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by

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Abstract

In this paper we discuss calibration problems under the dependent and independent elliptical family of distributions. In the dependent case, it is shown that the posterior distribution of the quantity of interest is robust with respect to the distributions in the elliptical family. In particular, the results obtained by Hoadley (1970) showing that the inverse estimator is a Bayes estimator under normal models with a Student-t prior also holds under the dependent elliptical family of distributions. In the independent case, the use of the the elliptical family allows the consideration of models which provide protection against possible outliers in the data. Section four presents a simulation study to illustrate the approach developed for the independent model.

Some key words: Calibration; Bayesian inference; Elliptical distributions ; Gibbs sampler.

1 Introduction

The Bayesian analysis of regression problems under elliptical distributions has received increasingly attention in the recent statistical literature. Essentially, in the case of the dependent model, it is shown that inference based on the posterior distribution using non informative priors for the scale parameter coincide with those obtained under the normal model. A pioneer work in this area is the paper by Zellner (1976), in which a study based on the multivariate Student-t distribution is considered. Extensions of those results are considered in Osiewalski and Steel (1993). The independent Student-t regression model is treated in Geweke (1993) by using a Bayesian perspective. A problem of great practical interest associated with the regression model is the calibration problem. Most of the studies concerning the Bayesian approach to calibration problems are restricted to normal models (Hoadley, 1970, Hunter and Lamboy, 1981, among other). A review of the main results concerning calibration problems are presented in Brown (1993).

In this paper, it is considered the linear calibration model specified by the equations

$$y_i = \beta_1 + \beta_2 x_i + \epsilon_i, \tag{1.1}$$

$i = 1, \ldots, n$, and

$$y_0 = \beta_1 + \beta_2 x_0 + \epsilon_0, \tag{1.2}$$

where $\epsilon_0, \epsilon_1, \ldots, \epsilon_n$ are uncorrelated random errors with $E[\epsilon_i] = 0$ and $Var[\epsilon_i] = \sigma^2$, $i = 1, \ldots, n$. It is assumed that $x_1, \ldots, x_n$ are known constants and that β_1, β_2, σ^2 and x_0 are unknown quantities (parameters). The main object is to make inference on x_0 (the calibration parameter) based on $y_0, y_1, \ldots, y_n$ and $x_1, \ldots, x_n$, which are observed. Bayesian solutions to the calibration problem under the dependent and independent elliptical models are developed. Here we use the notation $\mathbf{z} \sim El_p(\boldsymbol{\mu}, \Sigma; g)$ to indicate that $\mathbf{z}$ is a p-dimensional random vector elliptically distributed with location vector $\boldsymbol{\mu} \in \Re^p$ and a $p \times p$ (positive definite) dispersion matrix Σ and with density given by

$$f(\mathbf{z}|\boldsymbol{\mu}, \Sigma) = |\Sigma|^{-1/2} g[(\mathbf{z} - \boldsymbol{\mu})'\Sigma^{-1}(\mathbf{z} - \boldsymbol{\mu})],$$

for some density generator function $g(u)$, $u \geq 0$, that is,

$$\int_0^\infty u^{\frac{p}{2}-1} g(u) du = \frac{\Gamma(\frac{p}{2})}{\pi^{\frac{p}{2}}}, \tag{1.3}$$

which implies that g is a spherical p-dimensional density. (A comprehensive review of the properties and characterizations of elliptical distributions can be found in Fang et al.

(1990)). Thus, in (1.1)-(1.2) we have the dependent elliptical model by assuming that $\boldsymbol{\epsilon}_* = (\epsilon_0, \epsilon_1, \ldots, \epsilon_n)' \sim El_{n+1}(\mathbf{0}, \sigma^2\mathbf{I}_{n+1}; g)$, where $\mathbf{I}_{n+1}$ denotes the (n+1)-dimensional identity matrix, so that,

$$(1.4) \qquad f(\epsilon_0, \epsilon_1, \ldots, \epsilon_n|\sigma^2) = (1/\sigma^2)^{\frac{n+1}{2}} g(\sum_{i=0}^{n} \epsilon_i^2/\sigma^2).$$

Moreover, (1.4) implies that (provided they exist) $E[\boldsymbol{\epsilon}_*] = \mathbf{0}$ and $Var[\boldsymbol{\epsilon}_*] = a_g\sigma^2\mathbf{I}$, where a_g denotes the expectation associated with a positive random variable having density function $h(u) \propto u^{p/2-1}g(u)$. In the independent case, we assume in (1.1)-(1.2) that $\epsilon_i \overset{iid}{\sim} El_1(0, \sigma^2; g)$, $i = 0, 1, \ldots, n$, with g being now an univariate elliptical density generator, that is, g is such that (1.3) holds for $p = 1$. Thus, for this model $\epsilon_0, \epsilon_1, \ldots, \epsilon_n$ have joint density given by

$$(1.5) \qquad f(\epsilon_0, \epsilon_1, \ldots, \epsilon_n|\sigma^2) = (1/\sigma^2)^{\frac{n+1}{2}} \prod_{i=0}^{n} g(\epsilon_i^2/\sigma^2),$$

so that (provided they exist) $E[\epsilon_i] = 0$ and $Var[\epsilon_i] = a_g\sigma^2$, for $i = 0, 1, \ldots, n$.

Here we consider that the density generator g is known in both cases, namely, the dependent and independent elliptical models. In Section 2, we discuss Bayesian solutions to the linear calibration problem under dependent elliptical models. We show that inference does not depend of the density generator function when the prior distribution of the scale parameter is non informative. Solutions for the independent elliptical model are considered in Section 3. Finally, we present results of a simulation study in Section 4.

2 A Bayesian solution to the dependent case

In this section we present Bayesian solutions to the linear calibration problem under the dependent elliptical model. For simplicity the following notation is used in the sequel. Let $\mathbf{y} = (y_1, \ldots, y_n)'$ denote the response vector corresponding to the first step of the calibration experiment and $\mathbf{y}_* = (y_0, \mathbf{y}')'$. Similarly, we can define the vectors $\mathbf{x} = (x_1, \ldots, x_n)'$ and $\mathbf{x}_* = (x_0, \mathbf{x}')'$ and the matrices

$$\mathbf{X} = [\mathbf{1}_n \quad \mathbf{x}] \quad \text{and} \quad \mathbf{X}_* = [\mathbf{1}_{n+1} \quad \mathbf{x}_*],$$

where $\mathbf{1}_k$ denotes the k-dimensional vector of ones. Assuming that $\boldsymbol{\epsilon}_* \sim El_{n+1}(\mathbf{0}, \sigma^2\mathbf{I}_{n+1}; g)$, it follows from (1.1) and (1.2) that $\mathbf{y}_*|\mathbf{X}_*, \boldsymbol{\beta}, \sigma^2 \sim El_{n+1}(\mathbf{X}_*\boldsymbol{\beta}, \sigma^2\mathbf{I}_{n+1}; g)$, where $\boldsymbol{\beta} = (\beta_1, \beta_2)'$. Moreover, from (1.4) and from the fact that

$$(\mathbf{y}_* - \mathbf{X}_*\boldsymbol{\beta})'(\mathbf{y}_* - \mathbf{X}_*\boldsymbol{\beta}) = (\boldsymbol{\beta} - \hat{\boldsymbol{\beta}}_*)'\mathbf{X}_*'\mathbf{X}_*(\boldsymbol{\beta} - \hat{\boldsymbol{\beta}}_*) + (\mathbf{y}_* - \mathbf{X}_*\hat{\boldsymbol{\beta}}_*)'(\mathbf{y}_* - \mathbf{X}_*\hat{\boldsymbol{\beta}}_*),$$

where $\hat{\boldsymbol{\beta}}_* = (\mathbf{X}_*'\mathbf{X}_*)^{-1}\mathbf{X}_*'\mathbf{y}_*$, it follows that the likelihood function of the dependent case can be written as

$$f(\mathbf{y}_*|\boldsymbol{\beta},\sigma^2,\mathbf{x}_*) = (1/\sigma^2)^{\frac{(n+1)}{2}} g([Q_*(\boldsymbol{\beta},x_0) + (n-1)S_*^2]/\sigma^2), \tag{2.1}$$

where

$$Q_*(\boldsymbol{\beta},x_0) = (\boldsymbol{\beta}-\hat{\boldsymbol{\beta}}_*)'\mathbf{X}_*'\mathbf{X}_*(\boldsymbol{\beta}-\hat{\boldsymbol{\beta}}_*), \quad S_*^2 = (\mathbf{y}_* - \mathbf{X}_*\hat{\boldsymbol{\beta}}_*)'(\mathbf{y}_* - \mathbf{X}_*\hat{\boldsymbol{\beta}}_*)/(n-1). \tag{2.2}$$

In the following theorem we show that if $(\boldsymbol{\beta},\sigma^2,x_0)$ has a prior distribution of the form

$$p(x_0,\boldsymbol{\beta},\sigma^2|\mathbf{x}) \propto \frac{1}{\sigma^2}p(x_0,\boldsymbol{\beta}|\mathbf{x}), \tag{2.3}$$

then the posterior distribution of x_0 given $\mathbf{y}_*$ and $\mathbf{x}$ does not depend on g, whatever the prior distribution $p(x_0,\boldsymbol{\beta}|\mathbf{x}) = p(\boldsymbol{\beta}|\mathbf{x}_*)p(x_0|\mathbf{x})$ assigned to $(x_0,\boldsymbol{\beta})$.

Theorem 2.1. *Under the assumptions (2.1) and (2.3), the posterior density of x_0 is given by*

$$p(x_0|\mathbf{y}_*,\mathbf{x}) \propto p(x_0|\mathbf{x}) \int \left[\frac{1}{S_*^2}\right]^{\frac{n+1}{2}} \left[1 + \frac{Q_*(\boldsymbol{\beta},x_0)}{(n-1)S_*^2}\right]^{-\frac{n+1}{2}} p(\boldsymbol{\beta}|\mathbf{x}_*)d\boldsymbol{\beta},$$

for any elliptical density generator g, where $Q_*(\boldsymbol{\beta},x_0)$ and S_*^2 are as in (2.2).

Proof. Ordinary manipulations with conditional densities yield

$$p(x_0|\mathbf{y}_*,\mathbf{x}) \propto f(\mathbf{y}_*|\mathbf{x}_*)p(x_0|\mathbf{x}), \tag{2.4}$$

where $f(\mathbf{y}_*|\mathbf{x}_*)$ is the conditional density of $\mathbf{y}_*$ given $\mathbf{x}_*$. Since

$$f(\mathbf{y}_*|\mathbf{x}_*) = \int\int f(\mathbf{y}_*|\boldsymbol{\beta},\sigma^2,\mathbf{x}_*)p(\boldsymbol{\beta},\sigma^2|\mathbf{x}_*)\mathrm{d}\boldsymbol{\beta}d\sigma^2,$$

it suffices to show that $f(\mathbf{y}_*|\mathbf{x}_*)$ is independent of g. Now, using (2.1) and (2.3) we can write

$$p(\mathbf{y}_*|\mathbf{x}_*) \propto \int\int (1/\sigma^2)^{\frac{n+1}{2}+1} g([Q_*(\boldsymbol{\beta},x_0) + (n-1)S_*^2]/\sigma^2)d\sigma^2 p(\boldsymbol{\beta}|\mathbf{x})\mathrm{d}\boldsymbol{\beta}.$$

Taking $u = [Q_*(\boldsymbol{\beta},x_0) + (n-1)S_*^2]/\sigma^2$ and using (1.3), we have that

$$\int_0^\infty (1/\sigma^2)^{\frac{n+1}{2}+1} g([Q_*(\boldsymbol{\beta},x_0) + (n-1)\mathrm{S}_*^2]/\sigma^2)d\sigma^2$$

$$= \{Q_*(\boldsymbol{\beta},x_0) + (n-1)\mathrm{S}_*^2\}^{-\frac{n+1}{2}} \int_0^\infty u^{-\frac{n+1}{2}-1} g(u)du$$

$$\propto \{Q_*(\boldsymbol{\beta}, x_0) + (n-1)S_*^2\}^{-\frac{n+1}{2}},$$

which is independent of the function g, concluding the proof.

Notice that the last expression implies that $\boldsymbol{\beta}|\mathbf{y}_*, \mathbf{x}_* \sim t_2(\hat{\boldsymbol{\beta}}_*, \mathbf{S}_*^2(\mathbf{X}_*'\mathbf{X}_*)^{-1}; n-1)$, where $t_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}; \nu)$ denotes the p-dimensional t distribution with location vector $\boldsymbol{\mu}$, dispersion matrix $\boldsymbol{\Sigma}$ and ν degrees of freedom.

The above theorem shows that Bayesian inference on x_0 is independent of the particular elliptical distribution under consideration. This includes the fact that the results derived for the normal model can be used for any elliptical model. Following next, the posterior of x_0 for a special and important case of the elliptical family is considered as a special case of Theorem 2.1. The result is directly related to the results obtained by Hoadley (1970). Let $\hat{x}_{0,I}$ be the inverse estimator of x_0. As is well known, this estimator can be written as (see (3.7) in Hoadley, 1970)

$$\hat{x}_{0,I} = \Big(\frac{F}{n-2}\Big)\Big(1 + \frac{F}{n-2}\Big)^{-1}\hat{x}_0,$$

where

$$F = nS_x^2\hat{\beta}_2^2/S^2 \quad \text{and} \quad \hat{x}_0 = (y_0 - \hat{\beta}_1)/\hat{\beta}_2,$$

with $S_x^2 = \sum_{i=1}^n (x_i - \bar{x})^2/n$, $\bar{x} = \sum_{i=1}^n x_i/n$,

$$S^2 = (\mathbf{y} - \mathbf{X}\hat{\boldsymbol{\beta}})'(\mathbf{y} - \mathbf{X}\hat{\boldsymbol{\beta}})/(n-2), \quad \text{and} \quad \hat{\boldsymbol{\beta}} = (\hat{\beta}_1, \hat{\beta}_2)' = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}.$$

We call attention to the fact that, as in the normal model, under the dependent elliptical model (2.1), $\hat{\boldsymbol{\beta}}$ and $\hat{x}_0$ are the likelihood estimators of $\boldsymbol{\beta}$ and x_0, respectively.

Corollary 2.1. *If in (2.3), $p(x_0, \boldsymbol{\beta}|\mathbf{x}) = p(\boldsymbol{\beta}|\mathbf{x}_*)p(x_0|\mathbf{x})$ is such that $p(\boldsymbol{\beta}|\mathbf{x}_*) \propto$ constant and $x_0|\mathbf{x} \sim t(\bar{x}, (\frac{n+1}{n-3})S_x^2; n-3)$, $n > 3$, then*

$$x_0|\mathbf{y}_*, \mathbf{x} \sim t\Big(\hat{x}_{0,I}, \Big(\frac{n+1}{n-2}\Big)\Big\{\Big(1 + \frac{F}{n-2}\Big)^{-1} + \Big(\frac{n-2}{F}\Big)\hat{x}_{0,I}^2\Big\}; n-2\Big). \tag{2.5}$$

As a consequence of Corollary 2.1, it follows that the inverse estimator is a robust Bayes estimator in the family of the elliptical distributions. Moreover, Theorem 2.1 also illustrates the fact that this result depends on the prior distribution and not on the likelihood function (2.1), which is a member of the elliptical family. The prior specifications for x_0 indicates symmetry about $\bar{x}$, and the posterior density become more concentrated as n increases. This is a consequence of the fact that $Var[x_0|x] = (n+1)S_x^2/(n-5)^2$, $n > 5$. It is also true that the results obtained by Hunter and Lamboy (1981) and Kubokawa and Robert (1994) remain valid under the assumptions of Theorem 2.1.

3 A Bayesian solution for the independent case

In this section, it is considered that the error terms in equations (1.1)-(1.2) are independent with common distribution $El_1(0, \sigma^2; g)$, which yield the joint density given in (1.5). From (1.5) it follows that the likelihood for the independent elliptical model is given by

$$f(\mathbf{y}_*|\boldsymbol{\beta}, \sigma^2, \mathbf{x}_*) = (1/\sigma^2)^{\frac{n+1}{2}} \prod_{i=0}^{n} g[(y_i - \beta_1 - \beta_2 x_i)^2/\sigma^2], \tag{3.1}$$

$i = 1, \ldots, n$, where as before $\mathbf{y}_* = (y_0, \ldots, y_n)'$, $\mathbf{x}_* = (x_0, \ldots, x_n)'$ and $\boldsymbol{\beta} = (\beta_1, \beta_2)'$. In other words,

$$y_i|\boldsymbol{\beta}, \sigma^2, x_i \overset{ind}{\sim} El_1(\beta_1 + \beta_2 x_i, \sigma^2, g), \tag{3.2}$$

$i = 0, 1, \ldots, n$, so that, $y_0, \ldots, y_n$ are conditionally independent given $\boldsymbol{\beta}$, σ^2 and $\mathbf{x}_*$. We consider here the case where the function g is representable, that is,

$$g(u) = \int_0^\infty (2\pi v)^{-1/2} e^{-u/2v} dG(v), \tag{3.3}$$

for any distribution function G such that $G(0) = 0$. Representation (3.3) defines a subclass of the elliptical family, namely the compound normal distributions. Under this subclass the specification (3.2) can be represented equivalently by considering the following two steps:

(i) $y_i|\boldsymbol{\beta}, \sigma^2, x_i, v_i \overset{ind}{\sim} N(\beta_1 + \beta_2 x_i, \sigma^2 v_i)$, $\quad i = 1, \ldots, n$,

and

(ii) $v_i \overset{iid}{\sim} G$, $\quad i = 1, \ldots, n$, and independent of $\boldsymbol{\beta}$, σ^2 and $\mathbf{x}_*$.

Note that the correspondence between the class of the density generator functions g which are representable and the class of the distribution functions G such that $G(0) = 0$ is one to one so that the Bayesian analysis of the calibration problem under specification (3.1) is equivalent to the Bayesian analysis under specifications (i) and (ii), that is, under the heteroscedastic normal model. One advantage of this last specification is the fact that the derivation of the posterior distributions can be based on properties of the normal model, according to step (i). For step (ii) the main problem is the choice of the distribution function $G_*(\mathbf{v}_*) = \pi_{i=0}^{n} G(v_i)$ for the weight vector $\mathbf{v}_* = (v_0, v_1, \ldots, v_n)'$ and the numerical integration problem that results. Thus, under conditions (i) and (ii) in relation (2.6) we have that

$$f(\mathbf{y}_*|\mathbf{x}_*) = \int f(\mathbf{y}_*|\mathbf{x}_*, \mathbf{v}_*) dG_*(\mathbf{v}_*), \tag{3.4}$$

where

$$f(\mathbf{y}_*|\boldsymbol{x}_*, \mathbf{v}_*) = \int\int f(\mathbf{y}_*|\boldsymbol{\beta}, \sigma^2, \mathbf{x}_*, \mathbf{v}_*) p(\boldsymbol{\beta}, \sigma^2|\mathbf{x}_*) \mathrm{d}\boldsymbol{\beta} d\sigma^2, \tag{3.5}$$

with

$$f(\mathbf{y}_*|\boldsymbol{\beta}, \sigma^2, \mathbf{x}_*, \mathbf{v}_*) = (1/2\pi\sigma^2)^{\frac{n+1}{2}} |\mathbf{V}_*|^{-1/2} \exp\{-(\mathbf{y}_* - \mathbf{X}_*\boldsymbol{\beta})' \mathbf{V}_*^{-1} (\mathbf{y}_* - \mathbf{X}_*\boldsymbol{\beta})/2\sigma^2\}, \tag{3.6}$$

and $\mathbf{V}_* = \mathrm{diag}(v_0, v_1, \ldots, v_n)$. Notice that when the conditional posterior distribution of $p(\boldsymbol{\beta}, \sigma^2|\mathbf{x}_*, \mathbf{y}_*, \mathbf{v}_*)$ is available, (3.5) can be computed from the relation

$$f(\mathbf{y}_*|\mathbf{x}_*, \mathbf{v}_*) = \frac{f(\mathbf{y}_*|\boldsymbol{\beta}, \sigma^2, \mathbf{x}_*, \mathbf{v}_*) p(\boldsymbol{\beta}, \sigma^2|\mathbf{x}_*)}{p(\boldsymbol{\beta}, \sigma^2|\mathbf{x}_*, \mathbf{y}_*, \mathbf{v}_*)}.$$

If $p(\boldsymbol{\beta}, \sigma^2|\mathbf{x}_*)$ is an improper prior distribution, the posterior distribution $p(x_0|\mathbf{x}, \mathbf{y}_*)$ may not be a proper distribution. In this case it is necessary to study conditions under which such posterior is a proper distribution.

Two specifications for the prior distribution $p(\boldsymbol{\beta}, \sigma^2|\mathbf{x}_*)$ are considered next. Both are motivated by the Bayesian formulation of the usual regression problem. The first is the usual improper reference prior distribution:

$$\pi(\boldsymbol{\beta}, \sigma^2|\mathbf{x}_*) \propto \frac{1}{\sigma^2}. \tag{3.7}$$

The second is associated with the conjugate prior specification, that is, the normal/inverse-gamma prior distribution:

$$\boldsymbol{\beta}|\sigma^2, \mathbf{x}_* \sim N_2(\boldsymbol{\beta}_0, \sigma^2\mathbf{M}_0) \quad \text{and} \quad \sigma^2|\mathbf{x}_* \sim IG(n_0/2, r_0/2),$$

which has joint density such that

$$p(\boldsymbol{\beta}, \sigma^2|\mathbf{x}_*) \propto (1/\sigma^2)^{\frac{n_0}{2}+1} \exp\{-[r_0 + (\boldsymbol{\beta} - \mathbf{b}_0)' \mathbf{M}_0^{-1} (\boldsymbol{\beta} - \mathbf{b}_0)]/2\sigma^2\}. \tag{3.8}$$

The notation used to express this prior distribution is $(\boldsymbol{\beta}, \sigma^2)|\mathbf{x}_* \sim NIG(\mathbf{b}_0, \mathbf{M}_0, n_0, r_0)$. As usually, $\mathbf{b}_0$ represents a prior assignment to $\boldsymbol{\beta}$ with $\mathbf{M}_0$ expressing uncertain in that assignment.

In both cases, a proper prior distribution, $p(x_0|\mathbf{x})$, is considered for x_0, which however does not guarantee a proper posterior distribution for x_0 in the first case.

Results for the improper reference prior distribution given by (3.7) are considered first. The following notation will be used in the next proposition. Let

$$Q_*(\boldsymbol{\beta}, x_0, \mathbf{v}_*) = (\boldsymbol{\beta} - \hat{\boldsymbol{\beta}}_*(\mathbf{v}_*))' \mathbf{X}_*' \mathbf{V}_*^{-1} \mathbf{X}_* (\boldsymbol{\beta} - \hat{\boldsymbol{\beta}}_*(\mathbf{v}_*)), \tag{3.9}$$

and

$$q(\mathbf{v}_*; \mathbf{x}_*, \mathbf{y}_*) = [S_*^2(\mathbf{v}_*)]^{-\frac{(n-1)}{2}} |\mathbf{V}_*(\mathbf{X}_*'\mathbf{V}_*^{-1}\mathbf{X}_*)|^{-1/2}, \tag{3.10}$$

where

$$S_*^2(\mathbf{v}_*) = (\mathbf{y}_* - \mathbf{X}_*\hat{\boldsymbol{\beta}}_*(\mathbf{v}_*))'\mathbf{V}_*^{-1}(\mathbf{y}_* - \mathbf{X}_*\hat{\boldsymbol{\beta}}_*(\mathbf{v}_*))/(n-1), \tag{3.11}$$

and

$$\hat{\boldsymbol{\beta}}(\mathbf{v}_*) = (\mathbf{X}_*'\mathbf{V}_*^{-1}\mathbf{X}_*)^{-1}\mathbf{X}_*^{-1}\mathbf{V}_*^{-1}\mathbf{y}_*. \tag{3.12}$$

Proposition 3.1. *Let $p(x_0|\mathbf{x})$ a proper prior density and $p(\boldsymbol{\beta}, \sigma^2|\mathbf{x}_*)$ being the improper prior distribution given in (3.6). Suppose that for each $(\mathbf{x}, \mathbf{y}_*)$ there exists a real number $K(\mathbf{x}, \mathbf{y}_*)$ such that*

$$E_{G_*}[q(\mathbf{v}_*; \mathbf{x}_*, \mathbf{y}_*)] < K(\mathbf{x}, \mathbf{y}_*), \tag{3.13}$$

where $q(\mathbf{v}_*; \mathbf{x}_*, \mathbf{y}_*)$ is as defined in (3.10). Then,

$$p(x_0|\mathbf{x}, \mathbf{y}_*) \propto p(x_0|\mathbf{x}) \int q(\mathbf{v}_*; \mathbf{x}_*, \mathbf{y}_*) dG_*(\mathbf{v}_*),$$

which is a proper posterior distribution.

Proof. According to (2.4), (3.4) and (3.5) it is only necessary to evaluate the integral

$$\int\int f(\mathbf{y}_*|\boldsymbol{\beta}, \sigma^2, \mathbf{v}_*, \mathbf{x}_*) \frac{1}{\sigma^2} d\sigma^2 d\boldsymbol{\beta}.$$

Using (3.6) , it follows after some algebraic manipulations that

$$\int f(\mathbf{y}_*|\boldsymbol{\beta}, \sigma^2, \mathbf{v}_*, \mathbf{x}_*) d\sigma^2 \propto |\mathbf{V}_*|^{-1/2} (S_*^2(\mathbf{v}_*))^{-\frac{n+1}{2}} \left\{1 + \frac{Q_*(\boldsymbol{\beta}, x_0, \mathbf{v}_*)}{(n-1)S_*^2(\mathbf{v}_*)}\right\}^{-\frac{n+1}{2}}, \tag{3.14}$$

where $Q_*(\boldsymbol{\beta}, x_0, \mathbf{v}_*)$ and $S_*^2(\mathbf{v}_*)$ are as given in (3.9) and (3.11), respectively. Note that, given $\mathbf{v}_*$, $\mathbf{x}_*$ and $\mathbf{y}_*$, the last expression in (3.14) is the kernel density of the bivariate Student-t distribution $t_2(\hat{\boldsymbol{\beta}}(\mathbf{v}_*), S_*^2(\mathbf{v}_*)(\mathbf{X}_*'\mathbf{V}_*^{-1}\mathbf{X}_*)^{-1}; n-1)$. Thus, integrating (3.14) with respect to $\boldsymbol{\beta}$, we arrive at $q(\mathbf{v}_*; \mathbf{x}_*, \mathbf{y}_*)$ given in (3.10), so that

$$p(x_0|\mathbf{x}, \mathbf{y}_*) \propto p(x_0|\mathbf{x}) \int q(\mathbf{v}_*; \mathbf{x}_*, \mathbf{y}_*) dG_*(\mathbf{v}_*) = p(x_0|\mathbf{x}) E_{G_*}[q(\mathbf{v}_*; \mathbf{x}_*, \mathbf{y}_*)].$$

Thus, condition (3.13) implies that the above posterior distribution is proper and, moreover, that

$$\int p(x_o|\mathbf{x}) E_{G_*}[q(\mathbf{v}_*; \mathbf{x}_*, \mathbf{y}_*)] dx_0 < K(\mathbf{x}, \mathbf{y}_*).$$

Geweke (1993) reports results of a study on the posterior distribution of $\boldsymbol{\beta}$, showing that it is proper under the prior assumptions of Proposition 3.1. If his results were correct than it would no be difficult to show that the posterior distribution for x_0 obtained above would also be proper. However, it seems that Geweke's result is not correct. Using (3.14) above, Geweke(1993) mistakenly obtains the expression

$$(S_*^2(\mathbf{v}_*))^{1/2} |\mathbf{V}_*(\mathbf{X}_*' \mathbf{V}_*^{-1} \mathbf{X}_*)|^{-1/2}, \tag{3.15}$$

and not the expression given by (3.10) above, which is the expression that should result. Using (3.10) in place of (3.15) in Geweke's derivations, it seems that proving that the posterior distribution of $\boldsymbol{\beta}$ is proper, is still an open problem.

The next result is related to second case given by (3.8). We use the notation $t_p(\mathbf{z}; \boldsymbol{\mu}, \boldsymbol{\Sigma}, \nu)$ to denote the density of $\mathbf{z} \sim t_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \nu)$.

Proposition 3.2. *Let* $(\boldsymbol{\beta}, \sigma^2)|\mathbf{x}_* \sim NIG(\mathbf{b}_0, \mathbf{M}_0, n_0, r_0)$ *and* $p(x_0|\mathbf{x})$ *a proper prior density for* x_0. *Thus, the posterior density that follows is given by*

$$p(x_0|\mathbf{x}, \mathbf{y}_*) \propto p(x_0|\mathbf{x}_0) \int t_{n+1}(\mathbf{y}_*; \mathbf{X}_*\mathbf{b}_0, (r_o/n_0)(\mathbf{v}_* + \mathbf{X}_*\mathbf{M}_0\mathbf{X}_*'), n_0) dG_*(\mathbf{v}_*).$$

Proof. Let $\mathbf{z}_* = \mathbf{V}_*^{-1/2}\mathbf{y}_*$, $\mathbf{W}_* = \mathbf{V}_*^{-1/2}\mathbf{X}_*$, where as before $\mathbf{X}_* = [\mathbf{1}_{n+1} \quad \mathbf{x}_*]$, which is of dimension $(n+1) \times 2$. Thus, it follows that $\mathbf{Z}_*|\boldsymbol{\beta}, \sigma^2, \mathbf{v}_*, \mathbf{x}_* \sim N_{n+1}(\mathbf{V}_*\boldsymbol{\beta}, \sigma^2\mathbf{I}_{n+1})$. Now, since $(\boldsymbol{\beta}, \sigma^2)|\mathbf{x}_* \sim NIG(\mathbf{b}_0, \mathbf{M}_0, n_0, r_0)$, straightforward algebraic manipulations yield

$$\mathbf{z}_*|\mathbf{v}_*, \mathbf{x}_* \sim t_{n+1}(\mathbf{W}_*\mathbf{b}_0, (r_0/n_0)(\mathbf{I}_{n+1} + \mathbf{W}_*\mathbf{M}_0\mathbf{W}_*'), n_0).$$

Hence, simple transformation yields

$$\mathbf{y}_*|\mathbf{v}_*, \mathbf{x}_* \sim t_{n+1}(\mathbf{X}_*\mathbf{b}_0, (r_0/n_0)(\mathbf{v}_* + \mathbf{X}_*\mathbf{M}_0\mathbf{X}_*'), n_0),$$

from where it follows that

$$f(\mathbf{y}_*|\mathbf{x}_*) = \int_{\Re^n} t_{n+1}(\mathbf{y}_*; \mathbf{X}_*\mathbf{b}_0, (r_0/n_0)(\mathbf{v}_* + \mathbf{X}_*\mathbf{M}_0\mathbf{X}_*'), n_o) dG_*(\mathbf{v}_*),$$

which concludes the proof.

In the special case of G_* being degenerate at the point $\mathbf{v}_* = (1, \ldots, 1)$, the posterior density of x_0 can be written as

$$p(x_0|\mathbf{x}, \mathbf{y}_*) \propto p(x_0|\mathbf{x}) t_{n+1}(\mathbf{y}_*; \mathbf{X}_*\mathbf{b}_0, (r_0/n_0)(\mathbf{I}_{n+1} + \mathbf{X}_*\mathbf{M}_0\mathbf{X}_*'), n_0).$$

The next corollary presents an analytical expression for this posterior distribution, which extends the work of Hoadley (1970) to proper posterior distributions for $(\boldsymbol{\beta}, \sigma^2)$.

Corollary 3.1. *Suppose that*

$$e_i|\sigma^2 \stackrel{iid}{\sim} N(0, \sigma^2), i = 1, \ldots, n, \quad \text{and} \quad (\boldsymbol{\beta}, \sigma^2)|\mathbf{x}_* \sim NIG(\mathbf{b}_0, \mathbf{M}_0, n_0, r_0).$$

Then,

$$p(x_0|\mathbf{x}, \mathbf{y}_*) \propto p(x_0|\mathbf{x}) t(x_0),$$

where

$$t(x_0) \propto (1 + \mathbf{x}_0\mathbf{M}_1\mathbf{x}_0')^{-\frac{1}{2}} \left\{a(\mathbf{y}) + \frac{(y_0 - \mathbf{x}_0'\mathbf{b}_1)^2}{(1 + \mathbf{x}_0\mathbf{M}_1\mathbf{x}_0')}\right\}^{-\frac{(n_0+n+1)}{2}},$$

with $\mathbf{x}_0 = (1, x_0)'$,

$$a(\mathbf{y}) = n_0 + (\mathbf{y} - \mathbf{X}\mathbf{b}_0)'(\mathbf{I}_n + \mathbf{X}\mathbf{M}_0\mathbf{X}')^{-1}(\mathbf{y} - \mathbf{X}\mathbf{M}_0),$$

$$\mathbf{b}_1 = \mathbf{b}_0 + \mathbf{M}_0\mathbf{X}'(\mathbf{I}_n + \mathbf{X}\mathbf{M}_0\mathbf{X}')^{-1}(\mathbf{y} - \mathbf{X}\mathbf{b}_0)$$

and

$$\mathbf{M}_1 = \mathbf{M}_0 - \mathbf{M}_0\mathbf{X}'(\mathbf{I}_n + \mathbf{X}\mathbf{M}_0\mathbf{X}')^{-1}\mathbf{X}\mathbf{M}_0.$$

Proof. Since $\mathbf{y}_* = (y_0, \mathbf{y}')'$, the partition

$$\mathbf{X}_*\mathbf{b}_0 = \begin{pmatrix} \mathbf{x}_0'\mathbf{b}_0 \\ \mathbf{X}\mathbf{b}_0 \end{pmatrix}, \quad \mathbf{I}_{n+1} + \mathbf{X}_*\mathbf{M}_0\mathbf{X}_*' = \begin{pmatrix} 1 + \mathbf{x}_0'\mathbf{M}_0\mathbf{x}_0 & \mathbf{x}_0'\mathbf{M}_0\mathbf{X}' \\ \mathbf{X}\mathbf{M}_0\mathbf{x}_0' & \mathbf{I}_n + \mathbf{X}\mathbf{M}_0\mathbf{X}' \end{pmatrix}$$

and well known properties of the generalized Student-t distribution (see Arellano-Valle and Bolfarine, 1995) allow to write

$$y_0|\mathbf{x}_*, \mathbf{y} \sim t_1(m(\mathbf{y}), \mathbf{M}; a(\mathbf{y}), n_0 + n),$$

where

$$m(\mathbf{y}) = \mathbf{x}_0'\mathbf{b}_0 + \mathbf{x}_0'\mathbf{M}_0\mathbf{X}(\mathbf{I}_n + \mathbf{X}\mathbf{M}_0\mathbf{X}')^{-1}(\mathbf{y} - \mathbf{X}\mathbf{b}_0),$$
$$\mathbf{M} = 1 + \mathbf{x}_0'[\mathbf{M}_0\mathbf{X})'(\mathbf{I}_n + \mathbf{X}\mathbf{M}_0\mathbf{X}')^{-1}\mathbf{X}\mathbf{M}_0]\mathbf{x}_0'$$

and

$$a(\mathbf{y}) = n_0 + (\mathbf{y} - \mathbf{X}\mathbf{B}_0)'(\mathbf{I}_n + \mathbf{X}\mathbf{M}_0\mathbf{X}')^{-1}(\mathbf{y} - \mathbf{X}\mathbf{B}_0).$$

Thus,

$$p(y_0|\mathbf{x}_*, \mathbf{y}) \propto t(x_0)$$

and the result follows from the fact that $p(x_0|\mathbf{x}, \mathbf{y}_*) \propto p(x_0|x)p(y_0|\mathbf{x}_*, \mathbf{y})p(\mathbf{y}|\mathbf{x}_*)$.

The two propositions above imply the the posterior distribution of x_0 in the calibration problem depend on the generating density g only through G_*. Thus researchers with different g but the same prior specification will end up with similar inference on x_0. It is also worth noticing that it is not possible to obtain closed analytical expressions for the posterior distribution of x_0, except in the case of the normal distribution. One alternative in such situations is to employ Markov Chain Monte Carlo (MCMC) based approaches (Smith and Roberts, 1993). The approach allows the derivation of precise estimates of the posterior density of interest. To implement the Gibbs sampling algorithm it is required to derive the conditional posterior distribution of one parameter given the others. As shown, such posterior distributions involve well known and easy to simulate distributions.

Proposition 3.3. *Consider the likelihood function given in (3.6) and suppose that* $x_0 \sim N(m_0, w_0)$ *and* $(\beta, \sigma^2) \sim NGI(\mathbf{b}_0, \mathbf{M}_0, n_0, r_0)$ *are independent. Then,*

(i) $\boldsymbol{\beta}|\sigma^2, x_0, \mathbf{v}_*, \mathbf{y}_* \sim N_2(\hat{\boldsymbol{\beta}}_*(\mathbf{v}_*), \sigma^2\mathbf{M}_*(\mathbf{v}_*))$,

where $\hat{\beta}_*(v_*)$ *is given in (3.12) and* $\mathbf{M}_*(\mathbf{v}_*) = ([\mathbf{X}_*'\mathbf{V}_*^{-1}\mathbf{X}_*]^{-1} + \mathbf{M}_0^{-1})^{-1}$.

(ii) $\sigma^2|\beta, x_0, \mathbf{v}_*, \mathbf{y}_* \sim IG(n_o + n, r_1)$, *where*

$$r_1 = \sum_{i=0}^{n} \frac{(y_i - \beta_1 - \beta_2 x_i)^2}{v_i} + r_0;$$

(iii) $x_0|\boldsymbol{\beta}, \sigma^2, \mathbf{v}_*, \mathbf{y}_* \sim N(m_1, w_1)$, *where*

$$m_1 = w_1 \left[\frac{m_0}{w_0} + \frac{\beta_2(y_0 - \beta_1)}{\sigma^2 v_0}\right] \quad \text{and} \quad w_1 = (w_0^{-1} + \frac{\beta_2^2}{\sigma^2 v_0})^{-1}.$$

Proof. The results follow easily from ordinary computations with conjugate linear normal models.

Example 3.1 *The Student-t calibration model.* Considering that $y_i|\beta, \sigma^2, x_i \sim t_1(x_i\beta, \sigma^2; \nu)$, it follows that the density function $f(y_i|\beta, \sigma^2, \nu)$ is representable with mixing measure giving by the inverse gamma distribution, that is, $IG(\nu/2, \nu/2)$. Thus, the generating function can be written as

$$g(u) = \int_0^\infty (2\pi s)^{-n/2} e^{-1/2s} h(s) ds,$$

where $h(s) = (\nu/2)^{\nu/2}(1/s)^{\nu/2+1}e^{-\nu/2s}/\Gamma[\nu/2]$, which is the density function of the inverted gamma distribution. Thus, the assumption of independent Student-t errors is equivalent to considering a hierarchical formulation with heteroscedastic errors, by assuming that the weights $v_1, \ldots, v_n$ are independent with $v_i \sim \nu/\chi^2_\nu$, $i = 1, \ldots, n$. The heteroscedastic approach is considered in Geweke (1993) for estimating the regression line parameters. The posterior distribution of $\mathbf{v}_*$, which is required in the implementation of the Gibbs sampling algorithm in the Student-t model is given by

$$p(\mathbf{v}_*|\beta, \sigma^2, x_0, \mathbf{y}_*) = \prod_{i=o}^{n} p(v_i|\beta, \sigma^2, x_0, \mathbf{y}_*),$$

where

$$v_i|\beta, \sigma^2, x_0, \mathbf{y}_* \overset{ind}{\sim} a(y_i)/\chi^2_{\nu+1},$$

with $a(y_i) = \nu + (y_i - \beta_1 - \beta_2 x_i)^2/\sigma^2$, $i = 1, \ldots, n$.

4 Numerical illustration

In this section, results of a simulation study are presented to illustrate the behavior of the approach considered in the previous sections to estimate x_0, recalling that no explicit expression is obtained for its posterior distribution. Fixed values were considered for $x = (x_1, \ldots, x_n)$ so that $|x_i - x_{i-1}| = 1$ and $\bar{x} = 0$ and a value of x_0 from which simulated samples were obtained for $(y_0, y_1, \ldots, y_n)$, considering a Student-t distribution with ν degrees of freedom with location and scale parameters given, respectively, by βx_i and σ^2. The next tables were obtained through BUGS (Spiegelhalter et al., 1994) to $t = 500$ and $m = 100$ (the sample size) and the values presented represent 0.90 credibility regions and the posterior mean of x_0.

TABLE 4.1 : Credibility regions and posterior mean for several values of β
$(n = 21, x_0 = 0.5, \sigma^2 = 1)$

β	Degrees of freedom			
	2	5	10	30
0.5	(-8.657,17.68)	(-4.942,6.184)	(-3.049, 5.75)	(-2.929,4.174)
	4.461	0.6653	1.332	0.6398
1	(1.129,4.567)	(-3.03,2.23)	(-0.03616,4.16)	(-0.3528,2.5)
	2.851	-0.3512	2.014	1.09
3	(-0.1953,1.821)	(-0.1979,1.454)	(0.2835,1.656)	(-0.5092,0.7082)
	0.8388	0.6199	0.9702	0.09634

Table 4.1 illustrates the fact that by increasing the degrees of freedom parameter the credibility regions become more precise, a result somewhat expected. Moreover, the table also illustrates the fact that there is a strong influence of the values of β on the precision of the credibility regions. For example, for $\nu = 30$ the interval sizes go from 7 (for $\beta = 0.5$) to 1.2 (for $\beta = 3$). This result illustrates the fact that prediction becomes more precise as the regression line becomes steeper. Thus, uncertainty about x_0 decreases as β increases and some caution is recommended when the value of β is small.

TABLE 4.2 : Credibility regions and posterior mean for different sample sizes
$(x_0 = 0.5, \beta = 1, \sigma^2 = 1)$

n	Degrees of Freedom			
	2	5	10	30
7	(-4.96,8.26)	(-1.57,5.39)	(-1.545,1.729)	(-3.48,2.86)
	0.8828	1.541	0.07227	1.200
21	(1.129,4.567)	(-3.03,2.23)	(-0.03616,4.16)	(-0.3528,2.5)
	2.851	-0.3512	2.014	1.09
51	(-1.961,3.884)	(-1.237,2.89)	(-1.739,1.682)	(-2.197,0.7855)
	0.8238	0.8112	-0.05913	-0.7048

TABLE 4.3: Credibility regions and posterior mean for different values of x_0 (n=51 , β=1 , σ^2=1)

x_0	Degrees of Freedom			
	2	5	10	30
center	(-1.961,3.884)	(-1.04,2.863)	(-1.739,1.682)	(-2.197,0.7855)
(0.5)	0.8238	0.8112	-0.05913	-0.7048
border	(22.43,30)	(17.83,22.11)	(20.41,24.01)	(19.87,22.95)
(22.5)	25.97	19.95	22.2	21.42
outside	(26.2,33.77)	(28.37,32.3)	(27.34,31.02)	(27.4,30.52)
(30)	29.72	30.3	29.13	28.98

Tables 4.2 and 4.3 illustrate, respectively, the behavior of the credibility regions for varying sample sizes (n) and different positions for the value x_0 being estimated. As expected, the precision of the estimates increase by increasing the sample sizes. Moreover, as indicated by Table 3, the position of the value being estimated has little influence on the precision of the estimates.

5 Concluding remarks

The present paper considers Bayesian solutions to the linear calibration problem under two types of elliptical models. Under priors non informative for the scale parameter it is shown that the results obtained under the normal model remain valid under dependent elliptical models. Similar results can be obtained if we considerer a nonlinear calibration model. These result can be also extended with minor changes to multivariate calibration problems by considering an appropriate matrix-variate elliptical distribution for the random matrix of error terms. The robustness results derived for dependent elliptical model do not hold under independent elliptical models. Moreover, as in the case of a single covariable, closed form expressions for the posterior distribution of x_0 are difficult to obtain. However, in the representable case, the Gibbs sampler, provides a natural framework for obtaining approximations to the posterior distribution of x_0. As in the dependent case, these results can be extended to the multivariate calibration problem.

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