

Hard thermal loops in static background fields

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Abstract We discuss the high-temperature behavior of retarded thermal loops in static external fields. We employ an analytic continuation of the imaginary-time formalism and use a spectral representation of the thermal amplitudes. We show that, to all orders, the leading contributions of static hard thermal loops can be directly obtained by evaluating them at zero external energies and momenta.

1 Introduction

In thermal field theory, in order to deal with the infrared singularities which occur at finite temperature, it is necessary to put thermal masses into the zeroth order of a resummed perturbation theory. To this end, one must first calculate the hard thermal loops, where all the external energies and momenta are much smaller than the temperature T . These loops yield gauge-invariant contributions, which are in general non-local functionals of the external fields [1–4]. However, there are two special cases: the static and the long wavelength limits, when the hard thermal amplitudes become local functions, which are independent of the external energies and momenta [5, 6]. Nevertheless, these two limits give different functions [7, 8]. Of special interest is the evaluation in the above limits of causal thermal self-energy functions, which determine the high-temperature behavior of screening lengths and plasma frequencies [9].

The purpose of this work is to derive a simple method for calculating the leading contributions of retarded thermal loops in static external bosonic fields. We will show that these contributions can be directly obtained by evaluating the hard thermal loops at zero external energies and momenta. This result has been previously derived in gauge theories at one-loop level [5] and verified by explicit calculations at two-loops [10]. Here, we present an argument which is valid to all orders in thermal perturbation theory.

The above result may be more readily understood in the analytically continued imaginary-time formalism [11–13] which is well suited for the study of retarded Green's functions [14]. This formalism of thermal field theory defines the bosonic Green functions at integral values of $k_{j0}/2\pi iT$, where k_{j0} is the energy of the j th external particle. After performing the sums over the integral values (half-integral for fermions) of $Q_{l0}/2\pi iT$, where Q_{l0} is the energy of the l th internal particle, one arrives at bosonic (fermionic) thermal distribution functions of the form

$$N(k_0 + Q) = \frac{1}{e^{(k_0+Q)/T} \mp 1}, \quad (1)$$

where k_0 is some linear combination of external energies and Q is some combination of external and internal momenta.

It is worthwhile to note that if one would now analytically continue the external energies, one would get an analytic behavior when all k_{j0} and \vec{k}_j become small, leading to a well defined result in the limit $k_{j\mu} \rightarrow 0$. However, the analytic continuation constructed in this manner would yield, after performing the integrations over internal momenta, factors like $\exp(k_0/T)$ which are exponentially increasing for large values of k_0 . The proper procedure which avoids the appearance of such factors makes use of the relation

$$N(k_0 + Q) = N(Q), \quad (2)$$

which is valid before analytic continuation, since then, $k_0/2\pi iT$ is an integer.

In this way, the Green functions will be well behaved when k_{j0} are analytically continued to complex values, and various limits approaching the real axis from different directions may be taken. But this procedure, in contrast to the previous one, introduces non-analyticities in the thermal loops when $k_{j0} \rightarrow 0$ and $\vec{k}_j \rightarrow 0$. Nevertheless, the leading contributions of hard thermal loops in the static case $k_{j0} = 0$, $\vec{k}_j \rightarrow 0$, still agree with those obtained by setting directly in the loops all external energies and momenta $k_{j\mu} = 0$. This

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agreement occurs only in the static limit which entails that $k_0 = 0$, since then the condition (2) reduces to an identity. Thus, in the static case, analytic continuation preserves the form of the original thermal distribution functions (1), which lead to an analytic behavior when all $\vec{k}_j \rightarrow 0$.

We will first exemplify the above argument in the case of a two-loop thermal amplitude. Next, we shall present a more general approach, based on a spectral representation of the thermal Green functions [15–17], which allows to verify this argument to all orders.

2 Thermal self-energy at two-loops

Let us consider, for example, a two-loop diagram in the scalar $\lambda\phi^3$ theory, as shown in Fig. 1. The result for the one-loop sub-diagram in the renormalizable six-dimensional theory may be written in the form ($p \equiv |\vec{p}|$) [18]

$$\Sigma^{(1)}(p_0, \vec{p}) = \frac{1}{2}(p_0^2 - p^2)\Pi_T(p_0, p) + \sum_{n=0}^{\infty} \frac{P_{2(n+1)}(p_0, p)}{T^{2n}}, \quad (3)$$

where $P_{2(n+1)}$ is a polynomial of degree $2(n+1)$ in p_0 and p (when $n = 0$, there appear additional logarithmic terms) and $\Pi_T(p_0, p)$ is given by

$$\Pi_T(p_0, p) = \frac{\lambda^2}{48\pi} \left\{ \frac{T^2}{p^2} \left[1 - \frac{p_0}{2p} \log \left(\frac{p_0 + p}{p_0 - p} \right) \right] - \frac{1}{4\pi^2} \log \frac{T}{\mu} \right\}, \quad (4)$$

where μ is a renormalization mass scale. The above one-loop result for $\Pi_T(p_0, p)$ has branch points at $p_0 = \pm p$. When this term is inserted in the two-loop graph of Fig. 1, these branch points coincide with the poles at $p_0 = \pm p$ which arise from the other terms in this diagram. Therefore, in order to perform the p_0 integration in a well defined manner, it is necessary to regularize Π_T , which we do using dimensional regularization in $6 + 2\epsilon$ dimensions. This leads to the regularized form of $\Pi_T(p_0, p)$ given by

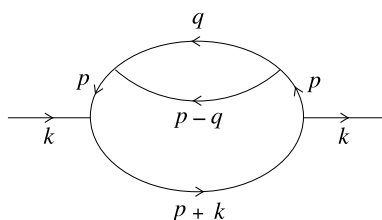


Fig. 1 A two-loop configuration of the thermal self-energy

$$\begin{aligned} \Pi_T(p_0, p, \epsilon) &= \frac{\lambda^2}{48\pi} \left\{ \frac{T^2}{p^2} \left[1 - (1 + \mathcal{O}(\epsilon)) \frac{p_0}{2} \int_0^\pi d\theta \frac{(\sin \theta)^{1+2\epsilon}}{p_0 - p \cos \theta} \right] \right. \\ &\quad \left. - \frac{1}{4\pi^2} \log \frac{T}{\mu} \right\}, \end{aligned} \quad (5)$$

which reduces to Eq. (4) in the limit $\epsilon \rightarrow 0$. In Eq. (5), θ denotes the angle between the momenta \vec{p} and \vec{q} shown in Fig. 1 and $\mathcal{O}(\epsilon)$ denote terms which are well behaved at $p_0 = \pm p$. Using this result in the two-loop graph of Fig. 1, and performing the sum over the integral values of $p_0/2\pi i T$ by contour integration, we obtain a leading thermal contribution of the form

$$\begin{aligned} \Sigma_T^{(2)}(k_0, \vec{k}) &= \frac{\lambda^2}{2} \int \frac{d^{5+2\epsilon} p}{(2\pi)^{5+2\epsilon}} \frac{1}{2\pi i} \oint_C dp_0 N(p_0) \\ &\quad \times \left[\frac{1}{p_0^2 - p^2} \frac{1}{(p_0 + k_0)^2 - (\vec{p} + \vec{k})^2} \Pi_T(p_0, p, \epsilon) \right. \\ &\quad \left. + (p_0 \rightarrow -p_0) \right], \end{aligned} \quad (6)$$

where the anticlockwise contour C , along the imaginary p_0 -axis, is closed in the right half p_0 plane. Evaluating the p_0 -integral in terms of the poles inside C , and using the fact that the leading contribution from the pole at $p_0 = p \cos \theta$ in $\Pi_T(p_0, p, \epsilon)$ vanishes, leads to the result ($\tilde{p} \equiv |\vec{p} + \vec{k}|$)

$$\begin{aligned} \Sigma_T^{(2)}(k_0, \vec{k}) &= \frac{\lambda^2}{8} \int \frac{d^{5+2\epsilon} p}{(2\pi)^{5+2\epsilon}} \frac{1}{p\tilde{p}} \\ &\quad \times \left[\frac{N(k_0 + \tilde{p})\Pi_T(k_0 + \tilde{p}, p, \epsilon) - N(p)\Pi_T(p, p, \epsilon)}{k_0 + \tilde{p} - p} \right. \\ &\quad \left. - \frac{N(k_0 + \tilde{p})\Pi_T(k_0 + \tilde{p}, p, \epsilon) + N(p)\Pi_T(p, p, \epsilon)}{k_0 + \tilde{p} + p} \right. \\ &\quad \left. + (k_0 \rightarrow -k_0) \right]. \end{aligned} \quad (7)$$

At this point, we note that if one would analytically continue (7) to complex values of k_0 , one would obtain an expression which is analytic as $k_\mu \rightarrow 0$, with the limit

$$\begin{aligned} \Sigma_T^{(2)}(k_0 = 0, \vec{k} = 0) &= \frac{\lambda^2}{4} \int \frac{d^{5+2\epsilon} p}{(2\pi)^{5+2\epsilon}} \frac{1}{p^2} \left\{ [N(p_0)\Pi_T(p_0, p, \epsilon)]'_{p_0=p} \right. \\ &\quad \left. - \frac{N(p)\Pi_T(p, p, \epsilon)}{p} \right\}, \end{aligned} \quad (8)$$

where the prime denotes a derivative with respect to p_0 . However, for the reasons mentioned previously, one must

first use in (7) the relation

$$N(k_0 + \tilde{p}) = N(\tilde{p}) \quad (9)$$

and then make the analytic continuation of k_0 . This procedure will introduce a non-analyticity when $k_\mu \rightarrow 0$. To study this, we remark that the leading contribution in T of the p -integral in (7) comes from the region where $p \sim T$. Thus, in order to find the high-temperature behavior of the hard thermal loop, one may assume that $|k_0|, |\vec{k}| \ll p$. Then, it is easy to see that the leading contribution obtained in the static limit agrees with the result given in (8)

$$\Sigma_T^{(2)}(k_0 = 0, \vec{k} \rightarrow 0) = \Sigma_T^{(2)}(k_0 = 0, \vec{k} = 0). \quad (10)$$

On the other hand, the leading contributions in the long wavelength limit would lead to a different result, namely

$$\begin{aligned} \Sigma_T^{(2)}(k_0 \rightarrow 0, \vec{k} = 0) \\ = \frac{\lambda^2}{4} \int \frac{d^{5+2\epsilon} p}{(2\pi)^{5+2\epsilon}} \frac{1}{p^2} N(p) \left[\Pi'_T(p_0, p, \epsilon) \Big|_{p_0=p} - \frac{\Pi_T(p, p, \epsilon)}{p} \right]. \end{aligned} \quad (11)$$

Let us now evaluate the leading static thermal contribution which arises from (10). In this case, there appear individual terms proportional to $1/\epsilon$ which exhibit a collinear singularity in the region where $p_0^2 = p^2$, with \vec{p} and \vec{q} being nearly parallel (see Fig. 1). However, such collinear singularities turn out to cancel so that in the present case, it is not necessary to resort to the KLN mechanism at finite temperature [19]. Using Eqs. (5) and (8), together with the relations

$$\int_0^\infty dp N(p) p = -\frac{1}{2} \int_0^\infty dp N'(p) p^2 = \frac{\pi^2 T^2}{6} \quad (12)$$

one then gets a leading $T^2 \log T$ contribution of the form

$$\Sigma_T^{(2)}(k_0 = 0, \vec{k} \rightarrow 0) = \frac{\lambda^4}{144} \frac{\pi}{4} \frac{1}{(2\pi)^5} T^2 \log \frac{T}{\mu}. \quad (13)$$

3 The self-energy to all orders

In order to verify to all orders that the leading static contributions agree with the result obtained by setting directly, in the self-energy, the external energy-momentum equal to zero, it is convenient to use a spectral representation of the analytically continued bosonic self-energy (see, for example, Chap. 3 in [15] and references therein). One then obtains for the analytic retarded self-energy the spectral form

$$\Sigma(k_0, \vec{k}) = \Sigma(\infty, \vec{k}) + \int_{-\infty}^{\infty} \frac{d\kappa_0}{2\pi} \frac{\sigma(\kappa_0, \vec{k})}{k_0 - \kappa_0 + i\epsilon}, \quad (14)$$

where k_0 is a real energy and, for simplicity, only the energy-momentum dependence has been written explicitly. Here, the spectral density $\sigma(\kappa_0, \vec{k}) = -\sigma(-\kappa_0, \vec{k})$ is related to the discontinuity of $\Sigma(\kappa_0, \vec{k})$ across the real axis and the second term approaches zero when $|k_0| \rightarrow \infty$. These features can be easily seen at one-loop in the scalar $\lambda\phi_0^3$ model, where $\Sigma(\infty, \vec{k}) = 0$ and

$$\begin{aligned} \sigma(\kappa_0, \vec{k}) \\ = -\frac{\lambda^2}{4\pi} \frac{1}{(2\pi)^6} \int d^6 p \int d^6 p' \rho_0(p) \rho_0(p') \\ \times \delta^6(p + p' + \kappa) [1 + N(p_0) + N(p'_0)]. \end{aligned} \quad (15)$$

Here, $\kappa_\mu = (\kappa_0, \vec{k})$ and the free spectral density $\rho_0(p)$ is given by

$$\rho_0(p) = 2\pi \epsilon(p_0) \delta(p_0^2 - p^2 - m^2), \quad (16)$$

where m is the ordinary mass of the scalar particles. One can check that, by substituting (15) into (14) and performing the κ_0 integration, one gets the one-loop thermal contribution

$$\begin{aligned} \Sigma_T^{(1)}(k_0, \vec{k}) \\ = \frac{\lambda^2}{8} \int \frac{d^5 p}{(2\pi)^5} \frac{1}{p \tilde{p}} \left[\frac{N(\tilde{p}) - N(p)}{k_0 + \tilde{p} - p} - \frac{N(\tilde{p}) + N(p)}{k_0 + \tilde{p} + p} \right. \\ \left. + (k_0 \rightarrow -k_0) \right], \end{aligned} \quad (17)$$

where $k_0 \rightarrow k_0 + i\epsilon$ is to be understood and we have neglected m with respect to p . This agrees with the result obtained from (7) by setting $\Pi_T = 1$ and employing the relation (9). Evaluating (17) in the static and long wavelength limits, leads to distinct terms of order T^2 .

Using the spectral representation (14), where we take for definiteness $\Sigma(\infty, \vec{k}) = 0$, we now consider the leading thermal contributions which arise in the long wavelength and static limits of the retarded self-energy

$$\Sigma_T(k_0 \rightarrow 0, \vec{k} = 0) = \frac{1}{2\pi} \lim_{k_0 \rightarrow 0} \int_{-\infty}^{\infty} \frac{d\kappa_0}{k_0 - \kappa_0 + i\epsilon} \sigma_T(\kappa_0, 0), \quad (18)$$

$$\Sigma_T(k_0 = 0, \vec{k} \rightarrow 0) = -\frac{1}{2\pi} \lim_{\vec{k} \rightarrow 0} \int_{-\infty}^{\infty} \frac{d\kappa_0}{\kappa_0 - i\epsilon} \sigma_T(\kappa_0, \vec{k}). \quad (19)$$

Next, let us compare these contributions with the result obtained by setting directly $k_0 = 0, \vec{k} = 0$ in the retarded thermal self-energy function

$$\Sigma_T(k_0 = 0, \vec{k} = 0) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{d\kappa_0}{\kappa_0 - i\epsilon} \sigma_T(\kappa_0, 0). \quad (20)$$

Since the integrand in $\Sigma_T(k_0, \vec{k} = 0)$ is not a uniformly continuous function of k_0 , we cannot take the limit $k_0 \rightarrow 0$ inside the integral (18). Thus, we infer that the leading thermal contribution (18) got in the long wavelength limit, would generally differ from the result given in (20).

On the other hand, we will argue that in the static case, the limit $\vec{k} \rightarrow 0$ can be taken inside the integral (19). To this end, using the fact that $\sigma_T(\kappa_0, \vec{k})$ is an odd function of κ_0 , it is convenient to write (19) in the alternative form (where \mathcal{P} denotes the principal value)

$$\Sigma_T(k_0 = 0, \vec{k} \rightarrow 0) = -\frac{1}{\pi} \lim_{\vec{k} \rightarrow 0} \mathcal{P} \int_0^\infty \frac{d\kappa_0}{\kappa_0} \sigma_T(\kappa_0, \vec{k}). \quad (21)$$

Consider now the integral

$$I(\vec{k}) = \mathcal{P} \int_0^\infty \frac{d\kappa_0}{\kappa_0} \sigma_T(\kappa_0, \vec{k}). \quad (22)$$

It is well known that if $I(\vec{k})$ converges uniformly, then the limit $\vec{k} \rightarrow 0$ can be taken inside the integral.

Such a convergence may be shown by considering the physical meaning of $\sigma_T(\kappa_0, \vec{k})$, which gives the imaginary part of the retarded self-energy. It yields the rates of processes occurring in a thermal plasma, such as particle creation/annihilation or scattering, in the presence of an external field [12]. At high temperatures, such that $|\vec{k}| \ll T$, the leading contributions to these rates have a smooth behavior when $\vec{k} \rightarrow 0$, in which case $\sigma_T(\kappa_0, \vec{k})$ would be a well behaved function in this limit. Furthermore, in consequence of unitarity (conservation of probability) such rates should decrease at large values of the energy κ_0 .

Assuming, for example, that $\sigma_T(\kappa_0, \vec{k})$ behaves for large κ_0 like $\kappa_0^2 \exp(-C\kappa_0/T)$, where C is a positive constant (which is consistent with (15)), one gets

$$\left| I(\vec{k}) - \mathcal{P} \int_0^E \frac{d\kappa_0}{\kappa_0} \sigma_T(\kappa_0, \vec{k}) \right| \propto \frac{E}{T} \exp - \frac{CE}{T} < \epsilon \quad (23)$$

for every ϵ , provided E/T is sufficiently large. A similar condition is obtained also for more general forms of the spectral density at large κ_0 , which lead to a T^2 behavior of the self-energy at high temperatures. Thus, $I(\vec{k})$ will be uniformly convergent so that the limit $\vec{k} \rightarrow 0$ can be taken inside (21) and, therefore, inside the integral (19).

Consequently, it follows that the leading static contribution (19) will agree to all orders with the result (20), got by calculating Σ_T at vanishing energy and momentum. This behavior is in agreement with the two-loop results given in Eqs. (10) and (13).

4 Generalization to n -point functions

According to the simple arguments given following Eqs. (1) and (2), the above result should hold as well in the case of higher point functions. We will now derive this property for the n -point Green functions, using a treatment which generalizes the previous method.

The spectral representation of the retarded n -point functions calculated in the imaginary-time formalism may be written in the form [14, 16]

$$\begin{aligned} \Gamma^{(n)}(\{k_{j0}, \vec{k}_j\}) = & \left(\frac{-1}{2\pi} \right)^{n-1} \int_{-\infty}^{\infty} d\kappa_{10} \dots d\kappa_{n0} \delta(\kappa_{10} + \dots + \kappa_{n0}) \left[\rho_{12\dots n}(\{\kappa_j\}, T) \right. \\ & \times \frac{i}{k_{20} + i\epsilon_2 + k_{30} + i\epsilon_3 + \dots + k_{n0} + i\epsilon_n - (\kappa_{20} + \kappa_{30} + \dots + \kappa_{n0})} \\ & \times \frac{i}{k_{30} + i\epsilon_3 + \dots + k_{n0} + i\epsilon_n - (\kappa_{30} + \dots + \kappa_{n0})} \times \dots \times \frac{i}{k_{n0} + i\epsilon_n - \kappa_{n0}} \\ & \left. + \text{all permutations of } (1, 2, 3, \dots, n) \right], \end{aligned} \quad (24)$$

where k_{j0} are real energy variables and $\kappa_{j\mu} = (\kappa_{j0}, \vec{k}_j)$. Setting in (24), for example, ϵ_l positive and all other epsilons negative such that $\sum \epsilon_j = 0$, defines the analytic l th retarded function. The spectral densities are the difference of two thermal Wightman functions which, in the case of pure bosonic fields, are given by

$$\rho_{12\dots n}(\{\kappa_j\}, T)$$

$$\begin{aligned} &= \frac{\text{Tr}\{e^{-H/T} \phi_1(\kappa_1) \phi_2(\kappa_2) \dots \phi_n(\kappa_n)\}}{\text{Tr}\{e^{-H/T}\}} \\ &- (-1)^n \frac{\text{Tr}\{e^{-H/T} \phi_n(\kappa_n) \dots \phi_2(\kappa_2) \phi_1(\kappa_1)\}}{\text{Tr}\{e^{-H/T}\}}. \end{aligned} \quad (25)$$

We assume that to leading order at high temperature, when all $|\vec{k}_j| \ll T$, these spectral densities (which may also de-

pend on the particles masses, etc.) are well behaved in the limit $\vec{k}_j \rightarrow 0$.

One may now consider the leading thermal contributions which arise from (24) in the static limit (when all $k_{j0} = 0$) which is well defined due to the analyticity properties of $\Gamma^{(n)}$. In this case, to leading order in T , one may next take the limits $\vec{k}_j \rightarrow 0$ and proceed similarly to the previous analysis. We then find a result which agrees with that obtained by setting in (24) all external energies and momenta equal to zero:

$$\begin{aligned} & \Gamma_T^{(n)}(\{k_{j0} = 0, \vec{k}_j \rightarrow 0\}) \\ &= \left(\frac{i}{2\pi}\right)^{n-1} \int_{-\infty}^{\infty} d\kappa_{10} \dots d\kappa_{n0} \delta(\kappa_{10} + \dots + \kappa_{n0}) \\ & \times \left[\rho_{T_{12\dots n}}(\{k_{j0}, \vec{k}_j = 0\}, T) \right. \\ & \times \frac{1}{\kappa_{20} + \kappa_{30} + \dots + \kappa_{n0} - i(\epsilon_2 + \epsilon_3 + \dots + \epsilon_n)} \\ & \times \frac{1}{\kappa_{30} + \dots + \kappa_{n0} - i(\epsilon_3 + \dots + \epsilon_n)} \times \dots \\ & \times \frac{1}{\kappa_{n0} - i\epsilon_n} \\ & \left. + \text{all permutations of } (1, 2, 3, \dots, n) \right]. \quad (26) \end{aligned}$$

We have explicitly verified this relation at two-loops order, by calculating the leading static thermal contributions of the three-point functions in gauge theories. Thus, we conclude

that to all orders in the static limit, the leading thermal contributions of retarded bosonic Green's functions may be directly obtained by evaluating them at zero external energies and momenta.

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