# ON THE REPRESENTATIONS OF LIE RING sl(2, Z)

A. Grishkov, A. Yuschenko

The theory of Lie rings is very different from that of Lie algebras. In this note we deal with the Lie ring of $2 \times 2$-matrices over an integral domain $R$ of characteristic zero. Let $F$ denote the field of fractions for $R$. A 3-dimensional free $R$-module $S$ in the Lie algebra $\mathrm{sl}(2, F)$ is called an *R-form* of $\mathrm{sl}(2, F)$ provided that $[S, S] \subseteq S$. Note that in this case $S \otimes_R F \simeq \mathrm{sl}(2, F)$.

The problem of describing all the $R$-forms of $\mathrm{sl}(2, F)$ is quite complicated and depends heavily on the ring $R$. A partial result in this direction was obtained by A.Yuschenko.

Let $R = \mathbf{Z}_p$ denote the ring of $p$-adic integers, where $p$ is an odd prime. Denote by $\mathbf{Q}_p$ the field of $p$-adic numbers. Let us fix three positive integers $n$, $m$, and $k$. Denote by $S_{n,m}$ the Lie algebra over $\mathbf{Z}_p$ with the basis $\{e, h, f\}$ and the multiplication:

$$[e, h] = p^n e, \qquad [e, f] = p^m h, \qquad [f, h] = -p^n f.$$

We will call such an algebra *diagonal*. We put $\Phi_{n,m,k} = \{S \subset \mathrm{sl}(2, \mathbf{Q}) \mid S = S_{a,b,c} = S_{n,m} + v\mathbf{Z}_p,\ v = (ae + bh + cf)/p^k,\ a, b, c \in \mathbf{Z}_p^*\}$. Clearly, the Lie algebras over $\mathbf{Z}_p$ belonging to $\Phi_{n,m,k}$ are $\mathbf{Z}_p$-forms of $\mathrm{sl}(2, \mathbf{Q}_p)$ if $n \geq k$ and $m \geq k$.

**Theorem.** *Let $n$, $m$, and $k$ be natural numbers such that $n \geq k$ and $m \geq k$. Then $\Phi_{n,m,k}$ contains just finitely many non-isomorphic Lie algebras over $\mathbf{Z}_p$.*

*A $\mathbf{Z}_p$-algebra $S(a, b, c)$ is diagonal if and only if $n = m$ and $\left(\frac{b^2 - 2ac}{p}\right) = 1$, where $\left(\frac{\cdot}{\cdot}\right)$ is the Lagrange symbol, or $n < m$.*

Now let us fix a diagonal form of $\mathrm{sl}(2, \mathbf{Q})$ isomorphic to $\mathrm{sl}(2, \mathbf{Z}) = \mathbf{Z}e \oplus \mathbf{Z}h \oplus \mathbf{Z}f$, where $[eh] = 2e$, $[ef] = h$, and $[fh] = -2f$. Denote by $\mathcal{O}$ the category of $S$-modules that are free and finite-dimensional as $\mathbf{Z}$-modules. If $V \in \mathcal{O}$, then $V$ is said to be *diagonal* if $V = \bigoplus_{i \in \mathbf{Z}} V_i$ where $V_i = \{v \in V \mid vh = iv\}$. One can easily show that every $S$-module $V \in \mathcal{O}$ includes a unique maximal diagonal submodule $V_d$. A module $V \in \mathcal{O}$ is called *irreducible* if $V$ is diagonal and the $\mathrm{sl}(2, \mathbf{Q})$-module $V \otimes_{\mathbf{Z}} \mathbf{Q}$ is irreducible. Every module $V \in \mathcal{O}$ includes a unique maximal submodule $V_m$ which is the direct sum of irreducible modules.

Now we will describe the structure of an irreducible $S$-module $V$. By definition, $V = \bigoplus_{i \in \Gamma} V_i$, where $\Gamma = \{-n, -n+2, \ldots, n - 2n\}$ and $V_i = \mathbf{Z}v_i$. If $v_i e = \alpha_i v_{i+2}$, $v_i f = \beta_i v_{i-2}$, and $\alpha_n = \beta_{-n} = 0$, then

$$\alpha_{n-2i}\beta_{n-2i+2} = -i(n - i + 1).$$

Hence, we can assume that $\alpha_i > 0$ and $\beta_i < 0$. Let $\overline{\alpha} = (\alpha_{-n}, \alpha_{-n+2}, \ldots, \alpha_{n-2}) \in \mathbf{N}^n$, where $\mathbf{N}^n$ stands for the set of the $n$-tuples of naturals. Denote by $V(\overline{\alpha})$ the $S$-module $V$ described above.

**Proposition.** *Every irreducible $S$-module $V$ is isomorphic to a module of type $V(\overline{\alpha})$ for some $\overline{\alpha}$. Two modules $V(\overline{\alpha})$ and $V(\overline{\gamma})$ are isomorphic if and only if $\overline{\alpha} = \overline{\gamma}$.*

Let $V(\overline{\alpha})$ and $V(\overline{\gamma})$ be irreducible $S$-modules and let $W = V(\overline{\alpha}) \otimes V(\overline{\gamma})$ be the diagonal $S$-module. In general, $W \neq W_m$. Our main goal is to compute the index $|W : W_m|$ of $W_m$ in $W$.

If $V(\overline{\alpha}) = \bigoplus_{i \in \Gamma} \mathbf{Z} v_i$ and $v = \sum_{i \in \Gamma} a_i v_i$ with $a_i \in \mathbf{Z}$ and $\Gamma \subset \mathbf{Z}$, then $|v| = a \in \mathbf{N}$ where $a_i / a \in \mathbf{Z}$, and $a$ is the maximal integer with such a property.

Let $\{v_n, v_{n-2}, \ldots, v_{-n}\}$ and $\{w_m, w_{m-2}, \ldots, w_{-m}\}$ be the bases of the modules $V(\overline{\alpha})$ and $V(\overline{\gamma})$ respectively. Then $W = V(\overline{\alpha}) \otimes V(\overline{\gamma})$ has the basis $\{v_i \otimes w_j \mid i = -n, \ldots, n; j = -m, \ldots, m\}$. Define in $W$ primitive vectors $u_{n+m}, u_{n+m_2}, \ldots, u_{n-m}$, $n \geq m$, as follows

$$u_{n+m-2k} = \sum_{0}^{k} x_i^k v_{n-2} \otimes w_{m-2k+2i},$$

where $u_{n+m-2k} e = 0$, $|u_{n+m-2k}| = 1$, $x_0^k > 0$, and $x_i^k \in \mathbf{Z}$.

Set $u_i^k = u_{n+m-2k} f^i$ and define the numbers $\Delta_N$, $N = 1, \ldots, [\frac{n+m}{2}]$ as follows:

$$\Delta_N = \prod_{j=1}^{N} (x_0^j j! (n+m-2j+3)^{j-1} (n+m-2j+2)^j) / \alpha_1^j,$$

if $N = 1, \ldots, m$ and

$$\Delta_N = (-1)^{N-m} C_N^{N-m} \beta_n \beta_{n-2} \ldots \beta_{n-2N+2m+2} \Delta_m,$$

if $N = m+1, \ldots, [\frac{n+m}{2}]$, where $\beta_{n-2i+2} = -i(n+1-i)/\alpha_{n-2i}$.

The main result is the following

**Theorem.**

$$|W : W_m| = \Delta \prod_{i=0}^{n+m-2k} \prod_{k=0}^{m} |u_k^i|^{-1} \prod_{N=1}^{[\frac{n+m-1}{2}]} \Delta_N^2,$$

*where* $\Delta = \Delta_{(n+m)/2}$ *if* $n+m$ *is even, and* $\Delta = 1$ *otherwise.*

Alexander Grishkov
Departamento de Matemática - IME
Universidade de São Paulo
CEP 05317-970, São Paulo, SP, Brasil
E-mail grishkov@ime.usp.br

Alexander Yuschenko
Department of Mathematics
Omsk State University
Prospekt Mira 55
Omsk 644077, Russia
E-mail yuschenk@univer.omsk.su