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A group theoretic proof of a compactness lemma and existence of nonradial solutions for semilinear elliptic equations

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Abstract

Symmetry plays a basic role in variational problems (settled, e.g., in $\mathbb{R}^n$ or in a more general manifold), for example, to deal with the lack of compactness which naturally appears when the problem is invariant under the action of a noncompact group. In $\mathbb{R}^n$, a compactness result for invariant functions with respect to a subgroup G of $\mathrm{O}(n)$ has been proved under the condition that the G action on $\mathbb{R}^n$ is compatible, see Willem (Minimax theorem. Progress in nonlinear differential equations and their applications, vol 24, Birkhäuser Boston Inc., Boston, 1996). As a first result, we generalize this and show here that the compactness is recovered for particular subgroups of the isometry group of a Riemannian manifold. We investigate also isometric action on Hadamard manifold (M, g) proving that a large class of subgroups of $\mathrm{Iso}(M, g)$ is compatible. As an application, we get a compactness result for "invariant" functions which allows us to prove the existence of nonradial solutions for a classical scalar equation and for a nonlocal fractional equation on $\mathbb{R}^n$ for $n = 3$ and $n = 5$, improving some results known in the literature. Finally, we prove the existence of nonradial invariant functions such that a compactness result holds for some symmetric spaces of noncompact type.

Keywords Compactness lemma · Existence of nonradial solutions · Symmetric spaces

Mathematics Subject Classification 35J20 · 35J61 · 22E60

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1 Introduction

It is known that many interesting partial differential equations in $\mathbb{R}^n$ are invariant under the orthogonal group $\mathrm{O}(n)$ so that it makes sense to find solutions which respect this symmetry, i.e., they are radial. These solutions are physically interesting, and indeed, in scalar field theory they are also called *particle-like*. Particularly interesting is the case when the equations are variational, i.e., a smooth functional (called *the energy functional*) on Banach or Hilbert space X can be defined in such a way that its critical points give exactly the solutions of the equations; very often the restriction of this functional to the subspace of radial functions $X_{\mathrm{O}(n)}$ is even "natural," in the sense of the *Palais' Criticality Symmetric Principle* [12]: One roughly speaking says that critical symmetric points are symmetric critical points. The advantage of working in the subspace $X_{\mathrm{O}(n)}$ is that its elements may have additional properties which enable to recover a compactness condition (as required by many abstract theorems in Critical Point Theory) that the functional has to satisfy in order to guarantee the existence of critical symmetric points (see below for a specific problem). From a functional point of view, this compactness is a consequence of the compact embedding of Sobolev spaces into Lebesgue spaces.

A natural problem that arises is then the search of nonradial solutions for such equations, and indeed, the main difficulty is exactly to guarantee that the solutions found are effectively nonradial.

This topic has attracted much attention, and the existence of nonradial solutions has been intensively investigated by many authors.

Particularly interesting for our purpose is the work by Bartsch and Willem [3] where the authors consider the following equation

$$-\Delta u + b(|x|)u = f(|x|, u) \quad \text{in } \mathbb{R}^n,\ n \geq 3 \tag{1.1}$$

under suitable assumptions on b and f and look for nonradial solutions. The approach of the authors consists in restricting the energy functional, let us say ϕ, which is naturally defined in the Sobolev space $H^1(\mathbb{R}^n)$, to the subspace $H^1_G(\mathbb{R}^n)$ of fixed points for a suitable group action G which does not contain radial functions (except of course the null function). Roughly speaking and without entering in details here, we can say that the group G is generated by

(i) functions which are "radial in groups of variables," that is, they are invariant for the action induced by the subgroup $\mathrm{O}(m)\times\mathrm{O}(m)\times\mathrm{O}(n-2m)$ of $\mathrm{O}(n)$,

(ii) and by functions which are invariant by a "suitable action" induced by

$$\tau\cdot(x_1,x_2,x_3) = (x_2,x_1,x_3),$$

where $(x_1,x_2,x_3)\in\mathbb{R}^m\times\mathbb{R}^m\times\mathbb{R}^{n-2m}$.

Note that $\tau^2 = e$, the identity. The success of the method is based on the fact that, as proved by Lions in [10], the set of functions in $H^1(\mathbb{R}^n)$ satisfying (i) has compact embedding into L^p spaces and the same holds for $H^1_G(\mathbb{R}^n)$, that is, when also the action of τ is taken into account. As a consequence, the energy functional restricted to $H^1_G(\mathbb{R}^n)$ satisfies the *Palais–Smale condition*: any sequence $\{u_k\}\subset H^1_G(\mathbb{R}^n)$ such that

$$\{\phi(u_k)\} \text{ is convergent and } \phi'(u_k)\to 0$$

has a convergent subsequence. However, in order to consider (ii) and then guarantee that there is no nontrivial radial function in $H^1_G(\mathbb{R}^n)$, the authors assume that $2 \leq m \leq n/2$ and $2m \neq n-1$ which forces $n=4$ or $n \geq 6$.

Another remarkable paper where nonradial solutions for an elliptic equation are found by means of a similar strategy is the one by d'Avenia [4]. Here the author is interested in a so called *Schrödinger–Poisson equation* in $\mathbb{R}^3$ and he restricts the energy functional to the set of functions in $\mathbb{R}^3$ which are radial in the first two variables and even in the third one, in order to guarantee that the solution found is not radial.

Looking at the group theoretic properties used in the previous papers and motivated also by [13, Definition 1.22 p.16] where Willem gives the definition of *compatible group* which permits to have the compact embedding of "invariant" functions into L^p, we try here to generalize and understand when the compact embedding of Sobolev spaces of "nonradial" functions into Lebesgue spaces holds.

Observe that in particular from [13, Definition 1.22 p.16] it follows that compatible groups with $\mathbb{R}^n$ are $G = \mathrm{O}(n)$ and $G = \mathrm{O}(N_1) \times \cdots \times \mathrm{O}(N_k)$, where $N_j \geq 2$, $j = 1, \ldots, k$ and $\sum_{j=1}^k N_j = n$, and the compact embedding results of Lions [10] are recovered.

1.1 Main results: general statements

Motivated by the cited papers, we generalize here the construction of the group action G given in [3] and investigate isometric actions on $(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$ and on its open G-invariant unbounded subsets (being interested in compact embeddings, we will consider just unbounded subsets). What we prove is that a large class of subgroups of $\mathrm{Iso}(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$ is compatible with $\mathbb{R}^n$ according to Definition 2.1. See Propositions 2.2, 3.1 and 3.2. Here $\langle \cdot, \cdot \rangle$ denotes the canonical scalar product. Actually the more general case of Riemannian manifold is treated.

As an application, we prove the existence of nonradial solutions for the problem in (1.1) for $n=3$ and $n=5$, thus extending the result of Bartsch and Willem in [3]. To show the generality of the method and the range of applicability of our abstract results, we show two more applications to systems of elliptic equations. The first one is to the existence of nonradial solutions for a nonlocal fractional equation on $\mathbb{R}^n$ extending again to the case $n=3$ and $n=5$ a result in [5] that was stated just for the cases $n=4$ and $n \geq 6$. The second one is the existence result of nonradial solutions in [4] to the case of $\mathbb{R}^n, n \geq 3$ (actually we show a multiplicity result): With our approach, we find nonradial solutions in $\mathbb{R}^3$ (since they are radial in the first two variables and periodic in the third one) and for $n \geq 4$ the compact embedding of our working space into L^p permits to have nonradial solutions, without any periodicity assumption.

However, since the statement of these theorems would imply many preliminary details and assumptions, we prefer do not state them here but refer the reader directly to Theorems 3.11, 3.12 and 3.13 in Sect. 3.1.

After the Euclidean case, we investigate subgroups of the isometry group of an Hadamard manifold (M, g). This means (M, g) is a simply connected complete Riemannian manifold of nonpositive sectional curvature, and so for any $p \in M$, $\exp_p : T_pM \longrightarrow M$ is a diffeomorphism (see, e.g., [7, 8]). By Cartan Theorem (see, e.g., [8]) a compact group G acting isometrically on (M, g) has a fixed point: Call it p. Then, $\exp_p : T_pM \longrightarrow M$ becomes G-equivariant, where the G action on T_pM is the isotropy representation. Since by Rauch Theorem (see [6]) the exponential map increases the distance, we prove that a large class of subgroups of $\mathrm{Iso}(M, g)$ are compatible (Proposition 4.2.)

As an application of this, we get a compactness result for invariants functions on an Hadamard manifold. We also point out that if a discrete and closed group G acting isometrically on a Riemannian manifold (M, g) is such that M/G is compact, then it is compatible. Finally, we prove that the basic tool used in [3] in order to prove the existence of a group action such that its fixed points have nonradial functions holds for some symmetric spaces of noncompact type. Roughly speaking, if $M = \mathrm{SL}(2n, \mathbb{R})/\mathrm{SO}(2n)$, or M is a the dual of the complex, real and quaternion projective space (see [8], then), we determine a compact subgroup $\widetilde{H}$ of the isometry group containing a closed subgroup of dimension bigger than one and of index two. This allows us to define an isometric action of $\widetilde{H}$ on $H^1(M)$ such that the invariant functions are not radial unless $u = 0$ and the embedding of the fixed points set with respect to $\widetilde{H}$ into $L^p(M)$ with $2 < p < 2^*$ is compact.

The paper is organized as follows.

In Sect. 2, we recall some facts on isometric actions on Riemannian manifolds and give the definition of compatible group. The main result here is the general Proposition 2.2.

In Sect. 3, we consider the special case of $\mathbb{R}^n$ where we give applications of our method to the existence of nonradial solutions for partial differential equations. A fundamental tool in order to show the main results, Theorems 3.11, 3.12 and 3.13, is the technical Lemma 3.7.

In Sect. 4, we consider the case of Hadamard manifold and in Sect. 5 the case of symmetric spaces of noncompact type.

2 Isometric actions on Riemannian manifolds

Let (M, g) be a connected Riemannian manifold. We may introduce the distance on M via the notion of length of curves, that we denote by d, and the topology of (M, d) coincides with the manifold topology, see, e.g., [6]. Let $\mathrm{Iso}(M, g)$ the group of isometries of (M, g). It is well known that any closed subgroup $G \subset \mathrm{Iso}(M, g)$ is a Lie group with respect to the compact open topology. In particular, $\mathrm{Iso}(M, g)$ is a Lie group, and if M is compact, then $\mathrm{Iso}(M, g)$ is compact as well, see [9]. Moreover, the map

$$G \times M \longrightarrow M, \qquad (f, p) \mapsto f(p),$$

is differentiable and so defines a differential action on M which is a proper action. This means that the map

$$G \times M \longrightarrow M \times M, \qquad (f, p) \mapsto (p, f(p)),$$

is proper. By a well-known results, it follows that for any $p \in M$, the isotropy subgroup

$$G_p = \{g \in G : \ gp = p\} \subset G$$

is compact, the orbit throughout p, i.e.,

$$G(p) = \{gp : \ g \in G\},$$

is a closed embedded submanifold of M, and the Slice Theorem and The Principal Orbit theorem hold. For these facts, we refer the reader, e.g., to [1].

Denote $\mathcal{B}_r(p) = \{q \in M : d(p, q) < r\}$, respectively, $\mathcal{S}_r(p) = \{q \in M : d(p, q) = r\}$, the open ball of center p and radius r, respectively, the sphere of center p and radius r.

A major role in what follows will be played by the next definition.

Definition 2.1 Let $G \subset \mathrm{Iso}(M, g)$ be a closed subgroup. For $y \in M$ and $r > 0$, we define

$$\mathrm{m}(y, r, G) = \sup\{n \in \mathbb{N} : \exists g_1, \dots, g_n \in G : j \neq k \Rightarrow \mathcal{B}_r(g_j y) \cap \mathcal{B}_r(g_k y) = \emptyset\}.$$

Let $\Omega \subset M$ be an open G-invariant unbounded subset. We say that Ω is compatible with G (or that G is compatible with Ω) if there exist $r > 0$ and $p \in M$ such that

$$\lim_{\substack{d(p,y) \to +\infty \\ d(\Omega, y) \leq r}} m(y, r, G) = +\infty.$$

Let $q \in M$. Since $d(q, y) \mapsto +\infty$ if and only if $d(p, y) \to +\infty$, the above definition does not depend on $p \in M$. Moreover, if $G \subset K$, then $\mathrm{m}(y, r, G) \leq \mathrm{m}(y, r, K)$. Hence, if Ω is compatible with G, then it is compatible with K.

Proposition 2.2 *Let* (M, g) *be a connected noncompact Riemannian manifold, and let* $G \subset \mathrm{Iso}(M, g)$ *be a closed and discrete subgroup with infinite elements. Then, there exist* $r > 0$ *and a G-invariant open unbounded domain* Ω *of* (M, g) *such that* $\mathrm{m}(z, r, G) = +\infty$ *for any* $z \in \Omega$. *Moreover, if M/G is compact, then there exists* $r > 0$ *such that* $\mathrm{m}(z, r, G) = +\infty$ *for any* $z \in M$ *and so any open G-invariant unbounded domain of M is compatible with G.*

Proof Let $p \in M$ be such that $G(p)$ is a principal orbit. We claim that $G_p = \{p\}$. Indeed, since the orbit throughout $p \in M$ is a discrete set, it follows that the slice representation coincides with the isotropy representation, i.e.,

$$\rho : G_p \longrightarrow \mathrm{O}(T_p M), \qquad g \mapsto \mathrm{d}g_p.$$

Since $G(p)$ is a principal orbit, we have that ρ is trivial, and so if $g \in G_p$, then $g(p) = p$ and $\mathrm{d}\, g_p = \mathrm{Id}$. This implies that $g = \mathrm{Id}_M$, see [6].

Applying the Slice Theorem, there exists a G-invariant neighborhood U of p such that for any $q \in U$, $G_q = \{e\}$ and

$$G(q) \cap U = G_p(q) = q.$$

We may assume that $U = \mathcal{B}_r(p) = \{q \in M : d(p, q) < r\}$ for some $r > 0$. We claim that for any $r' < r$ and for any $q \in \mathcal{B}_r(p)$, we have

$$d(gq, hq) > r',$$

whenever $g \neq h$. Indeed, otherwise there exists $g, h \in G$ such that

$$r > r' \geq d(gp, hp) = d(p, g^{-1}hp).$$

Hence, $g^{-1}h = \theta \in G_p = \{Id\}$ and so $h = g$. A contradiction.

Now, let any $p \in M$. Since G_p is compact, it follows that the cardinality of G_p is finite. Applying the Slice Theorem, there exists $r > 0$ such that for any $q \in \mathcal{B}_r(p)$ we have

$$G(q) \cap \mathcal{B}_r(p) = G_p(q),$$

and $G_q \subset G_p$. Therefore, if $gG_p \neq hG_p$, we have

$$d(gq, hq) > r',$$

for any $q \in \mathcal{B}_r(p)$ and for any $r' < r$. Indeed, as before, assume that there exists $g, h \in G$, such that $gG_p \neq hG_p$ and $d(gq, hq) \leq r'$. Therefore, $d(h^{-1}gq, q) \leq r' < r$ and so $h^{-1}g \in G_p$. A contradiction.

Now, let $p \in M$. Given $\epsilon > 0$, we denote by $B_\epsilon = \{z \in T_pM : \| z \| < \epsilon\}$ and by $S_\epsilon = \{z \in T_pM : \| z \| = \epsilon\}$, where $\| \cdot \| = \sqrt{g(p)(\cdot, \cdot)}$, the ball of radius r, respectively, the sphere of radius r, in T_pM. Let $\epsilon > 0$ such that $\exp_p : B_\epsilon \longrightarrow \mathcal{B}_\epsilon(p)$ is a diffeomorphism onto, see [6]. Let $\alpha < \epsilon$, and let $q \in S_\alpha(p) = \exp(S_\alpha)$. We have proved that there exists $r(q) > 0$ such that for any $z \in \mathcal{B}_{r(q)}(q)$, we have

$$d(gz, hz) > r(q),$$

whenever $gG_q \neq hG_q$. Since $\mathcal{S}_\alpha(p)$ is compact, there exists $q_1 \ldots, q_m \in \mathcal{S}_\alpha(p)$ and $r_1(q_1), \ldots r_m(q) > 0$ such that

$$\mathcal{S}_\alpha(r) \subset \mathcal{B}_{r_1(q)}(q_1) \bigcup \cdots \bigcup \mathcal{B}_{r_m(q)}(q_m)$$

, and for any $z \in \mathcal{B}_{r_j(q)}(q_j)$ we have

$$d(gz, hz) > r(q_j).$$

whenever $gG_{q_j} \neq hG_{q_j}$, for $j = 1, \ldots, m$. Applying the triangle inequality, we have $\mathcal{B}_{\frac{r}{2}}(gz) \cap \mathcal{B}_{\frac{r}{2}}(hz) = \emptyset$ for any $g, h \in G$ such that $gG_{g_j} \neq hG_{g_j}$ and so for infinite $g, h \in G$. Therefore, $\mathrm{m}(z, r/2, G) = +\infty$. This holds for any $z \in \mathcal{B}_{r_1(q)}(q_1) \bigcup \cdots \bigcup \mathcal{B}_{r_m(q)}(q_m)$.

Let $\Omega = G\big(\mathcal{B}_{r_1(q)}(q_1) \bigcup \cdots \bigcup \mathcal{B}_{r_m(q)}(q_m)\big)$. Then, Ω is a G-invariant unbounded domain of (M, g), i.e., $d : \Omega \times \Omega \longrightarrow \mathbb{R}$ is not bounded above, and finally, for any $z \in \Omega$, we have $\mathrm{m}(z, r/2, G) = +\infty$.

For the second part, assume that M/G is compact. Let $p \in M$. We have proved that there exists $r(p) > 0$ such that for every $q \in \mathcal{B}_r(p)$, we have

$$G(q) \cap \mathcal{B}_{r(p)}(p) = G_p(q),$$

and $d(gq, hq) > r$ whenever $gG_p \neq hG_q$. Therefore, $\Omega_p = G\mathcal{B}_{r(p)}(p)$ is an open G-invariant unbounded domain of M such that for any $z \in \Omega$, we have

$$d(gz, hz) > r,$$

for infinite $g, h \in G$. Since

$$M/G = \bigcup_{p \in M} \Omega_p/G,$$

and M/G is compact, there exist $p_1, \ldots, p_m \in M$ such that

$$M = \Omega_{p_1} \cup \cdots \cup \Omega_{p_m}.$$

Pick $r = \frac{1}{2}\min\{r(p_1), \ldots, r(p_m)\}$. Then, for every $z \in M$, we have

$$\mathrm{m}(z, r, G) = +\infty$$

which conclude the proof. □

Remark 2.3 If (M, g) is compact, then $\mathrm{Iso}(M, g)$ is compact and so any closed and discrete subgroup $G \subset \mathrm{Iso}(M, g)$ is finite. This means that the condition (M, g) is noncompact; it is not a technical condition.

3 The case of $\mathbb{R}^n$ and application to PDEs

Now, we investigate isometric action on $\mathbb{R}^n$ endowed by the canonical scalar product $\langle \cdot, \cdot \rangle$. In the sequel, we continue to denote by $d(x, y) = \| x - y \| = \sqrt{\langle x - y, x - y \rangle}$, i.e., the distance induced by $\langle \cdot, \cdot \rangle$ and by $B_r(p) = \{z \in \mathbb{R}^n : \| z - p \| < r\}$ the Euclidian open ball.

Given $\Omega \subset \mathbb{R}^n$, let $H^1(\Omega)$ be the usual Sobolev space and $H^1_0(\Omega)$ be the closure of $C^\infty_0(\Omega)$ in $H^1(\Omega)$.

Proposition 3.1 *Let G be a discrete nonfinite subgroup of* $\mathrm{Iso}(\mathbb{R}^n, \langle \cdot, \cdot \rangle) = \mathrm{O}(n) \ltimes \mathbb{R}^n$ *contained in* $\mathbb{R}^n$. *Then, there exists* $r > 0$ *such that* $\mathrm{m}(z, r, G) = +\infty$ *for any* $z \in \mathbb{R}^n$.

Proof Since $G \subset \mathbb{R}^n$ is discrete, there exists $\alpha > 0$ such that $G \cap B_\alpha(0) = \{0\}$. Let $g, h \in G$. Then, $gz = z + t(g)$ and $hz = z + t(h)$, respectively, and so

$$d(gz, hz) = \| t(g) - t(h) \| \geq \alpha.$$

Pick $r = \frac{1}{4}\alpha$. Then, for every $z \in \mathbb{R}^n$ and for every $g, h \in G$, we have

$$B_r(gz) \cap B_r(hz) = \emptyset,$$

whenever $g \neq h$ concluding the proof. □

Now, assume that G is a connected and closed subgroup of $\mathrm{Iso}(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$ of dimension bigger than one. Let $(\mathbb{R}^n)^G = \{p \in \mathbb{R}^n : G(p) = p\}$ be the fixed point set. It is well known that $(\mathbb{R}^n)^G$ is totally geodesic, see [1]. Hence, if it was not empty, then it would be a closed subspace. In the sequel, we assume that $(\mathbb{R}^n)^G$ is empty or is reduced to $\{0\}$. In particular, for any nonzero vector $y \in \mathbb{R}^n$, the orbit $G(y)$ has dimension at least one

Let $y \in \mathbb{R}^n \setminus \{0\}$ and let $\lambda \in \mathbb{R}$. Let $g \in G$ be such that $g \notin G_y$. Then, $g(q) = Aq + v$ for some $A \in \mathrm{O}(n)$ and some $v \in \mathbb{R}^n$. Assume firstly that $Ay \neq y$. Then,

$$\lim_{\lambda \mapsto +\infty} d(\lambda y, \lambda Ay + v)^2 = \lambda^2 \| Ay - y \|^2 - 2\lambda \langle Ay - y, v \rangle + \| v \|^2 = \infty.$$

Let $r > 0$, and let $n \in \mathbb{N}$. Let $\lambda_o > 0$ be such that for any $\lambda > \lambda_o$ we have $d(\lambda y, A\lambda y + v)^2 > (2nr)^2$. Since $G(\lambda y)$ is a closed submanifold, it follows $(G(\lambda y), \langle \cdot, \cdot \rangle)$ is a complete Riemannian manifold. By a Theorem of Hopf–Rinow [6], there exists at least one minimizing geodesic $\gamma : [0, l] \longrightarrow G(\lambda y)$ parameterized with the arc length and satisfying $\gamma(0) = \lambda y$, $\gamma(l) = g(\lambda y)$ and $L(\gamma) = l = d^{G(\lambda y)}(g(\lambda y), \lambda y) \geq d(\lambda y, g(\lambda y)) > 2nr$. Let

$$f : [0, l] \longrightarrow [0, d(\lambda y, g(\lambda y))], \qquad f(t) = d(\lambda y, \gamma(t)).$$

Then, f is continuous and surjective since the image is connected and it contains 0 and $d(\lambda y, g(\lambda y))$. Pick

$$t_1 = \sup\{t \in [0, l] : d(\lambda y, \gamma(t)) = 2r\}.$$

Then, $0 < t_1 < d(\lambda y, g(\lambda y))$, and for any $t \geq t_1$, we have $f(t) = d(\lambda y, \gamma(t)) > 2r$. Indeed, assume that there exists $t' > t_1$ such that $f(t') < 2r$. Since $f(l) = d(\lambda y, g(\lambda y)) > 2r$, it follows that there exists $t'' > t' > t_1$ such that $f(t'') = 2r$ which is a contradiction. Moreover, since

$$d(\lambda y, g(\lambda y)) \leq d(\lambda y, \gamma(t_1)) + d(\gamma(t_1), g(\lambda y)),$$

it follows

$$d(\gamma(t_1), g(\lambda y)) \geq d(\lambda y, g(\lambda y)) - 2r \geq 2(n-1)r.$$

Therefore, we are able to iterate this procedure at least $n-1$ times, proving that there exists $0 = t_0 < t_1 < \ldots < t_{n-1}$ such that

$$d(\gamma(t_i), \gamma(t_j)) \geq 2r,$$

whenever $i \neq j$. This implies, again applying the triangle inequality, $B_r(\gamma(t_i)) \cap B_r(\gamma(t_j)) = \emptyset$ for any $i \neq j$. Since γ is a minimizing geodesic, denoting $\gamma(t_i) = g_i y$ for $i = 0, \ldots, n-1$, it follows $\gamma(t_i) = g_i \lambda y \neq \gamma(t_j) = g_j \lambda y$ whenever $i \neq j$ and so $g_i \neq g_j$ whenever $i \neq j$. This proves

$$\mathrm{m}(\lambda y, r, G) \geq n.$$

Since it holds for any $n \in \mathbb{N}$, we get

$$\lim_{\|y\| + \infty} \mathrm{m}(y, r, G) = +\infty.$$

If $Ay = y$, then for any $\lambda \in \mathbb{R}$, we have

$$d(\lambda y, g^m(\lambda y))^2 = m \parallel v \parallel^2,$$

and so

$$\lim_{m \mapsto +\infty} d(\lambda y, g^m(\lambda y))^2 = +\infty,$$

for any $\lambda \in \mathbb{R}$. Hence, the above idea works as well and so

$$\lim_{\|y\| + \infty} \mathrm{m}(y, r, G) = +\infty.$$

Summing up, keeping in mind that $m(y, r, K) \leq m(y, r, G)$ whenever $K \subset G$, we have proved the following result.

Proposition 3.2 *Let $G \subset \mathrm{Iso}(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$ be a closed subgroup. Let G^o be the connected component containing e. Assume that G^o has dimension bigger than one and $(\mathbb{R}^n)^{G^o} = \{0\}$ or is empty. Then, for any $r > 0$ we have*

$$\lim_{\|y\| \mapsto +\infty} \mathrm{m}(y, r, G) = +\infty.$$

Therefore, any open G-invariant unbounded subset of $\mathbb{R}^n$ is compatible with G.

Remark 3.3 If G is connected and $(\mathbb{R}^n)^G$ is a closed nontrivial subspace, then the G action on $\mathbb{R}^n$ is not compatible. Indeed, if $v \in (\mathbb{R}^n)^G$ is nonzero, then $\mathrm{m}(\lambda v, r, G) = 1$ for any $r > 0$ and for any nonzero $\lambda \in \mathbb{R}$ and so G is not compatible.

Corollary 3.4 *Let G_1 be a closed subgroup of* $\mathrm{Iso}(\mathbb{R}^n, \langle\cdot,\cdot\rangle)$, *and let G_2 be a closed subgroup of* $\mathrm{Iso}(\mathbb{R}^m, \langle\cdot,\cdot\rangle)$. *Assume that both G_1 and G_2 satisfy the conditions in Proposition 3.2 or in Proposition 3.1. Then, any open $G_1 \times G_2$-invariant unbounded subset of $\mathbb{R}^{n+m}$ is compatible with respect to $G_1 \times G_2$.*

We state now the main result in the case of the Euclidean space.

Let $G \subset \mathrm{Iso}(\mathbb{R}^n, \langle\cdot,\cdot\rangle)$ and let $\Omega \subset \mathbb{R}^n$ any G-invariant open and unbounded subset of $\mathbb{R}^n$. Let $H^1(\Omega)$ and $H_0^1(\Omega)$ be the usual Sobolev spaces.

The action of G on $H_0^1(\Omega)$ is defined by

$$gu(x) := u(g^{-1}x), \quad g \in G,$$

and the subspace of fixed point for this action is

$$H^1_{0,G}(\Omega) := \left\{u \in H_0^1(\Omega) : gu = u, \ \forall g \in G\right\}.$$

Theorem 3.5 *Let $G \subset \mathrm{Iso}(\mathbb{R}^n, \langle\cdot,\cdot\rangle)$ be a closed subgroup. Let Ω be an unbounded open G-invariant subset of $\mathbb{R}^n$. Assume that one of the following conditions hold*:

(1) *G^o has dimension bigger than one and $(\mathbb{R}^n)^{G^o} = \{0\}$ or is empty*;
(2) *G is a discrete nonfinite closed subgroup of the translation group.*
(3) *$G = G_1 \times G_2 \subset \mathrm{Iso}(\mathbb{R}^{n+m}, \langle\cdot,\cdot\rangle)$ where $G_1 \subset \mathrm{Iso}(\mathbb{R}^n, \langle\cdot,\cdot\rangle)$, $G_2 \subset \mathrm{Iso}(\mathbb{R}^m, \langle\cdot,\cdot\rangle)$ and G_1 and G_2 satisfy condition* 1 *or* 2.

Then, the following embedding

$$H^1_{0,G}(\Omega) \hookrightarrow L^p(\Omega), \ 2 < p < 2^*,$$

is compact.

Proof Applying Proposition 3.2, Proposition 3.1 and Corollary 3.4, the proof follows like in [13, p.16-17]. □

Remark 3.6 If the condition $(\mathbb{R}^n)^{G^o} = \{0\}$ or empty is not satisfied, then $H^1_{0,G}(\Omega) \hookrightarrow L^p(\Omega)$, $2 < p < 2^*$, is not compact in general. Indeed, consider in $H^1(\mathbb{R}^3)$ the action of $G = O(2) \times \mathbb{Z}_2$. The fixed points for this action are the functions which are radial in the first two variables and odd in the third one, and $(\mathbb{R}^n)^{G^o} = \mathbb{R}$. However, they do not have compact imbedding into $L^p(\mathbb{R}^3)$, $2 < p < 6$ due to the invariance by translations in the third variable. A counterexample can be constructed in this way. Let $\phi, \psi \in C_0^\infty(\mathbb{R}) \setminus \{0\}$ with ψ odd and $\operatorname{supp} \psi = [-2,-1] \cup [1,2]$. Define for any $n \geq 1$:

$$u_n(x_1,x_2,x_3) = \begin{cases} \phi(x_1^2 + x_2^2)\psi(x_3 - n) & \text{if } x_3 \geq n, \\ 0 & \text{if } x_3 \in [0,n], \end{cases}$$

and let $\widetilde{u}_n$ be its oddness extension on $x_3 \leq 0$. Observe that $\widetilde{u}_n \rightharpoonup 0$ in $H^1(\mathbb{R}^3)$, but

$$\int_{\mathbb{R}^3} |\widetilde{u}_n|^p dx = \int_{\mathbb{R}^3} |\phi(x_1^2 + x_2^2)|^p |\psi(x_3)|^p dx =: c > 0$$

(with c independent on n) and so $\{\widetilde{u}_n\}$ does not converge to 0 strongly into $L^p(\mathbb{R}^3)$.

Our aim is to apply the previous results to the existence of nonradial solutions for some partial differential equations in $\mathbb{R}^n$. Since these equations are invariant for $\mathrm{O}(n)$, the main difficulty is exactly to exclude that the solutions found are radial. In order to do that, we need to consider the action of a suitable group in such a way that the subspace of the fixed point for this action has not radial functions (except of course the zero function). The next abstract result will be fundamental in order to construct this group.

Let G be a Lie group and let $H \subset G$ be a closed subgroup and let $\mathrm{N}(H) = \{g \in G : gHg^{-1} = H\}$.

The following lemma is easy to check, but for the sake of completeness we give the proof.

Lemma 3.7 *Assume that there exists $\tau \in \mathrm{N}(H)$ such that $\tau \notin H$ and $\tau^2 = e$. Then, the subgroup $\widetilde{H}$ generated by H and τ is closed, and so it is a Lie group, and H is a normal subgroup of index two. In particular, there exists a surjective homomorphism*

$$\rho : \widetilde{H} \longrightarrow \mathbb{Z}_2,$$

such that $\rho(H) = 1$ and $\rho(\tau) = -1$.

Proof It is easy to check that $\widetilde{H} = \{g\tau^i : g \in H, i = 0, 1\}$ and so H is a normal subgroup of $\widetilde{H}$ of index 2. In particular, the natural projection

$$\rho : \widetilde{H} \longrightarrow \widetilde{H}/H \cong \mathbb{Z}_2, \tag{3.1}$$

defines an homeomorphism satisfying $\rho(H) = 1$ and $\rho(\tau) = -1$. Finally, we prove that $\widetilde{H}$ is closed, and so it is a Lie group.

Let $\{h_n\} \subset \widetilde{H}$ be a sequence converging to some $h_0 \in G$. Up to subsequence, we may assume that $\rho(h_n) = 1$ or $\rho(h_n) = -1$. This means that $h_n \in H$ or $h_n = s_n\tau$, where $s_n \in H$ and so $h_0 \in \widetilde{H}$. □

To extend some results on existence of nonradial solutions for partial differential equations in $\mathbb{R}^n$, especially in the cases $n = 3$ and $n = 5$ (see the Introduction), the next two examples will be useful. They can be seen as an application of Lemma 3.7 and Lemma 5.1, to which the reader is explicitly referred. However, this last lemma will be proved in a more general situation in Sect. 5.

Example 3.8 Let $H = \mathrm{SO}(3) \times \mathrm{SO}(2) \subset \mathrm{O}(5)$ as follows

$$(A, B) \mapsto \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

Let $\tau : \mathbb{R}^5 \longrightarrow \mathbb{R}^5$ given by

$$\tau(x_1, x_2, x_3, x_4, x_5) = (-x_1, -x_2, -x_3, x_4, x_5).$$

Then, τ is an isometry satisfying $\tau^2 = \mathrm{Id}$ and $\tau H \tau = H$. By Lemma 3.7, the subgroup $\widetilde{H}$ generated by H and τ is closed and there exists a surjective homomorphism $\rho : \widetilde{H} \longrightarrow \mathbb{Z}_2$

such that $\rho(H) = 1$. Note that $\widetilde{H} = \mathrm{O}(3) \times \mathrm{SO}(2)$. By Lemma 5.1 (note that H is compatible by Theorem 3.5), the action

$$\widetilde{H} \times H^1(\mathbb{R}^5) \longrightarrow H^1(\mathbb{R}^5), \qquad (g, u) \mapsto \rho(g)u(g^{-1}\cdot)$$

is isometric, the embedding

$$H^1_{\widetilde{H}}(\mathbb{R}^5) \hookrightarrow L^p(\mathbb{R}^5),\ 2 < p < 2^*,$$

is compact, and finally, $H^1_{\widetilde{H}}(\mathbb{R}^5)$ does not contains radial functions, except the null one.

Example 3.9 Let $H = \mathrm{SO}(2) \times \mathbb{Z} \subset \mathrm{Iso}(\mathbb{R}^3, \langle \cdot, \cdot \rangle)$ being

$$(A, n) \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} + \begin{bmatrix} 0 \\ 0 \\ n \end{bmatrix}.$$

Pick $\tau : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ given by

$$\tau(x_1, x_2, x_3) = (-x_1, x_2, x_3).$$

It is easy to check $\tau^2 = \mathrm{Id}$ and $\tau H \tau = H$. Denote by $\widetilde{H}$ the subgroup generated by H and τ. Applying Lemmas 3.7 and 5.1 (note again that H is compatible by Theorem 3.5) the action

$$\widetilde{H} \times H^1(\mathbb{R}^3) \longrightarrow H^1(\mathbb{R}^3), \qquad (g, u) \mapsto \rho(g)u(g^{-1}\cdot)$$

is isometric, the embedding

$$H^1_{\widetilde{H}}(\mathbb{R}^3) \hookrightarrow L^p(\mathbb{R}^3),\ 2 < p < 2^*,$$

is compact, and finally, $H^1_{\widetilde{H}}(\mathbb{R}^3)$ does not contains radial functions, except of course the null function. Observe that the functions in $H^1_{\widetilde{H}}(\mathbb{R}^3)$ are periodic in the third variable.

Remark 3.10 Of course, the examples above are easily generalized to the fractional Sobolev space $H^s(\mathbb{R}^n)$ where $n \in \{3, 5\}$ and 2^* is replaced by the critical exponent $2^*_s = 2n/(n - 2s), n > 2s$.

3.1 Existence of nonradial solutions for some elliptic PDEs

We show now how our method permits to obtain multiplicity results of nonradial solutions for some scalar field equations which enjoy the radial symmetry. Known result is also recovered. Just to give an idea of how our method works, as we anticipated in Introduction, we limit ourselves to a few particular equations. Of course, many other equations can be treated with the same approach.

3.1.1 A classical scalar field equation

In particular, we are able to find nonradial solutions for an elliptic equation in the whole space $\mathbb{R}^n, n \geq 3$. Consider the problem

$$-\Delta u + b(|x|)u = f(|x|, u) \quad \text{in } \mathbb{R}^n,\ n \geq 3 \tag{3.2}$$

under the assumptions

(1) $b \in C([0, +\infty, \mathbb{R})$ is bounded from below by a positive constant a_0.
(2) $f \in C([0, +\infty) \times \mathbb{R}, \mathbb{R})$ and there are positive constants a_1, R and a constant $1 < q < (n+2)/(n-2)$ such that

$$|f(r, u)| \leq a_1 |u|^q \quad \text{for any } r \geq 0,\ |u| \geq R.$$

(3) There exists $\mu > 2$ such that

$$\mu F(r, u) := \mu \int_0^u f(r, v)dv \leq uf(r, u) \quad \text{for any } r \geq 0,\ u \in \mathbb{R}.$$

(4) There exists $K > 0$ such that $\inf_{r>0,|u|=K} F(r, u) > 0$.
(5) $f(r, u) = o(|u|)$ for $u \to 0$ uniformly in $r \geq 0$.
(6) f is odd in $u : f(r, -u) = -f(r, u)$ for any $r \geq 0, u \in \mathbb{R}$.

The above problem has been considered in $\mathbb{R}^n, n = 4$ or $n \geq 6$, in [3] which proved the existence of infinitely many solutions in $H^1(\mathbb{R}^n)$ which are not radial.

We are now able to achieve the same conclusion in $\mathbb{R}^n$ for any $n \geq 4$, extending [3, Theorem 2.1].

Theorem 3.11 *Under the previous assumptions, problem* (3.2) *possesses an unbounded sequence of solutions* $\{\pm u_k\}_{k\in\mathbb{N}}$ *in* $\mathbb{R}^n, n \geq 4$ *which are nonradial.*

It remains only to show the case $n = 5$. Let

$$\phi(u) = \frac{1}{2}\int_{\mathbb{R}^5} |\nabla u|^2 + \frac{1}{2}\int_{\mathbb{R}^{n5}} b(|x|)u^2 - \int_{\mathbb{R}^5} F(|x|, u)$$

be the energy functional associated with problem (3.2) which is well defined and C^1 on the subspace of $H^1(\mathbb{R}^5)$

$$X = \left\{ u \in H^1(\mathbb{R}^5) : \int_{\mathbb{R}^5} b(|x|)u^2 < +\infty \right\}.$$

This space is continuously embedded into $H^1(\mathbb{R}^5)$, then into $L^p(\mathbb{R}^5)$, but not compactly. However, for what we have seen before (see Example 3.8) the closed and infinite-dimensional space of fixed points of X for the group $\widetilde{H}$, $X_{\widetilde{H}}$, has compact embedding into $L^p(\mathbb{R}^5), p \in (2, 2^*)$, and then, being the functional ϕ invariant under the action of $\widetilde{H}$ (which is important to apply the Palais' Principle) the conclusion follows exactly as in [3].

As a consequence of Example 3.9 we have also that the problem in $\mathbb{R}^3$ has infinitely many solutions which are radially symmetric in the first two variables and periodic in the third one, hence nonradial in $\mathbb{R}^3$.

Of course, the fact that the above equation has infinitely many radial solutions in $\mathbb{R}^n, n \geq 3$ is well known and obtained by taking advantage of the compact embedding of the radial functions.

3.1.2 A nonlocal fractional scalar field equation

The above argument also works for the following system of fractional elliptic equations.

Given $\omega > 0$, $n \geq 4$, $\alpha \in (0, n)$, $s \in (0, 1)$ and $1 + \alpha/n < p < (n + \alpha)/(n - 2s)$, consider the problem

$$\begin{cases} (-\Delta)^s u + \omega u = \varphi |u|^{p-2} u, & \text{in } \mathbb{R}^n \\ (-\Delta)^{\alpha/2} \varphi = \gamma(\alpha)|u|^p, & \text{in } \mathbb{R}^n \end{cases} \tag{3.3}$$

where $\gamma(\alpha) := \frac{\pi^{n/2} 2^\alpha \Gamma(\alpha/2)}{\Gamma(n/2 - \alpha/2)}$, Γ is the gamma function and the unknowns u, φ are found in the fractional Sobolev spaces

$$u \in H^s(\mathbb{R}^n), \quad \varphi \in \dot{H}^{\alpha/2}(\mathbb{R}^n).$$

The search of solutions for such a problem is reduced to find critical points $u \in H^s(\mathbb{R}^n)$ of the following C^1 functional

$$E_\omega(u) = \int_{\mathbb{R}^n} |(-\Delta)^{s/2} u|^2 + \frac{\omega}{2} \int_{\mathbb{R}^n} u^2 - \frac{1}{2p} \int_{\mathbb{R}^n} \varphi_u |u|^p, \quad \text{where } \varphi_u = \frac{1}{|\cdot|^{n-\alpha}} * |u|^p.$$

Then, in the following we will speak also of “solution u” of (3.3) since φ_u is univocally determined by u in virtue of the second equation. See [5] for the details, where the problem is addressed and where, among other results, the multiplicity of nonradial solutions has been proved in the cases $n = 4$ and $n \geq 6$ by using the symmetric Mountain Pass Theorem.

We are able to obtain a similar result also in $\mathbb{R}^5$. Recall Remark 3.10 to deal with $H^s(\mathbb{R}^5)$. Let $\widetilde{H}$ as in Example 3.8 and $H^s_{\widetilde{H}}(\mathbb{R}^5)$ be the closed and infinite-dimensional subspace of fixed points for this action. It is easy to see that if $u \in H^s_{\widetilde{H}}(\mathbb{R}^5)$, then also φ_u enjoys the same symmetry and that the functional E_ω is invariant for this action (and then the Palais' Principle applies). Since $H^s_{\widetilde{H}}(\mathbb{R}^5)$ has compact embedding into $L^p(\mathbb{R}^5)$, $2 < p < 2^*_s$, we can prove the Palais–Smale condition and conclude exactly as in [5, Theorem 5.3] obtaining explicitly the following result covering the case $n = 5$.

Theorem 3.12 *Under the previous assumptions, problem* (3.3) *possesses an unbounded sequence of solutions* $\{\pm u_k\}_{k \in \mathbb{N}}$ *in* $\mathbb{R}^n$, $n \geq 4$ *which are nonradial.*

As before, in the case of $\mathbb{R}^3$ we can achieve existence of solutions radially symmetric in the first two variables and periodic in the third one.

Again the existence of infinitely many radial solutions in $\mathbb{R}^n$, $n \geq 3$ is known and obtained in a standard way by using the compact embedding of the radial functions.

3.1.3 A Schrödinger–Poisson system

With our approach, we are able to extend a result of d'Avenia [4] in $\mathbb{R}^n$, $n \geq 4$ with a simpler proof by using the compact embedding of our working space.

Let us start with the case $n = 5$. Following [4], we consider

$$\begin{cases} -\frac{1}{2}\Delta u + \omega u + \varphi u = |u|^{p-2} u, & \text{in } \mathbb{R}^5, \\ -\Delta \varphi = \gamma(2) u^2, & \text{in } \mathbb{R}^5 \end{cases}$$

in the unknowns $u, \varphi : \mathbb{R}^5 \to \mathbb{R}$ and $\omega > 0$. The constant $\gamma(2)$ is defined as above. Here $p \in (4, 6)$ is given. Then, after rescaling, the above problem is equivalent to find solutions

$$u \in H^1(\mathbb{R}^5), \quad \varphi \in D^{1,2}(\mathbb{R}^5) \quad \text{not radial, and} \quad \lambda > 0$$

for the system

$$\begin{cases} -\frac{1}{2}\Delta u + u + \varphi u = \lambda |u|^{p-2}u, & \text{in } \mathbb{R}^5 \\ -\Delta\varphi = \gamma(2)u^2, & \text{in } \mathbb{R}^5. \end{cases} \tag{3.4}$$

Equivalently the problem is reduced to find a critical point, which is not radially symmetric, of the functional

$$J(u) = \frac{1}{4}\int_{\mathbb{R}^5} |\nabla u|^2 + \frac{1}{2}\int_{\mathbb{R}^5} u^2 + \int_{\mathbb{R}^5} \varphi_u u^2, \quad \text{where} \quad \varphi_u = \frac{1}{|\cdot|^3} * u^2,$$

restricted to the sphere

$$S = \left\{ u \in H^1(\mathbb{R}^5) : \int_{\mathbb{R}^5} |u|^p = 1 \right\}$$

and λ is the associated Lagrange multiplier. For this reason (as before, since φ is uniquely determined by u), we will speak of "solutions u and λ" of (3.4).

With our approach, we can restrict the functional to the subspace of fixed points for the action considered in Example 3.8, i.e., $H^1_{\widetilde{H}}(\mathbb{R}^5)$, that we know it has compact embedding into $L^p(\mathbb{R}^5)$. Then, the set

$$S_{\widetilde{H}} = \left\{ u \in H^1_{\widetilde{H}}(\mathbb{R}^5) : \int_{\mathbb{R}^5} |u|^p = 1 \right\}$$

is weakly closed. It is easy to see that if $u \in H^1_{\widetilde{H}}$, then also $\varphi_u := \frac{1}{|\cdot|^3} * u^2$ enjoys the same symmetry (again, important to apply the Palais's Principle). Moreover, the energy functional restricted to $S_{\widetilde{H}}$ is bounded below and coercive (as proved in [4]), but also weakly lower semicontinuous and satisfies the Palais–Smale condition, as it is standard to see by using the compact embedding just stated.

Then, not only the existence of a minimizer is guaranteed, but also, since the functional is even, by the Ljusternick–Schnirelmann theory we deduce the following

Theorem 3.13 *Problem (3.4) has infinitely many solutions (u_n, λ_n) with u_n nonradial.*

The above theorem for the cases $n = 4$ and $n \geq 6$ can be easily obtained by the approach of [3].

Of course, our methods permits to have solutions in $\mathbb{R}^3$ which are radial in the first two variables and periodic in the third one.

Again, also in this case the existence of radially symmetric solutions is well known, for any $n \geq 3$.

4 The case of Hadamard manifolds

Let (M, g) be an Hadamard manifold of $\dim M = n$. This means (M, g) is a simply connected complete Riemannian manifold of nonpositive sectional curvature, see [7, 8]. Hence, if $p \in M$, then $\exp_p : T_pM \longrightarrow M$ is a diffeomorphism. Since M is simply connected, it is orientable. Let $\{U_i, \phi_i\}_{i\in I}$ be an orientable atlas. This means that for every $i \in I$, $U_i \subset \mathbb{R}^n$ and $\phi_i : U_i \longrightarrow \tilde{U}_i \subset M$ is a positive diffeomorphism. The Riemannian volume form v is the smooth n-form such that

$$\phi_i^* v = \sqrt{\det G}\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n,$$

where $G = \left(g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) \right)_{1\leq i,j\leq n}$ is the matrix associated with g with respect to the basis $\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\}$. It is easy to check that any isometry $T : M \longrightarrow M$ satisfies $T^* v = v$, i.e., T preserves the Riemannian volume form v.

Let $G \subset \mathrm{Iso}(M, g)$ and let $\Omega \subset M$ any G-invariant open and unbounded subset of M. Let $H^1(\Omega)$ and $H_0^1(\Omega)$ be the usual Sobolev spaces defined as for the case of $\mathbb{R}^n$, see [2].

The action of G on $H_0^1(\Omega)$ is defined by

$$gu(x) := u(g^{-1}x), \quad g \in G,$$

and the subspace of fixed point for this action is

$$H^1_{0,G}(\Omega) := \left\{ u \in H_0^1(\Omega) : gu = u, \ \forall g \in G \right\}.$$

Applying Proposition 2.2, we have the following result which gives the compactness of the Sobolev embedding in the case of a Riemannian manifold.

Theorem 4.1 *Let* (M, g) *be an Hadamard manifold, and let* $G \subset \mathrm{Iso}(M, g)$ *be a closed and discrete subgroup with infinite elements. If* M/G *is compact, then the following embedding*

$$H^1_{0,G}(M) \hookrightarrow L^p(M), \ 2 < p < 2^*,$$

is compact.

Proof Indeed, having Proposition 2.2 the proof follows as in [13, p.16-17]. □

Let $G \subset \mathrm{Iso}(M, g)$ be a compact subgroup. By a Theorem of Cartan, see [8], G has a fixed point. Let $p \in M^G = \{q \in M : G(q) = q\}$ be a fixed point. The isotropy representation

$$G \longrightarrow \mathrm{O}(T_pM), \qquad k \mapsto \mathrm{d}k_p.$$

is injective, and it satisfies

$$k(\exp_p(v)) = \exp(\mathrm{d}k_p(v)).$$

Therefore, the exponential map at p is G-equivariant, i.e., it interchanges the G action on M with the G action on T_pM. In the sequel, we also denote by $kv = \mathrm{d}k_p(v)$. It is well known that M^G is a totally geodesic submanifold of M and $T_pM^G = (T_pM)^G$, see [1]. Hence, if the

G action satisfies the first condition of Theorem 3.5, keeping in mind that any point can be joined by a unique minimizing geodesic, then $M^G = \{p\}$. The vice versa holds as well.

Proposition 4.2 *Let (M, g) be an Hadamard manifold, and let $G \subset \mathrm{Iso}(M, g)$ be a compact connected subgroup of dimension bigger than one. Assume that $M^G = \{p\}$. Then, for any $r > 0$, we have*

$$\lim_{d(p,z)\mapsto+\infty} \mathrm{m}(z, r, G) = +\infty.$$

Therefore, any G-invariant open unbounded subset of M is compatible.

Proof Since (M, g) has negative curvature, applying the Rauch Theorem [6], if $v, w \in T_pM$, then

$$\| x - y \| \leq d(\exp_p(v), \exp_p(w)).$$

Let $r > 0$, and let $n \in \mathbb{N}$. By Proposition, 3.2,

$$\lim_{\|y\|\mapsto+\infty} \mathrm{m}(y, r, G) = +\infty.$$

Let $y \in T_pM$, and let $n \in \mathbb{N}$ be such that there exists $g_1, \dots, g_n \in G$ such that $\| g_iy - g_jy \| > 2r$ for $i \neq j$ and so $B_r(g_iy) \cap B_r(g_iy) = \emptyset$ for $i \neq j$. Since

$$d(\exp_p(g_iy), \exp_p(g_jy)) = d(g_i \exp_p(y), g_j \exp_p(y)) \geq \| g_iy - g_jy \| > 2r,$$

it follows $\mathcal{B}_r(g_i \exp_p(y)) \cap \mathcal{B}_r(g_i \exp_p(y)) = \emptyset$ for $i \neq j$, and so, keeping in mind $d(p, \exp_p(y)) = \| y \|$, the result follows. □

As an application, we have the following compactness result for an Hadamard manifold.

Theorem 4.3 *Let (M, g) be an Hadamard manifold, and let $G \subset \mathrm{Iso}(M, g)$ be a compact subgroup of dimension bigger or equal than one. Assume that $M^G = \{p\}$. Let Ω be an unbounded G-invariant open subset of M containing p. Then, the following embedding*

$$H^1_{0,G}(\Omega) \hookrightarrow L^p(\Omega),\ 2 < p < 2^*,$$

is compact.

Proof It is similar to [13] p. 16 − 17. □

Let $p \in M$, and let $f : M \longrightarrow \mathbb{R}$ be a function. We say that f is a radial function (with respect to p) if $f(z) = f(w)$ whenever $d(p, z) = d(p, w)$. Since $\exp_p : T_pM \longrightarrow M$ is a diffeomorphism, and keeping in mind that $d(p, z) = \| \exp_p^{-1}(z) \|$, see [6], it follows that $f : M \longrightarrow \mathbb{R}$ is a radial function if and only if $f \circ \exp_p^{-1} : T_pM \longrightarrow \mathbb{R}$ is a radial function.

Let $G \subset G_p = \{g \in \mathrm{Iso}(M, g) : gp = p\}$ be a closed subgroup. Then, $H^1_{0,G}(M)$ contains the set of radial functions. Indeed, if $f : M \longrightarrow \mathbb{R}$ is a radial function, then it is G-invariant due to the fact that for any $k \in G$ we have

$$d(p, kz) = d(kp, kz) = d(p, z).$$

5 Symmetry and compactness for symmetric spaces of noncompact type

Let (M, g) be an Hadamard manifold. Let $p \in M$, and let $H \subset G_p$ be closed subgroup. Assume that there exists a closed subgroup $\widetilde{H} \subset G_p$ such that $H \subset \widetilde{H} \subset G_p$ and $\widetilde{H}/H$ has two elements. Then, H is a normal subgroup of $\widetilde{H}$ and the natural projection $\rho : \widetilde{H} \longrightarrow \widetilde{H}/H \cong \mathbb{Z}_2$ is a surjective homomorphism. By Lemma 3.7, this holds if there exists $\tau \in G_p$ such that $\tau \notin H$, $\tau^2 = e$ and $\tau H \tau = H$. Indeed, the subgroup $\widetilde{H}$ generated by H and τ is closed and $\widetilde{H}/H$ has two elements.

Let Ω be an unbounded open $\widetilde{H}$-invariant subset of M. Define

$$\widetilde{H} \times H^1(\Omega) \longrightarrow H^1(\Omega), \qquad (g, u) \mapsto \rho(g)u(g^{-1}\cdot) = gu.$$

It is easy to check that this map defines an isometric action of $\widetilde{H}$ on $H^1(\Omega)$. Denote by $H^1_{0,\widetilde{H}}(\Omega) = \{u \in H^1(\Omega) \,:\, gu = u \text{ for any } g \in \widetilde{H}\}$. Since $\rho(h) = 1$ if $h \in H$, then $H^1_{0,\widetilde{H}}(\Omega)$ is a closed subspace of $H^1_{0,H}(\Omega)$.

Lemma 5.1 *Under the above assumption, if*

$$H^1_{0,H}(\Omega) \hookrightarrow L^p(\Omega),\ 2 < p < 2^* \quad (\text{where } \dim M \geq 3)$$

is compact, then the embedding

$$H^1_{0,\widetilde{H}}(\Omega) \hookrightarrow L^p(\Omega),\ 2 < p < 2^*,$$

is compact and the set $H^1_{0,\widetilde{H}}(\Omega)$ does not contain radial functions unless $u = 0$.

Proof Since $H^1_{0,\widetilde{H}}(\Omega)$ is a closed subspace of $H^1_{0,H}(\Omega)$, it follows that the embedding

$$H^1_{0,\widetilde{H}}(\Omega) \hookrightarrow L^p(\Omega),\ 2 < p < 2^*,$$

is the restriction of a compact operator on a closed subspace, and so it is compact as well. Finally, pick $\tau \notin H$. If $u \in H^1_{0,\widetilde{H}}(\Omega)$, then

$$\tau u(\exp_p(v)) = -u(\exp_p(\tau(v)),$$

and so u is not radial unless it is zero. □

The aim of this section is to generalize the idea due to Bartch and Willem [3] in order to show the existence of group actions whose subspace of fixed points has nonradial functions.

5.1 Symmetric spaces of noncompact type of rank one

Let (M, g) be a symmetric space of noncompact type of rank one. Let $p \in M$, and let $G_p = \{g \in \mathrm{Iso}(M, g) \,:\, g(p) = p\}$. It is a well known G_p is a compact group acting transitively on the unit sphere of $(T_pM, g(p))$, see [8]. Hence, G_p satisfies the assumption on Theorem 4.3. We claim that $H^1_{0,G_p}(M)$ coincides with the set of radial functions of class

$H^1_0(M)$ that we denote by $H^1_{0,\mathrm{rad},p}(M)$. Indeed, let $u : M \longrightarrow \mathbb{R}$ be a G_p-invariant functions. It is enough to prove that $\widetilde{u} = u \circ \exp_p^{-1}$ is a radial function.

Let $v \in T_pM$, and let $w \in T_pM$ be such that $\| v \| = \| w \|$. Then, there exists $g \in G_p$ such that $g^{-1}(v) = w$. Hence,

$$\widetilde{u}(w) = u(\exp_p(w)) = u \exp_{g^{-1}p}(g^{-1}v)) = gu(\exp_p(v)) = \widetilde{u}(v).$$

Applying Theorem 4.3, we have proved the following result.

Theorem 5.2 *Let* (M, g) *be a symmetric space of noncompact type of rank one and let* $p \in M$. *Then, the following embedding*

$$H^1_{0,\mathrm{rad},p}(M) \hookrightarrow L^p(\Omega),\ 2 < p < 2^* \quad (\text{where } \dim M \geq 3)$$

is compact.

Let $\mathbb{H}^n$ be the hyperbolic space. $\mathbb{H}^n$ is a symmetric space of noncompact type of rank one [8]. One model is given as follows, see [6].

Let $\mathbb{R}^{n+1} = \mathbb{R}^n \oplus \mathbb{R}$ endowed by the Minkoswky inner product defined by the quadratic form

$$q(x, t) = \| x \|^2 - t^2,$$

where $\| \cdot \|$ is the norm with respect to the canonical scalar product on $\mathbb{R}^n$. Let

$$\mathbb{H}^n = \{(x, t) \in \mathbb{R}^{n+1} \ : \ q(x, t) = -1\}.$$

It is well known that the Minkowsky inner product induces on $\mathbb{H}^n$ a complete Riemannian metric $\langle \cdot, \cdot \rangle$ with constant sectional curvature -1. Moreover, $\mathrm{Iso}(\mathbb{H}^n, \langle \cdot, \cdot \rangle) = \mathrm{O}(1, n) = \{A \in \mathrm{Gl}(n+1, \mathbb{R}) \ : \ q(A(x, t)) = q(x, t)\}$ and $\mathrm{O}(1, n)_{e_{n+1}} = \mathrm{O}(n)$, where $e_{n+1} = (0, \dots, 0, 1)^{\mathrm{T}}$. Moreover, the slice representation, i.e., the $\mathrm{O}(n)$ action on $T_{e_{n+1}}\mathbb{H}^n = \mathbb{R}^n$, is the standard $\mathrm{O}(n)$ action on $\mathbb{R}^n$.

Assume that $n \geq 4$. Let $H = \mathrm{SO}(2) \times \mathrm{SO}(n-2) \subset \widetilde{H} = \mathrm{O}(2) \times \mathrm{SO}(n-2) \subset \mathrm{O}(n)$. Then, both H and $\widetilde{H}$ satisfy the condition of Theorem 4.3 and $\widetilde{H}/H \cong \mathbb{Z}_2$. Therefore, the action

$$\widetilde{H} \times H^1(\mathbb{H}^n) \longrightarrow H^1(\mathbb{H}^n), \qquad ((A, B), u) \mapsto \det(A)u((A^{-1}\cdot, B^{-1}\cdot)).$$

is an isometric action, the embedding

$$H^1_{0,\widetilde{H}}(\mathbb{H}^n) \hookrightarrow L^p(\mathbb{H}^n),\ 2 < p < 2^*,$$

is compact, and the set $H^1_{0,\widetilde{H}}(\mathbb{H}^n)$ does not contain radial functions unless $u = 0$.

Let $M = \mathrm{SU}(1, n)/\mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(n)), n \geq 2$, be the symmetric space of noncompact of type which is the dual of the complex projective space. It is a symmetric space of noncompact type of rank one [8]. The isotropy representation is given by the natural action of $\mathrm{U}(n)$ on $\mathbb{C}^n$.

Let $T \subset \mathrm{SU}(n)$ be a maximal torus. It is easy to check that $(\mathbb{C}^n)^T = \{0\}$, and so it satisfies the condition of Theorem 4.3. It is also well known that the Weyl group of $\mathrm{SU}(n)$ is isomorphic to the group of permutations of $n-1$ elements, see [11]. Therefore, there exists $\tau \in \mathrm{SU}(n) \setminus \{T\}$ such that $\tau^2 = \mathrm{Id}$ and $\tau T \tau = T$. Denote by $\widetilde{H}$ be the closed group generated by T and τ and by $\rho : \widetilde{H} \longrightarrow \widetilde{H}/T \cong \mathbb{Z}_2$ the natural projection. Then, the action

$$\widetilde{H} \times H^1(M) \longrightarrow H^1(M), \qquad (g) \mapsto \rho(g)u(g^{-1}\cdot).$$

is an isometric action, the embedding

$$H^1_{0,\widetilde{H}}(M) \hookrightarrow L^p(M),\ 2 < p < 2^*,$$

is compact, and the set $H^1_{0,\widetilde{H}}(M)$ does not contain radial functions unless $u = 0$.

Let $M = \mathrm{Sp}(1, n)/\mathrm{Sp}(1) \times \mathrm{Sp}(n))$ be the symmetric space of noncompact type which is the dual of the Quaternionic projective space. It is a symmetric space of noncompact type of rank one, see [8]. The isotropy representation is given by the natural action of $\mathrm{Sp}(1) \cdot \mathrm{Sp}(n)$ on $\mathbb{H}^n$.

Let $T \subset \mathrm{Sp}(n)$ be a maximal torus. It is easy to check that $(\mathbb{H}^n)^T = \{0\}$, and so it satisfies the condition of Theorem 4.3. It is also well known that the Weyl group of $\mathrm{Sp}(n)$ is isomorphic to $\mathrm{S}_n \ltimes \mathbb{Z}_2$, see [11]. Therefore, there exists $\tau \in \mathrm{Sp}(n) \setminus \{T\}$ such that $\tau^2 = \mathrm{Id}$ and $\tau T \tau = T$. Denote by $\widetilde{H}$ be the closed group generated by T and τ and by $\rho : \widetilde{H} \longrightarrow \widetilde{H}/T \cong \mathbb{Z}_2$ the natural projection. Then, the action

$$\widetilde{H} \times H^1(M) \longrightarrow H^1(M), \qquad (g) \mapsto \rho(g)u(g^{-1}\cdot).$$

is an isometric action, the embedding

$$H^1_{0,\widetilde{H}}(M) \hookrightarrow L^p(M),\ 2 < p < 2^*,$$

is compact, and the set $H^1_{0,\widetilde{H}}(M)$ does not contain radial functions unless $u = 0$.

5.2 The space $M = \mathrm{SL}(n, \mathbb{R})/\mathrm{SO}(n)$

The manifold $M = \mathrm{SL}(n, \mathbb{R})/\mathrm{SO}(n)$ is a symmetric space of noncompact type. For a sake of completeness, we briefly recall some well-known facts. The Cartan decomposition of the Lie algebra of $\mathrm{SL}(n, \mathbb{R})$ is given by

$$\mathfrak{sl}(n, \mathbb{R}) = \mathfrak{so}(n) \oplus \mathrm{Sym}_0(n),$$

where $\mathrm{Sym}_0(n) = \{A \in \mathfrak{gl}(n, \mathbb{R}) : A = A^{\mathrm{T}}, \mathrm{Tr}(A) = 0\}$ and $\mathfrak{so}(n)$ is the Lie algebra of $\mathrm{SO}(n)$. Therefore, we may identify $T_{[\mathrm{SO}(n)]}M$ with $\mathrm{Sym}_0(n)$ and the isotropy representation of $\mathrm{SO}(n)$ is given by

$$\sigma : \mathrm{SO}(n) \longrightarrow \mathrm{SO}(T_pM), \qquad A \mapsto \mathrm{Ad}(A),$$

where $\mathrm{Ad}(A)(X) = AXA^{\mathrm{T}}$, i.e., the adjoint action. This action is isometric with respect to scalar product $\langle X, Y \rangle := \mathrm{Tr}(XY) = \mathrm{Tr}(XY^{\mathrm{T}})$ defined on Sym_0. This allows us to define a Riemannian metric on M, which coincides with $\langle \cdot, \cdot \rangle$ at $[\mathrm{SO}(n)]$ by requesting that the left translation $L_g : M \longrightarrow M$, $h\mathrm{SO}(n)] \mapsto gh\mathrm{SO}(n)]$ is an isometry. We also denote by $\langle \cdot, \cdot \rangle$ this Riemannian metric, and $(M, \langle \cdot, \cdot \rangle)$ is a symmetric space of noncompact type of non-positive sectional curvature [8]. Let $\mathrm{SU}(n) \subset \mathrm{SO}(2n)$ being

$$A + iB \mapsto \begin{bmatrix} A & B \\ -B & A \end{bmatrix}.$$

This means that

$$\mathrm{SU}(n)=\{C\in \mathrm{SO}(2n)\,:\,CJC^{\mathrm{T}}=J\},$$

where $J=\begin{bmatrix}0 & -\mathrm{Id}_n\\ \mathrm{Id}_n & 0\end{bmatrix}$. We claim that $\mathrm{Sym}_0(2n)^{\mathrm{SU}(n)}=\{0\}$. Indeed, let $A\in \mathrm{Sym}_0(2n)^{\mathrm{SU}(n)}$. Since A is symmetric, it can be diagonalize. Since A is a $\mathrm{SU}(n)$ fixed point, it follows that any eigenspace is preserved by $\mathrm{SU}(n)$. Since $\mathrm{SU}(n)$ acts irreducibly on $\mathbb{R}^{2n}$, it follows that A must be a multiple of the identity matrix with trace 0 and so $A=\{0\}$.

Let $\tau=\begin{bmatrix}\mathrm{Id}_n & 0\\ 0 & -\mathrm{Id}_n\end{bmatrix}$. It is easy to see that $\tau^2=\mathrm{Id}$, $\tau\notin \mathrm{SU}(n)$ and $\tau\mathrm{SU}(2n)\tau=\mathrm{SU}(2n)$. Identifying τ with $\sigma(\tau)$, we define $\widetilde{\tau}=\exp\circ\tau\exp^{-1}$. Therefore, $\widetilde{\tau}$ is an isometry of M, due to the fact that τ lies in the image of the isotropy representation, satisfying $\widetilde{\tau}^2=\mathrm{Id}_M$. Moreover, if we consider H acting on M, we have and $\widetilde{\tau}H\widetilde{\tau}=H$. Indeed, denoting by $p=[\mathrm{SO}(2n)]$, for any $h\in H$, we have $\widetilde{\tau}h\widetilde{\tau}p=p$ and so $\widetilde{\tau}h\widetilde{\tau}$ is completely determine by its differential. Since $\tau\mathrm{d}h\tau\in H$, it follows that $\widetilde{\tau}h\widetilde{\tau}\in H$ as well.

By Lemma 3.7 the subgroup $\widetilde{H}$ generated by H and τ is closed, and there exists a surjective homomorphism $\rho:\widetilde{H}\longrightarrow\mathbb{Z}_2$ such that $\rho(H)=1$. By Lemma 5.1, the action

$$\widetilde{H}\times H^1(\mathrm{SL}(2n,\mathbb{R})/\mathrm{SO}(2n))\longrightarrow H^1(\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(2n)),\qquad (g,u)\mapsto\rho(g)u(g^{-1}\cdot)$$

is isometric, the embedding

$$H^1_{0,\widetilde{H}}(\mathrm{SL}(2n,\mathbb{R})/\mathrm{SO}(2n))\hookrightarrow L^p(\mathrm{SL}(2n,\mathbb{R})/\mathrm{SO}(2n)),\ 2<p<2^*,n\geq 2$$

is compact, and finally, $H^1_{0,\widetilde{H}}(\mathrm{SL}(2n,\mathbb{R})/\mathrm{SO}(2n))$ does not contains radial functions unless $u=0$.

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