ON PERIODIC SOLUTIONS OF SYSTEMS OF THE TYPE

$$\dot{\mathbf{x}} = \mathbf{H}(\mathbf{y}), \quad \dot{\mathbf{y}} = -\sum_{i=1}^{n} \mathbf{f}_i(\mathbf{x})\mathbf{H}_i(\mathbf{y}) - \mathbf{g}(\mathbf{x}).$$

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Abstract

We establish in this paper sufficient conditions for existence of periodic solutions for systems of the type

$$\dot{x} = H(y), \quad \dot{y} = -\sum_{i=1}^{n} f_i(x)H_i(y) - g(x).$$

1. Introduction

In [1] and [2] using, respectively, the positive definite functions

$$V_\alpha(x,y) = \int_0^y g(u)du + \int_0^y \frac{s}{\alpha s + 1} ds$$

and

$$V_\alpha(x,y) = \int_0^y g(u)du + \int_0^y \frac{s}{\alpha s^{p/q} + 1} ds,$$

we establish sufficient conditions for the existence of periodic solutions for the equations

$$\ddot{x} + f(x)\dot{x} + g(x) = 0 \quad \text{and} \quad \ddot{x} + f(x)\dot{x}^{p/q} + g(x) = 0,$$

where p and q are odd natural numbers, mutually prime, with $1 < \frac{p}{q} < 2$.

Now, we extend these conditions for systems of the type

$$\begin{cases} \dot{x} = H(y) \\ \dot{y} = -\sum_{i=1}^{n} f_i(x)H_i(y) - g(x) \end{cases} \tag{1.1}$$

where f_i, H_i, $g, H : \mathbf{R} \to \mathbf{R}$ are functions of class C^1 satisfying the following conditions:

I) $xg(x) > 0, \quad x \neq 0;$

II) $\int_0^{+\infty} g(x)dx = +\infty = \int_0^{-\infty} g(x)dx$

III) $yH_i(y) > 0 \ (i = 1, 2, \ldots, n)$ and $yH(y) > 0$ for $y \neq 0$;

IV) there are real numbers $M > 0$ and $r > 0$ such that $(i = 1, 2, \ldots, n)$

a) $\frac{H_i(y)}{H(y)} \leq M|y|$ and $\frac{1}{|H(y)|} \leq M|y|$ for $|y| \geq r$

or

b) $\frac{H_i(y)}{H(y)} \leq M$ and $\frac{1}{|H(y)|} \leq M$ for $|y| \geq r$.

2. Auxilliary Lemmas

From IV) it follows that, for all $j \ (j = 1, 2, \ldots, x)$ and $\alpha \geq 0$

$$\int_0^{+\infty} \frac{H(\tau)}{\alpha H_j(\tau) + 1} d\tau = +\infty.$$

Let $j \in \{1, 2, \ldots, n\}$. Suppose that, for all $\alpha > 0$, there is $s > 0$ such that

$$\alpha H_j(s) + 1 = 0, \quad \alpha H_j(y) + 1 > 0, \quad \text{for } y > s,$$

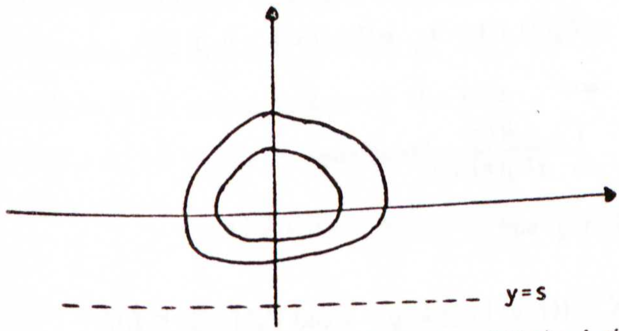
and

$$\int_0^{s} \frac{H(\tau)}{\alpha H_j(\tau) + 1} d\tau = +\infty.$$

It can be easily verified that the level curves of the positive definite function given by

$$V_{\alpha,j}(x, y) = \int_0^x g(u)du + \int_0^y \frac{H(\tau)}{\alpha H_j(\tau) + 1} d\tau, \quad y > s, \tag{1.2}$$

are all closed curves and $V_{\alpha,j}(x, 0)$ is strictly increasing in $[0, +\infty[$. Such curves show the following aspect:



It can also be immediately verified that the derivative of (1.2) relative to the system (1.1) is

$$\dot{V}_{\alpha,j}(x,y) = -\frac{H(y)H_j(y)[f_j(x) - \alpha g(x)] + \sum_{i=1,i\neq j}^{n} f_i(x)H_i(y)H(y)}{\alpha H_j(y) + 1}, \quad y > s. \tag{2.2}$$

We also consider the positive definite function

$$W(x,y) = \int_0^x g(u)du + \int_0^y H(\tau)d\tau, \ (x,y) \in \mathbf{R}^2. \tag{3.2}$$

It can be easily verified that all level curves of (3.2) are closed and $W(x,0)$ is strictly increasing in $[0,+\infty[$. The derivative of W relative to the system (1.1) is

$$\dot{W}(x,y) = -\sum_{i=1}^{n} f_i(x)H_i(y)H(y) \, . \tag{3.3}$$

It can also be easily verified that condition IV) ensures the solutions of

$$\frac{dy}{dx} = -\sum_{i=1}^{n} f_i(x)\frac{H_i(y)}{H(y)} - \frac{g(x)}{H(y)}, \ y \neq 0,$$

do not admit vertical asymptotes; consequently, the solutions of (1.1) do not admit either.

Lemma 1. Assume there are $\alpha > 0$, $b > 0$ and j such that, for all $x \geq b$,

$$f_j(x) \geq \alpha g(x) \quad \text{and} \quad f_i(x) \geq 0, \ i \neq j.$$

Assume, also, there is $s < 0$ such that

$$\alpha H_j(j) + 1 = 0, \quad \alpha H_j(y) + 1 > 0, \; y > s,$$

and

$$\int_0^s \frac{H(\tau)}{\alpha H_j(\tau) + 1} d\tau = +\infty.$$

Let $y_0 > 0$, $L = V_{\alpha,j}(b, y_0)$ and

$$K = \{(x, y) \mid x \geq b, \; y > s \text{ and } V_{\alpha,j}(x, y) \leq L\}.$$

Let $\gamma(t) = (x(t), y(t))$ be the solution of (1.1) so that $\gamma(t_0) = (b, y_1)$ with $0 < y_1 < y_0$. Then, there is $t_1 > t_0$ such that

$$\gamma(t) \in K, \quad t_0 \leq t \leq t_1, \quad \text{and} \quad \gamma(t_1) = (b, y_2)$$

with $s < y_2 < 0$.

Proof. Similar to the proof of Lemma 1 in [2], replacing V_α by $V_{\alpha,j}$. ∎

In a similar way, we can prove, using (3.2), the following lemma.

Lemma 2. Assume there exist $c < a < 0$ such that, for all $x \in [c, a]$ and $i = 1, 2, \ldots, n$,

$$f_i(x) \geq 0.$$

Let $r > 0$ be such that $W(c, 0) = W(a, -r) = L$. Consider the set

$$K = \{(x, y) \mid c \leq x \leq a \text{ and } W(x, y) \leq L\}.$$

Let $\gamma(t) = (x(t), y(t))$ be the solution of (1.1) such that $\gamma(t_0) = (a, y_0)$, with $-r < y_0 < 0$. Then, there is $t_1 > t_0$ such that

$$\gamma(t) \in K, \quad t_0 \leq t \leq t_1 \quad \text{and} \quad \gamma(t_1) = (a, y_1)$$

with $y_1 > 0$.

3. Sufficiente conditions for existence of periodic solutions

Theorem 1. Consider the system (1.1) where f_i, H_i, g and H $(i = 1, 2, \ldots, n)$ satisfy the conditions I) to IV) of section 1. Assume, also, that

1) there are $\alpha > 0$, $b > 0$ and j such that, for all $x \geq b$,

$$f_j(x) \geq \alpha g(x) \quad \text{and} \quad f_i(x) \geq 0, \quad i \neq j;$$

2) there exists $s < 0$ such that

$$\alpha H_j(s) + 1 = 0, \quad \alpha H_j(y) + 1 > 0, \quad y > s,$$

and

$$\int_0^s \frac{H(y)}{\alpha H_j(y) + 1} dy = +\infty;$$

3) the origin is repulsive;

4) there is $a < 0$ such that, for all $x \leq a$,

$$f_i(x) \geq 0 \quad (i = 1, 2, \ldots, x).$$

In these conditions, the system (1.1) admits at least one non trivial periodic solution.

Proof.

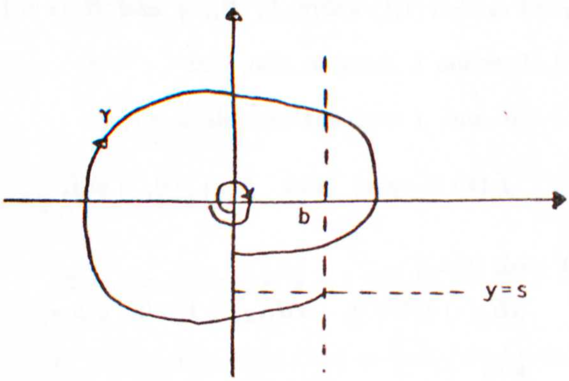
Let $\gamma(t) = (x(t), y(t))$ be the solution of (1.1) when at time $t = 0$ it is in the position

$$\gamma(0) = (b, s).$$

Because γ does not admit vertical asymptotes and the origin is repulsive, it follows that γ crosses the negative half axis of y at the point

$$\gamma(t_1) = (0, y_1), \quad y_1 < 0,$$

where $t_1 > 0$ is the smallest value of t for which γ crosses that axis. The assumptions 1) to 4) together with lemmas 1 and 2 ensure that $\gamma(t)$ will again cross the y negative half axis at the time $t_2 > t_1$ in a point $(0, y_2)$ with $y_1 < y_2 < 0$.



From theorem of Poincaré-Bendixson, the system (1.1) will admit at least one non trivial periodic solution. ∎

Remark 1. One sufficient condition for the origin to be repulsive is the existence of real numbers β_i $(i = 1, 2, \ldots, n)$ and $r_1 > 0$ such that

$$f_i(x) < \beta_i(x), \quad 0 < |x| < r_1,$$

because, in this case, there is $r_2 > 0$ such that $\dot{V}_{\beta_i}(x, y) < 0$ for $0 < |x| < r_1$ and $|y| < r_2$, where

$$V_{\beta_i}(x, y) = \int_0^x g(u)du + \int_0^y \frac{H(\tau)}{\sum_{i=1}^n \beta_i H_i(\tau) + 1} d\tau.$$

Observe that

$$\dot{V}_{\beta_i}(x, y) = -\frac{\sum_{i=1}^n [f_i(x) - \beta_i g(x)] H(y) H_i(y)}{\sum_{i=1}^n \beta_i H_i(y) + 1}$$

and there exists $r_3 > 0$ such that the level curves

$$V_{\beta_i}(x, y) = V_{\beta_i}(x_0, y_0), \quad |x_0| < r_3 \text{ and } |y_0| < r_3$$

are all closed curves.

Remark 2. Theorem 1 remains evidently valid if the assumptions 1, 2 and 4 are replaced by

1') there are $\alpha < 0$, $a < 0$ and j such that, for all $x \leq a$

$$f_j(x) \geq \alpha g(x) \quad \text{and} \quad f_i(x) \geq 0, \quad i \neq j;$$

2') there is $s > 0$ such that

$$\alpha H_j(s) + 1 = 0, \quad \alpha H_j(y) + 1 > 0, \quad y < s,$$

and

$$\int_0^s \frac{H(y)}{\alpha H_j(y) + 1} dy = +\infty;$$

4') there is $b > 0$ such that, for all $x \geq b$,

$$f_i(x) \geq 0.$$

Theorem 2. Consider the system (1.1) where f_i, H_i, g and H $(i = 1, 2, \ldots, n)$ satisfy the conditions I), II), III) and IV-b) of section 1. Assume, also, the hypotheses 1), 2) and 3) of theorem 1 and

4) there is $a < 0$ such that, for all $x \in [c, a]$,

$$f_i(x) \geq 0 \quad (i = 1, 2, \ldots, x)$$

where $W(c, 0) = W(a, -r)$, $r \geq -s + A(b - a)M_1$. $A \geq \max_{a \leq x \leq b} \left[\sum_{i=1}^{n} |f_i(x)| + |g(x)| \right]$ and M_1 is such that, for all $y \leq s$,

$$\frac{H_i(y)}{H(y)} \leq M_1 \quad \text{and} \quad \frac{1}{|H(y)|} \leq M_1 .$$

Then, the system (1.1) admits at least one non trivial periodic solution.

Proof.

Let $\gamma(t) = (x(t), y(t))$ be the solution of (1.1) when at the time $t = 0$ it is in the position

$$\gamma(0) = (b, s).$$

Since γ does not admit vertical asymptotes and the origin is repulsive, there is a smallest time $t_1 > 0$ such that

$$\gamma(t_1) = (a, y_1), \quad y_1 < 0, \quad \text{or} \quad \gamma(t_1) = (x_1, 0), \quad a < x_1 < 0.$$

It can be immediately shown that

$$s - A(b-a)M_1 < y_1 < 0.$$

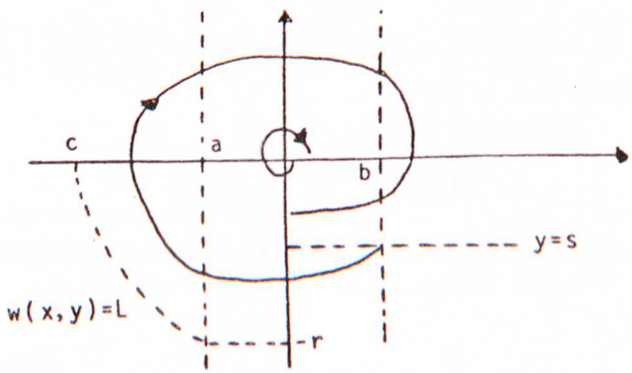
(Indeed: assuming $y_1 < s$, let $y = y(x)$ be the solution of

$$\frac{dy}{dx} = -\sum_{i=1}^{n} f_i(x)\frac{H_i(x)}{H(y)} - \frac{g(x)}{H(y)}$$

such that $y(a) = y_1$ and $y(b) = s$; there is $x_0 \in]a, b]$ such that $y(x_0) = s$ and $y(x) < s$, $a \leq x < x_0$; from mean value theorem

$$y(x_0) - y(a) < AM_1(x_0 - a).)$$

Let $t_2 > 0$ the smallest value of t for which γ crosses the y negative half axis: $\gamma(t_2) = (0, y_2)$, $y_2 < 0$. The hypotheses 1), 2), 3) and 4) together with lemmas 1 and 2 ensure that $\gamma(t)$ will again cross the y negative half axis at a point $(0, y_3)$ with $y_2 < y_3 < 0$.



$(W(x,y) = L = W(c,0) = W(a,-r))$. From theorem of Poincaré-Bendixson the system (1.1) admits at least one non trivial periodic solution. ■

Remark 3. Theorem 2 remains valid if the assumptions 1), 2) and 4) are replaced, respectively, by 1') and 2') of remark 2 and by

4') there is $b > 0$ such that, for all $x \in [b, c]$,

$$f_i(x) \geq 0 \quad (i = 1, 2, \ldots, n)$$

where $W(c, 0) = W(b, r)$, $r \geq s + A(b-a)M_1$, $A \geq \max_{a \leq x \leq b} \left[\sum_{i=1}^{n} |f_i(x)| + |g(x)|\right]$ and M_1 is such that, for $y \geq s$,

$$\frac{H_i(y)}{H(y)} \leq M_1 \quad \text{and} \quad \frac{1}{|H(y)|} \leq M_1.$$

Theorem 3. Consider the system (1.1) where f_i, H_i, g and H $(i = 1, 2, \ldots, n)$ satisfy the conditions I), II), III) and IV-b) of section 1. Assume, also, the assumptions 1), 2) and 3) of theorem 1 and

4) there are $a < 0$, $\beta_i \geq 0$ $(i = 1, 2, \ldots, n)$, with $\sum_{i=1}^{n} \beta_i > 0$, and $s_1 < 0$ such that

$$f_i(x) \geq \beta_i g(x), \quad x \leq a,$$

$$\sum_{i=1}^{n} \beta_i H_i(y) + 1 > 0, \quad y \geq s_1,$$

and

$$s_1 \leq s - A(b-a)M_1$$

where $A \geq \max_{a \leq x \leq b} \left[\sum_{i=1}^{n} |f_i(x)| + |g(x)|\right]$ and M_1 is such that, for all $y \leq s$,

$$\frac{H_i(y)}{H(y)} \leq M_1 \quad \text{and} \quad \frac{1}{|H(y)|} \leq M_1.$$

Then, the system (1.1) admits at least one non trivial periodic solution.

Proof.

Consider the positive definite function

$$V_{\beta_i}(x, y) = \int_0^x g(u)du + \int_0^y \frac{H(\tau)}{\sum_{i=1}^{n} \beta_i H_i(\tau) + 1} d\tau, \quad y \geq s_1.$$

The derivative of V_{β_i} relative to the system (1.1) is

$$\dot{V}_{\beta_i}(x,y) = -\frac{\sum_{i=1}^{n}[f_i(x) - \beta_i g(x)]H(y)H_i(y)}{\sum_{i=1}^{n}\beta_i H_i(y) + 1}, \quad y \geq s_1.$$

From

$$f_i(x) \geq \beta_i g(x), \quad x \leq a$$

it follows that

$$\dot{V}_{\beta_i}(x,y) \leq 0, \quad x \leq a \quad \text{and} \quad y \geq s_1. \tag{1.3}$$

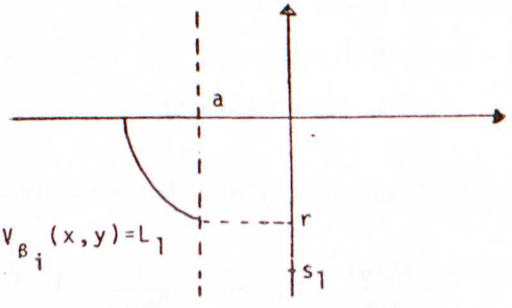
From $\int_0^{-\infty} g(x)dx = +\infty$, it follows that, for each $r \in [s_1, 0]$, there is $c < a$ such that

$$V_{\beta_i}(a,r) = V_{\beta_i}(c,0).$$

So, the arc

$$y \leq 0, \quad x \leq a \quad \text{and} \quad V_{\beta_i}(x,y) = V_{\beta_i}(a,r) = L_1$$

shows the following aspect:



Consider, now, the solution $\gamma(t) = (x(t), y(t))$ of (1.1) when at the time $t = 0$ it is in the position

$$\gamma(0) = (b, s).$$

By the same reasoning as in theorem 2, there will be a smallest value $t_1 > 0$ such that

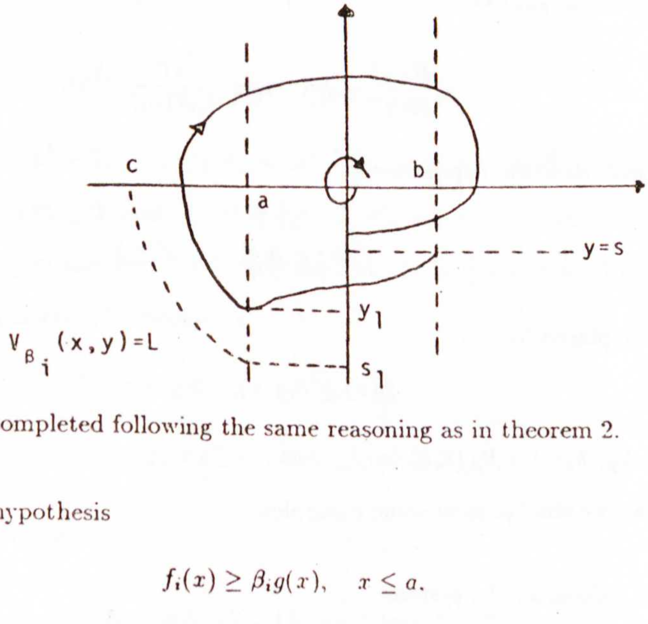
$$\gamma(t_1) = (x_1, 0), \quad a < x_1 < 0, \quad \text{and} \quad \gamma(t_1) = (a, y_1)$$

where $s_1 < y_1 < 0$. Suppose $\gamma(t_1) = (a, y_1)$. From (1.3), it follows that $\gamma(t)$ cannot leave the set

$$K = \{(x,y) \mid x \leq a,\ s_1 \leq y \leq 0 \text{ and } V_{\beta_i}(x,y) = V_{\beta_i}(a,s_1)\}$$

by crossing the arc

$$x \leq a, \quad s_1 \leq y \leq 0 \quad \text{and} \quad V_{\beta_i}(x,y) = V_{\beta_i}(a,s_1) = L\ .$$



The proof is completed following the same reasoning as in theorem 2. ∎

Remark 4. The hypothesis

$$f_i(x) \geq \beta_i g(x), \quad x \leq a.$$

can be evidently replaced by

$$f_i(x) \geq \beta_i g(x), \quad c \leq x \leq a,$$

where $V_{\beta_i}(a,s_1) = V_{\beta_i}(c,0)$.

Remark 5. Theorem 3 remains evidently valid if the hypotheses 1, 2 and 4 are replaced, respectively, by 1' and 2' of remark 2 and by

4') there are $b > 0$, $\beta_i \leq 0$ $(i = 1, 2, \ldots, n)$ with $\sum_{i=1}^{n} \beta_i < 0$ and $s_1 > 0$ such that

$$f_i(x) \geq \beta_i g(x), \quad x \geq b,$$

$$\sum_{i=1}^{n} \beta_i H_i(y) + 1 > 0, \quad y \leq s_1,$$

and

$$s_1 \geq s + A(b-a)M_1$$

where $A \geq \max_{a \leq x \leq b} \left[\sum_{i=1}^{n} |f_i(x)| + |g(x)| \right]$ and M_1 is such that, for all $y \geq s$,

$$\frac{H_i(y)}{H(y)} \leq M_1 \quad \text{and} \quad \frac{1}{|H(y)|} \leq M_1.$$

As in remark 4, the hypothesis

$$f_i(x) \geq \beta_i g(x), \quad x \geq b$$

can be replaced by

$$f_i(x) \geq \beta_i g(x), \quad b \leq x \leq c,$$

where $V_{\beta_i}(b, s_1) = V_{\beta_i}(c, 0)$.

To close, we shall present some examples.

Example 1. Consider the system

$$\begin{cases} \dot{x} = y^7 \\ \dot{y} = -f_1(x)y^5 - f_2(x)y - g(x) \end{cases}$$

where $f_1(x) = x^2 - 1$, $f_2(x) = x^6 - x^4 + x^3$ and $g(x) = x^3$. We have

1) $f_1(x) \geq 0$ and $f_2(x) \geq \alpha g(x)$ for $x \geq b$, with $\alpha = 1$ and $b = 2$;

2) the hypothesis 2) of theorem 1 is verified ($\alpha = 1$ and $s = -1$);

3) $f_1(x) < 0$ and $f_2(x) < x^3$, $0 < |x| < 1$, therefore, from remark 1, the origin is repulsive;

4) $f_1(x) \geq 0$ and $f_2(x) \geq 0$ for $x \leq -1$.

The conditions I) to IV) are evidently verified. From theorem 1, the system admits at least one non trivial periodic solution.

Example 2. From theorem 1, the system

$$\begin{cases} \dot{x} = y \\ \dot{y} = -(x^2-1)y^{5/3} - (x^6 - x^4 + x^3)y - x^3 \end{cases}$$

admits at least one non trivial periodic solution.

Example 3. Consider the system

$$\begin{cases} \dot{x} = y^5 \\ \dot{y} = -f_1(x)y^5 - f_2(x)y - g(x) \end{cases}$$

where $f_1(x) = x^3 + 10x^2 - 2x - 1$, $f_2(x) = 2x^6 - \frac{1}{64}$ and $g(x) = x^5$. We have

1) for all $x \geq \frac{1}{2}$, $f_1(x) \geq 0$ and $f_2(x) \geq \frac{1}{2}g(x)$. We set $b = \frac{1}{2}$ and $\alpha = \frac{1}{2}$.
2) The hypothesis 2) of theorem 2 is clearly verified. ($\alpha = \frac{1}{2}$ and $s = -2$).
3) The origin is repulsive. (Immediate.)
4) For every $x \in [-10, -\frac{1}{2}]$,

$$f_1(x) \geq 0 \quad \text{and} \quad f_2(x) \geq 0.$$

Let $a = -\frac{1}{2}$. We have

$$\max_{a \leq x \leq b} [|f_1(x)| + |f_2(x)| + |g(x)|] \leq 3.$$

Let $A = 3$ We can also take $M_1 = 1$. So, $r = -s + A(b-a)M_1 = 5$. Since

$$W(x,y) = \frac{x^6}{6} + \frac{y^6}{6}$$

it follows that

$$c^6 = a^6 + r^6 \quad (W(c,0) = W(a,-r)).$$

So, for all $x \in [c, -\frac{1}{2}]$, $f_1(x) \geq 0$ and $f_2(x) \geq 0$.

The conditions I), II), III) and IV-b) are clearly verified. Then, from theorem 2, the system admits at least one non trivial periodic solution.

Example 4. Consider the system

$$\begin{cases} \dot{x} = y^5 \\ \dot{y} = -f_1(x)y^3 - f_2(x)y - g(x) \end{cases}$$

where $f_1(x) = 7^{-5}(x-1)$, $f_2(x) = 2x^6 - 1$ and $g(x) = x^5$. We have

1) for all $x \geq 1$, $f_1(x) \geq 0$ and $f_2(x) \geq g(x)$. We set $b = 1$ and $\alpha = 1$.

2) The hypothesis 2 of theorem 3 is clearly verified ($\alpha = 1$ and $s = -1$).

3) It is immediate that origin is repulsive.

4) Let $a = -1$. We have

$$\max_{a \leq x \leq b} [|f_1(x)| + |f_2(x)| + |g(x)|] \leq 3.$$

Let $A = 3$. It can also be taken $M_1 = 1$. So,

$$s_1 = s - A(b-a)M_1 = -7.$$

Let $\beta = 7^{-4}$. Then, for all $y \geq -7$, $\beta y^3 + 1 > 0$. We have also, for all $x \leq -1$,

$$f_1(x) \geq \beta g(x) \quad \text{and} \quad f_2(x) \geq 0.$$

The conditions I), II), III) and IV-b) are clearly verified. Then, from the theorem 3, the system admits at least one non trivial periodic solution.

Remark 6. Suppose there exist $\alpha_i \geq 0$ $(i = 1, 2, \ldots, n)$, with $\sum_{i=1}^{n} \alpha_i > 0$, $s < 0$ and

j such that:

$$f_j(x) > \alpha_j g(x), \quad x \neq 0,$$

$$f_i(x) \geq \alpha_i g(x), \quad i \neq j \text{ and } x \neq 0,$$

$$\sum_{i=1}^{n} \alpha_i H_i(s) + 1 = 0,$$

$$\sum_{i=1}^{n} \alpha_i H_i(y) + 1 > 0, \quad y > s,$$

and

$$\int_0^s \frac{H(\tau)}{\sum_{i=1}^n \alpha_i H_i(\tau) + 1} d\tau = +\infty.$$

In these conditions, it can be easily verified that all solutions $\gamma(t)$ of (1.1) with

$$\gamma(0) = (x_0, y_0), \quad y_0 > s$$

approach to the origin when $t \to +\infty$. (It is enough to consider the positive definite function given by

$$V_{\alpha_i}(x,y) = \int_0^x g(u)du + \int_0^y \frac{H(\tau)}{\sum_{i=1}^n \alpha_i H_i(\tau) + 1} d\tau.)$$

Besides, if the condition: "there are $\beta_i \geq 0$ $(i = 1, 2, \ldots, n)$, $r > 0$ and k such that, for $0 < |x| < r$,

$$f_k(x) < \beta_k |g(x)| \quad \text{and} \quad f_i(x) \leq \beta_i |g(x)|, \quad i \neq k,\text{"}$$

is also verified, then all solutions $\gamma(t)$ of (1.1) with

$$\gamma(0) = (x_0, y_0), \quad y_0 > s,$$

approach, in spiral, to the origin when $t \to +\infty$.

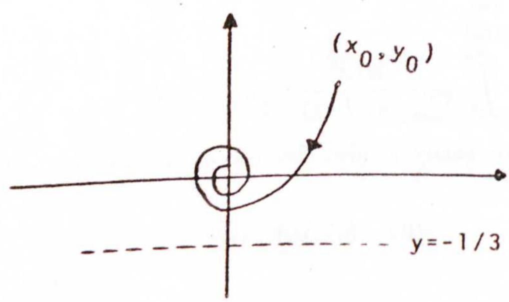
Example 5. Consider the system

$$\begin{cases} \dot{x} = y^5 \\ \dot{y} = -x^2 y^3 - 3xy - x. \end{cases}$$

From remark 6, all solutions $\gamma(t)$ with

$$\gamma(0) = (x_0, y_0), \quad y_0 > -\frac{1}{3}$$

approach, in spiral, to the origin when $t \to +\infty$.



References

[1] Guidorizzi, H.L. *On the existence of periodic solutions for the equation* $\ddot{x} + f(x)\dot{x} + g(x) = 0$. To appear in Boletim da Sociedade Brasileira de Matemática.

[2] Guidorizzi, H.L. *On the existence of periodic solutions for the equation* $\ddot{x} + f(x)\dot{x}^{p/q} + g(x) = 0$. 30º Seminário Brasileiro de Análise (1989), 69-79.