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# ON THE ORDER THEORY OF $\mathcal{C}^\infty$ -REDUCED $\mathcal{C}^\infty$ -RINGS AND APPLICATIONS

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## Abstract

In the present work we carry on the study of the order theory for ( $\mathcal{C}^\infty$ -reduced)  $\mathcal{C}^\infty$ -rings initiated in [16] (see also [4]). In particular, we apply some results of the order theory of  $\mathcal{C}^\infty$ -fields (*e.g.* every such field is real closed) to present another approach to the order theory of general  $\mathcal{C}^\infty$ -rings: “smooth real spectra” (see [5]). This suggests that a model-theoretic investigation of the class of  $\mathcal{C}^\infty$ -fields could be interesting and also useful to provide the first steps towards the development of the “Real Algebraic Geometry” of  $\mathcal{C}^\infty$ -rings.

**Keywords:** order theory,  $\mathcal{C}^\infty$ -rings, real spectrum

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## Introduction

Given a smooth manifold,  $M$ , the set  $\mathcal{C}^\infty(M, \mathbb{R})$  supports a far richer structure than just of an  $\mathbb{R}$ -algebra: it interprets not only the symbols for real polynomial functions but for all smooth real functions  $\mathbb{R}^n \rightarrow \mathbb{R}$ ,  $n \in \mathbb{N}$ . Thus,  $\mathcal{C}^\infty(M, \mathbb{R})$  is a natural instance of the algebraic structure called  $\mathcal{C}^\infty$ -ring.

It was not until the decades of 1970's and 1980's that a study of the abstract (algebraic) theory of  $\mathcal{C}^\infty$ -rings was made, mainly in order to construct topos models for "Synthetic Differential Geometry" ([12]). The interest in  $\mathcal{C}^\infty$ -rings gained strength in recent years, mainly motivated by the differential version of "Derived Algebraic Geometry" (see [11]).

In this paper we address the study of the order theory of  $\mathcal{C}^\infty$ -reduced  $\mathcal{C}^\infty$ -rings, presenting a useful characterization of the "natural order" of a  $\mathcal{C}^\infty$ -ring, introduced by Moerdijk and Reyes in [16]: given any  $\mathcal{C}^\infty$ -ring  $\mathfrak{A} = (A, \Phi)$ , this canonical strict partial order  $<$  is given by:

$$(a < b) \iff (\exists u \in A^\times)(b - a = u^2).$$

Since this natural binary relation given on a generic  $\mathcal{C}^\infty$ -ring involves invertible elements, we should first analyze these elements of a  $\mathcal{C}^\infty$ -ring. In order to do so, we shall restrict ourselves to the case of the  $\mathcal{C}^\infty$ -reduced  $\mathcal{C}^\infty$ -rings. This is carried out in two steps: first proving the results for finitely generated  $\mathcal{C}^\infty$ -rings and then proving them for arbitrary ones.

Since any  $\mathcal{C}^\infty$ -ring can be expressed as the quotient of a free object –  $\mathcal{C}^\infty(\mathbb{R}^E)$ , for some set  $E$  – by some (ring-theoretic) ideal, it is appropriate to characterize the equality between their elements by making use of these ring-theoretic ideals. We show that in this context the canonical strict partial order of a generic  $\mathcal{C}^\infty$ -ring, say  $\mathcal{C}^\infty(\mathbb{R}^E)/I$ , can be characterized by properties concerning filters of zerosets of functions in  $I$ .

In [16], Moerdijk and Reyes prove that every  $\mathcal{C}^\infty$ -field is real closed (cf. **Theorem 2.10**). This suggests that the class of  $\mathcal{C}^\infty$ -fields is "well behaved" with respect to its model theory.

We apply, in particular, some results on the order theory of  $\mathcal{C}^\infty$ -fields – *e.g.*, every such field is real closed (cf. **Theorem 2.10** in [16]). – to present another approach to the order theory of general  $\mathcal{C}^\infty$ -rings, introducing the so-called "smooth real spectra" (see [5]). This suggests that a model-theoretic study of the class of  $\mathcal{C}^\infty$ -fields could be interesting and also useful to provide the first steps towards the development of the "Real Algebraic Geometry" of  $\mathcal{C}^\infty$ -rings in the vein of [19].

**Overview of the paper:** In the first section we present some preliminary notions and results that are used (implicitly or explicitly) throughout the paper, such

as the basic concept of  $\mathcal{C}^\infty$ -ring and some features of the category of  $\mathcal{C}^\infty$ -rings,  $\mathcal{C}^\infty$ -fields,  $\mathcal{C}^\infty$ -rings of fractions,  $\mathcal{C}^\infty$ -radical ideals,  $\mathcal{C}^\infty$ -reduced  $\mathcal{C}^\infty$ -rings and some facts about the smooth Zariski spectrum. **Section 2** is devoted to present some results that (dually) connect subsets of  $\mathbb{R}^E$  to quotients of  $\mathcal{C}^\infty(\mathbb{R}^E)$ : we present the characterizations of equalities and inequalities between elements of  $\mathcal{C}^\infty$ -reduced  $\mathcal{C}^\infty$ -rings, *i.e.*,  $\mathcal{C}^\infty$ -rings of the form  $A = \frac{\mathcal{C}^\infty(\mathbb{R}^E)}{I}$  with  $\sqrt[I]{I} = I$ , by means of the filter of zerosets of functions of  $I$ , and we use a Galois connection between filters of zerosets of  $\mathbb{R}^E$  and ideals of  $\mathcal{C}^\infty(\mathbb{R}^E)$  to show that there are bijections between the set of maximal filters on  $\mathbb{R}^E$  and the set of maximal ideals of  $\mathcal{C}^\infty(\mathbb{R}^E)$ . In **Section 3** we develop a detailed study of the natural strict partial ordering  $<$  (introduced first by Moerdijk and Reyes) defined on a non-trivial  $\mathcal{C}^\infty$ -reduced  $\mathcal{C}^\infty$ -ring, with the aid of the results established in the previous sections (see **Theorem 3.7**). **Section 4** presents some interesting results on  $\mathcal{C}^\infty$ -fields based on the results from **Section 3**: for instance, every  $\mathcal{C}^\infty$ -field has  $<$  as its unique (strict) total ordering compatible with the operations  $+$  and  $\cdot$ , thus being a Euclidean field, cf. **Theorem 4.4**, (in fact, it is a real closed field); this is useful to analyse the concept of “real  $\mathcal{C}^\infty$ -spectrum of a  $\mathcal{C}^\infty$ -ring”, which seems to be the suitable notion to deal with a smooth version of Real Algebraic Geometry. Finally, **Section 5** brings some concluding remarks, pointing some possible applications of the order structure of a  $\mathcal{C}^\infty$ -reduced  $\mathcal{C}^\infty$ -ring to its model theory.

## 1 Preliminaries

In this section we present the ingredients of the theory of  $\mathcal{C}^\infty$ -rings needed in the sequel of this work for the reader’s convenience: we present the class of  $\mathcal{C}^\infty$ -rings as the class of models of an algebraic theory, and we describe the main notions of “Smooth Commutative Algebra of  $\mathcal{C}^\infty$ -rings”: smooth rings of fractions,  $\mathcal{C}^\infty$ -radicals,  $\mathcal{C}^\infty$ -saturation and the smooth Zariski spectra. The main references used here are [16], [17], [3], [4].

### 1.1 On the Algebraic Theory of $\mathcal{C}^\infty$ -Rings

In order to formulate and study the concept of  $\mathcal{C}^\infty$ -ring, we use a first order language  $\mathcal{L}$  with a denumerable set of variables ( $\mathbf{Var}(\mathcal{L}) = \{x_1, x_2, \dots, x_n, \dots\}$ ), whose nonlogical symbols are the symbols of all  $\mathcal{C}^\infty$ -functions from  $\mathbb{R}^m$  to  $\mathbb{R}^n$ , with  $m, n \in \mathbb{N}$ , *i.e.*, the non-logical symbols consist only of function symbols, described as follows.

For each  $n \in \mathbb{N}$ , we have the  $n$ -ary **function symbols** of the set  $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ , *i.e.*,  $\mathcal{F}_{(n)} = \{f^{(n)} \mid f \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})\}$ . Thus, the set of function symbols of our language

is given by:

$$\mathcal{F} = \bigcup_{n \in \mathbb{N}} \mathcal{F}_{(n)} = \bigcup_{n \in \mathbb{N}} \mathcal{C}^\infty(\mathbb{R}^n)$$

Note that our set of constants is  $\mathbb{R}$ , since it can be identified with the set of all 0-ary function symbols, *i.e.*,  $\mathbf{Const}(\mathcal{L}) = \mathcal{F}_{(0)} = \mathcal{C}^\infty(\mathbb{R}^0) \cong \mathcal{C}^\infty(\{*\}) \cong \mathbb{R}$ .

The terms of this language are defined in the usual way as the smallest set which comprises the individual variables, constant symbols and  $n$ -ary function symbols followed by  $n$  terms ( $n \in \mathbb{N}$ ).

Before we proceed, we give the following:

**Definition 1.1.** A  $\mathcal{C}^\infty$ -**structure** on a set  $A$  is a pair  $\mathfrak{A} = (A, \Phi)$ , where:

$$\begin{aligned} \Phi : \bigcup_{n \in \mathbb{N}} \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}) &\rightarrow \bigcup_{n \in \mathbb{N}} \text{Func}(A^n; A) \\ (f : \mathbb{R}^n \xrightarrow{\mathcal{C}^\infty} \mathbb{R}) &\mapsto \Phi(f) := (f^A : A^n \rightarrow A) \end{aligned}$$

that is,  $\Phi$  interprets the **symbols** of all smooth real functions of  $n$  variables as  $n$ -ary functions on  $A$ . Given two  $\mathcal{C}^\infty$ -structures,  $\mathfrak{A} = (A, \Phi)$  and  $\mathfrak{B} = (B, \Psi)$ , a  $\mathcal{C}^\infty$ -**structure homomorphism** is a function  $\varphi : A \rightarrow B$  such that for any  $n \in \mathbb{N}$  and any  $f : \mathbb{R}^n \xrightarrow{\mathcal{C}^\infty} \mathbb{R}$  the following diagram commutes:

$$\begin{array}{ccc} A^n & \xrightarrow{\varphi^{(n)}} & B^n \\ \Phi(f) \downarrow & & \downarrow \Psi(f) \\ A & \xrightarrow{\varphi} & B \end{array}$$

*i.e.*,  $\Psi(f) \circ \varphi^{(n)} = \varphi \circ \Phi(f)$ . The class of  $\mathcal{C}^\infty$ -structures and their morphisms compose a category that we denote by  $\mathcal{C}^\infty\mathbf{Str}$ .

We call a  $\mathcal{C}^\infty$ -structure  $\mathfrak{A} = (A, \Phi)$  a  $\mathcal{C}^\infty$ -**ring** if it preserves projections and all equations between smooth functions. Formally, we have the following:

**Definition 1.2.** Let  $\mathfrak{A} = (A, \Phi)$  be a  $\mathcal{C}^\infty$ -structure. We say that  $\mathfrak{A}$  (or, when there is no danger of confusion,  $A$ ) is a  $\mathcal{C}^\infty$ -**ring** if the following is true:

- Given any  $n, k \in \mathbb{N}$  and any projection  $p_k : \mathbb{R}^n \rightarrow \mathbb{R}$ , we have:

$$\mathfrak{A} \models (\forall x_1) \cdots (\forall x_n) (p_k(x_1, \dots, x_n) = x_k)$$

- For every  $f, g_1, \dots, g_n \in \mathcal{C}^\infty(\mathbb{R}^m, \mathbb{R})$  with  $m, n \in \mathbb{N}$ , and every  $h \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$  such that  $f = h \circ (g_1, \dots, g_n)$ , one has:

$$\mathfrak{A} \models (\forall x_1) \cdots (\forall x_m) (f(x_1, \dots, x_m) = h(g(x_1, \dots, x_m), \dots, g_n(x_1, \dots, x_m)))$$

Given two  $\mathcal{C}^\infty$ -rings,  $\mathfrak{A} = (A, \Phi)$  and  $\mathfrak{B} = (B, \Psi)$ , a  $\mathcal{C}^\infty$ -homomorphism is just a  $\mathcal{C}^\infty$ -structure homomorphism between these  $\mathcal{C}^\infty$ -rings. The category of all  $\mathcal{C}^\infty$ -rings and  $\mathcal{C}^\infty$ -ring homomorphisms make up a full subcategory of  $\mathcal{C}^\infty\mathbf{Str}$ , that we denote by  $\mathcal{C}^\infty\mathbf{Rng}$ .

**Remark 1.3** (cf. Sections 2, 3 and 4 of [3]). Since  $\mathcal{C}^\infty\mathbf{Rng}$  is a “variety of algebras” (it is a class of  $\mathcal{C}^\infty$ -structures which satisfy a given set of equations), it is closed under substructures, homomorphic images and products, by **Birkhoff’s HSP Theorem**. Moreover:

- $\mathcal{C}^\infty\mathbf{Rng}$  is a concrete category and the forgetful functor,  $U : \mathcal{C}^\infty\mathbf{Rng} \rightarrow \mathbf{Set}$  creates directed inductive colimits. Since  $\mathcal{C}^\infty\mathbf{Rng}$  is a variety of algebras, it has all (small) limits and (small) colimits. In particular, it has binary coproducts, that is, given any two  $\mathcal{C}^\infty$ -rings  $A$  and  $B$ , we have their coproduct  $A \xrightarrow{L_A} A \otimes_\infty B \xleftarrow{L_B} B$  again in  $\mathcal{C}^\infty\mathbf{Rng}$ ;

- Each set  $X$  freely generates a  $\mathcal{C}^\infty$ -ring,  $L(X)$ , as follows:

- for any finite set  $X'$  with  $\sharp X' = n$  we have  $L(X') = \mathcal{C}^\infty(\mathbb{R}^{X'}) \cong \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ , which is the free  $\mathcal{C}^\infty$ -ring on  $n$  generators,  $n \in \mathbb{N}$ ;

- for a general set,  $X$ , we take  $L(X) = \mathcal{C}^\infty(\mathbb{R}^X) := \varinjlim_{X' \subseteq_{\text{fin}} X} \mathcal{C}^\infty(\mathbb{R}^{X'})$ , with transition maps induced by restriction from  $\mathbb{R}^{X''}$  to  $\mathbb{R}^{X'}$  for  $X' \subseteq X'' \subseteq_{\text{fin}} X$ ;

- Given any  $\mathcal{C}^\infty$ -ring  $A$  and a set,  $X$ , we can freely adjoin the set  $X$  of variables to  $A$  with the following construction:  $A\{X\} := A \otimes_\infty L(X)$ . The elements of  $A\{X\}$  are usually called  $\mathcal{C}^\infty$ -polynomials;

- The congruences of  $\mathcal{C}^\infty$ -rings are classified by their “ring-theoretical” ideals;

- Every  $\mathcal{C}^\infty$ -ring is the homomorphic image of some free  $\mathcal{C}^\infty$ -ring determined by some set, being isomorphic to the quotient of a free  $\mathcal{C}^\infty$ -ring by some ideal.

Moreover, since  $\mathcal{C}^\infty\mathbf{Rng}$  is a variety of algebras, the **Fundamental Theorem of Homomorphism** holds (**Theorem 3** of [3]). We register this result here, for the benefit of the reader:

**Fact 1.4. (Fundamental Theorem of the  $\mathcal{C}^\infty$ -Homomorphism)** Let  $(A, \Phi)$  be a  $\mathcal{C}^\infty$ -ring and  $R \subseteq A \times A$  be a  $\mathcal{C}^\infty$ -congruence. For every  $\mathcal{C}^\infty$ -ring  $(B, \Psi)$  and for every  $\mathcal{C}^\infty$ -homomorphism  $\varphi : (A, \Phi) \rightarrow (B, \Psi)$  such that  $R \subseteq \ker(\varphi)$ , that is, such that:

$$(a, a') \in R \Rightarrow \varphi(a) = \varphi(a'),$$

there is a unique  $\mathcal{C}^\infty$ -homomorphism:

$$\tilde{\varphi} : \left( \frac{A}{R}, \overline{\Phi} \right) \rightarrow (B, \Psi)$$

such that the following diagram commutes:

$$\begin{array}{ccc}
 (A, \Phi) & \xrightarrow{\varphi} & (B, \Psi) \\
 q \downarrow & \nearrow \tilde{\varphi} & \\
 \left(\frac{A}{R}, \bar{\Phi}\right) & & 
 \end{array}$$

that is, such that  $\tilde{\varphi} \circ q = \varphi$ , where  $\bar{\Phi}$  is the canonical  $\mathcal{C}^\infty$ -structure induced on the quotient  $\frac{A}{R}$ .

Within the category of  $\mathcal{C}^\infty$ -rings we can perform a construction that is similar to the “ring of fractions” in Commutative Algebra, as well as define a suitable notion of “radical ideal”. We analyze these concepts in the following section.

## 1.2 On $\mathcal{C}^\infty$ -Rings of Fractions and $\mathcal{C}^\infty$ -Radical Ideals

In order to extend the notion of the ring of fractions to the category  $\mathcal{C}^\infty\mathbf{Rng}$ , we make use of the universal property a ring of fractions must satisfy in  $\mathbf{Ring}$ - except that we must deal with  $\mathcal{C}^\infty$ -rings and  $\mathcal{C}^\infty$ -homomorphisms instead of rings and homomorphisms of rings.

**Definition 1.5.** Let  $\mathfrak{A} = (A, \Phi)$  be a  $\mathcal{C}^\infty$ -ring and  $S \subseteq A$  be a subset. The  $\mathcal{C}^\infty$ -ring of fractions of  $A$  with respect to  $S$  is a  $\mathcal{C}^\infty$ -ring  $A\{S^{-1}\}$ , together with a  $\mathcal{C}^\infty$ -homomorphism  $\eta_S : A \rightarrow A\{S^{-1}\}$  satisfying the following properties:

- (1)  $(\forall s \in S)(\eta_S(s) \in (A\{S^{-1}\})^\times)$
- (2) If  $\varphi : A \rightarrow B$  is any  $\mathcal{C}^\infty$ -homomorphism such that for every  $s \in S$  we have  $\varphi(s) \in B^\times$ , then there is a unique  $\mathcal{C}^\infty$ -homomorphism  $\tilde{\varphi} : A\{S^{-1}\} \rightarrow B$  such that the following triangle commutes:

$$\begin{array}{ccc}
 A & \xrightarrow{\eta_S} & A\{S^{-1}\} \\
 & \searrow \varphi & \downarrow \tilde{\varphi} \\
 & & B
 \end{array}$$

By this universal property, the  $\mathcal{C}^\infty$ -ring of fractions is unique, up to (unique) isomorphisms.

The existence of smooth rings of fractions can be guaranteed by a combination of constructions:

- first consider the addition of  $\sharp S$ -variables to the  $\mathcal{C}^\infty$ -ring  $A$ :

$$A\{x_s | s \in S\} := A \otimes_\infty \mathcal{C}^\infty(\mathbb{R}^S),$$

and let  $j_S : A \rightarrow A\{x_s | s \in S\}$  be the (left) canonical morphism;

- now consider the ideal  $\langle \{x_s \cdot \iota_A(s) - 1 | s \in S\} \rangle$  of  $A$  generated by  $\{x_s \cdot \iota_A(s) - 1 | s \in S\}$ , and take the quotient:

$$A\{x_s | s \in S\} \xrightarrow{q_S} \frac{A\{x_s | s \in S\}}{\langle \{x_s \cdot \iota_A(s) - 1 | s \in S\} \rangle}.$$

Finally, define:

$$A\{S^{-1}\} := \frac{A\{x_s | s \in S\}}{\langle \{x_s \cdot \iota_A(s) - 1 | s \in S\} \rangle};$$

and

$$\eta_S := q_S \circ j_S : A \rightarrow A\{S^{-1}\}.$$

It is not difficult to see that such a construction satisfies the required universal property.

**Example 1.6.** *Let  $\varphi \in \mathcal{C}^\infty(\mathbb{R}^n)$  and consider the (closed) subset  $Z(\varphi) = \{\vec{x} \in \mathbb{R}^n : \varphi(\vec{x}) = 0\} \subseteq \mathbb{R}^n$ . Then  $\mathcal{C}^\infty(\mathbb{R}^n)\{\varphi^{-1}\} \cong \mathcal{C}^\infty(\mathbb{R}^{n+1})/\langle \{y \cdot \varphi - 1\} \rangle \cong \mathcal{C}^\infty(\mathbb{R}^n \setminus Z(\varphi))$  together with the restriction map  $\mathcal{C}^\infty(\mathbb{R}^n) \rightarrow \mathcal{C}^\infty(\mathbb{R}^n \setminus Z(\varphi))$  is a  $\mathcal{C}^\infty$ -homomorphism that satisfies the universal property of  $\eta_{\{\varphi\}}$ .*

Now we analyze the concept of the “ $\mathcal{C}^\infty$ -radical ideal” in the theory of  $\mathcal{C}^\infty$ -rings, which plays a similar role to the one played by radical ideals in Commutative Algebra. This concept was first presented by I. Moerdijk and G. Reyes in [16] in 1986, and explored in more detail in [17].

Unlike many notions in the branch of Smooth Rings such as  $\mathcal{C}^\infty$ -fields ( $\mathcal{C}^\infty$ -rings whose underlying rings are fields),  $\mathcal{C}^\infty$ -domains ( $\mathcal{C}^\infty$ -rings whose underlying rings are domains) and local  $\mathcal{C}^\infty$ -rings ( $\mathcal{C}^\infty$ -rings whose underlying rings are local rings), the concept of a  $\mathcal{C}^\infty$ -radical of an ideal cannot be brought from Commutative Algebra via the forgetful functor. This happens because when we take the localization of a  $\mathcal{C}^\infty$ -ring by an arbitrary prime ideal, it is not always true that we get a local  $\mathcal{C}^\infty$ -ring (see **Example 1.2** of [17]). In order to get a local  $\mathcal{C}^\infty$ -ring we must require an extra condition, that we are going to see later on.

Recall, from Commutative Algebra, that the radical of an ideal  $I$  of a commutative unital ring  $R$  is given by:

$$\sqrt{I} = \{x \in R \mid (\exists n \in \mathbb{N})(x^n \in I)\}.$$

There are several characterizations of this concept, among which we highlight the following ones:

$$\sqrt{I} = \bigcap \{\mathfrak{p} \in \text{Spec}(R) \mid I \subseteq \mathfrak{p}\} = \left\{ x \in R \mid \left( \frac{R}{I} \right) [(x + I)^{-1}] \cong 0 \right\}.$$

The latter equality is the one which motivates our next definition.

**Definition 1.7.** (cf. p. 329 of [16]) *Let  $A$  be a  $\mathcal{C}^\infty$ -ring and let  $I \subseteq A$  be a proper ideal. The  $\mathcal{C}^\infty$ -**radical** of  $I$  is given by:*

$$\sqrt[\infty]{I} := \left\{ a \in A \mid \left( \frac{A}{I} \right) \{(a + I)^{-1}\} \cong 0 \right\}$$

**Definition 1.8.** (cf. **Definition 2.1.5** of [2]) *Given a  $\mathcal{C}^\infty$ -ring  $A$  and a subset  $S \subseteq A$ , we define the  $\mathcal{C}^\infty$ -**saturation** of  $S$  by:*

$$S^{\infty\text{-sat}} := \{a \in A \mid \eta_S(a) \in A^\times\}.$$

**Example 1.9.** *Given  $\varphi \in \mathcal{C}^\infty(\mathbb{R}^n)$ , we have  $\{\varphi\}^{\infty\text{-sat}} = \{\psi \in \mathcal{C}^\infty(\mathbb{R}^n) \mid Z(\psi) \subseteq Z(\varphi)\}$ .*

The concept of  $\mathcal{C}^\infty$ -saturation is similar to the ordinary (ring-theoretic) concept of saturation in many aspects (for a detailed account of this concept, see **Section 3.1** of [4]). In particular, we use it to give a characterization of the elements of  $\mathcal{C}^\infty$ -radical ideals.

**Proposition 1.10.** [ **Proposition 3.48** of [4] ] *Let  $A$  be a  $\mathcal{C}^\infty$ -ring and let  $I \subseteq A$  be any ideal. We have the following equalities:*

$$\sqrt[\infty]{I} = \{a \in A \mid (\exists b \in I) \& (\eta_a(b) \in (A\{a^{-1}\})^\times)\} = \{a \in A \mid I \cap \{a\}^{\infty\text{-sat}} \neq \emptyset\}$$

where  $\eta_a : A \rightarrow A\{a^{-1}\}$  is the  $\mathcal{C}^\infty$ -homomorphism of fractions with respect to  $\{a\}$ .

In ordinary Commutative Algebra, given an element  $x$  of a ring  $R$ , we say that  $x$  is a nilpotent infinitesimal if and only if there is some  $n \in \mathbb{N}$  such that  $x^n = 0$ . Let  $A$  be a  $\mathcal{C}^\infty$ -ring and  $a \in A$ . D. Borisov and K. Kremnizer in [6] call  $a$  an  $\infty$ -infinitesimal if, and only if  $A\{a^{-1}\} \cong 0$ . The next definition describes the notion of a  $\mathcal{C}^\infty$ -ring being free of  $\infty$ -infinitesimals - which is analogous to the notion of “reducedness”, of a commutative ring.

**Definition 1.11.** A  $\mathcal{C}^\infty$ -ring  $A$  is  $\mathcal{C}^\infty$ -**reduced** if, and only if,  $\sqrt[\infty]{(0)} = (0)$ .

**Example 1.12.** The simplest example of  $\mathcal{C}^\infty$ -reduced  $\mathcal{C}^\infty$ -rings is the free  $\mathcal{C}^\infty$ -ring on any set of generators  $E$  (cf. **Proposition 4.47** of [4]).

Next we register some useful results on  $\mathcal{C}^\infty$ -radical ideals and  $\mathcal{C}^\infty$ -reduced  $\mathcal{C}^\infty$ -rings.

**Proposition 1.13 (Proposition 4.33, [4]).** Let  $A', B'$  be two  $\mathcal{C}^\infty$ -rings and  $j : A' \rightarrow B'$  be a monomorphism. If  $B'$  is  $\mathcal{C}^\infty$ -reduced, then  $A'$  is also  $\mathcal{C}^\infty$ -reduced.

**Proposition 1.14.** Let  $A$  be a  $\mathcal{C}^\infty$ -ring. We have:

- (a) An ideal  $J \subseteq A$  is a  $\mathcal{C}^\infty$ -radical ideal if, and only if,  $\frac{A}{J}$  is a  $\mathcal{C}^\infty$ -reduced  $\mathcal{C}^\infty$ -ring.
- (b) A proper prime ideal  $\mathfrak{p} \subseteq A$  is  $\mathcal{C}^\infty$ -radical if, and only if,  $\frac{A}{\mathfrak{p}}$  is a  $\mathcal{C}^\infty$ -reduced  $\mathcal{C}^\infty$ -domain.

*Proof.* See **Corollary 4.31** of [4]. □

Next we present some properties of  $\mathcal{C}^\infty$ -radical ideals of a  $\mathcal{C}^\infty$ -ring  $A$  regarding some “operations” such as the intersection, the directed union and the preimage by a  $\mathcal{C}^\infty$ -homomorphism of  $\mathcal{C}^\infty$ -radical ideals. To simplify the notation, given a  $\mathcal{C}^\infty$ -ring  $A$ , we denote by  $\mathfrak{I}_A^\infty$  the set of all its  $\mathcal{C}^\infty$ -radical ideals. The proofs of the results given in the next proposition can be found in [4].

**Proposition 1.15 (Proposition 4.42 of [4]).** The following results hold:

- (a) Suppose that  $(\forall \alpha \in \Lambda)(I_\alpha \in \mathfrak{I}_A^\infty)$ . Then  $\bigcap_{\alpha \in \Lambda} I_\alpha \in \mathfrak{I}_A^\infty$ , that is, if  $(\forall \alpha \in \Lambda)(I_\alpha \in \mathfrak{I}_A^\infty)$ , then:

$$\sqrt[\infty]{\bigcap_{\alpha \in \Lambda} I_\alpha} = \bigcap_{\alpha \in \Lambda} I_\alpha = \bigcap_{\alpha \in \Lambda} \sqrt[\infty]{I_\alpha}$$

- (b) Let  $\{I_\alpha | \alpha \in \Sigma\}$  an upwards directed family of elements of  $\mathfrak{I}_A^\infty$ . Then  $\bigcup_{\alpha \in \Sigma} I_\alpha \in \mathfrak{I}_A^\infty$ .
- (c) Let  $A, B$  be  $\mathcal{C}^\infty$ -rings,  $f : A \rightarrow B$  a  $\mathcal{C}^\infty$ -homomorphism and  $J \subseteq B$  any ideal. Then:

$$\sqrt[\infty]{f^{-1}[J]} \subseteq f^{-1}[\sqrt[\infty]{J}].$$

(d) Let  $A, B$  be  $\mathcal{C}^\infty$ -rings,  $f : A \rightarrow B$  be a  $\mathcal{C}^\infty$ -homomorphism and  $J \subseteq B$  be a  $\mathcal{C}^\infty$ -radical ideal. Then  $f^{-1}[J]$  is a  $\mathcal{C}^\infty$ -radical ideal of  $A$ .

(e) Given any two  $\mathcal{C}^\infty$ -radical ideals of a  $\mathcal{C}^\infty$ -ring  $A$ ,  $I, J \in \mathfrak{I}_A^\infty$ , we have:

$$\sqrt[\infty]{I \cdot J} = \sqrt[\infty]{I \cap J}$$

For a more comprehensive account of  $\mathcal{C}^\infty$ -radical ideals and  $\mathcal{C}^\infty$ -reduced  $\mathcal{C}^\infty$ -rings, we refer the reader to sections **3** and **4** of [4].

### 1.3 On the Smooth Zariski Spectrum

Recall that the spectrum of a commutative unital ring  $R$  consists of all prime ideals of  $R$ , together with a spectral topology - given by its “distinguished basic sets”, its Zariski topology. Recall, also, that in ordinary Commutative Algebra, every prime ideal is radical - and that the  $\mathcal{C}^\infty$ -version of this implication is false in the context of Smooth Commutative Algebra (not every prime ideal of a  $\mathcal{C}^\infty$ -ring is  $\mathcal{C}^\infty$ -radical). At this point it is natural to look for a  $\mathcal{C}^\infty$ -analog of the Zariski spectrum of a commutative unital ring. Keeping in mind the definitions of the previous subsection, we give the following definition, that can be found in [17]:

**Definition 1.16.** For a  $\mathcal{C}^\infty$ -ring  $A$ , we define:

$$\text{Spec}^\infty(A) = \{\mathfrak{p} \in \text{Spec}(A) \mid \mathfrak{p} \text{ is } \mathcal{C}^\infty\text{-radical}\}$$

equipped with the smooth Zariski topology defined by its basic open sets:

$$D^\infty(a) = \{\mathfrak{p} \in \text{Spec}^\infty(A) \mid a \notin \mathfrak{p}\}$$

**Proposition 1.17.** For every  $\mathcal{C}^\infty$ -ring  $A$ ,  $\text{Spec}^\infty(A)$  is a spectral space. Given two  $\mathcal{C}^\infty$ -rings  $A, A'$  and a  $\mathcal{C}^\infty$ -homomorphism  $f : A \rightarrow A'$ , The function:

$$\begin{aligned} f^* : \text{Spec}^\infty(A') &\rightarrow \text{Spec}^\infty(A) \\ \mathfrak{p} &\mapsto f^{-1}[\mathfrak{p}] \end{aligned}$$

is a spectral map.

*Proof.* For a detailed proof that  $\text{Spec}^\infty(A)$  is a spectral space for any  $\mathcal{C}^\infty$ -ring  $A$  we refer the reader to **Theorem 5.16** of [4]. For a detailed proof that for any  $\mathcal{C}^\infty$ -rings  $A, A'$  and any  $\mathcal{C}^\infty$ -homomorphism  $f : A \rightarrow A'$ , the map  $f^*$  is spectral, we refer the reader to **Proposition 5.19** of [4].  $\square$

**Theorem 1.18.** (*Separation Theorems*, [5]) *Let  $A$  be a  $\mathcal{C}^\infty$ -ring,  $S \subseteq A$  be a subset of  $A$  and  $I$  be an ideal of  $A$ . Denote by  $\langle S \rangle$  the multiplicative submonoid of  $A$  generated by  $S$ . We have:*

(a) *If  $I$  is a  $\mathcal{C}^\infty$ -radical ideal, then:*

$$I \cap \langle S \rangle = \emptyset \iff I \cap S^{\infty\text{-sat}} = \emptyset$$

(b) *If  $S \subseteq A$  is a  $\mathcal{C}^\infty$ -saturated subset, then:*

$$I \cap S = \emptyset \iff \sqrt[\infty]{I} \cap S = \emptyset$$

(c) *If  $\mathfrak{p} \in \text{Spec}^\infty(A)$ , then  $A \setminus \mathfrak{p} = (A \setminus \mathfrak{p})^{\infty\text{-sat}}$*

(d) *If  $S \subseteq A$  is a  $\mathcal{C}^\infty$ -saturated subset, then:*

$$I \cap S = \emptyset \iff (\exists \mathfrak{p} \in \text{Spec}^\infty(A))((I \subseteq \mathfrak{p}) \& (\mathfrak{p} \cap S = \emptyset)).$$

(e)  $\sqrt[\infty]{I} = \bigcap \{\mathfrak{p} \in \text{Spec}^\infty(A) \mid I \subseteq \mathfrak{p}\}$

*Proof.* See **Theorem 4.49** of [4]. □

A more detailed account of the smooth Zariski spectrum containing detailed proofs can be found in **Section 5.1** of [4].

## 2 On Smooth Spaces and Smooth Algebras

Every (finite dimensional) smooth manifold  $M$  can be embedded as a closed subspace of some  $\mathbb{R}^n$  (**Whitney's Theorem**) and determines a  $\mathcal{C}^\infty$ -ring,  $\mathcal{C}^\infty(M)$ . This mapping,  $M \mapsto \mathcal{C}^\infty(M)$ , extends to a full and faithful contravariant functor into the category of  $\mathcal{C}^\infty$ -rings (see for instance **Theorem 2.8** of [18]). In this section we present some results that (dually) connects subsets of  $\mathbb{R}^E$  and quotients of  $\mathcal{C}^\infty(\mathbb{R}^E)$ . More precisely, we present some very useful characterizations of equalities and inequalities between elements of  $\mathcal{C}^\infty$ -reduced  $\mathcal{C}^\infty$ -rings, *i.e.*,  $\mathcal{C}^\infty$ -rings of the form  $A = \frac{\mathcal{C}^\infty(\mathbb{R}^E)}{I}$  with  $\sqrt[\infty]{I} = I$ , by means of the filter of zerosets of functions of  $I$ .

## 2.1 The Finitely Generated Case

We start by recalling an important fact about the relation between closed subsets of  $\mathbb{R}^n$  and zerosets of  $\mathcal{C}^\infty$ -functions.

**Fact 2.1.** (essentially Lemma 1.4 of [18]) For each open subset  $U \subseteq \mathbb{R}^n$  there is a smooth function  $\chi : \mathbb{R}^n \rightarrow \mathbb{R}$  such that:

- $(\forall x \in \mathbb{R}^n)(\chi(x) \geq 0)$
- $(\forall x \in \mathbb{R}^n)((\chi(x) = 0) \iff (x \notin U)).$

**Definition 2.2 (smooth function).** Let  $X \subseteq \mathbb{R}^n$ . A function  $f : X \rightarrow \mathbb{R}$  defined over  $X$  is smooth if there is an open subset  $U \subseteq \mathbb{R}^n$  such that  $X \subseteq U$  and a  $\mathcal{C}^\infty$ -extension of  $f$ ,  $\tilde{f} : U \rightarrow \mathbb{R}$ , such that  $\tilde{f} \upharpoonright_X = f$ .

**Fact 2.3 (Smooth Tietze Theorem).** Let  $F \subseteq \mathbb{R}^n$  be a closed set and let  $f \in \mathcal{C}^\infty(F)$ . Then there is a smooth function  $\tilde{f} \in \mathcal{C}^\infty(\mathbb{R}^n)$  such that  $\tilde{f} \upharpoonright_F = f$ . Moreover:

- If  $(\forall x \in F)(f(x) \neq 0)$ , then we can choose a  $\mathcal{C}^\infty$ -extension  $\tilde{f}$  of  $f$  and an open subset  $U \subseteq \mathbb{R}^n$  such that  $F \subseteq U$  and  $(\forall x \in U)(\tilde{f}(x) \neq 0)$ .
- If  $(\forall x \in F)(f(x) > 0)$ , then we can choose a  $\mathcal{C}^\infty$ -extension  $\tilde{f}$  of  $f$  such that  $(\forall x \in \mathbb{R}^n)(\tilde{f}(x) > 0)$ .

**Proposition 2.4.** Let  $A = \frac{\mathcal{C}^\infty(\mathbb{R}^n)}{I}$  be a  $\mathcal{C}^\infty$ -reduced  $\mathcal{C}^\infty$ -ring, so  $\sqrt[\infty]{I} = I$ . Given  $f, g \in \mathcal{C}^\infty(\mathbb{R}^n)$ , we have:

$$(q_I(f) = f + I = g + I = q_I(g)) \iff (\exists \varphi \in I)(\forall x \in Z(\varphi))(f(x) = g(x)).$$

*Proof.* Suppose  $q_I(f) = f + I = g + I = q_I(g)$ , so  $g - f \in I$ . It suffices to take  $\varphi = g - f$ , so:

$$(\forall x \in Z(\varphi))(g(x) - f(x) = 0)$$

and

$$(\forall x \in Z(\varphi))(f(x) = g(x))$$

Conversely, if there is some  $\varphi \in I$  such that  $(\forall x \in Z(\varphi))(f(x) = g(x))$ , then  $Z(\varphi) \subseteq Z(g - f)$  and  $\varphi \upharpoonright_{\mathbb{R}^n \setminus Z(g-f)} \in \mathcal{C}^\infty(\mathbb{R}^n \setminus Z(g-f))^\times \cong \mathcal{C}^\infty(\mathbb{R}^n)\{(g-f)^{-1}\}^\times$  (see

**Example 1.6**). It follows that  $g - f \in \sqrt[\infty]{I} \subseteq I$  and  $f + I = g + I$ .  $\square$

Now we characterize the invertible elements of a  $\mathcal{C}^\infty$ -reduced  $\mathcal{C}^\infty$ -ring.

**Proposition 2.5.** *Let  $A = \frac{\mathcal{C}^\infty(\mathbb{R}^n)}{I}$  be a  $\mathcal{C}^\infty$ -reduced finitely generated  $\mathcal{C}^\infty$ -ring, so  $\sqrt[I]{I} = I$ . Given  $f \in \mathcal{C}^\infty(\mathbb{R}^n)$  we have:*

$$\left( q_I(f) = (f + I) \in \left( \frac{\mathcal{C}^\infty(\mathbb{R}^n)}{I} \right)^\times \right) \iff (\exists \varphi \in I)(\forall x \in Z(\varphi))(f(x) \neq 0).$$

*Proof.* Suppose, first, that  $(f + I) \in \left( \frac{\mathcal{C}^\infty(\mathbb{R}^n)}{I} \right)^\times$ , so there is some  $h + I \in \frac{\mathcal{C}^\infty(\mathbb{R}^n)}{I}$  such that:

$$(f + I) \cdot (h + I) = 1 + I$$

$$q_I(f) \cdot q_I(h) = q_I(1)$$

$$f \cdot h - 1 \in \ker(q_I)$$

$$\varphi = f \cdot h - 1 \in I,$$

One has:

$$(\forall x \in Z(\varphi))(f(x) \cdot h(x) = 1 \neq 0)$$

and thus:

$$(\forall x \in Z(\varphi))(f(x) = \frac{1}{h(x)} \neq 0)$$

Conversely, suppose that  $f \in \mathcal{C}^\infty(\mathbb{R}^n)$  is such that there is some  $\varphi \in I$  with  $(\forall x \in Z(\varphi))(f(x) \neq 0)$ . Since  $f$  is a continuous function, there is an open subset  $U \subseteq \mathbb{R}^n$  such that  $Z(\varphi) \subseteq U$  and  $(\forall x \in U)(f(x) \neq 0)$ .

We define:

$$\begin{aligned} g : U \subseteq \mathbb{R}^n &\rightarrow \mathbb{R} \\ x &\mapsto \frac{1}{f(x)} \end{aligned}$$

Thus  $g \upharpoonright_{Z(\varphi)}$  is smooth on  $Z(\varphi)$  and by **Smooth Tietze's Theorem (Fact 2.3)**, one is able to construct a  $\mathcal{C}^\infty$ -function  $\tilde{g} : \mathbb{R}^n \rightarrow \mathbb{R}$  such that  $\tilde{g} \upharpoonright_{Z(\varphi)} = g \upharpoonright_{Z(\varphi)}$ .

Since we have:

$$(\forall x \in Z(\varphi))(f(x) \cdot \tilde{g}(x) - 1 = 0)$$

it follows, by **Proposition 2.4**, that  $f \cdot \tilde{g} - 1 \in I$ , so:

$$(f + I) \cdot (\tilde{g} + I) = 1 + I$$

and  $f + I \in \left( \frac{\mathcal{C}^\infty(\mathbb{R}^n)}{I} \right)^\times$ . □

Combining the previous proposition with **Proposition 2.4**, we obtain the following:

**Corollary 2.6.** *Let  $A = \frac{\mathcal{C}^\infty(\mathbb{R}^n)}{I}$  be a finitely generated  $\mathcal{C}^\infty$ -reduced  $\mathcal{C}^\infty$ -ring, so  $\sqrt[I]{I} = I$ . Given  $f \in \mathcal{C}^\infty(\mathbb{R}^n)$ , are equivalent:*

1.  $(\exists u \in \mathcal{C}^\infty(\mathbb{R}^n))(((f + I) = (u^2 + I)) \& (u + I \in \left(\frac{\mathcal{C}^\infty(\mathbb{R}^n)}{I}\right)^\times))$
2.  $(\exists u \in \mathcal{C}^\infty(\mathbb{R}^n))(\exists \psi \in I)(\forall x \in Z(\psi))(f(x) = u^2(x) \neq 0)$
3.  $(\exists \psi \in I)(\forall x \in Z(\psi))(f(x) > 0)$

## 2.2 The General Case

We know that every closed subset of  $\mathbb{R}^n$  is the zeroset of some smooth function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  (see **Fact 2.1**). We now expand the notion of zero set to  $\mathbb{R}^E$ , where  $E$  is not necessarily a finite set.

**Definition 2.7.** *Let  $E$  be any set. Consider  $\mathbb{R}^E = \{v : E \rightarrow \mathbb{R} \mid v \text{ is a function}\}$  and denote  $\mathcal{F}(\mathbb{R}^E) := \{f : \mathbb{R}^E \rightarrow \mathbb{R} \mid f \text{ is a function}\}$ . For every  $D \subseteq E$ , we have the canonical projection:*

$$\begin{array}{ccc} \pi_{ED} : \mathbb{R}^E & \rightarrow & \mathbb{R}^D \\ v & \mapsto & v \upharpoonright_D : D \rightarrow \mathbb{R} \end{array}$$

and this induces a function:

$$\begin{array}{ccc} \mu_{DE} : \mathcal{F}(\mathbb{R}^D) & \rightarrow & \mathcal{F}(\mathbb{R}^E) \\ f & \mapsto & \mu_{DE}(f) := f \circ \pi_{ED} \end{array}$$

**Definition 2.8.** *Let  $E$  be any set. Define:*

$$\mathcal{C}^\infty(\mathbb{R}^E) := \{f \in \mathcal{F}(\mathbb{R}^E) \mid (\exists E' \subseteq_{\text{fin}} E)(\exists f' \in \mathcal{C}^\infty(\mathbb{R}^{E'}))(f = f' \circ \pi_{EE'})\}.$$

Thus, by a smooth function on  $\mathbb{R}^E$  we mean a function  $f : \mathbb{R}^E \rightarrow \mathbb{R}$  that factors through some projection  $\pi_{EE'} : \mathbb{R}^E \rightarrow \mathbb{R}^{E'}$  and a smooth function  $\tilde{f} \in \mathcal{C}^\infty(\mathbb{R}^{E'})$ , for some  $E' \subseteq_{\text{fin}} E$ . I.e., given  $f \in \mathcal{F}(\mathbb{R}^E)$  we have:

$$f \in \mathcal{C}^\infty(\mathbb{R}^E) \iff (\exists E' \subseteq_{\text{fin}} E)(\exists \tilde{f} \in \mathcal{C}^\infty(\mathbb{R}^{E'}))(f = \tilde{f} \circ \pi_{EE'}).$$

It is not hard to see that for every  $E', E'' \subseteq_{\text{fin}} E$  with  $E' \subseteq E''$ , the following diagram commutes:

$$\begin{array}{ccc}
 & \mathcal{C}^\infty(\mathbb{R}^E) & \\
 \mu_{E'E} \nearrow & & \nwarrow \mu_{E''E} \\
 \mathcal{C}^\infty(\mathbb{R}^{E'}) & \xrightarrow{\mu_{E'E''}} & \mathcal{C}^\infty(\mathbb{R}^{E''})
 \end{array}$$

Moreover, notice that  $\mathcal{C}^\infty(\mathbb{R}^E) \cong \varinjlim_{E' \subseteq_{\text{fin}} E} \mathcal{C}^\infty(\mathbb{R}^{E'})$ , where for every  $E', E'' \subseteq_{\text{fin}} E$  such that  $E' \subseteq E''$ , the following triangle commutes:

$$\begin{array}{ccc}
 & \varinjlim_{E' \subseteq_{\text{fin}} E} \mathcal{C}^\infty(\mathbb{R}^{E'}) & \\
 \ell_{E'} \nearrow & & \nwarrow \ell_{E''} \\
 \mathcal{C}^\infty(\mathbb{R}^{E'}) & \xrightarrow{\mu_{E'E''}} & \mathcal{C}^\infty(\mathbb{R}^{E''})
 \end{array}$$

where the morphisms  $\ell_{E'}, \ell_{E''}$  are defined as in **Section 3** of [3].

**Definition 2.9.** Let  $E$  be any set. A subset  $X \subseteq \mathbb{R}^E$  is a **zeroset** if there is some  $\varphi \in \mathcal{C}^\infty(\mathbb{R}^E)$  such that  $X = Z(\varphi)$ , where

$$Z(\varphi) := \varphi^{-1}[\{0\}] = \{\vec{x} \in \mathbb{R}^E : \varphi(\vec{x}) = 0\}$$

The set  $\mathcal{Z}_E := \{Z(\varphi) \in \wp(\mathbb{R}^E) : \varphi \in \mathcal{C}^\infty(\mathbb{R}^E)\}$  denotes the set of all zerosets in  $\mathbb{R}^E$ .

**Remark 2.10.** • Let  $E$  be an arbitrary set and  $\varphi \in \mathcal{C}^\infty(\mathbb{R}^E)$ . Select  $E' \subseteq_{\text{fin}} E$  and  $\varphi' \in \mathcal{C}^\infty(\mathbb{R}^{E'})$  such that  $\varphi = \varphi' \circ \pi_{EE'}$ . Then  $Z(\varphi) = \pi_{EE'}^{-1}[Z(\varphi')]$ .

• If  $E$  is a finite set, then by **Fact 2.1**,  $\mathcal{Z}_E = \text{Closed}(\mathbb{R}^E) \subseteq \wp(\mathbb{R}^E)$  thus it is stable under finite unions and arbitrary intersections; in particular,  $\emptyset = \bigcup \emptyset$  and  $\mathbb{R}^E = \bigcap \emptyset$  are in  $\mathcal{Z}_E$ .

• In general, for an arbitrary set  $E$ , the subset  $\mathcal{Z}_E \subseteq \wp(\mathbb{R}^E)$  is stable just under finite unions and finite intersections.

**Definition 2.11.** If  $I \subseteq \mathcal{C}^\infty(\mathbb{R}^E)$  is an ideal, then  $I' = \mu_{E'E}^{-1}[I]$  is an ideal of  $\mathcal{C}^\infty(\mathbb{R}^{E'})$ . We define:

$$\hat{I} = \{F \subseteq \mathbb{R}^E \mid (\exists E' \subseteq_{\text{fin}} E)(\exists f \in I' = \mu_{E'E}^{-1}[I])(F = \pi_{E'E}^{-1}[Z(f)])\}.$$

**Proposition 2.12.** Let  $A = \frac{\mathcal{C}^\infty(\mathbb{R}^E)}{I}$  be a  $\mathcal{C}^\infty$ -reduced  $\mathcal{C}^\infty$ -ring, so  $\sqrt[I]{I} = I$ . Given  $f, g \in \mathcal{C}^\infty(\mathbb{R}^E)$ , we have:

$$(q_I(f) = f + I = g + I = q_I(g)) \iff (\exists \varphi \in I)(\forall x \in Z(\varphi))(f(x) = g(x)).$$

*Proof.* Given  $f, g \in \mathcal{C}^\infty(\mathbb{R}^E)$  such that  $q_I(f) = f + I = g + I = q_I(g)$ , by definition there are finite subsets  $E_f, E_g \subseteq E$  and  $\hat{f} \in \mathcal{C}^\infty(\mathbb{R}^{E_f}), \hat{g} \in \mathcal{C}^\infty(\mathbb{R}^{E_g})$  such that  $f = \mu_{E_f}(\hat{f}) = \hat{f} \circ \pi_{E_f} \in \mathcal{C}^\infty(\mathbb{R}^{E_f})$  and  $g = \mu_{E_g}(\hat{g}) = \hat{g} \circ \pi_{E_g} \in \mathcal{C}^\infty(\mathbb{R}^{E_g})$ . Then  $E_f \cup E_g \subseteq_{\text{fin}} E$ . Let  $\tilde{f} = \mu_{E_f, E_f \cup E_g}(\hat{f}) \in \mathcal{C}^\infty(\mathbb{R}^{E_f \cup E_g})$  and  $\tilde{g} = \mu_{E_g, E_f \cup E_g}(\hat{g}) \in \mathcal{C}^\infty(\mathbb{R}^{E_f \cup E_g})$ . By hypothesis,  $f + I = g + I$ , so  $f - g \in I$  and  $\mu_{E_f \cup E_g, E}(\tilde{f}) - \mu_{E_f \cup E_g, E}(\tilde{g}) \in I$ . We have, thus,  $(\tilde{f} - \tilde{g}) \in \mu_{E_f \cup E_g, E}^{-1}[I] = \sqrt[\infty]{\mu_{E_f \cup E_g, E}^{-1}[I]}$ , since  $I$  is a  $\mathcal{C}^\infty$ -radical ideal (see **Proposition 1.15.(d)**). By the finitely generated case (**Proposition 2.4**), since  $\tilde{f}, \tilde{g} \in \mathcal{C}^\infty(\mathbb{R}^{E_f \cup E_g})$  and  $\tilde{f} + \mu_{E_f \cup E_g}^{-1}[I] = \tilde{g} + \mu_{E_f \cup E_g}^{-1}[I]$ , it follows that there is some  $\tilde{\varphi} \in \mu_{E_f \cup E_g}^{-1}[I]$  such that:

$$(\forall y \in Z(\tilde{\varphi}))(\tilde{f}(y) = \tilde{g}(y)).$$

Taking  $\varphi = \mu_{E_f \cup E_g, E}(\tilde{\varphi}) = \tilde{\varphi} \circ \pi_{E, E_f \cup E_g} \in I$ , we have:

$$(\forall x \in Z(\varphi))(f(x) = \tilde{f} \circ \pi_{E_f \cup E_g}(x) = \tilde{g} \circ \pi_{E_f \cup E_g}(x) = g(x)).$$

On the other hand, suppose  $f, g \in \mathcal{C}^\infty(\mathbb{R}^E)$  are such that  $(\exists \varphi \in I)(\forall x \in Z(\varphi))(f(x) = g(x))$ . Thus, for such  $\varphi$  there is a finite  $E_\varphi \subseteq E$  and  $\hat{\varphi} \in \mathcal{C}^\infty(\mathbb{R}^{E_\varphi})$  such that  $\varphi = \hat{\varphi} \circ \pi_{E, E_\varphi}$ , and there are also some finite  $E_f, E_g \subseteq E$  and some  $\hat{f} \in \mathcal{C}^\infty(\mathbb{R}^{E_f}), \hat{g} \in \mathcal{C}^\infty(\mathbb{R}^{E_g})$  such that  $f = \mu_{E_f, E}(\hat{f})$  and  $g = \mu_{E_g, E}(\hat{g})$ . Let  $\tilde{\varphi} = \mu_{E_\varphi, E_\varphi \cup E_f \cup E_g}(\hat{\varphi}), \tilde{f} = \mu_{E_f, E_\varphi \cup E_f \cup E_g}(\hat{f})$  and  $\tilde{g} = \mu_{E_g, E_\varphi \cup E_f \cup E_g}(\hat{g})$ . By the finitely generated case (**Proposition 2.4**), since  $\tilde{f}, \tilde{g}, \tilde{\varphi} \in \mathcal{C}^\infty(\mathbb{R}^{E_\varphi \cup E_f \cup E_g})$ ,  $(\forall x \in Z(\tilde{\varphi}))(\tilde{f}(x) = \tilde{g}(x))$  and

$$\sqrt[\infty]{\mu_{E_\varphi \cup E_f \cup E_g, E}^{-1}[I]} = \mu_{E_\varphi \cup E_f \cup E_g, E}^{-1}[I],$$

it follows that  $\tilde{f} - \tilde{g} \in \mu_{E_\varphi \cup E_f \cup E_g, E}^{-1}[I]$ , so  $f - g = \mu_{E_\varphi \cup E_f \cup E_g, E}(\tilde{f} - \tilde{g}) \in I$ , and  $f + I = g + I$ .  $\square$

**Proposition 2.13.** *Let  $E$  be any set and  $I \subseteq \mathcal{C}^\infty(\mathbb{R}^E)$  be a  $\mathcal{C}^\infty$ -radical ideal. We have, for every  $f \in \mathcal{C}^\infty(\mathbb{R}^E)$ :*

$$(f + I \in \left( \frac{\mathcal{C}^\infty(\mathbb{R}^E)}{I} \right)^\times) \iff (\exists \varphi \in I)(\forall x \in Z(\varphi))(f(x) \neq 0)$$

*Proof.* Given  $f \in \mathcal{C}^\infty(\mathbb{R}^E)$  such that  $q_I(f) = f + I$  is invertible, let  $h, \varphi \in \mathcal{C}^\infty(\mathbb{R}^E)$  such that

$$(f \cdot h - 1) = \varphi \in I.$$

As in the proof of previous proposition, we can select  $E' \subseteq_{\text{fin}} E$  and  $f', h', \varphi' \in \mathcal{C}^\infty(\mathbb{R}^{E'})$  such that  $f = \mu_{E'E}(f')$ ,  $h = \mu_{E'E}(h')$ ,  $\varphi = \mu_{E'E}(\varphi')$ . Then

$$(f' \cdot h' - 1) = \varphi' \in I' := \mu_{E'E}^{-1}[I].$$

Thus

$$(\forall x' \in \mathbb{R}^{E'})(x' \in Z(\varphi') \Rightarrow f'(x') \neq 0).$$

Since  $Z(\varphi) = \pi_{E'E'}^{-1}[Z(\varphi')]$  and  $f = f' \circ \pi_{E'E'}$ , then

$$(\forall x \in \mathbb{R}^E)(x \in Z(\varphi) \Rightarrow f(x) \neq 0).$$

Conversely, let  $f, \varphi \in \mathcal{C}^\infty(\mathbb{R}^E)$  such that  $\varphi \in I$  and

$$(\forall x \in \mathbb{R}^E)(x \in Z(\varphi) \rightarrow f(x) \neq 0).$$

Select  $E' \subseteq_{\text{fin}} E$  and  $f', \varphi' \in \mathcal{C}^\infty(\mathbb{R}^{E'})$  such that  $f = \mu_{E'E}(f')$ ,  $\varphi = \mu_{E'E}(\varphi')$ .

Then  $I' := \mu_{E'E}^{-1}[I]$  is a  $\mathcal{C}^\infty$ -radical ideal of  $\mathcal{C}^\infty(\mathbb{R}^{E'})$ ,  $\varphi' \in I'$  and

$$(\forall x' \in \mathbb{R}^{E'})(x' \in Z(\varphi') \rightarrow f'(x') \neq 0).$$

By the finitely generated case (**Proposition 2.5**),  $f' + I' \in (\mathcal{C}^\infty(\mathbb{R}^{E'})/I')^\times$ . Let  $h' \in \mathcal{C}^\infty(\mathbb{R}^{E'})$  such that

$$(f' + I')(h' + I') = 1 + I' \in \mathcal{C}^\infty(\mathbb{R}^{E'})/I'.$$

Now define  $h := \mu_{EE'}(h')$ . Then

$$(f + I)(h + I) = 1 + I \in \mathcal{C}^\infty(\mathbb{R}^E)/I$$

□

**Proposition 2.14.** *Let  $E$  be any set. If  $I \subseteq \mathcal{C}^\infty(\mathbb{R}^E)$  is an ideal, then:*

$$\hat{I} := \{X \in \wp(\mathbb{R}^E) \mid (\exists f \in I)(X = Z(f))\} \subseteq \wp(Z(\mathbb{R}^E))$$

*is a filter of zerosets in  $\mathbb{R}^E$ .*

*Proof.* Note first that

$$\hat{I} = \{F \subseteq \mathbb{R}^E \mid (\exists E' \subseteq_{\text{fin}} E)(\exists f' \in I' = \mu_{E'E}^{-1}[I])(F = \pi_{E'E'}^{-1}[Z(f')])\} \subseteq \wp(\mathcal{Z}(\mathbb{R}^E))$$

Note that  $\mathbb{R}^E$  is a zeroset:  $\mathbb{R}^E = Z(0_E)$ , where

$$0_E : \mathbb{R}^E \rightarrow \mathbb{R} \\ x \mapsto 0.$$

Note that  $0_E \in \mathcal{C}^\infty(\mathbb{R}^E)$ : choose any finite  $D \subseteq E$  and consider the  $\mathcal{C}^\infty$ -function

$$0_D : \mathbb{R}^D \rightarrow \mathbb{R} \\ x \mapsto 0,$$

so  $0_E = \mu_{DE}(0_D) \in \mathcal{C}^\infty(\mathbb{R}^E)$ .

Given  $G_1, G_2 \in \widehat{I} \subseteq \mathbb{R}^E$ , let  $g_1, g_2 \in I \subseteq \mathcal{C}^\infty(\mathbb{R}^E)$  such that  $G_i = Z(g_i)$  for  $i = 1, 2$ . There are finite  $E', E'' \subseteq E$ ,  $f_1 \in \mu_{E'E}^{-1}[I] \subseteq \mathcal{C}^\infty(\mathbb{R}^{E'})$ ,  $f_2 \in \mu_{E''E}^{-1}[I] \subseteq \mathcal{C}^\infty(\mathbb{R}^{E''})$  such that  $g_1 = f_1 \circ \pi_{EE'}$ ,  $g_2 = f_2 \circ \pi_{EE''}$ . Thus  $\pi_{E'E'}^{-1}[Z(f_1)] = G_1$  and  $\pi_{E'E''}^{-1}[Z(f_2)] = G_2$ , where  $\pi_{EE'} : \mathbb{R}^E \rightarrow \mathbb{R}^{E'}$  and  $\pi_{EE''} : \mathbb{R}^E \rightarrow \mathbb{R}^{E''}$  are the canonical projections (restrictions). Consider:

$$\begin{array}{ccc} & \mathbb{R}^{E' \cup E''} & \\ \pi_{E' \cup E'', E'} \swarrow & & \searrow \pi_{E' \cup E'', E''} \\ \mathbb{R}^{E'} & & \mathbb{R}^{E''} \end{array}$$

where:

$$\pi_{E' \cup E'', E'} : \mathbb{R}^{E' \cup E''} \rightarrow \mathbb{R}^{E'} \\ v \mapsto v \upharpoonright_{E'} : E' \rightarrow \mathbb{R}$$

and

$$\pi_{E' \cup E'', E''} : \mathbb{R}^{E' \cup E''} \rightarrow \mathbb{R}^{E''} \\ v \mapsto v \upharpoonright_{E''} : E'' \rightarrow \mathbb{R}$$

We have the commutative diagram:

$$\begin{array}{ccc} & \mathbb{R}^E & \\ \pi_{EE'} \swarrow & \downarrow \pi_{E, E' \cup E''} & \searrow \pi_{EE''} \\ & \mathbb{R}^{E' \cup E''} & \\ \pi_{E' \cup E'', E'} \swarrow & & \searrow \pi_{E' \cup E'', E''} \\ \mathbb{R}^{E'} & & \mathbb{R}^{E''} \end{array}$$

Define  $\tilde{f}_1 = f_1 \circ \pi_{E' \cup E'', E'} : \mathbb{R}^{E' \cup E''} \rightarrow \mathbb{R}$  and  $\tilde{f}_2 = f_2 \circ \pi_{E' \cup E'', E''} : \mathbb{R}^{E' \cup E''} \rightarrow \mathbb{R}$ , so  $\tilde{F}_1 = \pi_{E' \cup E'', E'}^{-1}[Z(f_1)] = \tilde{f}_1^{-1}[\{0\}] = Z(\tilde{f}_1) \subseteq \mathbb{R}^{E' \cup E''}$  and  $\tilde{F}_2 = \pi_{E' \cup E'', E''}^{-1}[Z(f_2)] = \tilde{f}_2^{-1}[\{0\}] = Z(\tilde{f}_2) \subseteq \mathbb{R}^{E' \cup E''}$  are zerosets.

Note that  $\tilde{F}_1 \cap \tilde{F}_2$  is also a zeroset, namely  $\tilde{F}_1 \cap \tilde{F}_2 = Z(\tilde{f}_1^2 + \tilde{f}_2^2)$ , with  $\tilde{f}_1^2 + \tilde{f}_2^2 \in \mu_{E' \cup E'', E}^{-1}[I]$ . In fact, we have the commutative diagram:

$$\begin{array}{ccc}
 & \mathcal{C}^\infty(\mathbb{R}^E) & \\
 \mu_{E'E} \nearrow & \uparrow \mu_{E' \cup E'', E} & \nwarrow \mu_{E''E} \\
 & \mathcal{C}^\infty(\mathbb{R}^{E' \cup E''}) & \\
 \mu_{E', E' \cup E''} \nearrow & & \nwarrow \mu_{E'', E' \cup E''} \\
 \mathcal{C}^\infty(\mathbb{R}^{E'}) & & \mathcal{C}^\infty(\mathbb{R}^{E''})
 \end{array}$$

Since the diagram commutes, we have  $\mu_{E', E' \cup E''}(f_1) = f_1 \circ \pi_{E' \cup E'', E'} = \tilde{f}_1 \in \mu_{E' \cup E''}^\dagger[I]$  and  $\mu_{E'', E' \cup E''}(f_2) = f_2 \circ \pi_{E' \cup E'', E''} = \tilde{f}_2 \in \mu_{E' \cup E'', E}^\dagger[I]$ , so  $\tilde{f}_1^2 + \tilde{f}_2^2 \in \mu_{E' \cup E'', E}^\dagger[I]$ .

Then  $g_1^2 + g_2^2 \in I$  and

$$\pi_{E, E' \cup E''}^\dagger[\tilde{F}_1 \cap \tilde{F}_2] = \pi_{E, E' \cup E''}^\dagger[Z(\tilde{f}_1^2 + \tilde{f}_2^2)] = Z(g_1^2 + g_2^2) = G_1 \cap G_2.$$

Let  $G \in \hat{I}$  and  $H \in \mathcal{Z}(\mathbb{R}^E)$  such that  $G \subseteq H$ . Then there are  $g \in I, h \in \mathcal{C}^\infty(\mathbb{R}^E)$  such that  $G = Z(g), H = Z(h) \in \mathcal{Z}(\mathbb{R}^E)$ . Now select  $E' \subseteq_{\text{fin}} E$  and  $g', h' \in \mathcal{C}^\infty(\mathbb{R}^{E'})$  such that  $\mu_{E'E}(g') = g, \mu_{E'E}(h') = h$ ; thus  $\pi_{E'E}^\dagger[Z(g')] \subseteq \pi_{E'E}^\dagger[Z(h')]$  and  $g' \in I' := \mu_{E'E}^\dagger[I]$ . Let  $G' = Z(g'), H' = Z(h') \in \mathcal{Z}(\mathbb{R}^{E'})$ , then  $G' \subseteq H'$ . Since we are dealing with  $\mathcal{C}^\infty(\mathbb{R}^{E'})$  with  $E'$  finite, the **Smooth Tietze Theorem (Fact 2.3)** gives us a smooth function,  $\chi_{H'} \in \mathcal{C}^\infty(\mathbb{R}^{E'})$  such that  $H' = Z(\chi_{H'})$ . We have  $Z(h') = H' = G' \cap H' = Z(g') \cap Z(\chi_{H'}) = Z(g' \cdot \chi_{H'})$  and, since  $I' = \mu_{E'E}^\dagger[I]$  is an ideal,  $g' \cdot \chi_{H'} \in I'$ .

Since  $H = \pi_{E'E'}^\dagger[H']$ ,  $H' = Z(g' \cdot \chi_{H'})$ ,  $g' \cdot \chi_{H'} \in I'$  and

$$\hat{I} = \{F \subseteq \mathbb{R}^E \mid (\exists E' \subseteq_{\text{fin}} E)(\exists f' \in I' = \mu_{E'E}^\dagger[I])(F = \pi_{E'E}^\dagger[Z(f')])\} \subseteq \wp(\mathcal{Z}(\mathbb{R}^E)),$$

we have  $H \in \hat{I}$ . □

**Proposition 2.15.** *Let  $E$  be any set. If  $\Phi \subseteq \wp(\mathcal{Z}(\mathbb{R}^E))$  is a filter of zerosets in  $\mathbb{R}^E$ , then:*

$$\check{\Phi} := \{f \in \mathcal{C}^\infty(\mathbb{R}^E) \mid Z(f) \in \Phi\} \subseteq \mathcal{C}^\infty(\mathbb{R}^E)$$

*is an ideal of  $\mathcal{C}^\infty(\mathbb{R}^E)$ .*

*Proof.* Note first that

$$\check{\Phi} = \{f \in \mathcal{C}^\infty(\mathbb{R}^E) \mid (\exists E' \subseteq_{\text{fin}} E)(\exists f' \in \mathcal{C}^\infty(\mathbb{R}^{E'}))((\mu_{E'E}(f') = f) \& \& (\pi_{E'}^\dagger[Z(f')] \in \Phi))\} \subseteq \mathcal{C}^\infty(\mathbb{R}^E).$$

It is easy to see that  $0_E \in \check{\Phi}$ . In fact,  $Z(0_E) = \mathbb{R}^E \in \Phi$ .

Given  $f \in \check{\Phi} \subseteq \mathcal{C}^\infty(\mathbb{R}^E)$  and  $h \in \mathcal{C}^\infty(\mathbb{R}^E)$ . Select  $E' \subseteq_{\text{fin}} E$  and  $f', h' \in \mathcal{C}^\infty(\mathbb{R}^{E'})$  such that  $\mu_{E'E}(f') = f, \mu_{E'E}(h') = h$ . Then  $h \cdot f = \mu_{E'E}(h' \cdot f') \in \mathcal{C}^\infty(\mathbb{R}^E)$  and  $Z(h \cdot f) = Z(h) \cup Z(f) \supseteq Z(f) \in \Phi$ . Thus  $h \cdot f \in \check{\Phi}$ .

Let  $f, g \in \check{\Phi}$ . Select  $E' \subseteq_{\text{fin}} E$  and  $f', g' \in \mathcal{C}^\infty(\mathbb{R}^{E'})$  such that  $\mu_{E'E}(f') = f, \mu_{E'E}(g') = g$ . Thus  $\pi_{E'}^\dagger[Z(f)], \pi_{E'}^\dagger[Z(g')] \in \Phi$  and  $f + g = \mu_{E'E}(f' + g') \in \mathcal{C}^\infty(\mathbb{R}^E)$ . Since  $Z(f + g) \supseteq Z(f) \cap Z(g) \in \Phi$ , we obtain  $f + g \in \check{\Phi}$ . □

**Proposition 2.16.** *Consider the partially ordered sets:*

$$\mathfrak{F} = (\{\Phi \subseteq \wp(\mathcal{Z}(\mathbb{R}^E)) \mid \Phi \text{ is a filter}\}, \subseteq)$$

and

$$\mathfrak{I} = (\{I \subseteq \mathcal{C}^\infty(\mathbb{R}^E) \mid I \text{ is an ideal of } \mathcal{C}^\infty(\mathbb{R}^E)\}, \subseteq)$$

The following functions:

$$\begin{aligned} \vee : \mathfrak{F} &\rightarrow \mathfrak{I} \\ \Phi &\mapsto \check{\Phi} \\ \wedge : \mathfrak{I} &\rightarrow \mathfrak{F} \\ I &\mapsto \hat{I} \end{aligned}$$

form a covariant Galois connection,  $\wedge \dashv \vee$ , that is:

- (a) Given  $\Phi_1, \Phi_2 \in \mathfrak{F}$  such that  $\Phi_1 \subseteq \Phi_2$ , then  $\check{\Phi}_1 \subseteq \check{\Phi}_2$ ;
- (b) Given  $I_1, I_2 \in \mathfrak{I}$  such that  $I_1 \subseteq I_2$  then  $\hat{I}_1 \subseteq \hat{I}_2$ ;
- (c) For every  $\Phi \in \mathfrak{F}$  and every  $I \in \mathfrak{I}$  we have:

$$\hat{I} \subseteq \Phi \iff I \subseteq \check{\Phi}$$

Moreover, the mappings  $(\vee, \wedge)$  establish a correspondence between:

- (1)  $\wp(\mathcal{Z}(\mathbb{R}^E))$  and  $\mathcal{C}^\infty(\mathbb{R}^E)$ ;
- (2) Proper filters of  $(\mathfrak{F}, \subseteq)$  and proper ideals of  $(\mathfrak{I}, \subseteq)$ .

*Proof.* Items (a), (b), (c) follows directly from the definitions.

Suppose that  $\Phi = \wp(\mathcal{Z}(\mathbb{R}^E))$ . Then  $\check{\Phi} = \{f \in \mathcal{C}^\infty(\mathbb{R}^E) \mid f \in \wp(\mathcal{Z}(\mathbb{R}^E))\}$ , thus  $\check{\check{\Phi}} = \mathcal{C}^\infty(\mathbb{R}^E)$ .

Suppose that  $I = \mathcal{C}^\infty(\mathbb{R}^E)$ . Then  $\hat{I} = \{Z(f) \in \wp(\mathcal{Z}(\mathbb{R}^E)) \mid f \in I\}$ , thus  $\hat{\hat{I}} = \wp(\mathcal{Z}(\mathbb{R}^E))$ .

Suppose that  $\Phi$  is a proper filter. If  $f \in \mathcal{C}^\infty(\mathbb{R}^E)$  is such that  $Z(f) = \emptyset \notin \Phi$ , then  $f \in \mathcal{C}^\infty(\mathbb{R}^E)^\times$  and  $f \notin \check{\Phi} \subseteq \mathcal{C}^\infty(\mathbb{R}^E)$ . Thus  $\check{\check{\Phi}} \subseteq \mathcal{C}^\infty(\mathbb{R}^E)$  is a proper ideal.

Suppose that  $I$  is a proper ideal. So  $f \notin I$  whenever  $f \in \mathcal{C}^\infty(\mathbb{R}^E)^\times$ , i.e. whenever  $Z(f) = \emptyset$ . Thus  $\emptyset \notin \hat{I}$ , i. e.  $\hat{I}$  is a proper filter.  $\square$

**Remark 2.17.** *As in any (covariant) Galois connection, we have automatically:*

- $I \subseteq \check{\hat{I}}; \Phi \supseteq \hat{\check{\Phi}}$
- $\hat{I} = \hat{\check{\hat{I}}}; \check{\Phi} = \check{\hat{\check{\Phi}}}$

The following result gives a more detailed information on these compositions.

**Proposition 2.18.** *Let  $I \subseteq \mathcal{C}^\infty(\mathbb{R}^E)$  be any ideal and  $\Phi \subseteq \wp(\mathcal{Z}(\mathbb{R}^E))$  be a filter of zerosets. Then:*

1.  $\hat{\check{\Phi}} = \{X \subseteq (\mathbb{R}^E) \mid \exists f \in \mathcal{C}^\infty(\mathbb{R}^E)(X = Z(f), Z(f) \in \Phi)\} = \Phi$ .
2.  $\check{\hat{I}} = \{g \in \mathcal{C}^\infty(\mathbb{R}^E) \mid \exists f \in \mathcal{C}^\infty(\mathbb{R}^E)(f \in I, Z(g) = Z(f))\} = \sqrt[3]{I}$ .

*Proof.* Item (1) and the first equality in item (2) follow directly from the definitions. We will show that

$$\{g \in \mathcal{C}^\infty(\mathbb{R}^E) \mid (\exists f \in \mathcal{C}^\infty(\mathbb{R}^E))(f \in I, Z(g) = Z(f))\} = \sqrt[3]{I}$$

Note that:  $\sqrt[3]{I} = \{g \in \mathcal{C}^\infty(\mathbb{R}^E) \mid (\exists f \in I)((\eta_g(f) \in \mathcal{C}^\infty(\mathbb{R}^E)\{g^{-1}\}^\times))\} = \{g \in \mathcal{C}^\infty(\mathbb{R}^E) \mid (\exists E' \subseteq_{\text{fin}} E)(\exists \tilde{g} \in \mathcal{C}^\infty(\mathbb{R}^{E'}))(\exists \tilde{f} \in \mu_{E'E}^{-1}[I])(g = \tilde{g} \circ \pi_{EE'}) \& (\eta_{\tilde{g}}(\tilde{f}) \in \mathcal{C}^\infty(\mathbb{R}^{E'})\{\tilde{g}^{-1}\}^\times)\}$

Given  $g \in \check{\hat{I}}$ , there is some finite  $E' \subseteq E$ , some  $\tilde{g} \in \mathcal{C}^\infty(\mathbb{R}^{E'})$  with  $g = \tilde{g} \circ \pi_{EE'}$  and some  $\tilde{f} \in \mu_{E'E}^{-1}[I]$  such that  $\pi_{EE'}^{-1}[Z(\tilde{g})] = Z(g) = Z(f) = \pi_{EE'}^{-1}[Z(f)]$ . Since  $\pi_{EE'} : \mathbb{R}^E \rightarrow \mathbb{R}^{E'}$  is surjective, we have  $Z(\tilde{g}) = \pi_{EE'}[\pi_{EE'}^{-1}[Z(\tilde{g})]] = \pi_{EE'}[\pi_{EE'}^{-1}[Z(f)]] = Z(f)$ . It follows that  $\tilde{f} \upharpoonright_{\mathbb{R}^{E'} \setminus Z(\tilde{g})} \in \mathcal{C}^\infty(\mathbb{R}^{E'} \setminus Z(\tilde{g}))^\times$  and, by **Example 1.6**,  $\eta_{\tilde{g}}(\tilde{f}) \in \mathcal{C}^\infty(\mathbb{R}^{E'})\{\tilde{g}^{-1}\}^\times$ . Since there is  $\tilde{f} \in \mu_{E'E}^{-1}[I]$  such that  $\eta_{\tilde{g}}(\tilde{f}) \in \mathcal{C}^\infty(\mathbb{R}^{E'})\{\tilde{g}^{-1}\}^\times$ , it follows that  $g \in \sqrt[3]{I}$ .

Conversely, given  $g \in \sqrt[\infty]{I}$ , there is some finite  $E' \subseteq_{\text{fin}} E$ , some  $\tilde{g} \in \mathcal{C}^\infty(\mathbb{R}^{E'})$  and some  $\tilde{f} \in \mu_{E'E}^{-1}[I]$  such that  $g = \tilde{g} \circ \pi_{E'}$  and  $\eta_{\tilde{g}}(\tilde{f}) \in \mathcal{C}^\infty(\mathbb{R}^{E'}) \setminus \{\tilde{g}^{-1}\}^\times$ . So  $\tilde{f} \upharpoonright_{\mathbb{R}^{E'} \setminus Z(\tilde{g})} \in \mathcal{C}^\infty(\mathbb{R}^{E'} \setminus Z(\tilde{g}))^\times$ ,  $Z(\tilde{f}) \subseteq Z(\tilde{g})$  and  $\pi_{E'E}^{-1}[Z(\tilde{f})] \subseteq \pi_{E'E}^{-1}[Z(\tilde{g})]$ . Since  $\pi_{E'E}^{-1}[Z(\tilde{f})] \in \hat{I}$  and  $\hat{I}$  is a filter, we have  $Z(g) = \pi_{E'E}^{-1}[Z(\tilde{g})] \in \hat{I}$ , so  $g \in \check{I}$ .  $\square$

**Remark 2.19.** *The item (2) in the previous proposition ensures that the  $\mathcal{C}^\infty$ -radical of any ideal of a  $\mathcal{C}^\infty$ -ring is an ideal.*

**Proposition 2.20.** *Let  $E$  be any set, and consider  $A = \mathcal{C}^\infty(\mathbb{R}^E)$ . The Galois connection  $\wedge \dashv \vee$  establishes bijective correspondences between:*

- (a) *The poset of all (proper) filters of zerosets of  $\mathbb{R}^E$  and the poset of all (proper)  $\mathcal{C}^\infty$ -radical ideals of  $\mathcal{C}^\infty(\mathbb{R}^E)$ ,  $\mathfrak{J}^\infty = \{I \in \mathfrak{J} \mid \sqrt[\infty]{I} = I\}$ ;*
- (b) *The set of all maximal filters of  $(\mathfrak{F}, \subseteq)$  and the set of all maximal ideals of  $(\mathfrak{J}, \subseteq)$ ;*
- (c) *The poset of all prime (proper) filters of  $(\mathfrak{F}, \subseteq)$  and the poset of all prime (proper)  $\mathcal{C}^\infty$ -radical ideals of  $(\mathfrak{J}, \subseteq)$ .*

*Proof.* We saw in **Proposition 2.16**, that the functions  $(\vee, \wedge)$  restricts to maps between proper filters of zerosets of  $\mathbb{R}^E$  and proper ideals of  $\mathcal{C}^\infty(\mathbb{R}^E)$ . Thus the additional parts in items (a) and (c) are automatic.

Ad (a): Let  $\Phi$  be a filter of zerosets in  $\mathbb{R}^E$ , then by **Proposition 2.18.(1)**  $\Phi = \hat{\Phi}$ . Let  $I$  be a  $\mathcal{C}^\infty$ -radical ideal in  $\mathcal{C}^\infty(\mathbb{R}^E)$ , then by **Proposition 2.18.(2)** and **Remark 2.17**,

$$\check{I} = \check{\check{I}} = \sqrt[\infty]{I} = I$$

Thus, since  $\hat{I} = \hat{\hat{I}}$  and  $\check{\Phi} = \check{\check{\Phi}}$ , the (increasing) mappings  $(\vee, \wedge)$  establish a bijective correspondence between the poset of all filters of zerosets of  $\mathbb{R}^E$  and the poset of all  $\mathcal{C}^\infty$ -radical ideals of  $\mathcal{C}^\infty(\mathbb{R}^E)$ .

Ad (b): First of all, note that, by a combination of previous results, if  $I$  is a proper ideal of  $\mathcal{C}^\infty(\mathbb{R}^E)$ , then  $\sqrt[\infty]{I}$  is also a proper ideal of  $\mathcal{C}^\infty(\mathbb{R}^E)$ . Thus if  $I$  is a (proper) maximal ideal of  $\mathcal{C}^\infty(\mathbb{R}^E)$ , then  $I = \sqrt[\infty]{I}$ .

Now, by item (a), the increasing mappings  $(\vee, \wedge)$  establishes a bijective correspondence between the poset of all proper filters of zerosets of  $\mathbb{R}^E$  and the poset of all proper  $\mathcal{C}^\infty$ -radical ideals of  $\mathcal{C}^\infty(\mathbb{R}^E)$ . Thus the mappings  $(\vee, \wedge)$  restrict to a pair of inverse bijective correspondence between the set of all maximal filters of zerosets of  $\mathbb{R}^E$  and the set of all maximal ideals of  $\mathcal{C}^\infty(\mathbb{R}^E)$ .

Ad (c): By the bijective correspondence in item (a), it is enough to show that the mappings  $(\vee, \wedge)$  restricts to a pair of mappings between the set of all prime filters of zerosets of  $\mathbb{R}^E$  and the set of all  $\mathcal{C}^\infty$ -radical prime ideals of  $\mathcal{C}^\infty(\mathbb{R}^E)$ .

Let  $\Phi$  be a prime filter of zerosets of  $\mathbb{R}^E$ . If  $f, g \in \mathcal{C}^\infty(\mathbb{R}^E)$  are such that  $f \cdot g \in \check{\Phi}$ , then  $Z(f \cdot g) = Z(f) \cup Z(g) \in \Phi$ , so we have  $Z(f) \in \Phi$  or  $Z(g) \in \Phi$ . Thus,  $f \in \check{\Phi}$  or  $g \in \check{\Phi}$ , so  $\check{\Phi}$  is a prime ideal of  $\mathcal{C}^\infty(\mathbb{R}^E)$ ; moreover, by item (a),  $\check{\Phi}$  is  $\mathcal{C}^\infty$ -radical.

Let  $I$  be a  $\mathcal{C}^\infty$ -radical prime ideal of  $\mathcal{C}^\infty(\mathbb{R}^E)$ , that is, if  $f, g \in \mathcal{C}^\infty(\mathbb{R}^E)$  are such that  $f \cdot g \in I$  then  $f \in I$  or  $g \in I$ . We need to show that  $\hat{I} = \{Z(h) | h \in I\}$  is a prime filter of zerosets of  $\mathbb{R}^E$ .

Let  $F = Z(f), G = Z(g), H = Z(h)$  be zerosets of  $\mathbb{R}^E$  such that  $F \cup G = H \in \hat{I}$ ,  $h \in I$ . Select  $E' \subseteq_{\text{fin}} E$  and  $f', g', h' \in \mathcal{C}^\infty(\mathbb{R}^{E'})$  such that  $f = \mu_{E'E}(f'), g = \mu_{E'E}(g'), h = \mu_{E'E}(h')$ . Let  $F' = Z(f'), G' = Z(g'), H' = Z(h') \subseteq \mathbb{R}^{E'}$  then  $F' \cup G' = H' \in \hat{I}'$ , where  $I' := \mu_{E'E}^{-1}[I]$ . Since  $I$  is a  $\mathcal{C}^\infty$ -radical prime ideal of  $\mathcal{C}^\infty(\mathbb{R}^E)$ , then  $I'$  is a  $\mathcal{C}^\infty$ -radical prime ideal of  $\mathcal{C}^\infty(\mathbb{R}^{E'})$ , see **Proposition 1.15.(d)**. If we show that  $\hat{I}'$  is a prime filter of zerosets of  $\mathbb{R}^{E'}$  then, we may assume w.l.o.g. that  $Z(f') = F' \in \hat{I}'$  and  $f' \in \sqrt[I']{I'} = I' = \mu_{E'E}^{-1}[I]$ ; thus  $f = \mu_{E'E}(f') \in I$  and  $F = Z(f) \in \hat{I}$ , finishing the proof.

We will prove that  $\hat{I}'$  is a prime filter. We have  $Z(f' \cdot g') = Z(f') \cup Z(g') = F' \cup G' = H' = Z(h')$ , where  $h' \in I'$ . Then,  $f' \cdot g' \in \sqrt[I']{I'} = I'$ . Since  $I'$  is a prime ideal,  $f' \in I'$  or  $g' \in I'$ . Thus  $F' = Z(f') \in \hat{I}'$  or  $G' = Z(g') \in \hat{I}'$ . □

### 3 The Order Theory of $\mathcal{C}^\infty$ -Reduced $\mathcal{C}^\infty$ -Rings

The results established in the previous section are fundamental to develop an order theory over a broader class of  $\mathcal{C}^\infty$ -rings. In fact, in order to get nice results, we need to assume some technical conditions: the  $\mathcal{C}^\infty$ -rings must be non-trivial (i.e.  $0 \neq 1$ ) and  $\mathcal{C}^\infty$ -reduced (see **Section 1**).

The fundamental notion here is the following (see p. 328 of [16]):

**Definition 3.1.** *Let  $A$  be a  $\mathcal{C}^\infty$ -ring. The canonical relation on  $A$  is*

$$\prec_A = \{(a, b) \in A \times A \mid (\exists u \in A^\times)(b - a = u^2)\}$$

**Remark 3.2.** *Note that the canonical relation is preserved by  $\mathcal{C}^\infty$ -homomorphism. In more detail: let  $A, A'$  be  $\mathcal{C}^\infty$ -rings and  $h : A \rightarrow A'$  be a  $\mathcal{C}^\infty$ -homomorphism. Then for each  $a, b \in A$ :*

$$a \prec_A b \Rightarrow h(a) \prec_{A'} h(b).$$

**Proposition 3.3.** *Let  $A$  be any  $\mathcal{C}^\infty$ -ring. The canonical relation on  $A$ ,  $<_A$ , is compatible with the sum and with the product of  $A$ .*

*Proof.* Let  $a, b \in A$  such that  $a < b$  and let  $c \in A^\times$  such that  $(b - a = c^2)$ .

Given any  $x \in A$ , we have:

$$(b + x) - (a + x) = b - a = c^2,$$

thus  $a + x < b + x$ .

Given  $x \in A$  such that  $0 < x$ , one has  $(\exists d \in A^\times)(x = d^2)$ . We have, thus:

$$b \cdot x - a \cdot x = (b - a) \cdot x = c^2 \cdot d^2 = (c \cdot d)^2.$$

Since both  $c$  and  $d$  are invertible, it follows that  $c \cdot d$  is invertible, and  $a \cdot x < b \cdot x$ .  $\square$

**Proposition 3.4.** *If  $A$  is a non trivial  $\mathcal{C}^\infty$ -ring, then  $<$ , defined as above, is ir-reflexive, that is,*

$$(\forall a \in A)(\neg(a < a)).$$

*Proof.* Suppose there is some  $a_0 \in A$  such that  $a_0 < a_0$ . By definition, this happens if, and only if there is some  $c \in A^\times$  such that  $0 = a_0 - a_0 = c^2$ , so 0 would be invertible and  $0 = 1$ .  $\square$

In order to obtain further information about the canonical relation  $<_A$ , we need to pass to its specific “spatial” characterizations, by the aid of the results developed in the previous section. We start this enterprise by the following:

**Proposition 3.5.** *Let  $A = \frac{\mathcal{C}^\infty(\mathbb{R}^n)}{I}$  be a finitely generated  $\mathcal{C}^\infty$ -reduced  $\mathcal{C}^\infty$ -ring. Then  $<$  equals:*

$$\left\{ (f + I, g + I) \in \frac{\mathcal{C}^\infty(\mathbb{R}^n)}{I} \times \frac{\mathcal{C}^\infty(\mathbb{R}^n)}{I} \mid (\exists h \in \left(\frac{\mathcal{C}^\infty(\mathbb{R}^n)}{I}\right)^\times)(g - f + I = h^2 + I) \right\}$$

$$(f + I < g + I) \iff ((\exists \varphi \in I)(\forall x \in Z(\varphi))(f(x) < g(x))).$$

*Proof.* Despite this is a direct application of **Corollary 2.6**, we register here a detailed proof, since it stresses the centrality of the  $\mathcal{C}^\infty$ -reducedness hypothesis.

Suppose  $f + I < g + I$ , so there is some  $h + I \in \left(\frac{\mathcal{C}^\infty(\mathbb{R}^n)}{I}\right)^\times$  such that  $g - f + I = h^2 + I$ . Since  $h + I$  is invertible, by **Proposition 2.5** there is some  $\psi \in I$  such that:

$$(\forall x \in Z(\psi))(h(x) \neq 0)$$

Since  $g - f + I = h^2 + I$ , by **Proposition 2.4**, there is some  $\phi \in I$  such that:

$$(\forall x \in Z(\phi))(g(x) - f(x) = h^2(x)),$$

that is, such that:

$$(\forall x \in Z(\phi))(g(x) = f(x) + h^2(x))$$

Taking  $\varphi = \phi^2 + \psi^2 \in I$  we have, for every  $x \in Z(\psi) \cap Z(\phi) = Z(\varphi)$  both:

$$g(x) - f(x) = h^2(x)$$

and

$$h^2(x) > 0$$

Hence,

$$(\forall x \in Z(\varphi))(f(x) < g(x))$$

Conversely, suppose  $f, g \in \mathcal{C}^\infty(\mathbb{R}^n)$  are such that there is some  $\varphi \in I$  with satisfying:

$$(\forall x \in Z(\varphi))(f(x) < g(x)).$$

Since  $f$  and  $g$  are continuous functions, there is an open subset  $U \subseteq \mathbb{R}^n$  such that  $Z(\varphi) \subseteq U$  and

$$(\forall x \in U)(f(x) < g(x)).$$

The  $\mathcal{C}^\infty$ -function:

$$\begin{aligned} m : \mathbb{R}^n &\rightarrow \mathbb{R} \\ x &\mapsto g(x) - f(x) \end{aligned}$$

is such that  $(\forall x \in U)(m(x) > 0)$ . Thus  $m \upharpoonright_{Z(\varphi)}$  is smooth and  $(\forall x \in Z(\varphi))(m(x) > 0)$ , so by **Fact 2.3** there is a smooth function  $\tilde{m} \in \mathcal{C}^\infty(\mathbb{R}^n)$  such that  $\tilde{m} \upharpoonright_{Z(\varphi)} = m \upharpoonright_{Z(\varphi)}$  and  $(\forall x \in \mathbb{R}^n)(\tilde{m}(x) > 0)$ .

Now the function  $h := \sqrt{\tilde{m}} : \mathbb{R}^n \rightarrow \mathbb{R}$  is a smooth function and since  $h \in (\mathcal{C}^\infty(\mathbb{R}^n))^\times$ , by **Proposition 2.5**,

Since  $I = \sqrt[\infty]{I}$ , by **Proposition 2.4** it follows that  $(g - f) + I = h^2 + I$  with  $h + I$  invertible. Thus,  $f + I < g + I$ .  $\square$

**Proposition 3.6.** *Given any  $\mathcal{C}^\infty$ -reduced  $\mathcal{C}^\infty$ -ring  $A$ . Then there is a directed system of its finitely generated  $\mathcal{C}^\infty$ -rings  $(A_i, \alpha_{ij} : A_i \rightarrow A_j)_{\substack{i \in I \\ i \leq j}}$  such that:*

1.  $A \cong \varinjlim_{i \in I} A_i$ ;
2. For each  $i, j$ , if  $i \leq j$  then  $\alpha_{ij} : A_i \rightarrow A_j$  and  $\alpha_i : A_i \rightarrow A$  are injective;
3. For each  $i \in I$ ,  $A_i$  is a  $\mathcal{C}^\infty$ -reduced  $\mathcal{C}^\infty$ -ring;
4. For each  $a, b \in A$ ,  $a <_A b$  iff  $\exists i \in I, \exists a_i, b_i \in A_i (\alpha_i(a_i) = a, \alpha_i(b_i) = b \text{ and } a_i <_{A_i} b_i)$ .

*Proof.* Note that any  $\mathcal{C}^\infty$ -reduced  $\mathcal{C}^\infty$ -ring can be presented as  $A \cong \mathcal{C}^\infty(\mathbb{R}^E)/I$ , where  $I = \sqrt[I]{I}$  and the latter can be described as a directed colimit of finitely generated  $\mathcal{C}^\infty$ -reduced  $\mathcal{C}^\infty$ -rings. Indeed, we have that

$$\mathcal{C}^\infty(\mathbb{R}^E)/I \cong \left( \varinjlim_{E' \subseteq_{\text{fin}} E} \mathcal{C}^\infty(\mathbb{R}^{E'}) \right) / I \cong \varinjlim_{E' \subseteq_{\text{fin}} E} (\mathcal{C}^\infty(\mathbb{R}^{E'}) / \mu_{E', E}^\perp[I]).$$

It is clear that  $\alpha_{E'} : \mathcal{C}^\infty(\mathbb{R}^{E'}) / \mu_{E', E}^\perp[I] \rightarrow \mathcal{C}^\infty(\mathbb{R}^E) / I$  is injective, for each  $E' \subseteq_{\text{fin}} E$ . Thus if  $E'' \subseteq E' \subseteq_{\text{fin}} E$ , then  $\alpha_{E'' E'} : \mathcal{C}^\infty(\mathbb{R}^{E''}) / \mu_{E'', E}^\perp[I] \rightarrow \mathcal{C}^\infty(\mathbb{R}^{E'}) / \mu_{E', E}^\perp[I]$  is injective too.

We combine the results in **Proposition 1.14.(a)** and **Proposition 1.15.(d)** to conclude that  $\mathcal{C}^\infty(\mathbb{R}^{E'}) / \mu_{E', E}^\perp[I]$  is a  $\mathcal{C}^\infty$ -reduced  $\mathcal{C}^\infty$ -ring.

Now item (4) follows directly from the definition of canonical relation, since for each  $f, g \in \mathcal{C}^\infty(\mathbb{R}^E)$ :

$$\exists u \in \mathcal{C}^\infty(\mathbb{R}^E) ((g - f) + I = u^2 + I; u + I \in (\mathcal{C}^\infty(\mathbb{R}^E) / I)^\times)$$

iff  $\exists E' \subseteq_{\text{fin}} E, \exists f', g', u' \in \mathcal{C}^\infty(\mathbb{R}^{E'}), \mu_{E', E}(f') = f, \mu_{E', E}(g') = g, \mu_{E', E}(u') = u$  such that:

$$(g' - f') + \mu_{E', E}^\perp[I] = u'^2 + \mu_{E', E}^\perp[I] \text{ and } u' + \mu_{E', E}^\perp[I] \in (\mathcal{C}^\infty(\mathbb{R}^{E'}) / \mu_{E', E}^\perp[I])^\times.$$

□

We are ready to state and proof the following (very useful) general characterization result of <:

**Theorem 3.7.** *Let  $A = \frac{\mathcal{C}^\infty(\mathbb{R}^E)}{I}$  be a “general”  $\mathcal{C}^\infty$ -reduced  $\mathcal{C}^\infty$ -ring. We have:  $(f + I < g + I) \iff ((\exists \varphi \in I)(\forall x \in Z(\varphi))(f(x) < g(x)))$ .*

*Proof.* Suppose  $f + I < g + I$ , so there is some  $h + I \in \left(\frac{\mathcal{C}^\infty(\mathbb{R}^n)}{I}\right)^\times$  such that  $g - f + I = h^2 + I$ . Since  $h + I$  is invertible, by **Proposition 2.13** there is some  $\psi \in I$  such that:

$$(\forall x \in Z(\psi))(h(x) \neq 0).$$

Since  $g - f + I = h^2 + I$ , by **Proposition 2.12**, there is some  $\phi \in I$  such that:

$$(\forall x \in Z(\phi))(g(x) - f(x) = h^2(x)),$$

Taking  $\varphi = \phi^2 + \psi^2 \in I$  we have, for every  $x \in Z(\psi) \cap Z(\phi) = Z(\varphi)$  both  $g(x) - f(x) = h^2(x)$  and  $h^2(x) > 0$ . Thus,

$$(\forall x \in Z(\varphi))(f(x) < g(x)).$$

Conversely, suppose  $f, g \in \mathcal{C}^\infty(\mathbb{R}^E)$  are such that there is some  $\varphi \in I$  with satisfying:

$$(\forall x \in Z(\varphi))(f(x) < g(x)).$$

Pick  $E' \subseteq_{\text{fin}} E$  and  $f', g', \varphi' \in \mathcal{C}^\infty(\mathbb{R}^{E'})$  such that  $f = \mu_{E'E}(f')$ ,  $g = \mu_{E'E}(g')$ ,  $\varphi = \mu_{E'E}(\varphi')$ .

Then  $I' := \mu_{E'E}^{-1}[I]$  is a  $\mathcal{C}^\infty$ -radical ideal of  $\mathcal{C}^\infty(\mathbb{R}^{E'})$ ,  $\varphi' \in I'$  and

$$\forall x' \in \mathbb{R}^{E'} (x' \in Z(\varphi') \rightarrow f'(x') < g'(x')).$$

By the finitely generated case (i.e. **Proposition 3.5**),

$$f' + I' < g' + I'.$$

By (the proof of) **Proposition 3.6.(4)** we obtain

$$f + I < g + I.$$

□

Now, having available a characterization of the canonical relation  $<$ , we can establish many of its properties.

**Proposition 3.8.** *Let  $A$  be any  $\mathcal{C}^\infty$ -reduced  $\mathcal{C}^\infty$ -ring. The canonical relation  $<_A$  is transitive.*

*Proof.* The  $\mathcal{C}^\infty$ -reduced  $\mathcal{C}^\infty$ -ring  $A$  can be presented as  $A \cong \mathcal{C}^\infty(\mathbb{R}^E)/I$ , for some set  $E$  and some  $\mathcal{C}^\infty$ -radical ideal  $I = \sqrt[\infty]{I} \subseteq \mathcal{C}^\infty(\mathbb{R}^E)$ . Thus let  $f, g, h \in \mathcal{C}^\infty(\mathbb{R}^E)$  such that  $f + I < g + I < h + I$ .

Now apply the characterization **Theorem 3.7** and consider  $\alpha, \beta \in I$  such that  $f(x) < g(x), \forall x \in Z(\alpha)$  and  $g(x) < h(x), \forall x \in Z(\beta)$ .

Then  $\gamma := \alpha^2 + \beta^2 \in I$  and  $Z(\gamma) = Z(\alpha) \cap Z(\beta)$ .

Thus  $f(x) < g(x) < h(x), \forall x \in Z(\gamma)$  and since  $\gamma \in I$ , applying again the **Theorem 3.7** we obtain  $f + I < h + I$ .  $\square$

**Proposition 3.9.** *Let  $A$  be any non-trivial  $\mathcal{C}^\infty$ -ring and  $a, b \in A$ . Then the relation  $<_A$  is asymmetric, i.e. it holds at most one of the following conditions:  $a < b$ ,  $b < a$ .*

*Proof.* Suppose that holds simultaneously both the conditions:  $a < b, b < a$ . Since  $<$  is transitive (**Proposition 3.8**) we have  $a < a$ , but this contradicts **Proposition 3.4**, since  $A$  is non-trivial.  $\square$

By a combination of **Propositions 3.4, 3.8** (and **3.9**), the canonical relation  $<$  on every  $\mathcal{C}^\infty$ -ring  $A$  that is non-trivial and  $\mathcal{C}^\infty$ -reduced is irreflexive, transitive (and asymmetric) binary relation on  $A$ : thus it defines a strict partial order. This motivates the following:

**Definition 3.10.** *Let  $A$  be a non-trivial,  $\mathcal{C}^\infty$ -reduced  $\mathcal{C}^\infty$ -ring. Then the canonical binary relation on  $A$ ,  $<_A$ , (**Definition 3.1**) will be called the “canonical strict partial order on  $A$ ”.*

Moreover, by **Proposition 3.3**, it holds:

**Theorem 3.11.** *Let  $A$  be any non-trivial  $\mathcal{C}^\infty$ -reduced  $\mathcal{C}^\infty$ -ring. The canonical partial order on  $A$ ,  $<_A$ , is compatible with the sum and with the product of  $A$ .*

Note that, due to the above result, to prove the trichotomy of  $<$  it suffices to prove that holds the “restricted form of trichotomy”: given any  $a \in A$  one has either  $a = 0, a < 0$  or  $0 < a$ . But, clearly, this is not true in general:

**Example 3.12.** *Let  $A = \mathcal{C}^\infty(\mathbb{R}^n)$  and consider the  $\mathcal{C}^\infty$ -function:*

$$f(x_1, \dots, x_n) := e^{(x_1 + \dots + x_n)} - 1.$$

*If  $x_1 + \dots + x_n > 0$ , then  $f(x_1, \dots, x_n) > 0$ ; if  $x_1 + \dots + x_n = 0$ , then  $f(x_1, \dots, x_n) = 0$  and if  $x_1 + \dots + x_n < 0$ , then  $f(x_1, \dots, x_n) < 0$ . Thus the assertion ( $f < 0$  or  $f = 0$  or  $0 < f$ ) is false.*

On the other hand, the restricted trichotomy holds for invertible members of some classes of  $\mathcal{C}^\infty$ -reduced  $\mathcal{C}^\infty$ -ring:

**Proposition 3.13.** *Given a  $\mathcal{C}^\infty$ -reduced  $\mathcal{C}^\infty$ -ring,  $A$ , one has:*

$$A^\times = (A^\times)^2 \cup (-A^\times)^2$$

*provided  $A$  satisfies some of the conditions below:*

1.  *$A$  is a free  $\mathcal{C}^\infty$ -ring;*
2.  *$A$  is a  $\mathcal{C}^\infty$ -reduced  $\mathcal{C}^\infty$ -domain.*

*Proof.* This holds trivially if  $0 = 1$ . We will prove that for a non trivial  $\mathcal{C}^\infty$ -reduced  $\mathcal{C}^\infty$ -ring,  $A$ , the non obvious inclusion:  $A^\times \subseteq (A^\times)^2 \cup (-A^\times)^2$  holds.

Item (1): First recall that any free  $\mathcal{C}^\infty$ -ring is  $\mathcal{C}^\infty$ -reduced (see **Example 1.12**). Let  $f \in \mathcal{C}^\infty(\mathbb{R}^E)^\times$ , then there is  $E' \subseteq_{\text{fin}} E$  and  $f' \in \mathcal{C}^\infty(\mathbb{R}^{E'})^\times$  such that  $f = \mu_{E'E}(f') = f' \circ \pi_{EE'}$ . Since  $f' : \mathbb{R}^{E'} \rightarrow \mathbb{R}$  is continuous and  $\mathbb{R}^{E'}$  is connected, then  $\text{range}(f) = \text{range}(f')$  is a connected subset of  $\mathbb{R}$ , so it is an interval. Since  $0 \notin \text{range}(f)$ , then exactly one of the following alternatives holds: (i)  $\text{range}(f) \subseteq ]-\infty, 0[$  or (ii)  $\text{range}(f) \subseteq ]0, \infty[$ . If (i) holds then  $f \in -(\mathcal{C}^\infty(\mathbb{R}^E)^\times)^2$  and if (ii) holds then  $f \in (\mathcal{C}^\infty(\mathbb{R}^E)^\times)^2$ .

Item (2): We take a presentation of  $A$  as  $A \cong \mathcal{C}^\infty(\mathbb{R}^E)/I$ , for some set  $E$  and some (ring theoretical) ideal  $I \in \text{Spec}^\infty(A)$ . Let  $f \in \mathcal{C}^\infty(\mathbb{R}^E)$  such that  $(f + I \in \frac{\mathcal{C}^\infty(\mathbb{R}^E)^\times}{I})$ . By **Proposition 2.13**, there exists  $\varphi \in I$  such that:

$$(\forall x \in \mathbb{R}^E)(x \in Z(\varphi) \rightarrow f(x) \neq 0).$$

Let  $E' \subseteq_{\text{fin}} E$  such that  $\varphi = \varphi' \circ \pi_{EE'}$  and  $f = f' \circ \pi_{EE'}$ , for some  $\varphi', f' \in \mathcal{C}^\infty(\mathbb{R}^{E'})$ . Then

$$(\forall x' \in \mathbb{R}^{E'})(x' \in Z(\varphi') \rightarrow f'(x') \neq 0).$$

Thus,

$$Z(\varphi') \subseteq [f' > 0] \cup [-f' > 0],$$

where:  $[\pm f' > 0] := \{x' \in \mathbb{R}^{E'} : \pm f'(x') > 0\}$ .

Note that:

$$Z(\varphi') \cap [\pm f' \geq 0] = Z(\varphi') \cap [\pm f' > 0]$$

Since  $f'$  is a continuous function,  $[\pm f' \geq 0] = (\pm f')^{-1}[[0, \infty[$  is a closed subset of  $\mathbb{R}^{E'}$ , and by **Fact 2.1**, there is some  $\chi_{\pm} \in \mathcal{C}^{\infty}(\mathbb{R}^{E'})$  such that

$$Z(\chi_{\pm}) = [\pm f' \geq 0].$$

Thus,

$$\begin{aligned} Z(\varphi') &= Z(\varphi') \cap \mathbb{R}^{E'} = Z(\varphi') \cap (Z(\chi_{-}) \cup Z(\chi_{+})) = \\ &= (Z(\varphi') \cap Z(\chi_{-})) \cup (Z(\varphi') \cap Z(\chi_{+})) = Z(\varphi'^2 + \chi_{-}^2) \cup Z(\varphi'^2 + \chi_{+}^2). \end{aligned}$$

Since  $I$  is a  $\mathcal{C}^{\infty}$ -radical (proper) prime ideal of  $\mathcal{C}^{\infty}(\mathbb{R}^E)$ , then  $I' := \mu_{E'E}^{-1}[I] \subseteq \mathcal{C}^{\infty}(\mathbb{R}^{E'})$  is a  $\mathcal{C}^{\infty}$ -radical (proper) prime ideal of  $\mathcal{C}^{\infty}(\mathbb{R}^{E'})$  (see **Proposition 1.15.(d)**).

By **Proposition 2.20**,  $I'$  corresponds to a prime filter (of zero sets)  $\hat{I}'$  and since  $Z(\varphi'^2 + \chi_{-}^2) \cup Z(\varphi'^2 + \chi_{+}^2) = Z(\varphi')$  and  $\varphi' \in I'$ , then some of the subsets  $Z(\varphi'^2 + \chi_{-}^2), Z(\varphi'^2 + \chi_{+}^2)$  belong to the  $\mathcal{C}^{\infty}$ -radical ideal  $I'$ . By **Proposition 2.18**:  $\check{I}' = \sqrt[\infty]{I'} = I'$ , thus some of the functions  $(\varphi'^2 + \chi_{-}^2), (\varphi'^2 + \chi_{+}^2)$  belongs to  $I'$ .

Now recall that:

$$Z(\varphi'^2 + \chi_{\pm}^2) = Z(\varphi') \cap Z(\chi_{\pm}) = Z(\varphi') \cap [\pm f' \geq 0] = Z(\varphi') \cap [\pm f' > 0]$$

and consider  $\alpha_{\pm} := (\varphi'^2 + \chi_{\pm}^2) \circ \pi_{EE'} \in \mathcal{C}^{\infty}(\mathbb{R}^E)$ .

Then some of the alternatives holds:

- (i)  $\alpha_{-} \in I$  and  $(\forall x \in Z(\alpha_{-}))(-f(x) > 0)$ ;
- (ii)  $\alpha_{+} \in I$  and  $(\forall x \in Z(\alpha_{+}))(f(x) > 0)$ .

Applying **Theorem 3.7**, if (i) holds then  $f + I \in -((\mathcal{C}^{\infty}(\mathbb{R}^E)/I)^{\times})^2$  and if (ii) holds then  $f + I \in ((\mathcal{C}^{\infty}(\mathbb{R}^E)/I)^{\times})^2$ .

This establishes the desired inclusion  $A^{\times} \subseteq (A^{\times})^2 \dot{\cup} (-A^{\times})^2$ .

□

There is another natural way to consider that a partial order  $\leq$  (where  $a \leq b$  iff  $a < b$  or  $a = b$ ) is compatible with sums: if  $0 \leq x$  and  $0 \leq y$ , then  $0 \leq x + y$ . This one also holds, as it follows (directly) from the results obtained below.

**Proposition 3.14.** *Given any  $\mathcal{C}^\infty$ -reduced  $\mathcal{C}^\infty$ -ring  $A$ , denote by  $(A^\times)^2 = A^2 \cap A^\times = (A^2)^\times$ . Then the following hold:*

1.  $(\sum A^2) \cap A^\times = (A^\times)^2$
2.  $(A^\times)^2 + \sum A^2 = (A^\times)^2$
3.  $\sum (A^\times)^2 = (A^\times)^2$

*Proof. First equality:*

One easily checks that:

$$\left(\sum A^2\right) \cap A^\times \supseteq A^2 \cap A^\times,$$

so we only need to prove the opposite inclusion.

We know that  $A \cong \frac{\mathcal{C}^\infty(\mathbb{R}^E)}{I}$  for some set  $E$  and some  $\mathcal{C}^\infty$ -radical ideal  $\sqrt[I]{I} = I \subseteq \mathcal{C}^\infty(\mathbb{R}^E)$ . Let  $f \in \mathcal{C}^\infty(\mathbb{R}^E)$  such that  $q_I(f) = f + I \in \left(\sum \frac{\mathcal{C}^\infty(\mathbb{R}^E)^2}{I}\right) \cap \frac{\mathcal{C}^\infty(\mathbb{R}^E)^\times}{I}$ .

Since  $q_I(f) \in \left(\frac{\mathcal{C}^\infty(\mathbb{R}^E)}{I}\right)^\times$ , by **Proposition 2.13**, there is some  $\varphi \in I$  such that:

$$(\forall x \in Z(\varphi))(f(x) \neq 0)$$

and since  $q_I(f) \in \sum \frac{\mathcal{C}^\infty(\mathbb{R}^E)^2}{I}$ , by **Proposition 2.12**, there are  $f_1, \dots, f_k \in \mathcal{C}^\infty(\mathbb{R}^E)$  and  $\psi \in I$  such that:

$$(\forall x \in Z(\psi))(f(x) = f_1(x)^2 + \dots + f_k(x)^2 \geq 0),$$

Thus  $\varphi^2 + \psi^2 \in I$  and

$$(\forall x \in Z(\varphi^2 + \psi^2)) = Z(\varphi) \cap Z(\psi)(f(x) = f_1^2(x) + \dots + f_k^2(x) > 0).$$

Applying **Theorem 3.7**, we have:

$$0 + I < f + I,$$

so  $f + I = u^2 + I$ , for some  $u \in \left(\frac{\mathcal{C}^\infty(\mathbb{R}^E)}{I}\right)^\times$ , establishing the equality in item (1).

**Second and third equalities:**

One easily checks that:

$$(A^\times)^2 + \sum A^2 \supseteq \sum (A^\times)^2 \supseteq (A^\times)^2,$$

so, to establish the items (2) and (3), we only need to prove that

$$(A^\times)^2 + \sum A^2 \subseteq (A^\times)^2.$$

Present  $A$  as  $A \cong \frac{\mathcal{C}^\infty(\mathbb{R}^E)}{I}$  for some set  $E$  and some  $\mathcal{C}^\infty$ -radical ideal  $\sqrt[I]{I} = I \subseteq \mathcal{C}^\infty(\mathbb{R}^E)$ . Let  $f \in \mathcal{C}^\infty(\mathbb{R}^E)$  such that  $q_I(f) = f + I \in \left( \frac{\mathcal{C}^\infty(\mathbb{R}^E)^\times}{I} \right)^2 + \left( \sum \frac{\mathcal{C}^\infty(\mathbb{R}^E)^2}{I} \right)$ . I.e., there are  $g, h_1, \dots, h_k \in \mathcal{C}^\infty(\mathbb{R}^E)$  such that  $f + I = (g^2 + h_1^2 + \dots + h_k^2) + I$  and  $g + I \in \left( \frac{\mathcal{C}^\infty(\mathbb{R}^E)^\times}{I} \right)$ .

Applying **Proposition 2.12** and **Proposition 2.13**, we conclude that there is  $\theta \in I$  such that:

$$(\forall x \in Z(\theta))(f(x) = g(x)^2 + h_1(x)^2 + \dots + h_k(x)^2 \text{ and } g(x) \neq 0)$$

Thus

$$(\forall x \in Z(\theta))(f(x) > 0).$$

Since  $\theta \in I$ , applying **Theorem 3.7**, we have:

$$0 + I < f + I,$$

so  $f + I = u^2 + I$ , for some  $u \in \left( \frac{\mathcal{C}^\infty(\mathbb{R}^E)}{I} \right)^\times$ , establishing the desired inclusion.  $\square$

From the second equality above, it follows directly the:

**Corollary 3.15.** *Every  $\mathcal{C}^\infty$ -ring  $A$  has the “weak bounded inversion property” (definition 7.1. in [7]), i.e.  $1 + \sum A^2 \subseteq A^\times$ .*

## 4 On the Order Theory of $\mathcal{C}^\infty$ -Fields and Applications

In this section we present concrete examples of finitely generated  $\mathcal{C}^\infty$ -fields and some important facts about general  $\mathcal{C}^\infty$ -fields. Then we use these facts, along

with the results presented in the previous section, to deepen the study of the order theory of  $\mathcal{C}^\infty$ -fields initiated by Moerdijk and Reyes in [16]. We address some results concerning  $\mathcal{C}^\infty$ -fields that will be useful in the sequel.

Now we consider two explicit situations where we have finitely generated  $\mathcal{C}^\infty$ -fields.

**Example 4.1.** *Let  $M$  be a smooth manifold. By **Theorem 2.3** of [18],  $\mathcal{C}^\infty(M) \cong \frac{\mathcal{C}^\infty(\mathbb{R}^k)}{J}$  for some  $k \in \mathbb{N}$  and some finitely generated ideal  $J$ . Thus, for any ideal  $I \subset \mathcal{C}^\infty(M)$ ,  $\mathcal{C}^\infty(M)/I$  is a finitely generated  $\mathcal{C}^\infty$ -ring.*

*Note that for each  $x \in M$ ,  $\mathfrak{m}_x = \{f \in \mathcal{C}^\infty(M) \mid f(x) = 0\}$  is a maximal ideal in  $\mathcal{C}^\infty(M)$ .*

*Now suppose that  $M$  is compact. If  $I \subset \mathcal{C}^\infty(M)$  is a maximal ideal of  $\mathcal{C}^\infty(M)$ , then  $I = \mathfrak{m}_x$  for some unique  $x \in M$ . In fact, given any ideal  $I \subseteq \mathcal{C}^\infty(M)$ , one has either  $I \subseteq \mathfrak{m}_x$  for some (unique)  $x \in M$  or  $I = \mathcal{C}^\infty(M)$ .*

*Suppose it is not the case that there is some  $x \in M$  such that  $I \subseteq \mathfrak{m}_x$ , i.e.,  $(\forall x \in M)(I \not\subseteq \mathfrak{m}_x)$ . For every  $x \in M$  we can find a function  $f_x \in I$  such that  $f_x(x) \neq 0$ . Consider the open covering  $\{M \setminus f_x^{-1}[\{0\}] \mid x \in M\}$  of  $M$ , which has a finite sub-covering, say  $\{M \setminus f_{x_1}^{-1}[\{0\}], \dots, M \setminus f_{x_r}^{-1}[\{0\}]\}$ . We obtain, thus, the function  $f = f_{x_1}^2 + \dots + f_{x_r}^2 \in I$  such that  $(\forall x \in M)(f(x) > 0)$ . Hence,  $f \in I \cap \mathcal{C}^\infty(M)^\times$  and  $I = \mathcal{C}^\infty(M)$ .*

*As for the uniqueness of  $x \in M$ , suppose that  $I \subset \mathfrak{m}_x$  and let  $y \in M$  be such that  $x \neq y$ . By the **Smooth Tietze's Theorem**, there is some  $f \in \mathcal{C}^\infty(M)$  such that  $f(x) = 0$  (so  $f \in I$  and  $f \in \mathfrak{m}_x$ ) and  $f(y) = 1$ , so  $f \notin \mathfrak{m}_y$ . Thus  $\mathfrak{m}_x \not\subseteq \mathfrak{m}_y$  and  $I \not\subseteq \mathfrak{m}_y$ .*

*It follows that whenever  $I$  is a maximal ideal of  $\mathcal{C}^\infty(M)$  - where  $M$  is a compact manifold - there is a unique  $x \in M$  such that  $I \subseteq \mathfrak{m}_x \subset \mathcal{C}^\infty(M)$ . Since  $I$  is a maximal ideal, then  $I = \mathfrak{m}_x$ .*

*Thus, for every maximal ideal  $I \subset \mathcal{C}^\infty(M)$ ,  $\mathcal{C}^\infty(M)/I \cong \mathbb{R}$  using the fact that the  $\mathcal{C}^\infty$ -homomorphism:*

$$\begin{array}{ccc} \text{ev}_x: \mathcal{C}^\infty(M) & \rightarrow & \mathbb{R} \\ f & \mapsto & f(x) \end{array}$$

*is surjective and the **Fundamental Theorem of the  $\mathcal{C}^\infty$ -Homomorphism** that:*

$$\frac{\mathcal{C}^\infty(M)}{\mathfrak{m}_x} = \frac{\mathcal{C}^\infty(M)}{\ker \text{ev}_x} \cong \mathbb{R}$$

*Hence, every  $\mathcal{C}^\infty$ -field obtained as a quotient  $\mathcal{C}^\infty(M)/I$  is isomorphic to  $\mathbb{R}$ .*

However, not every finitely generated  $\mathcal{C}^\infty$ -field is isomorphic to  $\mathbb{R}$ , as we see in the following:

**Example 4.2.** Consider  $\mathcal{C}^\infty(\mathbb{R})$  together with the ideal of all compactly supported functions:

$$I = \{f \in \mathcal{C}^\infty(\mathbb{R}) \mid \text{supp}(f) \subset \mathbb{R} \text{ is compact}\}$$

Naturally the constant function 1 does not belong to  $I$ , so there is a maximal ideal  $\hat{I} \subset \mathcal{C}^\infty(\mathbb{R})$  such that  $I \subseteq \hat{I}$ . Also, note that for every  $x \in \mathbb{R}$ ,  $I \not\subseteq \mathfrak{m}_x$ . In fact, for every  $x \in \mathbb{R}$ , the smooth characteristic function  $\chi_{]x-1, x+1[} : \mathbb{R} \rightarrow \mathbb{R}$  is a compactly supported function which does not belong to  $\mathfrak{m}_x$ . Since  $I \subset \hat{I}$ ,  $\hat{I} \neq \mathfrak{m}_x$  for every  $x \in \mathbb{R}$ .

Now, since  $\hat{I}$  is a maximal ideal different from  $\mathfrak{m}_x$  for every  $x \in \mathbb{R}$ ,  $\mathcal{C}^\infty(\mathbb{R})/\hat{I} \cong \mathbb{F}$  is a finitely generated  $\mathcal{C}^\infty$ -field that is different from  $\mathbb{R}$ .

An explicit description is given as follows. Let  $U \subset \wp(\mathbb{N})$  be a non-principal ultrafilter and let:

$$\hat{I} = \{f \in \mathcal{C}^\infty(\mathbb{R}) \mid \{n \in \mathbb{N} \mid f(n) = 0\} \in U\} \subset \mathcal{C}^\infty(\mathbb{R})$$

It is straightforward to check that  $\hat{I}$  is an ideal of  $\mathcal{C}^\infty(\mathbb{R})$ . Since  $U$  is a non principal ultrafilter,  $U$  contains all cofinite subsets of  $\mathbb{N}$ . Thus, given any  $f \in I$  - that is, any  $f \in \mathcal{C}^\infty(\mathbb{R})$  with compact support,  $K = \text{supp}(f)$ , since  $K$  is limited there is some  $n_0 \in \mathbb{N}$  such that  $K \subseteq [-n_0, n_0]$ , so  $(\forall n > n_0)(f(n) = 0)$ . Hence  $\{n \in \mathbb{N} \mid f(n) = 0\} \subset \mathbb{N}$  is cofinite and  $f \in \hat{I}$ . Thus  $I \subset \hat{I}$ .

Finally, in order to show that  $\hat{I}$  is a maximal ideal, we show that  $\mathcal{C}^\infty(\mathbb{R})/\hat{I}$  is a  $\mathcal{C}^\infty$ -field.

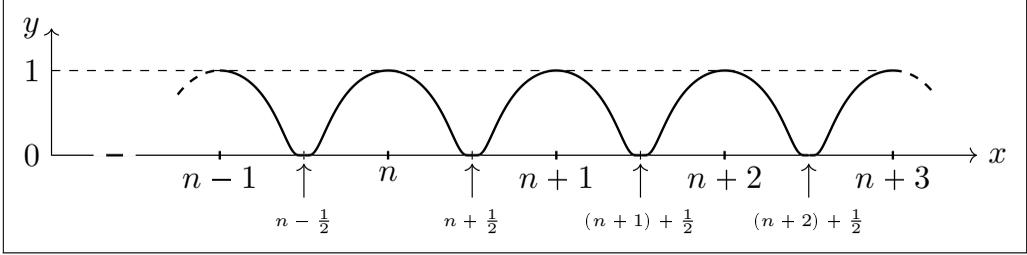
In fact, if  $f + \hat{I} \neq 0 + \hat{I}$ , then  $f \notin \hat{I}$  and  $\{n \in \mathbb{N} \mid f(n) = 0\} \notin U$ . Since  $U$  is an ultrafilter, we have  $\{n \in \mathbb{N} \mid f(n) \neq 0\} \in U$ .

Now, since  $\mathbb{N} \subset \mathbb{R}$  is discrete, we can take, for every  $n \in \mathbb{N}$  such that  $f(n) \neq 0$ , the open neighbourhood  $]n - \frac{1}{2}, n + \frac{1}{2}[$  with the smooth characteristic function:

$$\begin{aligned} \chi_{]n-\frac{1}{2}, n+\frac{1}{2}[} : \mathbb{R} &\rightarrow \mathbb{R} \\ x &\mapsto \begin{cases} e^{1-\frac{1}{1-4(x-n)^2}}, & \text{if } x \in ]n - \frac{1}{2}, n + \frac{1}{2}[ \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

and then glue them up to get the smooth function:

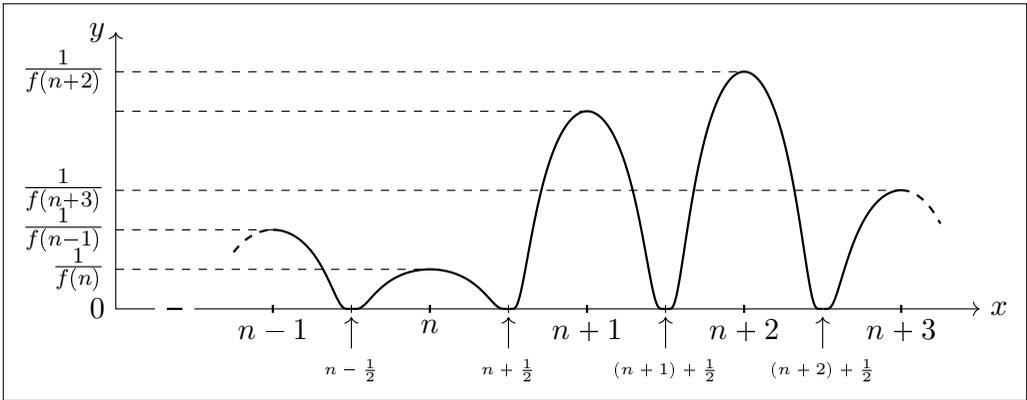
$$\begin{aligned} h : \mathbb{R} &\rightarrow \mathbb{R} \\ x &\mapsto \begin{cases} \chi_{]n-\frac{1}{2}, n+\frac{1}{2}[}(x), & \text{if } x \in ]n - \frac{1}{2}, n + \frac{1}{2}[ \text{ and } f(n) \neq 0 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$



Now consider:

$$g: \mathbb{R} \rightarrow \mathbb{R}$$

$$x \mapsto \begin{cases} \frac{h(x)}{f(n)}, & \text{if } x \in ]n - \frac{1}{2}, n + \frac{1}{2}[ \text{ and } f(n) \neq 0 \\ 0 & \text{otherwise} \end{cases}$$



and note that  $g \in \mathcal{C}^\infty(\mathbb{R})$ .

Also, note that for every  $n$  such that  $f(n) \neq 0$ , we have  $h(n) = 1$ , so  $g(n) = 1/f(n)$ . Thus, since  $\{n \in \mathbb{N} \mid f(n) \cdot g(n) - 1 = 0\} \in U \subset \wp(\mathbb{N})$  (for  $\{n \in \mathbb{N} \mid f(n) \cdot g(n) - 1 = 0\}$  is cofinite), it follows that:

$$f \cdot g - 1 \in \hat{I},$$

so  $f + \hat{I} \in (\mathcal{C}^\infty(\mathbb{R})/I)^\times$ . It follows that  $\mathcal{C}^\infty(\mathbb{R})/\hat{I}$  is a finitely generated  $\mathcal{C}^\infty$ -field which is not isomorphic to  $\mathbb{R}$ .

**Proposition 4.3.** *Let  $A$  be a  $\mathcal{C}^\infty$ -ring.*

1. *If  $A$  is a  $\mathcal{C}^\infty$ -field, then  $A$  is a  $\mathcal{C}^\infty$ -reduced  $\mathcal{C}^\infty$ -domain (see **Proposition 4.6** of [4]).*

2. If  $A$  is a  $\mathcal{C}^\infty$ -reduced  $\mathcal{C}^\infty$ -domain, then  $A\{(A\setminus\{0\})^{-1}\}$  is a  $\mathcal{C}^\infty$ -field,  $\eta_{A\setminus\{0\}} : A \rightarrow A\{(A\setminus\{0\})^{-1}\}$  is an injective  $\mathcal{C}^\infty$ -homomorphism and  $\eta_{A\setminus\{0\}}$  is universal among the injective  $\mathcal{C}^\infty$ -homomorphisms from  $A$  into some  $\mathcal{C}^\infty$ -field (see **Proposition 4.51** of [4]).
3.  $A$  is isomorphic to a  $\mathcal{C}^\infty$ -subring of a  $\mathcal{C}^\infty$ -field if, and only if  $A$  is a  $\mathcal{C}^\infty$ -reduced  $\mathcal{C}^\infty$ -domain (see **Corollary 4.36** and **Proposition 4.51** of [4]).
4. For every proper prime ideal  $\mathfrak{p} \subseteq A$  that is  $\mathcal{C}^\infty$ -radical, we have a canonical  $\mathcal{C}^\infty$ -field  $k_{\mathfrak{p}}(A) := \frac{A}{\mathfrak{p}}\{q_{\mathfrak{p}}[A\setminus\mathfrak{p}]^{-1}\}$  and a canonical  $\mathcal{C}^\infty$ -morphism with kernel  $\mathfrak{p}$ ,  $A \xrightarrow{q_{\mathfrak{p}}} \frac{A}{\mathfrak{p}} \xrightarrow{\eta_{q_{\mathfrak{p}}[A\setminus\mathfrak{p}]}} k_{\mathfrak{p}}(A)$  (see p. 102 of [4]).

Now we are ready to turn our attention to the order theory of  $\mathcal{C}^\infty$ -fields.

By **Corollary 3.15**, every  $\mathcal{C}^\infty$ -reduced  $\mathcal{C}^\infty$ -ring  $A$  satisfies the relation  $1 + \sum A^2 \subseteq A^\times$ . In particular, every  $\mathcal{C}^\infty$ -field  $A$  is formally real, i.e.  $-1 \notin \sum A^2$ , thus it can be endowed with some linear order relation compatible with its sum and product. In fact, since a  $\mathcal{C}^\infty$ -field is a non-trivial  $\mathcal{C}^\infty$ -reduced  $\mathcal{C}^\infty$ -ring, we have a distinguished linear order relation in  $A$  that is compatible with its sum and product:

**Theorem 4.4.** *Let  $A$  be a  $\mathcal{C}^\infty$ -field and  $<_A$  be the canonical strict partial order on  $A$ , cf. **Definition 3.10**. Then  $(A, \leq)$  is a totally/linearly ordered field, i.e.,  $\leq$  is a reflexive, transitive and anti-symmetric binary relation in  $A$  that is compatible with sum and product and, moreover, it holds the trichotomy law, i.e., for every  $a, b \in A$  we have exactly one of the following  $a = b$  or  $a < b$  or  $b < a$ . Moreover,  $0 \leq a$  iff  $a = b^2$  for some  $b \in A$ .*

*Proof.* Since  $a \leq b$  iff  $a < b$  or  $a = b$ , it follows directly from **Theorem 3.11** that  $\leq$  is a reflexive, transitive and anti-symmetric binary relation in  $A$  that is compatible with sum and product.

By the compatibility of  $<$  with the sum, to obtain the trichotomy law it is enough to show that for every  $f \in A\setminus\{0\}$  we have either  $(0 < f)$  or  $(f < 0)$ .

Since  $A$  is a  $\mathcal{C}^\infty$ -field,  $A^\times = A\setminus\{0\}$  and the result follows directly from **Proposition 4.3.(1)** and **Proposition 3.13.(2)**:  $A^\times = (A^\times)^2 \cup (-A^\times)^2$ . □

In general, a field could support many linear orders compatible with its sum and product. A field is called **Euclidean** if it has a unique (linear, compatible with  $+, \cdot$ ) ordering: these fields are precisely the ordered fields such that every positive member has a square root in the field. It is clear from the definition of  $<$  and by **Theorem 4.4** that every  $\mathcal{C}^\infty$ -field is Euclidean.

Now recall that a totally ordered field  $(\mathbb{F}, \leq)$  is **real closed** if it satisfies the following two conditions:

- (a)  $(\forall x \in \mathbb{F})(0 \leq x \rightarrow (\exists y \in \mathbb{F})(x = y^2))$  (i.e. it is an Euclidean field);
- (b) every polynomial of odd degree has, at least, one root;

Equivalently, a totally ordered field  $(\mathbb{F}, \leq)$  is real closed if, and only if it satisfies the conclusion of **intermediate value theorem** for all *polynomial* functions  $h : \mathbb{F} \rightarrow \mathbb{F}$ .

As pointed out in **Theorem 2.10** of [16], it holds the following:

**Fact 4.5.** *Every  $\mathcal{C}^\infty$ -field,  $A$ , together with its canonical order  $<$  is a real closed field.*

In fact, a stronger property holds for every  $\mathcal{C}^\infty$ -field. We have the  $\mathcal{C}^\infty$ -analog of the notion of “real closedness”:

**Fact 4.6 (Theorem 2.10’ of [16]).** *Let  $(\mathbb{F}, <)$  be a  $\mathcal{C}^\infty$ -field. Then  $(\mathbb{F}, <)$  is  $\mathcal{C}^\infty$ -real closed. I.e., it holds:*

$$(\forall f \in \mathbb{F}\{x\})((f(0) \cdot f(1) < 0) \& (1 \in \langle \{f, f'\} \rangle \subseteq \mathbb{F}\{x\}) \rightarrow (\exists \alpha \in ]0, 1[ \subseteq \mathbb{F})(f(\alpha) = 0))$$

Note that the class of  $\mathcal{C}^\infty$ -fields is an  $\mathcal{L}$ -elementary (proper) class, where all structures have cardinality at least  $2^{\aleph_0}$ .

The notion of  $\mathcal{C}^\infty$ -field is also useful to analyze the order theory of a  $\mathcal{C}^\infty$ -reduced  $\mathcal{C}^\infty$ -ring:

**Remark 4.7.** *Consider  $A = \mathcal{C}^\infty(\mathbb{R}^n)$ . The inclusion  $i : \mathcal{C}^\infty(\mathbb{R}^n) \hookrightarrow \text{Func}(\mathbb{R}^n, \mathbb{R}) = \mathbb{R}^{\mathbb{R}^n}$  obviously preserves and reflects the equality relation ( $=$ ) and the canonical strict partial order ( $<$ ). Note that, by **Example 4.1**, this inclusion can be factored through the  $\mathcal{C}^\infty$ -homomorphism:*

$$A \xrightarrow{(q_{\mathfrak{m}})_{\mathfrak{m}}} \prod_{\mathfrak{m} \in \text{Max}(A)} \frac{A}{\mathfrak{m}}$$

Thus, the family of all  $\mathcal{C}^\infty$ -fields  $\left\{ \frac{A}{\mathfrak{m}} : \mathfrak{m} \in \text{Max}(A) \right\}$  encodes the canonical relation  $<_A$  on  $A$ .

The family of all  $\mathcal{C}^\infty$ -fields  $\{k_{\mathfrak{p}}(A) : \mathfrak{p} \in \text{Spec}^\infty(A)\}$  also encodes the canonical relation  $<_A$  on  $A$ .

Consider the canonical  $\mathcal{C}^\infty$ -homomorphism  $c_A : A \rightarrow \prod_{\mathfrak{p} \in \text{Spec}^\infty(A)} k_{\mathfrak{p}}(A)$ , given by:

$$A \xrightarrow{(q_{\mathfrak{p}})_{\mathfrak{p}}} \prod_{\mathfrak{p} \in \text{Spec}^\infty(A)} \frac{A}{\mathfrak{p}} \xrightarrow{(\eta_{q_{\mathfrak{p}}[A/\mathfrak{p}]})_{\mathfrak{p}}} \prod_{\mathfrak{p} \in \text{Spec}^\infty(A)} k_{\mathfrak{p}}(A)$$

By **Proposition 4.3.(4)**,  $\ker(c_A) = \bigcap_{\mathfrak{p} \in \text{Spec}^\infty(A)} \mathfrak{p}$ . Thus, by **Theorem 1.18.(e)**,  $\ker(c_A) = \sqrt[0]{0} = \{0\}$  and  $c_A$  is an injective  $\mathcal{C}^\infty$ -homomorphism, i.e., it preserves and reflects the equality relation. We will see that  $c_A$  also preserves and reflects the canonical relation  $<$ .

The  $\mathcal{C}^\infty$ -homomorphism  $c_A$  preserves  $<$  (see **Remark 3.2**). Note that, to establish that  $c_A : A \rightarrow \prod_{\mathfrak{p} \in \text{Spec}^\infty(A)} k_{\mathfrak{p}}(A)$  reflects  $<$  it suffices to guarantee that:

$$A \xrightarrow{(q_{\mathfrak{m}})_{\mathfrak{m}}} \prod_{\mathfrak{m} \in \text{Max}(A)} \frac{A}{\mathfrak{m}}$$

reflects  $<$ . In fact, since  $(\forall \mathfrak{m} \in \text{Max}(A))(\frac{A}{\mathfrak{m}} \cong k_{\mathfrak{m}}(A))$ , the inclusion  $\text{Max}(A) \subseteq \text{Spec}^\infty(A)$  (this holds by **Proposition 4.3.(1)**) induces a canonical “projection”:

$$\pi_A : \prod_{\mathfrak{p} \in \text{Spec}^\infty(A)} k_{\mathfrak{p}}(A) \twoheadrightarrow \prod_{\mathfrak{m} \in \text{Max}(A)} \frac{A}{\mathfrak{m}}$$

and, obviously,

$$(q_{\mathfrak{m}})_{\mathfrak{m}} = \pi_A \circ c_A,$$

Thus, if  $a, b \in A$  are such that  $a \not\prec b$ , implies  $(q_{\mathfrak{m}})_{\mathfrak{m}}(a) \not\prec (q_{\mathfrak{m}})_{\mathfrak{m}}(b)$ , then also holds  $c_A(a) \not\prec c_A(b)$ .

Now we will apply the results on  $\mathcal{C}^\infty$ -fields to describe another approach of the order theory of (general)  $\mathcal{C}^\infty$ -rings.

**Definition 4.8.** Let  $A$  be an arbitrary  $\mathcal{C}^\infty$ -ring. Let  $\mathcal{F}$  be the (proper) class of all the  $\mathcal{C}^\infty$ -homomorphisms of  $A$  to some  $\mathcal{C}^\infty$ -field, that is:

$$\mathcal{F} = \bigcup_{\mathbb{F} \in \text{Obj}(\mathcal{C}^\infty\text{-Fld})} \text{Hom}_{\mathcal{C}^\infty\text{-Rng}}(A, \mathbb{F})$$

We define the following relation  $\mathcal{R}$ : given  $h_1 : A \rightarrow \mathbb{F}_1$  and  $h_2 : A \rightarrow \mathbb{F}_2$ , we say that  $h_1$  is related with  $h_2$  if, and only if, there is some  $\mathcal{C}^\infty$ -field  $\tilde{\mathbb{F}}$  and some

$\mathcal{C}^\infty$ -fields homomorphisms  $\mathcal{C}^\infty$   $f_1 : \mathbb{F}_1 \rightarrow \tilde{\mathbb{F}}$  and  $f_2 : \mathbb{F}_2 \rightarrow \tilde{\mathbb{F}}$  such that the following diagram commutes:

$$\begin{array}{ccccc}
 & & \mathbb{F}_1 & & \\
 & \nearrow^{h_1} & & \searrow_{f_1} & \\
 A & & & & \tilde{\mathbb{F}} \\
 & \searrow_{h_2} & & \nearrow_{f_2} & \\
 & & \mathbb{F}_2 & & 
 \end{array}$$

The relation  $\mathcal{R}$  defined above is symmetric and reflexive.

The above considerations prove the following:

**Proposition 4.9** (see **Proposition 6.11 of [4]**). *If  $h_1 : A \rightarrow \mathbb{F}_1$  and  $h_2 : A \rightarrow \mathbb{F}_2$  be two  $\mathcal{C}^\infty$ -homomorphisms from the  $\mathcal{C}^\infty$ -ring  $A$  to the  $\mathcal{C}^\infty$ -fields  $\mathbb{F}_1, \mathbb{F}_2$  such that  $(h_1, h_2) \in \mathcal{R}$ , then  $\ker(h_1) = \ker(h_2)$ .*

**Definition 4.10.** *Let  $A$  be a  $\mathcal{C}^\infty$ -ring. A  $\mathcal{C}^\infty$ -ordering in  $A$  is a subset  $P \subseteq A$  such that:*

(O1)  $P + P \subseteq P$ ;

(O2)  $P \cdot P \subseteq P$ ;

(O3)  $P \cup (-P) = A$

(O4)  $P \cap (-P) = \mathfrak{p} \in \text{Spec}^\infty(A)$

**Fact 4.11.** *Let  $\Sigma(A) := \{(\mathfrak{p}, Q) : \mathfrak{p} \in \text{Sper}^\infty(A), Q \in \text{Spec}^\infty(k_{\mathfrak{p}}(A))\}$ . The mapping  $P \in \text{Spec}^\infty(A) \mapsto (\mathfrak{p}_P, Q_P) \in \Sigma(A)$ , where  $\mathfrak{p}_P := P \cap (-P)$  (or simply  $\mathfrak{p}$ ) and  $Q_P := \{\eta_{q_{\mathfrak{p}}[A \setminus \mathfrak{p}]}(a + \mathfrak{p}) \cdot (\eta_{q_{\mathfrak{p}}[A \setminus \mathfrak{p}]}(b + \mathfrak{p}))^{-1} : b \notin \mathfrak{p}, a, b \in P\}$  is a bijection (this is the analog of **Proposition 5.1.1** in [13]).*

**Definition 4.12.** *Let  $A$  be a  $\mathcal{C}^\infty$ -ring. Given a  $\mathcal{C}^\infty$ -ordering  $P$  in  $A$ , the  $\mathcal{C}^\infty$ -support of  $A$  is given by:*

$$\text{supp}^\infty(P) := \mathfrak{p}_P = P \cap (-P)$$

**Definition 4.13.** *Let  $A$  be a  $\mathcal{C}^\infty$ -ring. The  $\mathcal{C}^\infty$ -real spectrum of  $A$  is given by:*

$$\text{Sper}^\infty(A) = \{P \subseteq A \mid P \text{ is an ordering of the elements of } A\}$$

together with the (spectral) topology generated by the sets:

$$H^\infty(a) = \{P \in \text{Sper}^\infty(A) \mid a \in P \setminus \text{supp}^\infty(P)\}$$

for every  $a \in A$ . The topology generated by these sets will be called “smooth Harrison topology”, and will be denoted by  $\text{Har}^\infty$ .

**Remark 4.14.** *The suitable notion of prime spectrum of a  $\mathcal{C}^\infty$ -ring  $A$ ,  $\text{Spec}^\infty(A)$ , appeared for the first time in [17]: this is the main spatial notion to develop “Smooth Algebraic Geometry”. On the other hand, in [2] was introduced the notion of smooth real spectrum of a  $\mathcal{C}^\infty$ -ring  $A$ ,  $\text{Sper}^\infty(A)$ : this seems to be the suitable spatial notion for the development of “Smooth Real Algebraic Geometry”.*

**Fact 4.15.** *Given a  $\mathcal{C}^\infty$ -ring  $A$ , we have a function given by:*

$$\begin{array}{ccc} \text{supp}^\infty : (\text{Sper}^\infty(A), \text{Har}^\infty) & \rightarrow & (\text{Spec}^\infty(A), \text{Zar}^\infty) \\ P & \mapsto & P \cap (-P) \end{array}$$

which is spectral, and thus continuous, since given any  $a \in A$ , we have:

$$\text{supp}^{\infty-1}[D^\infty(a)] = H^\infty(a) \cup H^\infty(-a).$$

Unlike what happens in ordinary Commutative Algebra, we have the following (and stronger) result in “Smooth Commutative Algebra”, as a consequence of the fact that every  $\mathcal{C}^\infty$ -field is  $(\mathcal{C}^\infty)$ -real closed<sup>1</sup> and some separation theorems (see [5] or **Theorem 6.22** of [4]):

**Theorem 4.16.** *For each  $\mathcal{C}^\infty$ -ring  $A$ , the mapping*

$$\text{supp}^\infty : (\text{Sper}^\infty(A), \text{Har}^\infty) \rightarrow (\text{Spec}^\infty(A), \text{Zar}^\infty)$$

*is a (spectral) bijection.*

## 5 Concluding Remarks and Future Works

**Remark 5.1.** *It is natural to ask if the class of  $\mathcal{C}^\infty$ -fields is model-complete in the language of  $\mathcal{C}^\infty$ -rings or even admits elimination of quantifiers (possibly in the language expanded by a unary predicate for the positive cone of an ordering). If the former holds, then the relation  $\mathcal{R}$  between pairs of morphisms with the same source and target  $\mathcal{C}^\infty$ -fields, that encodes  $\text{Sper}^\infty$ , is already a transitive relation (as it occurs in the algebraic case).*

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<sup>1</sup>In fact, to obtain this result it is enough to know that every  $\mathcal{C}^\infty$ -field is *Euclidean*.

**Remark 5.2.** *If the class of  $\mathcal{C}^\infty$ -fields admits quantifier elimination (over a reasonable language), then it is possible to adapt the definition and results provided in [19] on “Model-theoretic Spectra” and describe “logically” the spectral topological spaces  $\text{Spec}^\infty(A)$  and/or  $\text{Sper}^\infty(A)$  as certain equivalence classes of homomorphisms from  $A$  into models of a “nice” theory  $T$ . Moreover, since the techniques in this work provide structural sheaves of “definable functions”, we could compare them with other ones previously defined and determine other new natural model-theoretic spectra in  $\mathcal{C}^\infty$ -structures.*

**Remark 5.3.** *Another evidence that a systematic model-theoretic analysis of  $\mathcal{C}^\infty$ -rings, (not only under real algebra perspective but also under differential algebra perspective), should be interesting and deserves a further attention is indicated in [10]. In that work the first steps were taken towards a model-theoretic connection between three kinds of structures:  $o$ -minimal structures, Hardy fields and smooth rings. This triple is related to another one – Hardy fields, surreal numbers and transseries – studied in [1]: these are linked by the notion of  $H$ -field which provides a common framework for these structures. They present a model-theoretic analysis of the category of  $H$ -fields, e.g. the theory of  $H$ -closed fields is model complete, and relate these results with the latter triple, that according the authors M. Aschenbrenner, L. v. Dries and J. v. Hoeven ([1]): “...are three ways to enrich the real continuum by infinitesimal and infinite quantities. Each of these comes with naturally interacting notions of ordering and derivative”.*

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