Russian Physics Journal, Vol. 54, No. 8, January, 2012 (Russian Original No. 8, August, 2011)

STRUCTURE OF THE ELECTROMAGNETIC FIELD ALLOWING EXACT SOLUTION OF THE SCHRÖDINGER EQUATION IN SUPERPOSITION WITH AN AHARONOV–BOHM FIELD

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UDC 539.12; 530.145

A charged particle is considered in a complex external electromagnetic field. The field is a superposition of an Aharonov–Bohm field and some additional field. Here we describe all additional fields known up to the present time that allow exact solution of the Schrödinger equation in a complex field.

Keywords: Schrödinger equation, exact solutions, Aharonov–Bohm effect

INTRODUCTION

A model of electromagnetic potential that describes a singular Aharonov–Bohm (solenoid) field was proposed in pioneering work [1]. Evidence for this phenomenon was deduced for the first time in [2]. With this, the research field of the phenomenon called the Aharonov–Bohm effect was initiated. A quite important peculiarity of the model potential in such a field was the fact that it allowed exact solutions of the Schrödinger, Klein–Gordon, and Dirac equations to be found. A number of articles appeared (for example, see [3–10]) in which special features of the solution and the physical outcomes of these features were studied. Another essential fact is the possibility of finding exact solutions of the above-mentioned equations for complex fields that are a superposition of a singular Aharonov–Bohm field and some additional electromagnetic field. Solutions of the Schrödinger equation for the superposition of Aharonov–Bohm and constant uniform magnetic fields were first obtained by R. R. Lewis [11]. Another significant fact is the possibility of finding exact solutions of the Schrödinger, Klein–Gordon, and Dirac equations for the combined fields representing a superposition of a singular field of the Aharonov–Bohm field and some additional electromagnetic field. In the first work in this direction [11], the Schrödinger equations were obtained for the superposition field of Aharonov–Bohm with a constant and uniform magnetic field. In subsequent papers [12–17], the exact solutions of the Dirac and Klein–Gordon equations in such combined fields were obtained and studied in detail. A manifestation of the Aharonov–Bohm effect in synchrotron radiation was analyzed based on the above-mentioned solutions. The exact solution of the Schrödinger, Dirac and Klein–Gordon equations in a combination of the Aharonov–Bohm field, the Coulomb field, and a magnetic monopole field were obtained in a series of publications [18–25], and some physical effects in this field were discussed. Finally, in [25–28] the exact solutions in different combinations of the Aharonov–Bohm field with additional fields were studied. Thus far there is a collection of various external electromagnetic fields, which in combination with the Aharonov–Bohm field allow exact solutions of the quantum mechanics equations.

In this paper, a general analysis of methods for obtaining exact solutions of the Schrödinger equation for the combined fields, which are superpositions of a singular Aharonov–Bohm field and some additional electromagnetic field, is made given all the known exact solutions and ways of obtaining new exact solutions are identified.

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1. CHARACTERISTIC OF THE AHARONOV–BOHM FIELD AND SOME GENERAL PROPERTIES OF THE WAVE FUNCTIONS

In items 1 and 2 we use Cartesian coordinates $\boldsymbol{r}=(x,y,z)$ and cylindrical coordinates r,φ in the plane x,y $(x=r\cos\varphi,\ \ y=r\sin\varphi)$.

Let us call the Aharonov–Bohm (solenoid) field an electromagnetic field for which the electric field is zero, and the intensity (induction) of the magnetic $\boldsymbol{H}$ field is given as

$$B_x=B_y=0,\ \ B_z=\Phi\,\delta(x)\delta(y)=\frac{\Phi}{\pi r}\delta(r),\ \ r^2=x^2+y^2,\ \ \Phi=\text{const}. \tag{1.1}$$

For field (1.1), the corresponding electromagnetic potential $A_\nu^{(0)}$ has the form

$$A_\nu^{(0)}=\left(A_0^{(0)},\ \ -\boldsymbol{A}^{(0)}\right),\ \ \boldsymbol{A}^{(0)}=\left(A_x^{(0)},\ A_y^{(0)},\ A_z^{(0)}\right),$$

$$A_0^{(0)}=A_z^{(0)}=0,\ \ A_x^{(0)}=-\frac{\Phi}{2\pi}\frac{y}{r^2}=\frac{\Phi}{2\pi}\frac{\partial\varphi}{\partial x},\ \ A_y^{(0)}=\frac{\Phi}{2\pi}\frac{x}{r^2}=\frac{\Phi}{2\pi}\frac{\partial\varphi}{\partial y},$$

$$\boldsymbol{A}^{(0)}=\frac{\Phi}{2\pi r}\boldsymbol{e}_\varphi,\ \ \boldsymbol{e}_\varphi=-\sin\varphi\,\boldsymbol{i}+\cos\varphi\,\boldsymbol{j}. \tag{1.2}$$

We use the following notation: $\boldsymbol{i},\boldsymbol{j},\boldsymbol{k}$ are the unitary vectors for the Cartesian coordinates x,y,z; vectors $\boldsymbol{e}_\varphi,\ \boldsymbol{e}_r$ are the unitary vectors for the cylindrical coordinates $\varphi,\ r$.

It is impossible to remove the terms containing $\frac{\partial\varphi}{\partial x},\frac{\partial\varphi}{\partial y}$, in potentials (1.2) using gauge transformations for all x,y, because the function

$$\varphi(x,y)=i\ln\frac{x-iy}{\sqrt{x^2+y^2}} \tag{1.3}$$

is not defined at the point $x=0,\ y=0$, and at this point the corresponding gauge transformation loses its sense, which originates $\delta(r)$ in Eq. (1.1).

Suppose that an arbitrary two-dimensional simply connected domain (surface) S is located in the plane $z=\text{const}$, is limited by a smooth closed contour l, and the point $x=0,\ \ y=0$ is inside it. For the magnetic flux $\Phi_S^{(0)}$ of field (1.1) through this surface, we have

$$\Phi_S^{(0)}=\iint_S(\boldsymbol{B}d\boldsymbol{S})=\oint_l\ (\boldsymbol{A}^{(0)}d\boldsymbol{l})=\Phi, \tag{1.4}$$

which defines the meaning of quantity Φ.

Singular solenoid field (1.1) and (1.2) can be obtained as (generalized) limit of arbitrary smooth magnetic field. Indeed, consider a function $f(q)$ defined and continuously differentiable for $q\geq 0$. Let $f(q)$ additionally satisfy the conditions

$$f(0)=1,\ \ f'(0)=0,\ \ \lim_{q\to\infty}f(q)=\lim_{q\to\infty}qf(q)=0,\ \ \lim_{q\to+0}f(q)/q=\alpha,\ \ |\alpha|<\infty. \tag{1.5}$$

Consider the vector potential

$$A_a = \Phi \frac{1-f(q)}{2\pi r} e_\varphi, \quad q = \frac{r}{a}, \quad 0 < a = \text{const}, \tag{1.6}$$

which defines the magnetic field

$$H_a(r) = \left(0, 0, H(a, r)\right), \quad H(a, r) = -\frac{\Phi}{2\pi a^2} \frac{f'(q)}{q}. \tag{1.7}$$

Potentials (1.6) and field (1.7) are nonsingular for $r \geq 0$, and the function $H(a, r)$ approaches zero for $r \to \infty$ $r \to \infty$ (no slower than $\sim r^{-2}$).

We chose a circle of radius R in Eq. (1.4) as a surface S with the center at the point $x = 0, \quad y = 0$. Than the magnetic flux $\overline{\Phi}_a(R)$ of field (1.7) through this surface calculated according to Eq. (1.4) has the form

$$\overline{\Phi}_a(R) = \Phi\left[1 - f(R/a)\right], \quad \lim_{R\to\infty} \overline{\Phi}_a(R) = \lim_{a\to+0} \overline{\Phi}_a(R) = \Phi. \tag{1.8}$$

When $a \to +0$,$a \to +0$ potential (1.6) turns into (1.2), and filed (1.7) tends to field (1.1):

$$A^{(0)} = \lim_{a\to+0} A_a, \quad B = \lim_{a\to+0} H_a(R). \tag{1.9}$$

We consider the motion of charged particles in the combined external electromagnetic field, which is a superposition of Aharonov–Bohm filed (1.1), (1.2) and some additional arbitrary electromagnetic field. The electromagnetic potential for the combined field $\mathbf{A}_\nu$ is given in the form

$$\mathbf{A}_\nu = A_\nu + A_\nu^{(0)} = \left(A_0, \quad -A - A^{(0)}\right), \tag{1.10}$$

where A_ν is the electromagnetic potential of the additional field, and $A_\nu^{(0)}$ is the potential of Aharonov–Bohm field (1.2). The kinetic momentum operator $\hat{P}$ is given in the form

$$\hat{P} = \hat{p} - \frac{e}{c}\left(A + A^{(0)}\right), \quad \hat{p} = -i\hbar\nabla, \tag{1.11}$$

where $\hat{p}$ is the generalized momentum operator, $\hbar$ is Planck's constant, c is speed of light, e is the algebraic particle charge, and ∇ is the gradient operator.

The aim of this paper is to find the structure of electromagnetic potentials A_ν for additional fields that admit exact solutions of the Schrödinger equation:

$$\hat{\mathbf{S}}\Psi = 0, \quad \hat{\mathbf{S}} = i\hbar\frac{\partial}{\partial t} - eA_0 - \frac{\hat{P}^2}{2m_0}, \tag{1.12}$$

where m_0 is the particle mass. One of the most important characteristic of the Aharonov–Bohm field is the mantissa μ of magnetic flux Φ, which is defined as follows. For the covariant components of the electromagnetic potential field of solenoid (1.2), we find

$$\frac{e}{c} A_\nu^{(0)} = -\hbar \frac{e}{|e|} \frac{\Phi}{\Phi_0} \partial_\nu \varphi, \quad \Phi_0 = \frac{2\pi c\hbar}{|e|}. \tag{1.13}$$

The quantity Φ_0, introduced by Dirac, is called the quantum of magnetic flux. Obviously, we can always choose an integer l_0 such that

$$\Phi = -\frac{e}{|e|}(l_0 + \mu)\Phi_0, \quad 0 \le \mu < 1, \tag{1.14}$$

then we obtain

$$\frac{e}{c}A_\nu^{(0)} = \left(0, \quad -\frac{e}{c}\boldsymbol{A}^{(0)}\right) = \hbar(l_0 + \mu)\partial_\nu\varphi, \quad \frac{e}{c}\boldsymbol{A}^{(0)} = -\hbar\frac{l_0 + \mu}{r}\boldsymbol{e}_\varphi. \tag{1.15}$$

Thus the dimensionless mantissa μ defines the fractional part of the magnetic flux Φ measured in units of Φ_0.

In some physical problems, it can be convenient to use a different definition of the mantissa. Namely, we can always choose an integer $\bar{l}_0$ such that

$$\Phi = -\frac{e}{|e|}(\bar{l}_0 - \bar{\mu})\Phi_0, \quad 0 \le \bar{\mu} < 1, \tag{1.16}$$

with this we have

$$\frac{e}{c}A_\nu^{(0)} = \hbar(\bar{l}_0 - \bar{\mu})\partial_\nu\varphi, \quad \frac{e}{c}\boldsymbol{A}^{(0)} = -\hbar\frac{\bar{l}_0 - \bar{\mu}}{r}\boldsymbol{e}_\varphi. \tag{1.17}$$

Comparing Eqs. (1.14) and (1.16), we find $\bar{l}_0 = l_0 + 1$, $\bar{\mu} = 1 - \mu$, this way the quantity $\bar{\mu}$ can also be used for a definition of the magnetic field flux mantissa.

The special importance of the mantissa is emphasized by the following fact. In quantum theory equations (in particular, Eq. (1.2)), electromagnetic potentials enter only into the kinetic momentum operator. We seek solutions of the equations in the form

$$\Psi = \exp(-il_0\varphi)\tilde{\Psi}. \tag{1.18}$$

It is easy to verify that equations for the function $\tilde{\Psi}$ (in particular, Eq. (1.12)) maintain their forms when the potential $\boldsymbol{A}^{(0)}$ is substituted by $\tilde{\boldsymbol{A}}^{(0)}$, where

$$\frac{e}{c}\tilde{\boldsymbol{A}}^{(0)} = -\hbar\frac{\mu}{r}\boldsymbol{e}_\varphi. \tag{1.19}$$

Thus the equation for the function $\tilde{\Psi}$ contains only the mantissa of the solenoid field and does not depend on the integer part of the magnetic flux Φ. If the mantissa is equal to 0, then the Aharonov–Bohm field is completely eliminated from the quantum mechanics equations by transformation (1.18). Hence it follows that $|\Psi|^2$ depends only on the mantissa of the magnetic flux Φ, but not on the Φ itself.

When definition (1.16) is used, formulas (1.18) and (1.19) are preserved with substitutions $l_0 \to \bar{l}_0$ and $\mu \to -\bar{\mu}$.

All presently known exact solutions of equations (1.12) can be found only for fields that allow complete separation of variables. In accordance with the foregoing, we consider only the Aharonov–Bohm filed with a nonzero mantissa. In this case, cylindrical and spherical spatial coordinates are physically isolated, as follows from Eqs. (1.2), (1.6), and (1.9). Only for these coordinate choices it is possible to separate variables in Eq. (1.12) and to obtain exact solutions for some additional fields.

2. STRUCTURE OF ADDITIONAL ELECTROMAGNETIC FIELDS. CYLINDRICAL COORDINATES

We define the electromagnetic potential A_ν for the additional field in the form (below we use, as in relativistic theory, the variable $x_0 = ct$ instead of the variable t, which is convenient due to equal dimensions of variables x_0 and $\boldsymbol{r}$)

$$\frac{e}{c\hbar}A_\nu^{(0)} = f_0(r,z,x_0), \quad \frac{e}{c\hbar}\boldsymbol{A} = \frac{f_2(r)}{r}\boldsymbol{e}_\varphi - f_1(r,z,x_0)\boldsymbol{k}, \tag{2.1}$$

where f_k $(k = 0,1,2)$ are arbitrary functions of the specified arguments. Functions $f_0(r,\, z,\, x_0)$ and $f_1(r,\, z,\, x_0)$ have dimension of the inverse length, and function $f_2(r)$ is dimensionless. Thus the potential of Aharonov–Bohm field (1.2) can be considered as a particular case of additional field potential (2.1) when $f_0(r,\, z,\, x_0) = f_1(r,\, z,\, x_0) = 0$ and $f_2(r) = -(l_0 + \mu) = \text{const}$.

From Eq. (2.1) we obtain for intensities of electric ($\boldsymbol{E}$) and magnetic fields ($\boldsymbol{H}$)

$$\frac{e}{c\hbar}\boldsymbol{E} = -\partial_r f_0(r,\, z,\, x_0)\boldsymbol{e}_r + [\partial_0 f_1(r,\, z,\, x_0) - \partial_z f_0(r,\, z,\, x_0)]\boldsymbol{k},$$

$$\frac{e}{c\hbar}\boldsymbol{H} = \partial_r f_1(r,\, z,\, x_0)\boldsymbol{e}_\varphi + \frac{f_2'(r)}{r}\boldsymbol{k}; \quad \boldsymbol{e}_r = \cos\varphi\,\boldsymbol{i} + \sin\varphi\,\boldsymbol{j}. \tag{2.2}$$

In these fields, z-component of the angular momentum operator $L_z = [\boldsymbol{r}\hat{\boldsymbol{p}}]_z = -i\hbar\partial_\varphi$ is an integral of motion for the Schrödinger equation. We seek solutions of Eq. (1.12), which are eigenfunctions of the operator L_z with eigenvalues $\hbar(l - l_0)$, where l is an integer. For the Schrödinger equations, such solutions have the form

$$\Psi(x^\nu) = \exp\left[i(l - l_0)\varphi\right]\psi(r,z,x_0), \tag{2.3}$$

where the function $\psi(r,z,x_0)$ is a solution of the equation

$$\left\{2m\pi_0 + \partial_r^2 + \frac{1}{r}\partial_r - \frac{[f_2(r) - l - \mu]^2}{r^2} - \pi_3^2\right\}\psi(r,z,x_0) = 0. \tag{2.4}$$

In Eq. (2.4) we have used the notation

$$\pi_0 = i\partial_0 - f_0(r,z,x_0), \quad \pi_3 = i\partial_z - f_1(r,z,x_0), \quad m = \frac{m_0 c}{\hbar}. \tag{2.5}$$

Exact solutions of Eq. (2.4) are known only for two specific types of function $f_2(r)$. Under assumption that additional fields depend on the constant m, i.e., on the particle mass, these two types of function $f_2(r)$ are the following:

$$a)\, f_2(r) = r(\gamma + 2m\varepsilon r)\,,\ b)\, f_2(r) = r^2(\gamma + 2m\varepsilon r^2), \tag{2.6}$$

where γ and ε are arbitrary real constants. The structure of the exact solutions depends also on the special features of the functions $f_1(r,z,x_0)$ of all fields that allow exact solutions, which can be divided into two types.

Fields of the first type

Fields of the first type satisfy the following condition: $\partial_r f_1(r,z,x_0) \neq 0$, i.e., for this field type, the function $f_1(r,z,x_0)$ depends on *r*. One could obtain exact solutions of Eq. (2.4) only when functions f_0, f_1 do not depend on the *z*-variable: $f_0 = f_0(r,x_0)$, $f_1 = f_1(r,x_0)$. In this case, the solutions may be eigenfunction of the operator $i\partial_z$ with eigenvalue k_3.

a) If $f_2(r) = r(\gamma + 2m\varepsilon r)$, then exact solutions of the Schrödinger equation can be found for

$$f_0(r,x_0) = \frac{\alpha}{r} + \frac{\delta}{r^2} + \frac{2\lambda}{r^3}\left(\beta - \frac{m\lambda}{r}\right) - \frac{1}{m}\left(\frac{\beta}{r} - \frac{2m\lambda}{r^2}\right)\phi(x_0) - 2\varepsilon r(\gamma + m\varepsilon r),$$

$$f_1(r,x_0) = \frac{\beta}{r} - \frac{2m\lambda}{r^2} + \phi(x_0), \tag{2.7}$$

where α, β, δ, and λ are arbitrary real constants, and $\phi(x_0)$ is an arbitrary real function of variable x_0.

In the examined case, it is possible to obtain solutions $\psi(r,z,x_0)$ of Eq. (2.4) that depend on two quantum numbers k_0 and k_3 for arbitrary function $\phi(x_0)$ in the form

$$\psi(r,z,x_0) = \exp\{-i[k_0 x_0 + k_3 x_3 + \vartheta(x_0)]\}\,\psi(r), \quad \vartheta(x_0) = \frac{1}{2m}\int[\phi^2(x_0) - 2k_3\phi(x_0)]dx_0. \tag{2.8}$$

The function $\psi(r)$ in Eq. (2.8) can be expressed through the Laguerre functions $I_{s,\,n}(x)$ (*A* and *B* are arbitrary constants):

$$\psi(r) = AI_{s,\,n}(x) + BI_{n,\,s}(x), \tag{2.9}$$

where the following notation has been introduced:

$$x = 2r\sqrt{E}, \quad s = \frac{b}{\sqrt{E}} - \frac{1}{2} + \sqrt{a}, \quad n = \frac{b}{\sqrt{E}} - \frac{1}{2} - \sqrt{a}, \quad E = \gamma^2 + k_3^2 - 2mk_0 - 4m(l+\mu)\varepsilon,$$

$$a = (l+\mu)^2 + \beta^2 + 2m\delta + 4m\lambda k_3, \quad b = \gamma(l+\mu) + \beta k_3 - \alpha m. \tag{2.10}$$

The Laguerre functions $I_{s,\,n}(x)$ are related to the confluent hypergeometric functions $\Phi(\alpha,\gamma;x)$ (see [32], pp. 1072–1073) through the expression

$$I_{s,\,n}(x) = \sqrt{\frac{\Gamma(1+s)}{\Gamma(1+n)}}\frac{\exp(-x/2)}{\Gamma(1+s-n)}x^{\frac{s-n}{2}}\Phi(-n, s-n+1; x). \tag{2.11}$$

For integer nonnegative *n*, the Laguerre functions $I_{s,\,n}(x)$ are expressed through the Laguerre polynomials $L_n^{\alpha}(x)$ (see [32], Eq. (8.970.1)):

$$I_{s,\,n}(x) = \sqrt{\frac{\Gamma(1+n)}{\Gamma(1+s)}}\exp\left(\frac{x}{2}\right)x^{\frac{s-n}{2}}L_n^{s-n}(x), \quad n = 0, 1, 2, 3, \ldots \tag{2.12}$$

The Laguerre functions are solutions of the second-order ordinary linear differential equation

$$4x^2 I''_{s,\,n}(x) + 4x I'_{s,\,n}(x) - [x^2 - 2x(1+n+s) + (n-s)^2] I_{s,\,n}(x) = 0. \tag{2.13}$$

For $a \geq 0$ and integer nonnegative $n = 0, 1, 2, \ldots$, functions (2.9) for $B = 0$ are square-integrable, and the quantity k_0 is quantized

$$\psi(r) = A\exp(-x/2)x^{\sqrt{a}} L_n^{2\sqrt{a}}(x),$$

$$k_0 = \frac{1}{2m}\left[\gamma^2 + k_3^2 - 4m(l+\mu)\varepsilon - \frac{4b^2}{(2n+1+2\sqrt{a})^2}\right]. \tag{2.14}$$

b) $f_2(r) = r^2(\gamma + 2m\varepsilon r^2)$. Here it is possible to obtain exact solutions of Eq. (2.4) for

$$f_0(r) = \alpha r^2 + \frac{\beta}{r^2} - 2m\left(\frac{\lambda^2}{r^4} + \delta^2 r^4\right) + 2\left(\frac{\lambda}{r^2} + \delta r^2\right)\phi(x_0) - 2\varepsilon r^4(\gamma + m\varepsilon r^2),$$

$$f_1(r) = -2m\left(\frac{\lambda}{r^2} + \delta r^2\right) + \phi(x_0)\,, \tag{2.15}$$

where, as in the previous case, α, β, δ, and λ are arbitrary real constants, and $\phi(x_0)$ is an arbitrary real function of variable x_0. It is possible to express general solution of Eq. (2.4) again by Eqs. (2.8) and (2.9); however, instead of Eq. (2.10) here it is necessary to introduce the following notation:

$$x = \sqrt{E}\,r^2,\quad s = \frac{b}{4\sqrt{E}} - \frac{1}{2} + \frac{\sqrt{a}}{2},\quad n = \frac{b}{4\sqrt{E}} - \frac{1}{2} - \frac{\sqrt{a}}{2},\ E = \gamma^2 + 2\alpha m + 4m\delta k_3 - 4m\varepsilon(l+\mu),$$

$$a = (l+\mu)^2 + 2\beta m + 4m\lambda k_3,\quad b = 2mk_0 + 2\gamma(l+\mu) - k_3^2 - 8\delta\lambda m^2. \tag{2.16}$$

If $a \geq 0$ and $E > 0$, than for integer nonnegative n, the quantity k_0 is quantized, and functions $\psi(r)$ are square-integrable:

$$k_0 = \frac{1}{2m}\left[2\sqrt{E}(2n+1+\sqrt{a}) + k_3^2 - 2\gamma(l+\mu) + 8\delta\lambda m^2\right],$$

$$\psi(r) = A\exp(-x/2)x^{\frac{\sqrt{a}}{2}} L_n^{\sqrt{a}}(x). \tag{2.17}$$

If we require that the additional field is independent of *m*, then for case (a) we must chose $\varepsilon = \lambda = \phi(x_0) = 0$ in Eq. (2.7); for fields of case (b), correspondingly, $\varepsilon = \lambda = \delta = 0$ in Eq. (2.15). This demonstrates the absence of fields of the first type in case (b), for which the intensity does not depend on *m*.

Fields of the second type

We refer the fields to the second type when the following condition is satisfied: $\partial_r f_1(r, z, x_0) = 0$, i.e., for this field type, the function $f_1(r, z, x_0) = f_1(z, x_0)$ depends significantly only on variables z and x_0.

One could obtain exact solutions of Eq. (2.4) only when the function $f_0(r, z, x_0)$ has the following structure:

$$f_0(r,z,x_0)=\phi_0(z,\ x_0)+\chi(r), \tag{2.18}$$

where the functions $\phi_0(z,\ x_0)$ and $\chi(r)$ depend only on the specified arguments. In this case, solutions of Eq. (2.4) for fields of the second type can be written in the form (the variable r can be separated)

$$\psi(r,z,x_0)=w(z,\ x_0)\psi(r), \tag{2.19}$$

where the function $\psi(r)$ is a solution of the following second-order ordinarily linear differential equation:

$$\left\{\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{[f_2(r)-l-\mu]^2}{r^2}-2m\chi(r)+2mq\right\}\psi(r)=0, \tag{2.20}$$

and the function $w(z,\ x_0)$ is a solution of the equation

$$\left\{2m[i\partial_0-q-\phi_0(z,\ x_0)]-[i\partial_z-f_1(z,\ x_0)]^2\right\}w(z,\ x_0)=0. \tag{2.21}$$

In Eq. (2.20), the quantity q is the separation constant (integral of motion).

For Eq. (2.20), exact solutions are known only in two cases (2.6).

a) If $f_2(r)=r(\gamma+2m\varepsilon r)$, then exact solutions $\psi(r)$ of Eq. (2.20) are possible for

$$\chi(r)=\frac{\alpha}{r}+\frac{\beta}{r^2}-2\varepsilon r(\gamma+m\varepsilon r), \tag{2.22}$$

where α and β are arbitrary real constants. These solutions have form (2.9), where instead of Eq. (2.10), the following notations have to be introduced:

$$x=2r\sqrt{E},\quad s=\frac{b}{\sqrt{E}}-\frac{1}{2}+\sqrt{a},\quad n=\frac{b}{\sqrt{E}}-\frac{1}{2}-\sqrt{a},$$

$$E=\gamma^2-2mq-4m(l+\mu)\varepsilon,\quad a=(l+\mu)^2+2m\beta,\quad b=\gamma(l+\mu)-\alpha\, m. \tag{2.23}$$

Here, as in the previous case, for $a\geq 0$ and nonnegative integer $n=0,\ 1,\ 2,\ \ldots$, the functions $\psi(r)$ defined in Eqs. (2.9) and (2.23) are square-integrable at $B=0$, and the quantity q is quantized:

$$\psi(r)=A\exp(-x/2)x^{\sqrt{a}}L_n^{2\sqrt{a}}(x),$$

$$q=\frac{1}{2m}\left[\gamma^2-4m(l+\mu)\varepsilon-\frac{4b^2}{(2n+1+2\sqrt{a})^2}\right]. \tag{2.24}$$

b) $f_2(r)=r^2(\gamma+2m\varepsilon r^2)$. Here exact solution of Eq. (2.20) can be found for the following choice:

$$\chi(r)=\alpha r^2+\frac{\beta}{r^2}-2\varepsilon r^4(\gamma+m\varepsilon r^2), \tag{2.25}$$

where α and β are arbitrary real constants. These solutions have form (2.9), where instead of Eq. (2.10), the following notation should be introduced:

$$x = \sqrt{E}\, r^2, \quad s = \frac{b}{4\sqrt{E}} - \frac{1}{2} + \frac{\sqrt{a}}{2}, \quad n = \frac{b}{4\sqrt{E}} - \frac{1}{2} - \frac{\sqrt{a}}{2},$$

$$E = \gamma^2 + 2\alpha m - 4m\varepsilon(l+\mu), \quad a = (l+\mu)^2 + 2\beta m, \quad b = 2mq + 2\gamma(l+\mu). \tag{2.26}$$

If $a \geq 0$ and $E > 0$, then for integer nonnegative n, the quantity q is quantized, and functions $\psi(r)$ are square-integrable:

$$\psi(r) = A\exp(-x/2)x^{\frac{\sqrt{a}}{2}} L_n^{\sqrt{a}}(x), \quad q = \frac{1}{2m}\left[2\sqrt{E}(2n+1+\sqrt{a}) - 2\gamma(l+\mu)\right]. \tag{2.27}$$

When additional fields of the second type do not depend on m, then we should put $\varepsilon = 0$ in Eq. (2.6) and all the formulae of these item.

Now consider Eq. (2.21). If for example, $\phi_0(z, x_0) = f(z)$ and $f_1(z, x_0) = 0$, then the function $w(z, x_0)$ can be the eigenfunction of the $i\partial_0$ operator:

$$i\partial_0 w(z, x_0) = k_0 w(z, x_0) \Rightarrow w(z, x_0) = \exp(-ik_0 x_0) w(z), \tag{2.28}$$

and taking into account Eq. (2.28), from Eq. (2.21) we obtain the following equation for the function $w(z)$:

$$\left\{\frac{d^2}{dz^2} + 2m[k_0 - q - f(z)]\right\} w(z) = 0, \tag{2.29}$$

which coincides (to within the letter designations) with the one-dimensional Schrödinger equation. All the exact solutions for this equation can be found, for example, in [29].

If $\phi_0(z, x_0) = 0$ and $f_1(z, x_0) = f(x_0)$ (uniform electric field arbitrarily dependent on time), then the function $w(z, x_0)$ can be the eigenfunction of the $i\partial_z$ operator:

$$i\partial_z w(z, x_0) = k_3 w(z, x_0) \Rightarrow w(z, x_0) = \exp(-ik_3 z) w(x_0), \tag{2.30}$$

and taking into account Eq. (2.30), from Eq. (2.21) we obtain the following equation for the function $w(x_0)$:

$$\left\{2im\partial_0 - 2mq - [k_3 - f(x_0)]^2\right\} w(x_0) = 0. \tag{2.31}$$

The exact solution for this equation can be found for arbitrary $f(x_0)$ by quadrature

$$w(x_0) = \exp[-iS(x_0)], \quad S(x_0) = qx_0 + \int \frac{[k_3 - f(x_0)]^2}{2m} dx_0. \tag{2.32}$$

In the general case of an arbitrary dependence of the function $f_{0,1}(z, x_0)$ on both variables z and x_0, we can take advantage of the following transformation for $w(z, x_0)$:

$$w(z, x_0) = \exp[-i\varphi(z, x_0)]\psi(z, x_0), \quad \varphi(z, x_0) = qx_0 + \int f_1(z, x_0) dz, \tag{2.33}$$

and reduce Eq. (2.21) to the one-dimensional time-dependent Schrödinger equation with the potential depending on time

$$i\partial_0\psi(z,\,x_0)=\mathbf{H}\psi(z,\,x_0),\quad \mathbf{H}=-\frac{1}{2m}\frac{\partial^2}{\partial z^2}+U(z,\,x_0),$$

$$U(z,\,x_0)=f_0(z,\,x_0)-\int\left[\partial_0 f_1(z,\,x_0)\right]dz. \tag{2.34}$$

Exact solutions of Eqs. (2.29) and (2.32), naturally, can be obtained from Eq. (2.34). Up to now, no other exact solutions of Eq. (2.34) have been found. There is no doubt that new types of functions $f_{0,\,1}(z,\,x_0)$ that allow exact solutions can be found, since the technique for constructing such solutions has already been developed [30, 31]. The application of this technique and the discovery of new exact solutions of Eq. (2.34) may be the subject of further study.

This exhausts the additional electromagnetic fields for which exact solutions of the Schrödinger equation are known in spatial cylindrical coordinates.

3. STRUCTURE OF ADDITIONAL ELECTROMAGNETIC FIELDS. SPHERICAL COORDINATES

In spherical coordinates $r,\,\theta,\,\varphi$, we have

$$x=r\sin\theta\cos\varphi,\quad y=r\sin\theta\sin\varphi,\quad z=r\cos\theta\,, \tag{3.1}$$

and the electromagnetic potential of Aharonov–Bohm field (1.2) has the form

$$A_\nu^{(0)}=\left(A_0^{(0)},\quad -\boldsymbol{A}^{(0)}\right),\quad \boldsymbol{A}^{(0)}=\frac{\Phi}{2\pi r\sin\theta}\boldsymbol{e}_\varphi,\quad \boldsymbol{e}_\varphi=-\sin\varphi\,\boldsymbol{i}+\cos\varphi\,\boldsymbol{j}. \tag{3.2}$$

We define the electromagnetic potential A_ν of the additional field in the following form:

$$\frac{e}{c\hbar}A_0=f_0(r),\quad \frac{e}{c\hbar}\boldsymbol{A}=\frac{f_1(\cos\theta)}{r\sin\theta}\boldsymbol{e}_\varphi, \tag{3.3}$$

where f_k $(k=0,1)$ is an arbitrary function of the given arguments. In this case, Aharonov–Bohm potential (3.2) can be considered as a particular case of additional field potential (3.3) for $f_0(r)=0$ and $f_1(\cos\theta)=\frac{\Phi}{2\pi}=\text{const}$.

For the electric ($\boldsymbol{E}$) and magnetic ($\boldsymbol{H}$) intensities of the additional fields, from Eq. (3.3) we obtain that

$$\frac{e}{c\hbar}\boldsymbol{E}=-f_0'(r)\boldsymbol{e}_r,\quad \frac{e}{c\hbar}\boldsymbol{H}=-\frac{f_1'(\cos\theta)}{r^2}\boldsymbol{e}_r,\quad \boldsymbol{e}_r=\sin\theta(\cos\varphi\,\boldsymbol{i}+\sin\varphi\,\boldsymbol{j})+\cos\theta\,\boldsymbol{k}. \tag{3.4}$$

The procedure of complete separation of variables and finding of a complete set of integrals of motion for the Schrödinger equation in the fields defined by potentials (3.3) is described in detail, for example, in [29]. Exact solutions of the Schrödinger equation can be found for

$$f_1(\cos\theta)=\alpha\cos\theta,\quad f_0(r)=\frac{\beta}{r}+\frac{\gamma}{r^2}, \tag{3.5}$$

where α, β, and γ are real constants. The electromagnetic fields in this case are a combination of the Aharonov–Bohm field, the Coulomb field, the central symmetric electric field whose strength is $\sim r^{-3}$, and the magnetic monopole field. There is extensive literature (e.g. [18–25]) in which solutions in these fields are studied in detail (the most in depth study of the properties of these exact solutions is presented in [19]). We refer the reader to this literature.

CONCLUSIONS

The possibility of obtaining explicit exact solutions of the quantum mechanics equations for a combination of the Aharonov–Bohm field with the additional external electromagnetic field is of undoubted physical interest, because the additional field can separate or reinforce some specific manifestation of the Aharonov–Bohm effect. For example, exact solutions for the case when the additional field is a constant and uniform magnetic field allow the prediction of the nontrivial manifestation of the Aharonov–Bohm effect in synchrotron radiation.

To date, only three types of exact solutions considered here have been studied in details: 1) additional fields are absent; only a solenoid magnetic field is present, 2) the additional field is a constant and uniform magnetic field (the magnetic field vector is parallel to the axis of the Aharonov–Bohm solenoid), and 3) the additional field is a static central symmetric electric (Coulomb) field and a magnetic monopole field.

Here we have demonstrated that in addition to the three previously known cases, there are broad classes of additional external fields that admit exact solutions in the relativistic and nonrelativistic cases. The additional electric fields that act at a finite time or are concentrated in a finite region of space may be of interest; exact solutions of the Schrödinger equation exist for homogeneous and isotropic electric fields with arbitrarily time dependence.

We have demonstrated the possibility of exact investigation of the Aharonov–Bohm effect manifestations in combination with a broad class of additional external electromagnetic fields.

V. G. Bagrov thanks Special Federal Program "Scientific and Pedagogical Personnel of Innovative Russia" (Contract No. P789) and Brazilian Fund FAPESP for their financial support. D. M. Gitman thanks Brazilian Funds FAPESP and CNPq for constant financial support.

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