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Abstract. Explicit formulae are given identifying the adjoint representation for the classical Lie groups $O(n)$, $U(n)$, $Sp(n)$ and related groups as an element in the representation ring of the group.

1 Introduction

The adjoint representation Ad of a Lie group G into the linear automorphisms of its Lie algebra g is of fundamental importance in the theory and applications of Lie groups. Being a real representation it determines an element of the real representation ring $RO(G)$, which we always regard as a subring of the complex representation ring $R(G)$. In this note we shall identify this element in $R(G)$ for the classical (compact) Lie groups. For background material on representations of Lie groups we refer the reader to [2], [4], and [6] .

It is important to note that there are two distinct approaches to constructing $R(G)$. The "algebraic" approach uses the weights of the representations, while the "topological" approach uses the characters obtained by restriction to a maximal torus. Indeed let G be a compact simply connected simple Lie group, T a maximal torus and $W = W(G,T)$ the Weyl group of G with respect to T. Then it is a classical result due to H.Weyl that the RG is a polynomial ring over the integers. In the representation theoretic approach, algebra generators are taken as the (irreducible) fundamental representations. In the topological approach, RG is usually described in terms of W-invariants of RT. In this description, it is more convenient to use as algebra generators of RG certain W-invariant elements in RT which may not correspond to a fundamental representation. (Compare [1], Prop. 1.)

A comparision of the two approaches can be found in [8]. In this note

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we wish to obtain a description of the class of Ad in RG. The formulae for Ad in terms of the weights, i.e. in the algebraic approach, are well understood and appear in a number of texts (cf. [4], p.185). We give these in the table below, where the notations and labelling of roots and weights are as in [8]. For a dominant integral weight λ, $V(\lambda)$ denotes the irreducible representation with highest weight λ.

G	Type of G	$\dim G$	longest root	Ad
$SU(n)$ $(n \geq 2)$	A_{n-1}	$n^2 - 1$	$\varepsilon_1 - \varepsilon_n$	$V(\pi_1 + \pi_{n-1})$
$Spin(2n+1)$ $(n \geq 2)$	B_n	$2n^2 + n$	$\varepsilon_1 + \varepsilon_2$	$V(\pi_2)$
$Sp(n)$ $(n \geq 3)$	C_n	$2n^2 + n$	$2\varepsilon_1$	$V(2\pi_1)$
$Spin(2n)$ $(n \geq 4)$	D_n	$2n^2 - n$	$\varepsilon_1 + \varepsilon_2$	$V(\pi_2)$

Indeed, Ad is a fundamental representation for the classical simple groups $SO(2n+1), n \geq 2$ and $SO(2n), n \geq 4$. This is not so for $SU(n)$ or $Sp(n), n \geq 3$. However the formulae for Ad in the topological approach do not seem to be explicitly given in the literature and this note will do so. Standard sources for the representation rings via the topological approach are [2] Chapter 7, [4] Chapter 6, and [6] Chapter 7.

In terms of the usual notation for the generators of these rings (e.g. λ_j denotes the j'th exterior power representation) our main theorem is now stated.

Theorem 1.1: (i) For $G = O(n)$, $Ad = \lambda_2$,

(ii) For $G = U(n)$, $Ad = \lambda_1 \overline{\lambda_1}$,

(iii) For $G = Sp(n)$, $Ad = \lambda_1^2 - \lambda_2$.

As a corollary one immediately has similar formulae for the related Lie groups such as $SO(n), Spin(n), SU(n)$, etc., details are given below. One immediate application of this approach is in determining the induced map $B_{Ad} : BG \to BO(m)$ (where m is the dimension of the Lie group G) on the classifying spaces, and its action in the cohomology of these classifying spaces. An example of such a formula can be found in [5] §4. Another application appears in the relation of Ad to the normal bundle of a homogeneous space, a relation that goes back to the work of Borel and Hirzebruch [3] (or cf. [5]). For example, Theorem 1.1 shows up implicitly in the work of K. Y. Lam [7], §4 and §5 where Ad is called λ^2 in the real case and μ^2 in the complex case (corresponding respectively to (i), (ii) in Theorem 1.1).

2 The Orthogonal Group

Let $G = O(n)$. Its Lie algebra $o(n)$ consists of the real skew symmetric $n \times n$ matrices. As a basis (real or complex) for $o(n)$ we use the $\binom{n}{2}$ skew symmetric matrices $a_{jk} := e_{jk} - e_{kj}$, where e_{jk} is the unit matrix with a 1 in the j, k position and 0 elsewhere. The transpose of a matrix A will be denoted A^T. In this case a direct calculation of the adjoint representation for all $h \in O(n)$ turns out to be quite simple so we proceed this way rather than using the maximal torus (of $SO(n)$, which would involve a number of special cases, e.g. depending on whether n is even or odd).

So let $h \in O(n)$, then $h^{-1} = h^T$. For any $X \in o(n)$, the definition of the adjoint representation implies $Ad(h)(X) = hXh^{-1} = hXh^T$. It will be convenient to write $h = [x_{ij}]$, $h^T = [y_{ij}]$, so $y_{ij} = x_{ji}$. Then

$$Ad(h)(a_{jk}) = h(e_{jk} - e_{kj})h^T = he_{jk}h^T - he_{kj}h^T$$

is a skew symmetric matrix with (il)-term (say for $i < l$) clearly equal to

$x_{ij}y_{kl} - x_{ik}y_{jl} = x_{ij}x_{lk} - x_{ik}x_{lj}$.

It follows that

$$Ad(h)(a_{jk}) = \sum_{i<l}(x_{ij}x_{lk} - x_{ik}x_{lj})a_{il} \qquad (2.1)$$

On the other hand, the action of $\lambda_2 h$ in the vector space $\lambda_2\mathbf{R}^n$ is given by

$$\begin{aligned}\lambda_2 h(e_j \wedge e_k) = he_j \wedge he_k &= (\textstyle\sum_i x_{ij}e_i) \wedge (\textstyle\sum_l x_{lk}e_l) \\ &= \textstyle\sum_{i<l}(x_{ij}x_{lk} - x_{lj}x_{ik})e_i \wedge e_l. \qquad (2.2)\end{aligned}$$

Comparing (2.1) and (2.2) it is clear that the linear isomorphism $o(n) \to \lambda_2\mathbf{R}^n$ given by $a_{jk} \mapsto e_j \wedge e_k$ is in fact a G-isomorphism, and this proves Theorem 1.1 (i).

3 The Unitary Group

Let $G = U(n)$. As usual A^* denotes the Hermitian transpose or a matrix. A basis for its Lie algebra $u(n)$ (as a real vector space) is as follows (where $u(n) = \{A : A^* = -A\}$ consists of the $n \times n$ skew Hermitian matrices, of real dimension n^2). For notational clarity we henceforth reserve i for $\sqrt{-1}$.

$$\begin{aligned} a_j : &= ie_{jj}, \\ b_{jk} : &= e_{jk} - e_{kj}, \\ c_{jk} : &= ie_{jk} + ie_{kj}. \end{aligned}$$

Here $1 \leq j \leq n$ (for a_j) and $1 \leq j < k \leq n$ (for b_{jk}, c_{jk}). We can and do use the same basis for the complexification of this real vector space, so now we regard it as a complex vector space with complex dimension n^2 and having the same basis. For any $h \in U(n)$ and $X \in u(n)$, the adjoint action is given by $Ad(h)(X) = hXh^{-1}$, where here $h^{-1} = h^*$. To find the characters of this representation we take $h \in T$ for a maximal torus T, in particular let T be the diagonal matrices of form $D = \text{diag}(z_1, ..., z_n)$ where each z_j is a complex number of modulus 1 and let $h = D$. Write $z_j\overline{z_k} = x_{jk} + iy_{jk}$, a simple matrix multiplication shows

$$\begin{aligned} Ad(h)a_j &= a_j, \\ Ad(h)b_{jk} &= x_{jk}b_{jk} + y_{jk}c_{jk}, \\ Ad(h)c_{jk} &= -y_{jk}b_{jk} + x_{jk}c_{jk}. \end{aligned}$$

Taking the trace gives us the character

$$\chi Ad = n + 2\sum_{j<k} x_{jk} = n + \sum_{j<k}(z_j\overline{z_k} + z_k\overline{z_j}). \quad (3.1)$$

Now the characters of the exterior powers (of the fundamental representation of $U(n)$ on $\mathbf{C}^n$) are the elementary symmetric polynomials in $z_1, ..., z_n$, in particular $\chi\lambda_1 = z_1 + ... + z_n$ and $\chi\overline{\lambda_1} = \overline{z_1} + ... + \overline{z_n}$. Multiplying these last two gives the identical expression as the right hand side of (3.1), so we conclude $Ad = \lambda_1\overline{\lambda_1}$ as stated in Theorem 1.1 (ii).

4 The Symplectic Group

Let $G = Sp(n)$. As above A^T, A^* will denote the (usual) transpose and the Hermitian transpose of a matrix, respectively. The Lie group $Sp(n)$ is taken to be the matrices in $U(2n)$ having the form

$$\begin{bmatrix} A & B \\ -\overline{B} & \overline{A} \end{bmatrix}.$$

Its Lie algebra $sp(n)$, of (real) dim $\binom{2n+1}{2}$, will be all matrices (with complex entries) having the form

$$\begin{bmatrix} X & Y \\ -\overline{Y} & \overline{X} \end{bmatrix},$$

where $X + X^* = 0$, $Y = Y^T$, i.e. X is skew Hermitian and Y is (ordinary) symmetric. The maximal torus consists of diagonal matrices of the form $\mathrm{diag}(z_1, ..., z_n, \overline{z}_1, ..., \overline{z_n}) := h$, of course $h^{-1} = h^*$ and the z_i are unit complex numbers as before. As basis for this Lie algebra $sp(n)$ we take (using the convenient notation $j' = j + n$)

$$\begin{aligned}
a_j &= ie_{jj} - ie_{j'j'}, \\
b_{jk} &= e_{jk} - e_{kj} + e_{j'k'} - e_{k'j'}, \\
c_{jk} &= i(e_{jk} + e_{kj} - e_{j'k'} - e_{k'j'}), \\
d_j &= e_{jj'} - e_{j'j}, \\
d'_j &= i(e_{jj'} + e_{j'j}), \\
f_{jk} &= e_{jk'} + e_{kj'} - e_{j'k} - e_{k'j}, \\
g_{jk} &= i(e_{jk'} + e_{kj'} + e_{j'k} + e_{k'j}).
\end{aligned}$$

As usual $j < k$ here, and indices vary in $1, ..., n$.

It will be convenient to introduce the notation

$$\begin{aligned}
z_j\overline{z}_k &= x_{jk} + iy_{jk} = x + iy, \; j < k, \\
z_j z_k &= u_{jk} + iv_{jk} = u + iv, \; j \leq k.
\end{aligned}$$

In terms of this notation, straightforward matrix multiplications as in the $U(n)$ case give

$$\begin{aligned}
ha_jh^{-1} &= a_j, \\
hb_{jk}h^{-1} &= x_{jk}b_{jk} + y_{jk}c_{jk}, \\
hc_{jk}h^{-1} &= -y_{jk}b_{jk} + x_{jk}c_{jk}, \\
hd_jh^{-1} &= u_{jj}d_j + v_{jj}d'_j, \\
hd'_jh^{-1} &= -v_{jj}d_j + u_{jj}d'_j, \\
hf_{jk}h^{-1} &= u_{jk}f_{jk} + v_{jk}g_{jk}, \\
hg_{jk}h^{-1} &= -v_{jk}f_{jk} + u_{jk}g_{jk}.
\end{aligned}$$

¿From these it is trivial to calculate the trace, thus giving the character of Ad. The result is

$$\chi(Ad) = n+\Sigma_{1\le j<k\le n}(z_j\overline{z}_k+\overline{z}_j z_k)+\Sigma_{j=1}^n(z_j^2+\overline{z}_j^2)+\Sigma_{1\le j<k\le n}(z_j z_k+\overline{z}_j\overline{z}_k).$$

Now according to [2] Chap 7 the character of the r'th exterior power representation λ_r of the standard representation of $Sp(n)$ acting on C^{2n}, is the r'th symmetric polynomial in $z_1, ..., z_n, \overline{z}_1, ...\overline{z}_n$. Thus

$$\begin{aligned} \chi(\lambda_1) &= \Sigma_{j=1}^n(z_j+\overline{z}_j), \\ \chi(\lambda_2) &= n+\Sigma_{1\le j<k\le N}(z_j+\overline{z}_j)(z_k+\overline{z}_k). \end{aligned}$$

Using this, and the fact that χ is a ring homomorphism, a straightforward computation shows that $\chi(\lambda_1^2-\lambda_2)=\chi(Ad)$, so we conclude the equality of these two representations.

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