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Artin algebras which are equivalent to a hereditary algebra modulo preprojectives

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ARTIN ALGEBRAS WHICH ARE EQUIVALENT TO A HEREDITARY ALGEBRA MODULO PREPROJECTIVES

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M. Auslander and I. Reiten studied in [1] , 1972, the situation in which two artin algebras Λ and Λ' are stably equivalent or equivalent modulo projectives. In particular, they found necessary and sufficient conditions for Λ to be stably equivalent to a hereditary algebra Λ'. One of their classical results is that any radical-squared-zero algebra is stably equivalent to a hereditary algebra. The Auslander-Reiten conditions mentioned above were stated as properties of the non-projective, torsionless Λ-modules.

I. Reiten studied in [4] , 1982, the more general equivalences of the form $\frac{\text{mod } \Lambda}{\underline{\underline{V}}} \xrightarrow{\alpha} \frac{\text{mod } \Lambda'}{\underline{\underline{V}}'}$ which were introduced in [2] , 1977, and [3] , 1978. She imposed some restrictions on α, as were suitable for the case of selfinjective algebras.

M. Auslander and S. O. Smalø developped in [5], 1980, their theory of preprojective and preinjective partitions and the notions of preprojective and preinjective modules which, in some sense, generalize the projective and injective ones. On the other hand, their definitions amount to a generalization of the preprojective and preinjective modules introduced by Dlab and Ringel, [6] , 1976, for the case of a hereditary algebra.

In this paper we try a way of generalizing the Auslander-Reiten research of 1972 to the case of equivalences α, as above, where $\underline{\underline{V}}$ and $\underline{\underline{V}}'$ are the categories of preprojective modules up to some level n. However, instead of using categorical methods, our techniques are based mainly on the properties of Auslander-Reiten sequences and irreducible maps. And our conditions are expressed mainly in terms of properties of the components of the Auslander-Reiten quiver, Γ_{Λ} , of Λ which contain the projective modules. In case that Λ is indecomposable and of infinite representation type, our results allow for a complete description of Γ_{Λ} , except for the component containing the projectives, in terms of the Auslander-Reiten quiver of the hereditary algebra Λ'. On the other hand, by imposing that Λ' is of finite representation type, we obtain a new criterium for an artin algebra being of finite representation type (see THEOREM 3, below).

In order to clarify the statements of our main results we explain first our notations and terminology.

Λ, Λ' stand for artin algebras. $\underline{\underline{V}}$ (resp. $\underline{\underline{V}}'$) is the full additive subcategory of mod Λ (resp. mod Λ') generated by the preprojective modules of $\underline{\underline{P}}^n(\Lambda)$ (resp. $\underline{\underline{P}}^n(\Lambda')$) (see 1.3). $\frac{\text{mod }\Lambda}{\underline{\underline{V}}}$ and $\frac{\text{mod }\Lambda'}{\underline{\underline{V}}'}$ are the respective categories of modules modulo preprojectives (see 2.1). $\underline{\underline{I}}$ is the category of indecomposable $\underline{\underline{V}}$-injective modules (see Def.3.1).

Let $\underline{\underline{C}}$ be an additive category. Then Comod $\underline{\underline{C}}$ is the category whose objects are the morphisms $A \xrightarrow{f} B$ of $\underline{\underline{C}}$, and whose morphisms are the pairs of $\underline{\underline{C}}$-morphisms (a,b) making commutative the squares

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[d, "b"] \\ C \arrow[r, "g"] & D \end{tikzcd}$$

modulo the pairs (a',b'), with the same property, such that $a'=cf$ for some $B \xrightarrow{c} C$ in $\underline{\underline{C}}$.

We will say that the morphisms f,g are $\underline{\underline{C}}$-equivalent (or, equivalent with respect to $\underline{\underline{C}}$) if they are isomorphic as objects of Comod $\underline{\underline{C}}$. And we will say that f is $\underline{\underline{C}}$-indecomposable (or, indecomposable with respect to $\underline{\underline{C}}$), if it is indecomposable as an object of Comod $\underline{\underline{C}}$. In particular, if $\underline{\underline{C}}$ is the category of injective Λ-modules, Comod $\underline{\underline{C}}$ is equivalent to mod Λ through the functor $f \longrightarrow \ker f$.

THEOREM 1. *Let* $\dfrac{\text{mod } \Lambda}{\underline{\underline{V}}} \xrightarrow{\alpha} \dfrac{\text{mod } \Lambda'}{\underline{\underline{V}}'}$ *be an equivalence of categories. Then*

(i) $\underline{\underline{I}}$ *is open to the right in* $\Gamma_\Lambda \setminus \text{ind } \underline{\underline{V}}$;

(ii) $\underline{\underline{I}}$ *has no oriented cycles;*

(iii) *If* $I \xrightarrow{f} J$ *is an indecomposable morphism with respect to* $(\text{add } \underline{\underline{I}})/\underline{\underline{V}}$ *which is not a monomorphism, then there is an exact sequence*

$$0 \longrightarrow M \xrightarrow{i} I \xrightarrow{f} J \longrightarrow 0$$

where M *is in* $\text{ind}_{\underline{\underline{V}}}(\Lambda)$ *and* $M \xrightarrow{i} I$ *is the* $\underline{\underline{I}}$-envelope *of* M *(see 1.2) and* f *is equal to* cok i *modulo* $\underline{\underline{V}}$.

If, furthermore, all ring components of Λ' *are of infinite representation type, then* Λ' *is Morita equivalent to*

$$\Lambda^* = \frac{\text{End}_\Lambda(T)}{P(T,T)}$$

where T *is the direct sum of the objects of* $\underline{\underline{I}}$. *In this case, the following conditions are also satisfied.*

(iv) mod Λ^* *is equivalent to* $\text{Comod}((\text{add } \underline{\underline{I}})/\underline{\underline{V}})$;

(v) *The category* $\underline{\underline{V}}^*$ *of* mod Λ^* *corresponding, under the equivalence of* (iv), *to the morphisms* f *which are monomorphisms of* $\dfrac{\text{mod } \Lambda}{\underline{\underline{V}}}$, *is exactly* $\text{add}(\underline{\underline{P}}^n(\Lambda^*))$;

(vi) *No connected component of* $\underline{\underline{I}}$ *is a Dynkin diagram.*

Finally, if furthermore $\underline{\underline{V}}$ *contains no injective* Λ-*modules, then*

(vii) $\underline{\underline{I}}$ *contains the left boundary of* $\underline{\underline{V}}$.

THEOREM 2. *Let* $\underline{\underline{I}}$ *be without oriented cycles and open to the right in* $\Gamma_\Lambda \setminus \text{ind}\ \underline{\underline{V}}$, *and assume that condition (iii) of* THEOREM 1 *is satisfied. Let* Λ^* *and* $\underline{\underline{V}}^*$ *be defined as in* THEOREM 1 *and suppose that one of the following conditions is true.*

(vii) $\underline{\underline{I}}$ *contains the left boundary of* $\underline{\underline{V}}$;

(vii)' *all* M *in* $\text{ind}_{\underline{\underline{V}}}(\Lambda)$ *appear as kernels of a morphim* *f* *as in* (iii) *of* THEOREM 1.

Then Λ^* *is hereditary and there is an equivalence of categories*

$$\frac{\text{mod}\ \Lambda}{\underline{\underline{V}}} \xrightarrow{\alpha} \frac{\text{mod}\ \Lambda^*}{\underline{\underline{V}}^*} .$$

THEOREM 3. *Let* $\underline{\underline{I}}$ *be without oriented cycles, open to the right in* $\Gamma_\Lambda \setminus \text{ind}\ \underline{\underline{V}}$ *and containing the left boundary of* $\underline{\underline{V}}$. *If condition (iii) of* THEOREM 1 *is satisfied and every connected component of* $\underline{\underline{I}}$ *is a Dynkin diagram, then* Λ *is of finite representation type*.

The paper is organized in the following way. In section 1, we give the definitions and notations in full detail, together with basic facts about covers, envelopes, preprojective modules and so on which will be needed later on. The quotient categories $\frac{\text{mod}\ \Lambda}{\underline{\underline{V}}}$ are introduced in section 2, together with their properties concerning irreducible morphisms and Auslander-Reiten sequences. Section 2 includes also the proofs of some statements about existence of kernels and cokernels, in case of Λ being equivalent to a hereditary algebra modulo preprojectives.

In section 3, the $\underline{\underline{V}}$-injective modules are introduced and some of their properties are proved. One of the main results in this section is Prop.3.5 which gives necessary and sufficient conditions for Λ^* to be hereditary. With the results included in this section we have all the material needed to prove our theorems. This is done in section 4.

Section 5 treats the particular case of stable equivalence $(n = 0)$ and we obtain again, by different techniques, the classical conditions of Auslander and Reiten (see [1]).

1. Introduction.

In this section we fix our basic context, terminology and notations We also state, for the reader's convenience, the basic properties of preprojective modules which will be needed in the rest of the paper.

(1.1) Let Λ be an artin algebra, and mod Λ the category of finitely generated, left Λ-modules. If X, Y are in mod Λ, $\Lambda(X,Y)$ will denote the group of homomorphisms from X to Y. For a subcategory of mod Λ we mean a full subcategory which is closed under direct summands. If $\underline{\underline{V}}$ is such a subcategory, ind $\underline{\underline{V}}$ denotes the category of indecomposable objects (we write ind Λ instead of ind(mod Λ)), and we say that $\underline{\underline{V}}$ is finite (resp. of finite type) if $\underline{\underline{V}}$ (resp. ind $\underline{\underline{V}}$) has a finite number of non-isomorphic objects. We use the notation $\mathrm{mod}_{\underline{\underline{V}}}(\Lambda)$ for the subcategory of modules X which do not have components in $\underline{\underline{V}}$.

If there is no danger of misunderstanding, we will consider isomorphic objects as identical objects. For example, we will say ind Λ is the set of vertices of the Auslander-Reiten quiver Γ_Λ of Λ. The symbol Γ_Λ is used also to represent the Auslander-Reiten species of Λ. And, is $\underline{\underline{I}}$ is a set of indecomposable Λ-modules, we will use the same symbol $\underline{\underline{I}}$ to denote the corresponding subquiver or subspecies of Γ_Λ.

The notation τ is used for the Auslander-Reiten translation DTr, and τ' for its inverse TrD.

(1.2) Let $\underline{\underline{V}}$ be a subcategory of mod Λ which is of finite type. Then mod Λ is functorially finite over $\underline{\underline{V}}$ (see [5] ,p. 82) if the following two conditions are satisfied.

(CTVF) For each M in $\operatorname{mod}\Lambda$ there is a morphism $P \xrightarrow{p} M$, with P in $\operatorname{add}(\underline{\underline{V}})$, such that every morphism $Q \xrightarrow{q} M$ with Q in $\operatorname{add}(\underline{\underline{V}})$ factors through p, i.e. there exists a $Q \xrightarrow{q'} P$ such that $q = pq'$.

We will say that $P \xrightarrow{p} M$ is a $\underline{\underline{V}}$-cover of M, and we will write $P = C_{\underline{\underline{V}}}(M)$.

(COVF) For eah M in $\operatorname{mod}\Lambda$, there is a morphism $M \xrightarrow{i} P$, with P in $\operatorname{add}(\underline{\underline{V}})$, such that every morphism $M \xrightarrow{j} Q$ with Q in $\underline{\underline{V}}$ factors through i.

In this case we will say that $M \xrightarrow{i} P$ is a $\underline{\underline{V}}$-envelope of M, and we will write $P = E_{\underline{\underline{V}}}(M)$.

Since $\underline{\underline{V}}$ is closed for taking direct summands, among the covers and envelopes there are minimal ones, which are determined uniquely up to isomorphism (see [5]).

(1.3) Let $\underline{\underline{C}}$ be a subcategory of $\operatorname{mod}\Lambda$. Then $\underline{\underline{P}}_0(\underline{\underline{C}})$ is the category of the split-projectives of $\operatorname{ind}\underline{\underline{C}}$. In other words, an indecomposable module P lies in $\underline{\underline{P}}_0(\underline{\underline{C}})$ if it is in $\underline{\underline{C}}$ and if every epimorphism $C \longrightarrow P$ with C in $\operatorname{add}(\underline{\underline{C}})$ splits. If $\underline{\underline{C}}$ is $\operatorname{mod}\Lambda$ or $\operatorname{ind}\Lambda$ we use $\underline{\underline{P}}_0(\Lambda)$ instead of $\underline{\underline{P}}_0(\operatorname{mod}\Lambda)$ or $\underline{\underline{P}}_0(\operatorname{ind}\Lambda)$. The category $\underline{\underline{I}}_0(\underline{\underline{C}})$ of the split-injectives of $\underline{\underline{C}}$ is defined dually. Notice that $\underline{\underline{P}}_0(\Lambda)$ (resp. $\underline{\underline{I}}_0(\Lambda)$) are just the categories of indecomposable projective (resp. injective) Λ-modules.

The preprojective partition of Λ (or of $\operatorname{ind}\Lambda$), $(\underline{\underline{P}}_n(\Lambda))_{n\in\mathbb{N}\cup\{\infty\}}$ is uniquely determined by the following conditions.

(P1) $(\underline{\underline{P}}_n(\Lambda))_{n\in\mathbb{N}\cup\{\infty\}}$ is a partition of $\operatorname{ind}\Lambda$ (in the generalized sense that allows some members of the family to be empty).

(P2) If n is in $\mathbb{N}$, $\underline{\underline{P}}_n(\Lambda)$ is finite.

(P3) If n is in $\mathbb{N}$, $\underline{\underline{P}}_n(\Lambda)$ is a minimal cover of ind $\Lambda \setminus \underline{\underline{P}}^{n-1}(\Lambda)$

In the last condition, the notation $\underline{\underline{P}}^n(\Lambda)$ stands for the union of the family $\underline{\underline{P}}_i(\Lambda)$ $(i=0,\dots,n)$. The condition means that, for every indecomposable Λ-module M not in $\underline{\underline{P}}^{n-1}(\Lambda)$, there is an epimorphism $P \longrightarrow M$ with P in add$(\underline{\underline{P}}_n(\Lambda))$, and that $\underline{\underline{P}}_n(\Lambda)$ is minimal with respect to this property (see [5] for the details). The modules in $\underline{\underline{P}}_n(\Lambda)$ are called preprojective of level n.

The preinjective partition and preinjective modules are defined dually.

We write down now some basic, well known properties of preprojective modules for easy reference. The reader can consult [5] for details.

Prop. 1.1. The following propositions are equivalent for P in ind Λ.

(i) P is preprojective.

(ii) There is a finite subcategory $\underline{\underline{A}}$ of mod Λ such that if $B \longrightarrow P$ is an epimorphism, then B has a component in $\underline{\underline{A}}$.

(iii) There is a (finite) path $Q_o \longrightarrow Q_1 \longrightarrow \dots \longrightarrow Q_n = P$ in Γ_Λ with Q_o projective.

As a matter of fact, more is true than what is stated in (iii): if P is preprojective of positive level n then there is an arrow such as $Q \longrightarrow P$ where Q is preprojective of level less than n.

Prop.1.2. Let P be a preprojective Λ-module of level n, and let M be in $\text{mod}_{\underline{P}^n(\Lambda)}\Lambda$. Then, there is no epimorphism from M to P. And, if M is in $\text{mod}_{\underline{P}^{n-1}(\Lambda)}\Lambda$, then every epimorphism $M \longrightarrow P$ splits.

Prop. 1.3. Let $P \longrightarrow M$ be an arrow of Γ_Λ where P is preprojective of level n. If M is in $\text{mod}_{\underline{\underline{P}}^{n+1}(\Lambda)}\Lambda$, then τM is in $\underline{\underline{P}}^{n-1}\Lambda$

PROOF (compare to p.108in [5]). Let $X \longrightarrow M$ be an epimorphism with X in $\text{add}(\underline{\underline{P}}_{n+1}(\Lambda))$ and let $B \longrightarrow M$ be the right almost split map associated to M. Then, there is an apimorphism from $\tau M \oplus X$ onto B. Since P is a component of B, we get an epimorphism $\tau M \oplus X \longrightarrow P$. This leads to a contradiction because τM cannot be isomorphic to P.

We will use the following notation (where n is a natural number). For a Λ-module M, $T_n(M)$ (or, simply, $T(M)$ if there is no possibility of misunderstanding) denotes the sum of all images $f(A)$ for all morphisms $A \xrightarrow{f} M$ with A in $\text{mod}_{\underline{\underline{P}}^{n}(\Lambda)}\Lambda$.

Prop. 1.4. Let M be a Λ-module and X a submodule of M which is not contained in $T(M)$. Then X has a component in $\underline{\underline{P}}^{n}(\Lambda)$.

PROOF. Otherwise, X will be in $\text{mod}_{\underline{\underline{P}}^{n}(\Lambda)}\Lambda$, and hence in $T(M)$, by definition of $T(M)$.

If Λ is hereditary there are some special properties regarding to preprojective modules. They are listed below.

Prop. 1.5. Let Λ be hereditary. The following propositions are equivalent, for an indecomposable Λ-module P.

(i) P is preprojective.

(ii) There is a natural number m such that $\tau^m P$ is projective.

(iii) Up to isomorphism, the set of the indecomposable Λ-modules M such that $\Lambda(M,P)\neq 0$ is finite.

Prop. 1.6. Let Λ be hereditary. The following propositions are equivalent, for n a natural number greater than 0.

(i) P is in $\underline{P}_n(\Lambda)$.

(ii) P is not in $\underline{P}_{n-1}(\Lambda)$ and there is an arrow $Q \longrightarrow P$ in Γ_Λ with Q in $\underline{P}_{n-1}(\Lambda)$.

(iii) There is a path $P_m \longrightarrow P_{m-1} \longrightarrow \dots \longrightarrow P_o = P$ in Γ_Λ with P_m projective, and the minimum possible value for m is n.

PROOF. See [6] .

(1.4) We end up this introduction with some additional properties which will be needed later.

Let $\underline{\underline{V}}$ be a subcategory of mod Λ and let $\underline{\underline{C}}$ be a subquiver of Γ_Λ containing ind $\underline{\underline{V}}$. We say that $\underline{\underline{V}}$ (or ind $\underline{\underline{V}}$) is <u>open to the left in</u> $\underline{\underline{C}}$ if: for each arrow $A \longrightarrow B$ of $\underline{\underline{C}}$, if B is in $\underline{\underline{V}}$ then A is also in $\underline{\underline{V}}$. For example, as a consequence of Prop. 1.6 we have that if Λ is hereditary then $\underline{P}^n(\Lambda)$ is open to the left in Γ_Λ.

In a similar way it is defined what does it mean that $\underline{\underline{V}}$ is open to the right in $\underline{\underline{C}}$.

<u>Prop. 1.7.</u> Let Λ be an artin algebra and let $\underline{\underline{V}}$ be a subcategory of mod Λ, of finite type, open to the left in Γ_Λ and with no oriented cycles. Then we have.

(1) If $\Lambda(M,P) \neq 0$ for M,P in Γ_Λ , and P in $\underline{\underline{V}}$, then M is also in $\underline{\underline{V}}$.

(2) If $0 \longrightarrow Q \longrightarrow M \longrightarrow P \longrightarrow 0$ is an exact sequence with P,Q in $\underline{\underline{V}}$, then M is also in $\underline{\underline{V}}$.

PROOF. It is enough to indicate how to prove (1), since (2) is an easy consequence of (1).

The hypothesis implies that every chain of morphisms going out to the left from a vertex in $\underline{\underline{V}}$ must terminate, so that it is a path.

Hence, the maximal paths of this form end up at a simple projective module. Then the statement follows easily by induction on the maximum of the lengths of those paths beginning at P.

__Prop. 1.8.__ Let Λ be a hereditary artin algebra, indecomposable as a ring. Then the following propositions are equivalent.

(i) Λ is of finite representation type.

(ii) $\underline{\underline{P}}(\Lambda) \cap \underline{\underline{I}}(\Lambda) \neq \emptyset$.

Where $\underline{\underline{P}}(\Lambda)$ (resp. $\underline{\underline{I}}(\Lambda)$) is the set of all preprojective (resp. preinjective) Λ-modules.

PROOF. It is well known that an artin algebra Λ is of finite representation type if and only if $\underline{\underline{I}}(\Lambda) \subset \underline{\underline{P}}(\Lambda)$. When Λ is hereditary, given the arrow $A \longrightarrow B$ of Γ_Λ , B is in $\underline{\underline{P}}(\Lambda)$ if and only if A is in $\underline{\underline{P}}(\Lambda)$. From these remarks, the statement follows easily.

2. Categories of modules modulo preprojectives.

In this section we define the categories $\dfrac{\mathrm{mod}\,\Lambda}{\underline{\underline{V}}}$, looking mainly to properties of morphisms with respect to being split, irreducible and to having kernels and cokernels.

(2.1) Let $\underline{\underline{V}}$ be a subcategory of $\mathrm{mod}\,\Lambda$. Given X and Y in $\mathrm{mod}\,\Lambda$ we denote by $P(X,Y)$ the set of all morphisms $X \xrightarrow{f} Y$ that factor through an element of $\underline{\underline{V}}$ (i.e., such that there are morphisms $X \xrightarrow{g} P \xrightarrow{h} Y$, with P in $\underline{\underline{V}}$ and verifying $f = hg$) .

We define the category of Λ-modules modulo $\underline{\underline{V}}$, $\dfrac{\mathrm{mod}\,\Lambda}{\underline{\underline{V}}}$, as the category with the same objects as $\mathrm{mod}\,\Lambda$ and with morphisms given by $\dfrac{\Lambda}{\underline{\underline{V}}}(X,Y) = \dfrac{\Lambda(X,Y)}{P(X,Y)}$.

These categories were studied initially in [2] and [3] . We summarize below some basic, well known properties (see [3] for details).

(1) There is an obvious "quotient" functor $\mathrm{mod}\,\Lambda \longrightarrow \dfrac{\mathrm{mod}\,\Lambda}{\underline{\underline{V}}}$ which we indicate by bars:

$M \longrightarrow \underline{M}$ (for objects)

$f \longrightarrow \underline{f}$ (for morphisms).

(2) Let f be a morphism in $\mathrm{mod}_{\underline{\underline{V}}}\Lambda$. Then:

(i) f is an isomorphism if and only if $\underline{f}$ is;

(ii) f is a split epimorphism (resp. monomorphism) if and only if $\underline{f}$ is;

(iii) f is irreducible if and only if $\underline{f}$ is.

(3) Let $M \oplus P \xrightarrow{f} N$ be a morphism with M and N in $\mathrm{mod}_{\underline{\underline{V}}}\Lambda$ and P in $\underline{\underline{V}}$. If f is minimal right almos split, then $\underline{f}$ is minimal right almost split. Conversely, if $\underline{f}$ is minimal right almost split, then $\underline{f}$ may be represented by an f

as above wich is minimal right almost split.

The dual statement, for minimal left almost split maps, is also true.

If there is no possibility of confusion, we will take profit of property (2) for using the same symbol M to denote a module without components in $\underline{\underline{V}}$ and its image $\underline{M}$ in the quotient category.

We will use heavily also another property which is proven in [4], 1982. Let Λ, Λ' be artin algebras and $\underline{\underline{V}}$, $\underline{\underline{V}}'$ subcategories of mod Λ, mod Λ' , respectively, and let us assume that there is an equivalence $\frac{\text{mod}\ \Lambda}{\underline{\underline{V}}} \xrightarrow{\alpha} \frac{\text{mod}\ \Lambda'}{\underline{\underline{V}}'}$.

(4) Let $\underline{\underline{V}}$, $\underline{\underline{V}}'$ be as above and let us assume further that they are categories of finite type containing the projectives of mod Λ, mod Λ' , respectively. If $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is an Auslander-Reiten sequence in mod Λ with A, B, C not in $\underline{\underline{V}}$, then $0 \longrightarrow \alpha(A) \longrightarrow B' \longrightarrow \alpha(C) \longrightarrow 0$ is an Auslander-Reiten sequence in mod Λ' with B' not in $\underline{\underline{V}}'$.

(2.2) Now we give a list of properties which are true, in particular, in the case of hereditary algebras and when $\underline{\underline{V}} = \text{add}(\underline{\underline{P}}^n(\Lambda))$, which is the situation that interests us more in this paper.

Prop. 2.1. Let Λ be an artin algebra and $\underline{\underline{V}}$ a subcategory of mod Λ which is of finite type and open to the left in Γ_Λ and without oriented cycles. Then, if f is a morphism in $\text{mod}_{\underline{\underline{V}}}\Lambda$, $f = 0$ if and only if $\underline{f} = 0$. As a consequence, $\frac{\text{mod}\ \Lambda}{\underline{\underline{V}}}$ is equivalent to $\text{mod}_{\underline{\underline{V}}}\Lambda$.

PROOF. Let $M \xrightarrow{f} N$ be a morphism where M, N have no components in $\underline{\underline{V}}$. If f factors in the form $f = hg$ with $M \xrightarrow{g} P$ and P in $\underline{\underline{V}}$ then $g = 0$ (see Prop. 1.7,(1)), hence $f = 0$. This means that if $\underline{f} = 0$, then $f = 0$. The converse is obvious.

Prop. 2.2. Let Λ be an artin algebra and $\underline{\underline{V}}$ a subcategory of $\operatorname{mod}\Lambda$ which is of finite type and open to the left in Γ_Λ and without oriented cycles. Then $\dfrac{\operatorname{mod}\Lambda}{\underline{\underline{V}}}$ has cokernels.

PROOF. Let $M \xrightarrow{f} N$ be a morphism in the equivalent category $\operatorname{mod}_{\underline{\underline{V}}}\Lambda$ (see Prop. 2.1), and let $N \xrightarrow{g} L$ be the cokernel of f in $\operatorname{mod}\Lambda$. Then, by Prop. 2.1,(1), L is in $\operatorname{mod}_{\underline{\underline{V}}}\Lambda$. It follows easily that g is also the cokernel of f in $\operatorname{mod}_{\underline{\underline{V}}}\Lambda$.

Remark. Let Λ be an artin algebra and let $\underline{\underline{V}}$ be a subcategory of finite type. Then, for each morphism $M \xrightarrow{f} N$ in $\operatorname{mod}_{\underline{\underline{V}}}\Lambda$, there is a morphism $K \xrightarrow{i} M$ which is an "almost" kernel of $\underline{f}$. In fact, let

$$\begin{tikzcd} & P \\ K \arrow[ur, "p'"] \arrow[dr, "i"'] & \\ & M \end{tikzcd}$$

be the pull-back of

$$\begin{tikzcd} P \arrow[dr, "p"] & \\ & N \\ M \arrow[ur, "f"'] & \end{tikzcd}$$

where $P \xrightarrow{p} N$ is the $\underline{\underline{V}}$-cover of N (see (1.2)). Then, if $X \xrightarrow{g} M$ is such that $\underline{f}\,\underline{g} = 0$, there is a $X \xrightarrow{q} P$ such that $fg = pq$. Hence, there is a unique $X \xrightarrow{g'} K$ such that $g = ig'$ and $q = p'g'$. In particular, we see that $\underline{f}\,\underline{g} = 0$ if and only if $\underline{g} = \underline{i}\,\underline{g}'$ for some $X \xrightarrow{g'} K$.

As a result, if $\underline{i}$ were a monomorphism, then $\underline{f}$ would have a kernel, which would be $K \xrightarrow{i} M$ (unique up to isomorphism). This is precisely what happens when Λ is hereditary and when $\underline{\underline{V}} = \operatorname{add}(\underline{\underline{P}}^n(\Lambda))$, as the following lemma shows.

Prop. 2.3. Let Λ be an artin algebra and let $\underline{\underline{V}}$ be a subcategory of $\operatorname{mod}\Lambda$ which is of finite type and open to the left in Γ_Λ and without oriented cycles. Then $\frac{\operatorname{mod}\Lambda}{\underline{\underline{V}}}$ has kernels.

PROOF. Using the notations of the preceding remark, we consider the following commutative diagram.

$$\begin{tikzcd}
0 \arrow[r] & \ker p \arrow[r] & P \arrow[r, "p"] & N \arrow[r] & 0 \\
0 \arrow[r] & \ker p \arrow[r] \arrow[u, equal] & K \arrow[r, "i"] \arrow[u, "p'"] & M \arrow[r] \arrow[u, "f"'] & 0
\end{tikzcd}$$

(Notice that the $\underline{\underline{V}}$-cover, p, is always an epimorphism.) In order to show that $\underline{i}$ is a monomorphism, we construct its "almost" kernel using the following commutative diagram where Q is the $\underline{\underline{V}}$-cover of M.

$$\begin{tikzcd}
0 \arrow[r] & \ker p \arrow[r] & K \arrow[r, "i"] & M \arrow[r] & 0 \\
0 \arrow[r] & \ker p \arrow[r] \arrow[u, equal] & K' \arrow[r] \arrow[u, "i'"'] & Q \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

Then, by Prop.1.7,(1), ker p is in $\underline{\underline{V}}$ (because $\ker p \hookrightarrow P$), and, by Prop.1.7,(2), K' is also in $\underline{\underline{V}}$. Hence, $\underline{i}' = 0$ and, therefore, $\underline{i}$ is a monomorphism.

The following is used in the proof of THEOREM 1,(iii).

<u>Prop. 2.4.</u> Let Λ be an artin algebra and let $\underline{\underline{V}}$ be a subcategory of $\operatorname{mod}\Lambda$ which is of finite type and open to the left in Γ_Λ and without oriented cycles. Let us assume further that Λ is hereditary and that $I \xrightarrow{f} J$ is a morphism in $\operatorname{add}(\underline{\underline{I}}_0(\Lambda))$ such $\ker \underline{f} \neq 0$ (see Prop. 2.3). Then f is the direct sum of two morphisms $I' \xrightarrow{f'} J'$, $I'' \xrightarrow{f''} J''$ with the following properties.

(i) $\ker \underline{f} = \ker \underline{f}'$ and $\ker \underline{f}'' = 0$.

(ii) $0 \longrightarrow \ker f' \overset{i'}{\hookrightarrow} I' \xrightarrow{f'} J' \longrightarrow 0$ is an exact sequence in $\frac{\operatorname{mod}\Lambda}{\underline{\underline{V}}}$.

(iii) $\ker f' \xhookrightarrow{i'} I'$ is the injective enveloppe of $\ker f'$ in $\operatorname{mod} \Lambda$.

PROOF. Let M be in $\operatorname{mod} \Lambda$ such that $M = \ker \underline{f}$ and let $M \xhookrightarrow{i'} I'$ and $I' \xrightarrow{f'} J'$ be, respectively, the injective envelope of M and the cokernel of i'. We consider the diagram associated to $\ker \underline{f}'$.

$$\begin{tikzcd}
0 \arrow[r] & \ker p' \arrow[r] & P' \arrow[r, "p'"] & J' \arrow[r] & 0 \\
0 \arrow[r] & \ker p' \arrow[r] \arrow[u, equal] & K' \arrow[r, "j'"] \arrow[u, "p''"] & I' \arrow[r] \arrow[u, "f'"] & 0 \\
 & & M \arrow[r, equal] \arrow[u, "j'"] & M \arrow[u, "i'"] & \\
 & & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

where P' is the $\underline{\underline{V}}$-cover of J' and $K' \xrightarrow{j'} I'$ represents the kernel of f' modulo $\underline{\underline{V}}$. We decompose K' in the form $K' = Q' \oplus L'$ where Q' is in $\underline{\underline{V}}$ and L' is in $\operatorname{mod}_{\underline{\underline{V}}} \Lambda$. By Prop.1.7,(1), L' is contained in $M = \ker j'$, so that $M = (Q' \cap M) \oplus L'$. But, since $Q' \cap M$ is in $\underline{\underline{V}}$, and since M is in $\operatorname{mod}_{\underline{\underline{V}}} \Lambda$, we get $L' = M = \ker \underline{f}$

Now, since Λ is hereditary, $\operatorname{cok} f$ is injective and a direct summand of J, so that we may assume f to be an epimorphism. Let us consider the diagram associated to $\ker \underline{f}$.

$$\begin{tikzcd}
0 \arrow[r] & \ker p \arrow[r] & P \arrow[r, "p"] & J \arrow[r] & 0 \\
0 \arrow[r] & \ker p \arrow[r] \arrow[u, equal] & K \arrow[r, "j"] \arrow[u, "p'"] & I \arrow[r] \arrow[u, "f"] & 0
\end{tikzcd}$$

Writing, as before, $K = Q \oplus L$ with Q in $\underline{\underline{V}}$ and L in $\operatorname{mod}_{\underline{\underline{V}}} \Lambda$, we obtain $\ker f = (Q \cap M) \oplus L$ and $\underline{M} = \underline{K} = \underline{L} = \underline{\ker f}$, which shows that $L = M$.

Now the result follows easily from the facts above.

3. The category of $\underline{\underline{V}}$-injective modules.

In this sectio we introduce the indecomposable modules which behave as the principal injective modules of $\frac{\text{mod }\Lambda}{\underline{\underline{V}}}$. We assume throughout the section that Λ is an artin algebra and that $\underline{\underline{V}}$ is a subcategory of $\text{mod }\Lambda$ of finite type.

(3.1)

Definition 3.1. We say that I in $\text{ind}_{\underline{\underline{V}}}\,\Lambda$ is $\underline{\underline{V}}$-injective if one of the following conditions is satisfied.

(I1) I is in $\underline{\underline{I}}_o(\Lambda)$.

(I2) Ƨ'I is in $\underline{\underline{V}}$.

(I3) for every arrow $I \longrightarrow P$ in Γ_Λ , P is in $\underline{\underline{V}}$.

We will denote by $\underline{\underline{I}}(\Lambda/\underline{\underline{V}})$ or, simply, by $\underline{\underline{I}}$ the category (resp. subquiver of Γ_Λ) defined by the $\underline{\underline{V}}$-injective modules. When condition (I3) is satisfied, we say that I is a simple $\underline{\underline{V}}$-injective module.

The definition above is justified by the following fact.

Proposition 3.1. Let Λ, Λ' be hereditary artin algebras with Λ' hereditary, and let $\underline{\underline{V}}$, $\underline{\underline{V}}'$ be subcategories of finite type of $\text{mod }\Lambda$, $\text{mod }\Lambda'$, respectively, containing the indecomposable projective modules Let us assume further that $\underline{\underline{V}}'$ is open to the left in Γ_Λ and that there is an equivalence of categories

$$\frac{\text{mod }\Lambda}{\underline{\underline{V}}} \xrightarrow{\ \alpha\ } \frac{\text{mod }\Lambda'}{\underline{\underline{V}}'} .$$

Then $\alpha(\underline{\underline{I}}) = \underline{\underline{I}}_o(\Lambda') \setminus \underline{\underline{V}}'$.

PROOF. We observe that property (2.1),(4) is applicable here in both directions.If I is $\underline{\underline{V}}$-injective, then $\alpha(I)$ must be injective.

Otherwise, we would have the Auslander-Reiten sequence

$$0 \longrightarrow \alpha(I) \longrightarrow B' \longrightarrow C' \longrightarrow 0$$

and, by the assumptions on $\underline{\underline{V}}'$, B', C' would not be in $\underline{\underline{V}}'$. By property (4), we would get an Auslander-Reiten sequence

$$0 \longrightarrow I \longrightarrow B \longrightarrow \alpha^{-1}(C') \longrightarrow 0$$

with B, $\alpha^{-1}(C')$ not in $\underline{\underline{V}}$, a contradiction to the fact that I is in $\underline{\underline{I}}$. Conversely, if I is not in $\underline{\underline{I}}$, there is an Auslander-Reiten sequence $0 \longrightarrow I \longrightarrow B \longrightarrow C \longrightarrow 0$ with B and C outside of $\underline{\underline{V}}$. Hence there is an Auslander-Reiten sequence beginning with $\alpha(I)$, and $\alpha(I)$ is not injective.

Remark. If $\underline{\underline{V}}'$ has no oriented cycles and is open to the left in Γ_Λ and of finite type, then ind $\underline{\underline{V}}'$ consists of preprojective Λ'-modules. It follows then, from Prop. 1.8, that *if all ring components of Λ' are of infinite representation type* then $\alpha(\underline{\underline{I}}) = \underline{\underline{I}}_0(\Lambda')$. In this case Λ' is completely determined by $\underline{\underline{I}}$ up to Morita equivalence. To see this, put $T = \bigoplus_{I \in \underline{\underline{I}}} I$ and $T' = \bigoplus_{I' \in \underline{\underline{I}}_0(\Lambda')} I'$, so that $\alpha(T) = T'$. Then we have:

$$\Lambda' \cong \mathrm{End}_{\Lambda'}(T') \cong \frac{\Lambda'}{\underline{\underline{V}}'}(T',T') \cong \frac{\Lambda}{\underline{\underline{V}}}(T,T),$$

if Λ' is basic.

Corollary. Under the hypothesis of Prop. 3.1, we have:

(1) $\underline{\underline{I}}$ has no oriented cycles.

(2) $\underline{\underline{I}}$ is open to the right in $\Gamma_\Lambda \setminus \underline{\underline{V}}$.

(3.2) Let $\underline{\underline{V}}$ be a subcategory of modΛ of finite type. We define the *left boundary* of $\underline{\underline{V}}$ in Γ_Λ as the set of all vertices M in $\mathrm{ind}_{\underline{\underline{V}}}\Lambda$

such that there is an arrow $M \longrightarrow P$ with P in $\underline{\underline{V}}$. In case $\underline{\underline{I}}$ contains the left boundary of $\underline{\underline{V}}$ in Γ_Λ, we can give another characterization for the $\underline{\underline{V}}$-injective modules.

Proposition 3.2. Let Λ be an artin algebra and $\underline{\underline{V}} = \mathrm{add}(\underline{\underline{P}}^n(\Lambda))$, for some natural number n. If $\underline{\underline{I}}$ contains the left boundary of $\underline{\underline{V}}$ in Γ then the following are satisfied.

(i) If P is in $\underline{\underline{V}}$, then $T(P)$ (see (1.3)) is in add $\underline{\underline{I}}$.

(ii) For every I in $\underline{\underline{I}} \setminus \underline{\underline{I}}_o(\Lambda)$ there is a P in ind $\underline{\underline{V}}$ such that I is a component of $T(P)$.

PROOF. For the proof of (i), let I be a component of $T(P)$ which is not $\underline{\underline{V}}$-injective. We have then a commutative diagram

$$\begin{tikzcd}
I \arrow[rr, "f"] \arrow[dr, "i"'] & & E \arrow[dl, "f'"] \\
& P &
\end{tikzcd}$$

where f is the minimal left almos split map. Since i is not a split monomorphism, Im f' is not contained in $T(P)$. Therefore, there is an epimorphism from E onto some module in $\underline{\underline{V}}$ (see Prop. 1.4)), which gives us a contradiction with the assumption that I is not in $\underline{\underline{I}}$.

The proof of (ii) is even easier. If I is in $\underline{\underline{I}}$, there is at least one arrow $I \longrightarrow P$ with P in $\underline{\underline{V}}$ (unless I is injective). Since the indecomposable of $\underline{\underline{V}}$ are preprojective, this arrow must correspond to a monomorphism. Hence, I is contained in $T(P)$ and, by the definition of an irreducible map, must be a component of $T(P)$.

Corollary. Under the hypothesis of Prop. 3.2, let $M \xrightarrow{f} P$ be a morphism with M in $\mathrm{mod}_{\underline{\underline{V}}} \Lambda$ and P in $\underline{\underline{V}}$. If $M \xrightarrow{i} I$ is the $\underline{\underline{I}}$-envelope of M (see (1.2)) then f factors through i (i.e., there is an $I \xrightarrow{f'} P$ such that $f = f'i$). (Notice that, in this case, i is a monomorphism.)

Now we are going to investigate conditions such that, when Λ is equivalent to a hereditary algebra modulo preprojectives, $\underline{\underline{I}}$ contains the left boundary of $\underline{\underline{V}}$. We fix a natural number n and consider artin algebras Λ, Λ' such that Λ' is hereditary and there is an equivalence of categories $\frac{\operatorname{mod} \Lambda}{\underline{\underline{V}}} \xrightarrow{\alpha} \frac{\operatorname{mod} \Lambda'}{\underline{\underline{V}}'}$, where $\underline{\underline{V}}$ is the category $\operatorname{add}(\underline{\underline{P}}^n(\Lambda))$ and $\underline{\underline{V}}'$ is $\operatorname{add}(\underline{\underline{P}}^n(\Lambda'))$.

Lemma 3.1. If M is in $\operatorname{ind}_{\underline{\underline{V}}}\Lambda$ and M is in $\underline{\underline{V}}$ then $\alpha(M)$ is in $\underline{\underline{P}}_{n+1}(\Lambda') \cup \underline{\underline{P}}_{n+2}(\Lambda')$. Furthermore, if there is an arrow $X \to M$ with X not in $\underline{\underline{V}}$, then $\alpha(X)$ is in $\underline{\underline{P}}_{n+1}(\Lambda')$.

PROOF. $\alpha(M)$ cannot be projective, so that we may consider the almost split sequence $0 \to \tau\alpha(M) \to B \to \alpha(M) \to 0$. If $\tau\alpha(M)$ is not in $\underline{\underline{V}}'$, it follows from (2.1),(4) that $\tau(M)$ is not in $\underline{\underline{V}}$. Hence, $\tau\alpha(M)$ is in $\underline{\underline{V}}'$. On the other hand, if $\alpha(M)$ is in some $\underline{\underline{P}}_m(\Lambda')$ with m greater than $n+2$, then, by Prop. 1.6, the components of B are at least in $\underline{\underline{P}}_{n-1}(\Lambda')$ and $\tau\alpha(M)$ would not be in $\underline{\underline{V}}'$. This shows the first part of the statement. Let us suppose now that there is an arrow $X \to M$ with X not in $\underline{\underline{V}}$. Then, by Prop. 1.6, $\alpha(X)$ is in a $\underline{\underline{P}}_m(\Lambda')$ with m less than $n+2$. This completes the proof.

Lemma 3.2. Let us assume that all ring components of Λ' are of infinite representation type and that $\underline{\underline{V}}$ does not contain injective modules. Then, $\operatorname{ind} \underline{\underline{V}}$ does not contain τ-periodic modules.

PROOF. Let m be minimum such that there is a P in $\underline{\underline{P}}_m(\Lambda) \cap \underline{\underline{V}}$ which is τ-periodic. We observe first that all translates $\tau^s(P)$ $(s \in \mathbb{Z})$ are in $\underline{\underline{V}}$. Otherwise, for some s, we would have that $\tau^{s+1}(P)$ is in $\underline{\underline{I}}$. But, since $\alpha(I)$ is injective, I could not be τ-periodic (as a consequence of (2.1),(4)).

Let us consider an arrow $Q \to P$ in Γ_Λ with Q in $\underline{\underline{P}}_k(\Lambda)$ for some k less than m, and let us consider the translates $\tau'^s(Q)$

for s positive. If all those translates were defined, Q would be τ-periodic, a contradiction. Hence, for some s, $\tau'^{s}Q$ is injective. But, since there is an epimorphism $\tau'^{s}Q \longrightarrow \tau'^{s}P$, it follows that $\tau'^{s}Q$ is in $\underline{\underline{V}}$. And this is again a contradiction to the hypothesis that $\underline{\underline{V}}$ does no contain injective modules.

Lemma 3.3. Let us assume that all ring components of Λ' are of infinite representation type and that $\underline{\underline{V}}$ does not contain injective modules. If there is an arrow $P \longrightarrow M$ in Γ_{Λ} with P in $\underline{\underline{V}}$ and M not in $\underline{\underline{V}}$, then $\alpha(M)$ is preprojective.

PROOF. If $\alpha(M)$ were not preprojective, we would have in Γ_{Λ}:

$$\begin{tikzcd}
 & P \arrow[dr] & \\
\tau M \arrow[ur] \arrow[dr] & & M \\
 & X \arrow[ur] &
\end{tikzcd}$$

with $\alpha(M)$, $\alpha(\tau M)$ and $\alpha(X)$ not preprojectives. We distinguish two cases.

First case: $n = 0$. We have that $\tau' P$ is in $\operatorname{ind}_{\underline{\underline{V}}} \Lambda$ so that, by lemma 3.1, $\alpha(M)$ is preprojective.

Second case: $n > 0$. Here, P is not projective and we have a translated arrow $\tau P \longrightarrow \tau M$. If τP is not in $\underline{\underline{V}}$, it is in $\underline{\underline{I}}$, so that τM is in $\underline{\underline{I}}$ (see Cor. to Prop. 3.1), a contradiction. Repeating the argument we deduce that P must be τ-periodic (because $\operatorname{ind} \underline{\underline{V}}$ is finite), in contradiction to lemma 3.2.

Corollary. Under the hypothesis of lemma 3.3, if there is an arrow $P \longrightarrow M$ in Γ_{Λ} with P in $\underline{\underline{V}}$ and M not in $\underline{\underline{V}}$, then M is not in $\underline{\underline{I}}$.

Proposition 3.3. Let Λ, Λ' be artin algebras with Λ' hereditary and having no ring components of finite representation type. Let $\underline{\underline{V}} = \operatorname{add}(\underline{\underline{P}}^n(\Lambda))$ and $\underline{\underline{V}}' = \operatorname{add}(\underline{\underline{P}}^n(\Lambda'))$ and suppose that there is an equivalence of categories $\frac{\operatorname{mod}\Lambda}{\underline{\underline{V}}} \xrightarrow{\alpha} \frac{\operatorname{mod}\Lambda'}{\underline{\underline{V}}'}$. If there are no injective modules in $\underline{\underline{V}}$, then $\underline{\underline{I}}$ contains the left boundary of $\underline{\underline{V}}$ in Γ_Λ.

PROOF. Let M be in the left boundary of $\underline{\underline{V}}$, so that there is an arrow $M \to P$ in Γ_Λ with P, say, in $\underline{\underline{P}}_m(\Lambda)$ $(m \leqslant n)$. If m is strictly less than n, it follows from Prop. 1.3 that $\tau' M$ is in $\underline{\underline{V}}$ and M is in $\underline{\underline{I}}$, as has to be proved. So, we may assume that P is in $\underline{\underline{P}}_n(\Lambda)$. Since P is not τ-periodic (lemma 3.2) and since there are no injectives in $\underline{\underline{V}}$, there is a first natural number s $(s > 1)$ such that $\tau'^s P$ is not in $\underline{\underline{V}}$. It follows from lemma 3.1 that $\alpha(\tau'^s M)$ lies in $\underline{\underline{P}}_{n+1}(\Lambda)$, so that $\tau'^{(s-1)}M$ is in $\underline{\underline{V}}$. This implies that s is at least 2 and that $\tau'^{(s-2)}M$ is also in $\underline{\underline{V}}$ (otherwise, it will be in $\underline{\underline{I}}$, a contradiction to the corollary of lemma 3.3). By repeating the argument, we deduce that M is in $\underline{\underline{I}}$.

The rest of this section is devoted to establishing some properties of the $\underline{\underline{V}}$-injective modules related to when they co-generate a hereditary algebra. These properties, which are interesting in themselves, are needed for the proof of THEOREM 2.

(3.3) Let R be a finite quiver. We say that R is oriented from left to right if there are no subquivers of the form $\cdot \longrightarrow \cdot \longleftarrow \cdot$. Then, for each vertex x there is in R a unique maximal path $y \to \cdots \to x$ beginning at a source of R. Such an R has as many connected components as it has sources.

Let R be a finite quiver oriented from left to right. By a representation of R in Γ_Λ we mean a quiver map which associates to each vertex x of R an indecomposable Λ-module M_x, and to each arrow

arrow $x \xrightarrow{\alpha} y$ an irreducible morphism f_α from M_x to M_y. We say that such a representation is admissible if, whenever we have

$$\begin{tikzcd}
 & M_{y_1} \\
M_x \arrow[ur, "f_{\alpha_1}"] \arrow[r, "f_{\alpha_2}"] \arrow[dddr, "f_{\alpha_k}"] & M_{y_2} \\
 & \vdots \\
 & \\
 & M_{y_k}
\end{tikzcd}$$

with $M_{y_1} \cong M_{y_2} \cong \ldots \cong M_{y_k} \cong N$, then $f_{\alpha_1}, f_{\alpha_2}, \ldots, f_{\alpha_k}$ (viewed as a subset of $\dfrac{\operatorname{rad}(M_x,N)}{\operatorname{rad}^2(M_x,N)}$) is linearly independet with respect to $\operatorname{End}_\Lambda(N)/\operatorname{rad}(\operatorname{End}_\Lambda(N))$.

To each admissible representation $\rho = ((M_x)_{x \in R_0}; (f_\alpha)_{\alpha \in R_1})$ is associated a morphism f_ρ of mod Λ in the following form.

If R is connected with source s, let $s \xrightarrow{\alpha_1} x_1 \xrightarrow{\alpha_2} x_2 \cdots\cdots \xrightarrow{\alpha_k} x_k = x$ be the unique path in R going from s to the sink x. Then define

$$M_s \xrightarrow{f_\rho = (f_x)_x} \bigoplus_{\substack{x \text{ sink} \\ \text{of } R}} M_x$$

where each component f_x is defined by $f_x = f_{\alpha_k} f_{\alpha_{k-1}} \ldots f_{\alpha_1}$. If R is not connected, f_ρ is the direct sum of the morphisms associated by the previous definition to the representation of each connected component of R.

In order to simplify the exposition we will say simply that f_ρ is a morphism with support R.

Proposition 3.4. Let us assume that $\underline{I}$ is open to the right in $\Gamma_\Lambda \setminus \operatorname{ind} \underline{V}$ and that it does not have oriented cycles. Let R be a finite quiver, oriented from left to right. If ρ is an admissible representation

$$\rho = ((I_x)_{x \in R_0};\ (f_\alpha)_{\alpha \in R_1})$$

with values in $\underline{\underline{I}}$, then $\underline{f}_\rho$ is an epimorphism of $\dfrac{\text{mod } \Lambda}{\underline{\underline{V}}}$.

PROOF. We will consider several cases.

First case: R is a path.

We see at once that it is enough to consider the simple case $R = \cdot \longrightarrow \cdot$. Let $M \xrightarrow{f} N$ be the corresponding representation and let

$$M \xrightarrow{\begin{pmatrix} f \\ g \end{pmatrix}} \begin{matrix} N \\ \oplus \\ L \end{matrix}$$

be the minimal left almost split map associated to M. Since we have to show that $\underline{f}$ is an epimorphism, let us suppose that there is a morphism $N \xrightarrow{h} X$ such that $\underline{h}\,\underline{f} = 0$. This means that for some P in $\underline{\underline{V}}$ we have a commutative diagram

$$\begin{tikzcd}
 & P \arrow[dr, "h'"] & \\
M \arrow[ur, "f'"] \arrow[dr, "f"'] & & X \\
 & N \arrow[ur, "h"'] &
\end{tikzcd}$$

Since f' is not split monic, there is a map $\begin{matrix} N \\ \oplus \\ L \end{matrix} \xrightarrow{(f'', g'')} P$ such that $f' = f''f + g''g$. Therefore, we have:

$$hf = h'f' = h'f''f + h'g''g$$

or

$$(h'f'' - h)f + h'g''g = 0$$

or

$$(h'f'' - h,\ h'g'')\begin{pmatrix} f \\ g \end{pmatrix} = 0 \ .$$

Now, if M were injective, $\begin{pmatrix} f \\ g \end{pmatrix}$ would be epic and we would have $h'f'' - h = 0$, from which follows the desired conclusion $\underline{h} = 0$. On the other hand, if M is not injective, $\begin{pmatrix} h'f''-h \\ hg' \end{pmatrix}$ factors through $\zeta' M$ which is in $\underline{\underline{V}}$ and, again, we get $\underline{h'f'' - h} = 0$, or $\underline{h} = 0$.

Second case: general case.

The conditions imposed for defining f , together with the fact that $\underline{\underline{I}}$ is finite and does not contain oriented cycles, imply that there is a bound on the number of vertices of R. Hence, we can proceed by decreasing induction on this number $|R_o|$. We assume therefore that R produces a minimal counter-example to the proposition, the statement being true for quivers with more vertices than R.

Let h be a morphism such that $\underline{h}\ \underline{f} = 0$ and $\underline{h} \neq 0$. This morphism goes from the direct sum $\bigoplus_{\substack{x \text{ sink} \\ \text{of } R}} I_x$ to some module L, so that it is given by its componentes $h_x \colon I_x \longrightarrow L$.

First possibility: h_x is not split monic.

If I_x is a simple injective module, $h_x = 0$.

If I_x is not injective but is $\underline{\underline{V}}$-simple (see (3.1)), then $\underline{h}_x$ is 0.

In the remaining cases, we have the minimal left almost split map

$$\begin{tikzcd}
 & \bigoplus_j P_j \\
I_x \arrow[ur, "(g_j')_j"] \arrow[dr, "(g_i)_i"'] & \oplus \\
 & \bigoplus_{i=1}^{t} Y_i
\end{tikzcd}$$

where the P_j's are in $\underline{\underline{V}}$ and the Y_i's are in $\underline{\underline{I}}$ and where $t > 0$. It follows that there are maps $Y_i \xrightarrow{h_i'} L$ such that $\underline{h}_x = \sum_i \underline{h}_i' g_i$. In this case, we consider the quiver R' obtained adjoining to R t arrows going out from the R-source x:

$$\begin{tikzcd}
 & y_1 \\
x \arrow[ur, "\beta_1"] \arrow[dr, "\beta_t"'] & \vdots \\
 & y_t
\end{tikzcd}$$

and we extend the representation ρ of R to a representation ρ' of R' by putting Y_i at y_i and the morphism g_i corresponding to the arrow β_1. Let h' be the morphism going from the direct sum of the modules at the sinks of R' which coincides with h at sinks other than x (of R) and with components h'_i at the sinks y_i, then we have $\underline{h}'\,\underline{f}_{\rho'} = \underline{h}\,\underline{f}_{\rho} = 0$. By the induction hypothesis we get that $\underline{h}' = 0$, and this implies that $\underline{h}_x = 0$.

Second possibility: h_x is split monic.

Let y be the vertex of R which is linked to the sink x by an arrow: $y \longrightarrow x = x_1$, and let $y \longrightarrow x_i$ $(i \geqslant 2)$ be the other arrows going out from y. Let $I_i \xrightarrow{f_i} Ix_i$ be the morphisms corresponding to these arrows in the representation ρ. Finally, let f' be the morphism associated to the path $s \rightarrow \ldots \longrightarrow y$ of R (where s is the source of the connected component of x). Then

$$\begin{pmatrix} f_1 \\ \vdots \\ f_t \end{pmatrix} f'$$

is one of the components of f_ρ. By the definition of f_ρ, the f_i's for which I_{x_i} is isomorphic to $I_{x_1} = I_x$ are all linearly independent, modulo the radical squared, with respect to scalars from $\mathrm{End}_\Lambda(I_x)/\mathrm{rad}(\mathrm{End}_\Lambda(I_x))$. We observe also that it follows from the first case of this proof that $\underline{f}'$ is an epimorphism.

Since we are supposing that h_x is split monic, there is a morphism $L \xrightarrow{h'_x} I_x$ such that $h'_x h_x$ is the identity of I_x. Let us consider the following composition.

$$\begin{tikzcd}
 & & \bigoplus\limits_i I_{x_i} \arrow[dr, "{(h_{x_1},\ldots,h_{x_t})}"] & & \bigoplus\limits_i I_{x_i} \\
I_s \arrow[r, "f'"] & I_y \arrow[ur, "{\begin{pmatrix} f_1 \\ \vdots \\ f_t \end{pmatrix}}"] \arrow[dr, "g"'] & \oplus & L \arrow[ur, "{\begin{pmatrix} h'_x \\ 0 \\ \vdots \\ 0 \end{pmatrix}}"] \arrow[dr, "0"'] & \oplus \\
 & & Y \arrow[ur, "h_Y"'] & & Y
\end{tikzcd}$$

Since, by assumption, $\underline{h}\,\underline{f} = 0$, we obtain that:

$$\begin{pmatrix} \underline{h}'_x \\ 0 \\ \vdots \\ 0 \end{pmatrix} (\underline{h}_{x_1}, \dots, \underline{h}_{x_t}) \begin{pmatrix} \underline{f}_1 \\ \vdots \\ \underline{f}_t \end{pmatrix} \underline{f}' = 0 \quad .$$

Remembering that $\underline{f}'$ is an epimorphism, we get:

$$\begin{pmatrix} 1 & \dots & \underline{h}'_x \underline{h}_{x_t} \\ 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} \underline{f}_1 \\ \vdots \\ \underline{f}_t \end{pmatrix} = \begin{pmatrix} \underline{f}_1 + \dots + \underline{h}'_x \underline{h} x_t \underline{f}_t \\ 0 \\ \vdots \\ 0 \end{pmatrix} = 0$$

which means that $f_1 + \dots + h'_x h_{x_t} f_t$ is in $P(I_y, I_x) \subset \mathrm{rad}^2(I_y, I_x)$. But this is impossible and the proof is complete.

Corollary. Let $\underline{\underline{I}}$ be open to the right in Γ_Λ ind $\underline{\underline{V}}$ and let us suppose that it does not have oriented cycles. If $I \xrightarrow{f} M$ is the minimal left almost split map corresponding to an indecomposable module I of $\underline{\underline{I}}$, then $\underline{f}$ is an epimorphism.

Proposition 3.5. Let us assume that $\underline{\underline{I}}$ is open to the right in ind $\underline{\underline{V}}$ and let $\Lambda^* = \dfrac{\Lambda(T,T)}{P(T,T)}$, where $T = \bigoplus\limits_{I \in \underline{\underline{I}}} I$. Then the following propositions are equivalent.

(1) Λ^* is hereditary.

(2) $\underline{\underline{I}}$ has no oriented cycles.

(3) If $I \xrightarrow{f} M$ is the minimal left almost split map associated to an indecomposable module I in $\underline{\underline{I}}$, then $\underline{f}$ is an epimorphism of $\dfrac{\mathrm{mod}\ \Lambda}{\underline{\underline{V}}}$.

PROOF. As it is well known, the indecomposable injective Λ^*-modules corresponde in a one-to-one way to the elements of $\underline{\underline{I}}$, and the ordinary quiver of Λ^* may be identified with $\underline{\underline{I}}$. Hence, (1) implies (2). Prop. 3.4 means that (2) implies (3), so that we only have to show that (3) implies (1). In order to do this, it will be convenient to identify the category mod Λ^* with the category $\underline{\underline{C}}$ of finitely generated, covariant functors from $\dfrac{\text{add}\ \underline{\underline{I}}}{\underline{\underline{V}}}$ to the category of abelian groups. Let

$$\frac{\Lambda}{\underline{\underline{V}}}(K,-) \xrightarrow{g^*} \frac{\Lambda}{\underline{\underline{V}}}(I,-) \longrightarrow F \to 0$$

be a presentation of an object F of $\underline{\underline{C}}$. To prove that Λ^* is hereditary, it is enough to show that the projective dimension of F is $\leqslant 1$ when F is assumed to be simple. Hence, we suppose that F is simple, so that we may take I to be an element of $\underline{\underline{I}}$. K is an element of add $\underline{\underline{I}}$ and g^* is induced by a map $I \xrightarrow{g} K$, modulo $P(I,K)$.

We distinguish two cases.

First case: I is simple modulo $\underline{\underline{V}}$ (see (3.1)).

In this case, $\underline{g}$ is either 0 or a split monomorphism. If $\underline{g} = 0$, $F \cong \dfrac{\Lambda}{\underline{\underline{V}}}(I,-)$, and, if $\underline{g}$ is split monic, g^* is epi and F is 0.

Second case: I is not $\underline{\underline{V}}$-simple.

We may as well suppose that g is not a split monomorphism. Let $I \xrightarrow{f} M$ be the minimal left almost split map associated to I. Since, by assumption, $\underline{f}$ is epic, the following sequence is exact.

$$0 \to \frac{\Lambda}{\underline{\underline{V}}}(J,-) \xrightarrow{f^*} \frac{\Lambda}{\underline{\underline{V}}}(I,-) \longrightarrow \text{cok } f^* \longrightarrow 0$$

where J is the module in add $\underline{\underline{I}}$ which is equal to M modulo $\underline{\underline{V}}$. We show first that cok f^* is a simple functor of $\underline{\underline{C}}$ and, then, that it is isomorphic to F. With this, the proof will be complete.

In order to show that cok f^* is simple, we prove that any morphism $G \xrightarrow{\xi} \dfrac{\Lambda}{\underline{\underline{V}}}(I,-)$ of $\underline{\underline{C}}$ whose image is not contained in the image of f^*, is an epimorphism. Clearly, we can suppose that G is a projective object: $G = \dfrac{\Lambda}{\underline{\underline{V}}}(X,-)$, so that ξ is a map h^* induced by an $I \xrightarrow{h} X$, modulo $\underline{\underline{V}}$. Our restrictions mean that there is a morphism $I \xrightarrow{a} Y$, such that we have $\underline{a} = \underline{b}\ \underline{h}$ for some $X \xrightarrow{b} Y$, with the property that, for all $M \xrightarrow{a'} Y$, we have $\underline{a} \neq \underline{a'}\,\underline{f}$. Hence, remembering that f is minimal left almost split, we see that $\underline{a}$ must be a split monomorphism, which implies that $\underline{h}$ is also a split monomorphism and, therefore, that h^* is an epimorphism.

To show that $\operatorname{cok} f^*$ is isomorphic to F, we write $\underline{g}$ in the form $\underline{g} = \underline{g'}\,\underline{f}$, and we consider the following commutative diagram.

$$\begin{tikzcd}
 & \frac{\Lambda}{\underline{\underline{V}}}(K,-) \arrow[r, "g^*"] \arrow[d, "g'^*"] & \frac{\Lambda}{\underline{\underline{V}}}(I,-) \arrow[r] \arrow[d, equal] & F \arrow[r] \arrow[d, "\alpha"] & 0 \\
0 \arrow[r] & \frac{\Lambda}{\underline{\underline{V}}}(J,-) \arrow[r, "f^*"] & \frac{\Lambda}{\underline{\underline{V}}}(I,-) \arrow[r] & \operatorname{cok} f^* \arrow[r] & 0
\end{tikzcd}$$

Since F and $\operatorname{cok} f^*$ are both simple, they must be isomorphic.

Lemma 3.4. Let n be some natural number and let $\underline{\underline{V}} = \operatorname{add}(\underline{\underline{P}}^n(\Lambda))$. Let us assume that $\underline{\underline{I}}$ is open to the right in $\Gamma_\Lambda \setminus \operatorname{ind} \underline{\underline{V}}$ and that it does not have oriented cycles. Let $I \xrightarrow{f} M$ be a morphism with I, M indecomposable Λ-modules and with I in $\underline{\underline{I}}$. Then, if $\underline{f} \neq 0$, $\underline{f}$ is an epimorphism and M is in $\underline{\underline{I}}$.

PROOF. Let t be the maximum of the lengths of paths $I \to \dots \to M$ of $\underline{\underline{I}}$ beginning with I. We proceed by induction on t. If f is not an isomorphism, we may write $\underline{f} = \underline{f'}\underline{g}$ where $I \xrightarrow{g = (g_i)_i} \bigoplus_i M_i$ is the minimal left almost split map associated to I. By the assumption, since at least one $\underline{f'_i}$ is different from 0, we deduce that M is in $\underline{\underline{I}}$. To finish the proof we observe that each morphism in $\operatorname{add} \underline{\underline{I}} / \underline{\underline{V}}$ may be thought as a morphism between injectives of $\operatorname{mod} \Lambda^*$ where $\Lambda^* = \operatorname{End}_\Lambda(T)/P(T,T)$ is hereditary (by Prop. 3.5). Hence, $\underline{f}$ is an epimorphism.

Lemma 3.5. Let us assume that each indecomposable morphism $I \xrightarrow{f} J$

of add $\underline{\underline{I}} / \underline{\underline{V}}$ has a kernel and that, if $\underline{f}$ is not a monomorphism, the corresponding sequence $0 \to M \to I \to J \to 0$ is exact in add $\underline{\underline{I}}/\underline{\underline{V}}$

Let us suppose that $\underline{\underline{I}}$ is open to the right in Γ_Λ ind $\underline{\underline{V}}$, that $\underline{\underline{I}}$ contains the left boundary of $\underline{\underline{V}}$, and that it does not contain oriented cycles. Let M be in $\mathrm{ind}_{\underline{\underline{V}}}\Lambda$ and let $M \xrightarrow{i} I$ be the minimal $\underline{\underline{I}}$-envelope of M (see (1.2)). Then, there is a morphism $I \xrightarrow{\underline{f}} J$ such that $M \xrightarrow{\underline{i}} I$ is the kernel of $\underline{f}$ in $\dfrac{\mathrm{mod}\ \Lambda}{\underline{\underline{V}}}$.

PROOF. Let $I \xrightarrow{g} N$ be the cokernel of i. If N is equal to J modulo $\underline{\underline{V}}$, we have that $\underline{g} = \underline{f}$ is of the required type $I \xrightarrow{\underline{f}} J$. Let us consider the commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \ker p \arrow[r] & P \arrow[r, "p"] & N \arrow[r] & 0 \\
0 \arrow[r] & \ker p \arrow[r] \arrow[u, equal] & K \arrow[r, "g'"] \arrow[u, "p'"] & I \arrow[r] \arrow[u] & 0 \\
& & M \arrow[u, "i'"] \arrow[r, equal] & M \arrow[u, "i"] & \\
& & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

where $P \xrightarrow{p} N$ is the $\underline{\underline{V}}$-cover of N and where $K \begin{array}{l}\nearrow P\\ \searrow I\end{array}$ is the pull-back of $\begin{array}{l}P \searrow\\ I \nearrow\end{array} N$. Let us decompose $K = Q \oplus L$ with Q in $\underline{\underline{V}}$ and L in $\mathrm{mod}_{\underline{\underline{V}}}\Lambda$. Then, i', g' decompose, respectively, as follow:

$$\begin{matrix}Q\\ \oplus\\ L\end{matrix} \xrightarrow{(c,d)} P \qquad \text{and} \qquad M \xrightarrow{\binom{a}{b}} \begin{matrix}Q\\ \oplus\\ L\end{matrix} .$$

By the corollary to Prop. 3.2, $a = a'i$ for some $I \xrightarrow{a'} Q$. Hence, $i = g'i' = (c,d)\binom{a}{b} = ca + db = ca'i + db$ and $(1-ca')i = db$. Since ca' lies in $\mathrm{rad}(\mathrm{End}_\Lambda(I))$, $1-ca'$ is an automorphim F^{-1} of I. Hence we have that $i = Fdb$ so that b must be a monomorphism. We may consider

$$P = \frac{Q \oplus L}{\{(a(m),b(m)) \ / \ m \in M\}}$$

as the pushout:

$$\begin{tikzcd}
 & Q \arrow[dr, "\alpha"] & \\
M \arrow[ur, "a"] \arrow[dr, "-b"'] & & P \\
 & L \arrow[ur, "\beta"'] &
\end{tikzcd}$$

and form the following commutative diagram.

$$\begin{tikzcd}
0 \arrow[r] & Q \arrow[r, "\alpha"] & P \arrow[r] & \frac{L}{M} \arrow[r] & 0 \\
0 \arrow[r] & M \arrow[u, "a"] \arrow[r, "-b"] & L \arrow[u, "\beta"'] \arrow[r] & \frac{L}{M} \arrow[u, equal] \arrow[r] & 0
\end{tikzcd}$$

Since $a = a'i = a'Fdb = (-a'Fd)(-b)$, it follows that the upper exact sequence splits. This implies that $\frac{L}{M}$, being a summand of P, is in $\underline{\underline{V}}$; and, since it is an image of L which is in $\text{mod}_{\underline{\underline{V}}}\Lambda$, it has to be zero. Hence, $L=M$, $\underline{K} = \underline{L} = \underline{M}$ and $\underline{i} = \underline{d}\ \underline{b}$ is isomorphic to $\underline{d} = \underline{g}'$.

We observe that $\underline{i} \neq 0$ because, otherwise, we would have g factoring through P, and it would follow that $K = \ker p \oplus I$. Since K has only the indecomposable component M outside of $\underline{\underline{V}}$, this would imply that $M=I$, a contradiction. Since $\underline{g}\ \underline{i} = 0$, $\underline{g}$ is not a monomorphism. By the assumption, $\underline{g}$ has a kernel which must then be a summand of K, modulo $\underline{\underline{V}}$. From this, we get that $\ker \underline{g} = M$, as was to be proved.

4. Proofs of the main theorems.

Theorem 1.

Conditions (i) and (ii) are the items (1) and (2) of the corollary to Prop. 3.1. Condition (iii) coincides essencially with lemma 2.4, and the statement concerning Λ^* is proved in the remark following Prop. 3.1. Conditions (iv) and (v) are also immediate consequences of these facts. As for condition (vi), it coincides with Prop. 3.3.

Theorem 2.

The fact that Λ^* is hereditary is part of Prop. 3.5. In order to define the equivalence $\frac{\operatorname{mod}\Lambda^*}{\underline{\underline{V}}^*} \xrightarrow{\alpha} \frac{\operatorname{mod}\Lambda}{\underline{\underline{V}}}$ we consider the functor $\underline{f} \longrightarrow \ker \underline{f}$ which associates to each indecomposable morphism $\underline{f}$ in add $\underline{\underline{I}} / \underline{\underline{V}}$ the object 0, if $\underline{f}$ is monic, or the kernel M of the exact sequence $0 \longrightarrow M \xrightarrow{i} I \xrightarrow{f} J \longrightarrow 0$ of condition (iii). Considering the definition of $\underline{\underline{V}}^*$, this gives a functor α from $\operatorname{mod}\Lambda^* / \underline{\underline{V}}^*$ to $\operatorname{mod}\Lambda / \underline{\underline{V}}$. Using the fact that $M \xrightarrow{i} I$ is an $\underline{\underline{I}}$-envelope of M, we see that α is full, and using that $\underline{f}$ is the cokernel of $\underline{i}$, it follows that α is faithfull. Finally, applying Prop. 3.6, we obtain that α is dense.

Theorem 3 is clearly a corollary of theorem 2.

5. The case of stable equivalence $(n = 0)$.

In the classical paper [1] (1972) Auslander and Reiten studied the notion of stable equivalence (or equivalence modulo projectives) for artin algebras. In particular, they considered the following question:

When is an artin algebra Λ stably equivalent to a hereditary algebra?

And they gave, for it, the following necessary and sufficient conditions:

1) Every torsionless module M in $\operatorname{ind} \Lambda$ is either projective or simple.

2) If M, in $\operatorname{ind} \Lambda$, is torsionless not projective, then M is in the top of some indecomposable injective Λ-module.

In this section, we apply our techniques to get another proof of this result. So, throughout this section we make the additional assumption that $n=0$ (i.e., $\underline{\underline{V}}$ is the additive category generated by the projective Λ-modules and $\underline{V}'$ the category of projective Λ'-modules).

(5.1)

In this particular case, the minimal $\underline{\underline{V}}$-cover of a Λ-module M is just the projective cover of M, so that it is an essencial epimorphism. As an immediate consequence we obtain the following.

1. If f is a morphism in $\operatorname{mod}_{\underline{\underline{V}}} \Lambda$, f is an epimorphism if and only if $\underline{f}$ is (see [1], p.34).

2. If I is in $\underline{\underline{I}}$ and I is not injective, then I is simple.
PROOF. The proof is easy by contradiction. If I is not simple, we have an epimorphism $I \xrightarrow{f} S$ with S simple, not in $\underline{\underline{V}}$. On the other hand, since $\tau' I$ cannot be in $\underline{\underline{V}}$, I is $\underline{\underline{V}}$-simple and we get

$\underline{f} = 0$, a contradiction to 1. .

3. Let us suppose that Λ is stably equivalent to a hereditary algebra Λ': there is an equivalence $\frac{\text{mod } \Lambda}{\underline{\underline{V}}} \xrightarrow{\alpha} \frac{\text{mod } \Lambda'}{\underline{\underline{V}}'}$. If M is a submodule of a module P of $\underline{\underline{V}}$ (i.e., P is projective) and M is in $\text{ind}_{\underline{\underline{V}}} \Lambda$, then M is simple and lies at the top of an injective module.

PROOF. We show that, if M is contained in P, then M is in $\underline{\underline{I}}$. Otherwise, $\alpha(M)$ would not be injective and we would have $\alpha(M) \xrightarrow{i'} I'$, the injective envelope of $\alpha(M)$, with $\underline{i}' \neq 0$. Hence, there would be a map $M \xrightarrow{i} I$, with I injective and $\underline{i} \neq 0$. But then we get a contradiction because i has to factor through the inclusion $M \hookrightarrow P$.

Let now $I \to M$ be an irreducible map. If I were not injective, we would have a map $M \longrightarrow \tau' I$, in contradiction to the fact that M is in $\underline{\underline{I}}$ and is $\underline{\underline{V}}$-simple. Therefore, I is injective and M is at the top of I.

(5.2)

In this section we show that Auslander-Reiten conditions 1) and 2) are sufficient for Λ to be stably equivalent to a hereditary algebra. We do this by showing directly that THEOREM 2 (with condition ii; (vii)) is applicable here.

We begin with the remark that stable equivalence is a self-dual notion, so that we may formulate 1) and 2) as follows.

1)' If M in $\text{ind}_{\underline{\underline{V}}} \Lambda$ is a quotient of an injective module, then M is injective or simple.

2)' In the latter case of 1)', M is in the socle of an indecomposable projective module.

4. If 1)' and 2)' are satisfied, then the category add $\underline{\underline{I}}$ / $\underline{\underline{V}}$ is equivalent to add $\underline{\underline{I}}$.

PROOF. Let us consider the commutative diagram

$$\begin{tikzcd} I \arrow[rr, "f"] \arrow[dr, "g"'] & & J \\ & P \arrow[ur, "h"'] & \end{tikzcd}$$

with I, J in add $\underline{\underline{I}}$ and P in $\underline{\underline{V}}$. Then g(I) sits in soc(P), so that f = hg = 0.

5. Let us suppose that 1)' and 2)' are satisfied, and let M be an indecomposable Λ-module outside of $\underline{\underline{V}} \cup \underline{\underline{I}}$. Then, there is an indecomposable morphism $I \xrightarrow{f} J$, in add $\underline{\underline{I}}$, such that putting M= ker f then $i\colon M \hookrightarrow I$ is the injective envelope of M and $\underline{i}$ is the kernel of $\underline{f}$.

PROOF. Let $M \xhookrightarrow{i} I$ be the injective envelope of M and let f be the cokernel of i. Then, clearly, f is indecomposable as a morphism of mod Λ. Let us show that ker $\underline{f}$ = $\underline{i}$, by considering the pull-back of

$$\begin{tikzcd} I \arrow[r, "f"] & J \\ P \arrow[ur, "p"'] & \end{tikzcd}$$

, where p is the projective cover of J:

$$\begin{tikzcd}
0 \arrow[r] & \ker p \arrow[r] & P \arrow[r, "p"] & J \arrow[r] & 0 \\
0 \arrow[r] & \ker p \arrow[r] \arrow[u, equal] & K \arrow[r, "f'"] \arrow[u, "p'"'] & I \arrow[r] \arrow[u, "f"'] & 0 \\
& & M \arrow[r, equal] \arrow[u] & M \arrow[u, "i"'] & \\
& & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

We notice that p' is an epimorphism, so that $K = P \oplus M$ and $\underline{f}'$ may be identified to $\underline{i}$. Hence, we only have to show that $\underline{i}$ is a monomorphism and, for this, we consider the pull back of

$$\begin{tikzcd} K \arrow[r, "f'"] & I \\ Q \arrow[ur, "q"'] & \end{tikzcd}$$

where q is the projective cover of I:

$$\begin{tikzcd}
0 \arrow[r] & \ker p \arrow[r] & K \arrow[r, "f'"] & I \arrow[r] & 0 \\
0 \arrow[r] & \ker p \arrow[r] \arrow[u, equal] & K' \arrow[r] \arrow[u, "f''"'] & Q \arrow[r] \arrow[u, "q"'] & 0
\end{tikzcd}$$

Again, we have that $K' = Q \oplus \ker p$ and a simple computation shows that $\underline{f}'' = 0$.

6. Let M be in $\mathrm{ind}_{\underline{\underline{V}}}\underline{\Lambda}$ such that M is contained in a projective module P. If 1)' and 2)' are satisfied, then M is in $\underline{\underline{I}}$. As a consequence, we see that $\underline{\underline{I}}$ contains the left boundary of $\underline{\underline{V}}$, that $\underline{\underline{I}}$ is open to the right in $\Gamma_\Lambda \setminus \mathrm{ind}\ \underline{\underline{V}}$, and that $\underline{\underline{I}}$ does not have oriented cycles.

PROOF. If M were not in $\underline{\underline{I}}$, we can apply 5. to conclude that if i is the injective envelope of M then $\underline{i} \neq 0$. But this is a contradiction because i must factor through the inclusion $M \hookrightarrow P$.

7. If 1)' and 2)' are satisfied, then condition (iii) of Theorem 1 is satisfied.

PROOF. We observe that, by lemma 3.4 and 6., if $I \xrightarrow{\underline{f}} M$ is a non-zero morphism with I in $\underline{\underline{I}}$, then $M = N \oplus P$ with N in add $\underline{\underline{I}}$ and P in $\underline{\underline{V}}$. And, if g is the N-component of f, then g and $\underline{g}$ are epimorphisms (see 1.). Hence, if $\underline{f}$ is indecomposable in add $\underline{\underline{I}}$ / $\underline{\underline{V}}$ we may write $M = J$, an object of add $\underline{\underline{I}}$, and assume that f and $\underline{f}$ are epimorphisms.

Let us consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "i"] & I \arrow[r, "f"] & J \arrow[r] & 0 \\
0 \arrow[r] & M \arrow[r, "i'"] \arrow[u, equal] & I' \arrow[r, "f'"] \arrow[u, "g"'] & I' \arrow[r] \arrow[u, "g'"'] & 0
\end{tikzcd}$$

where i is the kernel of f and where i' is the injective envelope of M. We decompose I in the form $I = I_1 \oplus I_2$, where I_1 is the semisimple part of I. Since, by lemma 3.4, g is zero on I_1, it is easy to see that the lower sequence splits off from the upper one. Since f is indecomposable, we conclude that (g,g') is an isomorphism. Hence, M is an indecomposable module which is not in $\underline{\underline{I}}$. By property 5., $\underline{i}$ is the kernel of $\underline{f}$ and we only have to show that $\underline{f} = \operatorname{cok} \underline{i}$.

Let us suppose that we have a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[rr, "i"] \arrow[dr] \arrow[dd, "h"'] & & I \arrow[rr, "f"] \arrow[dr] \arrow[dd, "k"] & & J \arrow[r] \arrow[dr, equal] & 0 & \\
& & M' \arrow[rr, "i'"] \arrow[dl, "h'"] & & I' \arrow[rr, "f'"] \arrow[dl, "k'"] & & J \arrow[r] \arrow[dlll, "a"] & 0 \\
& P \arrow[rr] & & X & & & &
\end{tikzcd}$$

in order to show that $\underline{k}\,\underline{i} = 0$ implies that $\underline{k}$ factors through $\underline{f}$. We can assume that X is indecomposable and (by lemma 3.4) that it is in $\underline{\underline{I}}$. Also, since $\underline{i}$ is not zero, h is not injective. Putting $M' = M/\ker h$ and $I' = I/\ker h$ we get factorizations of h and k thorugh M' and I', respectively. Since M' is isomorphic to a submodule of P, M' is in add $\underline{\underline{I}}$ (see 6.) so that, applying 4., we deduce that $k'i' = 0$. Therefore, k' factors through $f' = \operatorname{cok} i'$ and it follows readily that k factors through f, as was to be proved.

8. If 1)' and 2)' are satisfied, then Λ is stably equivalent to a hereditary algebra.

PROOF. From properties 5., 6. and 7. follows that Theorem 2 (with condition (vii)) is applicable to Λ, so that $\frac{\operatorname{mod} \Lambda}{\underline{\underline{V}}}$ is equivalent to $\frac{\operatorname{mod} \Lambda^*}{\underline{\underline{V}}^*}$. This equivalence of categories is given by the functor that maps an indecomposable morphism $\underline{f}$ of add $\underline{\underline{I}} / \underline{\underline{V}}$ to $\ker \underline{f}$ and $\underline{\underline{V}}^*$ is the category of all such morphisms which are monomorphisms of $\frac{\operatorname{mod} \Lambda}{\underline{\underline{V}}}$. The only thing that has to be proved is that $\underline{\underline{V}}^*$ coincides with the category of the projective objects of Comod(add $\underline{\underline{I}} / \underline{\underline{V}}$). But now, using property 4. and the fact, easily proved, that such an $\underline{f}$ is a monomorphism if and only if ker f is projective, the completion of the proof is an exercise which may be left to the reader.

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