Λ-module that satisfies the n-V property. Then $\Omega^n(M)[-\delta(n)]$ and $\Omega^{n-1}(\mathbf{r}M)[-\delta(n)]$ are Koszul modules which satisfy the n-V property.*

Proof. Apply the proof of Proposition 4.1. □

As in Section 4, we can apply this result to obtain the following two results.

Theorem 5.2 *Let Λ be a δ-resolution determined algebra which satisfies the n^{th}-Veronese property and let M be a Λ-module which satisfies the n^{th}-Veronese property. Let $E(\Lambda)^{[n]}$ be the n^{th}-Veronese subring of $E(\Lambda)$ and $E(M)^{[n]} = \bigoplus_{k\geq 0} Ext_\Lambda^{nk}(\Lambda_0, \Lambda_0)$. Then $E(M)^{[n]}$ is a Koszul $E(\Lambda)^{[n]}$-module after regrading.* □

Corollary 5.3 *Keeping the notation of Theorem 5.2, $E(\Lambda)^{[n]}$ is a Koszul algebra.* $\square$

The next result shows that an algebra that satisfies the n-V property is a δ-Koszul algebra.

Proposition 5.4 *If Λ is a δ-resolution determined algebra that satisfies the n-V property then $E(\Lambda)$ is finitely generated.*

Proof. We have seen that δ is strictly increasing. Fix i with $0 < i < n$. Consider $A_i = \{r : 0 < r < \delta(n)$ and there is t such that $\delta(tn+i) \equiv r(\text{mod}(\delta(n))\}$. Note that for each i, A_i is a finite set. Hence, for each i, there an integer $\alpha(i)$ such that for each $t > \alpha(i)$, there is an integer s, with $s < t$, such that $\delta(tn+i) \equiv \delta(sn+i)(\text{mod}(\delta(n))$. Since Λ satisfies the n-V property and since δ is increasing, we see that $t\delta(n) < \delta(tn+i) < (t+1)\delta(n)$, and $s\delta(n) < \delta(sn+i) < (s+1)\delta(n)$. Now $\delta(tn+i) - \delta(sn+i)$ is a multiple of $\delta(n)$. The above two inequalities imply that

$$t\delta(n) - (s+1)\delta(n) < \delta(tn+i) - \delta(sn+i) < (t+1)\delta(n) - s\delta(n).$$

Thus we conclude that $\delta(tn+i) = \delta((t-s)n) + \delta(sn+i)$.

Setting $\alpha = \max\{n, \alpha(1), \ldots, \alpha(n-1)\}$ we see that if if $u > \alpha$, writing $u = tn+i$, there is some integer $s < t$ such that $\delta(u) = \delta(tn+i) = \delta((t-s)n) + \delta(sn+i)$. By Theorem 3.6, we conclude that $E(\Lambda)$ is finitely generated. $\square$

We end this section with some remarks about the growth of algebras that satisfy the n-V property. We refer the reader to [4] for both definitions and further details. Assume that Λ does not have finite global dimension. Define

$$\text{rate}(\Lambda) = \sup_{m>1} \frac{\delta(m)-1}{m-1}.$$

If $m = kn$ then

$$\frac{\delta(m)-1}{m-1} = \frac{k\delta(n)-1}{kn-1} \leq \frac{\delta(n)-1}{n-1},$$

where the last equality can be shown to true by considering the derivative of the function $\frac{x\delta(n)-1}{xn-1}$.

Next, consider $m = nt+i$, with $0 < i < n$. Then

$$\frac{\delta(m)-1}{m-1} \leq \frac{\delta(n(t+1)}{nt} = \frac{t+1}{t}\frac{\delta(n)}{n} \leq \frac{2\delta(n)}{n}.$$

Since $n \geq 2$, we have that $\frac{2\delta(n)}{n} \geq \frac{\delta(n)-1}{n-1}$. It follows that $\text{rate}(\Lambda) \geq \frac{2\delta(n)}{n}$. By Backelin's theorem [4], we conclude that the Veronese subring $\Lambda^t = \bigoplus_{k\geq 0} \Lambda_{kt}$ is Koszul if $t \geq \frac{2\delta(n)}{n}$.

If Λ is a δ-resolution determined algebra, then Λ has finite rate if and only if $\{\frac{\delta(n)-1}{n-1} \geq 2\}$ is bounded. In this case, if $t \geq \sup\{\frac{\delta(n)-1}{n-1}\}$ then $\Lambda^{[n]}$ is Koszul. In our example $\Lambda = K\langle x, y, z, w\rangle/(xyz, zwz)$, we have $\delta(n) = 2n - 1$ for $n \geq 1$. Therefore, $\text{rate}(\Lambda) = \sup\{\frac{2(n-1)}{n-2}\} = 2$. Hence the Veronese subalgebras, $\Lambda^{[d]} = \oplus_{k\geq 0}\Lambda_{kd}$ are always Koszul algebras for $d \geq 2$.

6 Examples and questions

We begin by presenting a class of δ-Koszul algebras which are neither a Koszul algebra nor a D-Koszul algebra

Let $A = KQ/I$ where I is an ideal generated by a set of paths of length $2N$ such that all overlaps are length $3N$ and there is at least one overlap occurs among the elements of I. Recall that we say path p *overlaps* path q if there is a path r such that $r = qq' = p'p$ for some paths p' and q' and length $q >$ length p'. For example, if $KQ = K\langle x, y, u, v\rangle$, the free algebra on 4 noncommuting indeterminants, and $I = (xyuv, uvxy)$. For this example, $N = 2$.

For an algebra A in this general class of algebras, A is δ-resolution determined where $\delta(0) = 0, \delta(1) = 1$, and, for $k \geq 2$, $\delta(k) = kN$. This can be seen using the results in [9]. For such a δ, for $i, j \geq 2$, $\delta(i + j) = \delta(i) + \delta(j)$. By Theorem 3.6, A is is a δ-Koszul algebra. Applying the results of [10], it can be shown that A can be generated in degrees 0,1,2, and 3 but not in degrees 0, 1, and 2. Thus, A is not a Koszul nor a D-Koszul algebra. We also can obtain this from Theorem 3.6, noting that $kN = (k - 2)N + 2N$ and, in this case, $k - 2 \geq 2$.

We end the paper with some open questions.

Question 1: For which functions $\delta : \mathbf{N} \to \mathbf{N}$ is there a δ-resolution determined algebra?

Question 2: For which functions $\delta : \mathbf{N} \to \mathbf{N}$ is there a δ-Koszul algebra?

Question 3: Is there a bound N such that if A is a δ-Koszul algebra, then the Ext-algebra, $E(A)$, is generated in degrees 0 to N.

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