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Existence, uniqueness, variation-of-constant formula and controllability for linear dynamic equations with Perron Δ-integrals

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In this paper, we investigate the existence and uniqueness of a solution for a linear Volterra-Stieltjes integral equation of the second kind, as well as for a homogeneous and a nonhomogeneous linear dynamic equations on time scales, whose integral forms contain Perron Δ-integrals defined in Banach spaces. We also provide a variation-of-constant formula for a nonhomogeneous linear dynamic equations on time scales and we establish results on controllability for linear dynamic equations. Since we work in the framework of Perron Δ-integrals, we can handle functions not only having many

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discontinuities, but also being highly oscillating. Our results require weaker conditions than those in the literature. We include some examples to illustrate our main results.

Keywords: Perron Δ-integrals; Perron-Stieltjes integrals; existence; uniqueness; variation-of-constant formula; dynamic equation on time scales; Volterra-Stieltjes integral equations; controllability.

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1. Introduction

Calculus on time scales, introduced in 1988 by Stefan Hilger, allows us to describe continuous, discrete and hybrid systems which have several applications (see [2, 24]). One of the main concepts of the time scale theory is the delta derivative, which is a generalization of the classical time derivative in the continuous time and the finite forward difference in the discrete time. As a consequence, differential equations as well as difference equations are naturally accommodated in this theory (see [3, 4]).

The best known results on the existence and uniqueness of a solution for a nonhomogeneous linear dynamic equation of the form

$$x^{\Delta} = a(t)x + f(t) \tag{1.1}$$

and for its corresponding homogeneous equation

$$x^{\Delta} = a(t)x \tag{1.2}$$

on a time scale $\mathbb{T}$, take into account that a is a regressive and rd-continuous $n \times n$-matrix-valued function and $f : \mathbb{T} \to \mathbb{R}^n$ is rd-continuous. Moreover, the integrals appearing in the solutions of the dynamic equations (1.1) and (1.2) are in the sense of the Riemann Δ-integral (see [1, 3, 4, 21], for instance). In the general Banach space setting, [6, Theorem 5.32] ensures the existence and uniqueness of a solution of the nonlinear dynamic integral equation

$$y(t) = y(t_0) + \int_{t_0}^{t} h(y^*(s), s)\Delta s$$

on a time scale $\mathbb{T}$, where $t_0 \in \mathbb{T}$, $[t_0, +\infty)_{\mathbb{T}} = [t_0, +\infty) \cap \mathbb{T}$, the integral on the right-hand side is in the sense of Perron Δ-integral, X is a Banach space and $h : X \times [t_0, +\infty)_{\mathbb{T}} \to X$ is a function satisfying the following conditions:

(H1) for all $t_1, t_2 \in [t_0, +\infty)_{\mathbb{T}}$ and all regulated function $y : [t_0, +\infty)_{\mathbb{T}} \to X$, the Perron Δ-integral $\int_{t_1}^{t_1} h(y(s), s)\Delta s$ exists;

(H2) there exists a locally Perron Δ-integrable function $M : [t_0, +\infty)_{\mathbb{T}} \to \mathbb{R}$ such that

$$\left\| \int_{t_1}^{t_1} h(y(s), s)\Delta s \right\| \leq \int_{t_1}^{t_2} M(s)\Delta s$$

holds for every $t_1, t_2 \in [t_0, +\infty)_{\mathbb{T}}$ and every regulated function $y : [t_0, +\infty)_{\mathbb{T}} \to X$ such that $\sup_{s \in [t_0, +\infty)_{\mathbb{T}}} e^{-(s-t_0)} \|y(s)\| < \infty$.

(H3) there exists a locally Perron Δ-integrable function $L : [t_0, +\infty)_{\mathbb{T}} \to \mathbb{R}$ such that

$$\left\| \int_{t_1}^{t_1} [h(y(s), s) - h(z(s), s)]\Delta s \right\| \leq \|y - z\|_{[t_0,+\infty)_{\mathbb{T}}} \int_{t_1}^{t_2} L(s)\Delta s$$

holds for every $t_1, t_2 \in [t_0, +\infty)_{\mathbb{T}}$ and every regulated functions $y, z : [t_0, +\infty)_{\mathbb{T}} \to X$ such that $\sup_{s\in[t_0,+\infty)_{\mathbb{T}}} e^{-(s-t_0)}\|y(s)\| < \infty$, $\sup_{s\in[t_0,+\infty)_{\mathbb{T}}} e^{-(s-t_0)}\|z(s)\| < \infty$ and $\|y-z\|_{[t_0,+\infty)_{\mathbb{T}}} = \sup_{s\in[t_0,+\infty)_{\mathbb{T}}} e^{-(s-t_0)}\|y(s)-z(s)\|$.

If, in [6, Theorem 5.32], we take $h : X \times [t_0, +\infty)_{\mathbb{T}} \to X$ as $h(x, s) = a(s)x$ for all $(x, s) \in X \times [t_0, +\infty)_{\mathbb{T}}$, where $a : \mathbb{T} \to L(X)$ is a given function and $L(X)$ is the Banach space of continuous linear mappings $T : X \to X$, then, clearly, we have an existence and uniqueness result for the linear dynamic integral equation

$$y(t) = y(t_0) + \int_{t_0}^{t} a(s)y(s)\Delta s,$$

where a and y may be discontinuous functions with highly oscillating behavior. This is due to the choice of the integral.

We point out that the theorems on the existence and uniqueness of solutions presented in [1, 3, 4, 21] are not a particular case of [6, Theorem 5.32]. Indeed, we notice that there exist a special time scale $\mathbb{T}$ and an rd-continuous function $a : \mathbb{T} \to L(\mathbb{R})$ such that condition (H2) is not fulfilled and condition (H1) holds. Consider the time scale $\mathbb{T} = \mathbb{Z}$ and define $a : \mathbb{T} \to L(\mathbb{R})$ by

$$a(t)s = \left(\frac{(-1)^t}{(1+t)}\right) s, \quad (t, s) \in \mathbb{T} \times \mathbb{R}. \tag{1.3}$$

Then, since $\mathbb{Z}$ does not contain any right-dense point, a is rd-continuous and $[0, +\infty)_{\mathbb{T}} = \mathbb{Z}^+ = \{t \in \mathbb{Z}; t \geq 0\}$. Moreover, for $y : \mathbb{Z}^+ \to \mathbb{R}$ being the constant function equal to 1, we have

$$\int_{s}^{t} a(s)y(s)\Delta s = \sum_{k=s}^{t-1} a(k)$$

for all $s, t \in \mathbb{Z}^+$ (see [3, Table 1.3]). Note that for all $t \in \mathbb{T}$, we get

$$\int_{t-1}^{t} a(s)y(s)\Delta s = \frac{(-1)^{t-1}}{t}.$$

Therefore, for all $T \in \mathbb{Z}^+$, we obtain the finite alternating harmonic series

$$\int_{0}^{T} a(s)y(s)\Delta s = \sum_{t=1}^{T} \int_{t-1}^{t} a(s)y(s)\Delta s = \sum_{t=1}^{T} \frac{(-1)^{t-1}}{t}.$$

Because the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ converges, the Riemann Δ-integral $\int_0^N a(s)y(s)\Delta s$ exists for a large $N \in \mathbb{Z}^+$. Assume that condition (H1) holds. Then, for all

$t_1, t_2 \in \mathbb{Z}^+$,

$$\left|\int_{t_1}^{t_2} a(s)y(s)\Delta s\right| \leq \int_{t_1}^{t_2} M(s)\Delta s$$

and for a large $N \in \mathbb{Z}^+$,

$$\int_0^N M(s)\Delta s = \sum_{t=1}^N \int_{t-1}^t M(s)\Delta s \geq \sum_{t=1}^N \left|\int_{t-1}^t a(s)y(s)\Delta s\right| = \sum_{t=1}^N \left|\frac{(-1)^{t-1}}{t}\right| = \sum_{t=1}^N \frac{1}{t},$$

which leads to a contraction, since the series $\sum_{n=1}^\infty \frac{1}{n}$ diverges.

It is well known that the study of the convergence of infinite series is widely used in physics and the position of the source charges are given as functions of time, as in the problem of determining of electrical potential and electrical force exerted by infinite point electrical charges. As an example, if we consider a distribution, where the positive and the negative electrical charges are placed in $1, -3, -5, \ldots$ and $-2, -4, -6, \ldots$, respectively, then the electric potential, in the origin, is given by the series $\frac{q}{4\pi\varepsilon_0}\sum_{n=1}^\infty \frac{(-1)^{n-1}}{n}$, where ε_0 is a constant and q is the modulo of the electrical changes. See [11, Chap. 2]. Therefore, the function a defined by (1.3) has an application in physics. So, it is interesting that a good theory encompasses this situation as well. This is one reason why, in this paper, we sought to suppress condition (H2) from [6, Theorem 5.32] (in the linear case), and we assume that the functions a and f satisfy conditions (H1) and (H3) among others, instead of being rd-continuous. As a matter of fact, we prove that all rd-continuous function satisfies all our hypotheses (see Lemma 4.24) and since we are not considering condition (H2), our main result on the existence and uniqueness of a solution for dynamic equations on time scales is not a particular case of [6, Theorem 5.32].

Motivated by these features, we are interested in proving existence and uniqueness of a solution for linear homogeneous and nonhomogeneous dynamic equations on time scales, where their integral forms contain Banach spaces-valued Perron Δ-integrals. To this end, we prove that the linear Volterra-Stieltjes integral equation of the second kind

$$y(t) = \int_{t_0}^t d[A(s)]y(s) + h(t), \tag{1.4}$$

admits a unique solution, where $J \subset \mathbb{R}$ is an interval containing t_0, X and Y are Banach spaces, $L(X, Y)$ is the Banach space of all continuous operator $T : X \to Y$, $A : J \to L(X, Y)$ and $h : J \to X$ are functions of locally bounded variation and A is left-continuous (see Theorem 4.4). In [28, Propositions 6.2–6.4], the reader can find a proof of the existence of a solution of the linear Volterra-Stieltjes equation (1.4) when $X = Y = \mathbb{R}^n$, $J = [a, b]$ and $A : [a, b] \to L(\mathbb{R}^n)$ is such that A is of bounded variation and satisfies:

(S1) $I - [A(t) - A(t^-)]$ is invertible for all $t \in (t_0, b]$;
(S2) $I + [A(t^+) - A(t)]$ is invertible for all $t \in [a, t_0)$,

where $A(t^-) = \lim_{s\to t^-} A(s)$, $A(t^+) = \lim_{s\to t^+} A(s)$ and $I : \mathbb{R}^n \to \mathbb{R}^n$ is the identity operator.

We notice that condition (S1) is used, in the proof of [28, Proposition 6.4], to guarantee that $I - [A(t) - A(t^-)]$ is one to one. In the case of infinite dimensional spaces, the fact that $I - [A(t) - A(t^-)]$ is invertible does not imply that $I - [A(t) - A(t^-)]$ is one to one. Therefore, we replace condition (S1) by the left-continuity of A. In addition, our result (Theorem 4.4), neither requires condition (S2) nor any additional hypothesis on A.

Concerning a variation-of-constant formula for the nonhomogeneous dynamic equation (1.1), we notice that the result presented in [1, 3, 4, 21] also requires that a is a regressive and rd-continuous $n\times n$-matrix-valued function and $f : \mathbb{T} \to \mathbb{R}^n$ is rd-continuous. Here, we provide a variation-of-constant formula, where the functions a and f take values in an arbitrary Banach space and satisfy conditions (H1) and (H3) among others. We emphasize that Lemma 4.24 in the sequel shows that every rd-continuous functions a and f satisfy all the hypotheses of our results. Therefore, our results on the existence and uniqueness of a solution for a linear nonhomogeneous dynamic equation and for a linear homogeneous dynamic equation, as well as the variation-of-constant formula, generalize the results from [1, 3, 4, 21] (see comments following Lemma 4.24 and Theorem 4.25). Moreover, since we are considering Perron Δ-integrals instead of Riemann Δ-integrals, our integrands may be highly oscillating and have many discontinuities.

In order to give an application of our results, we obtain a characterization of controllability for a nonhomogeneous dynamic equation on time scales. It is worth highlighting that since we are considering Perron Δ-integrable functions defined in Banach spaces, our main results, namely Theorems 5.2 and 5.3, yield Corollary 5.4 which, in turn, encompasses [5, Theorem 4.3; 9, Theorem 2.2; 25, Theorem 1].

Our paper is organized as follows. Section 2 is devoted to presenting basic concepts and results concerning vector integrals. In order to prove the existence and uniqueness of a solution of dynamic equations on time scales, we recall some basic facts of the time scales theory, in Sec. 3, and we present a relation between Perron-Stieltjes integrals and Perron Δ-integrals. In Sec. 4, we prove our main results. In Sec. 4.1, we prove the existence and uniqueness of a solution of a Volterra-Stieltljes integral equation of the second kind and in Sec. 4.2, we consider homogeneous and nonhomogeneous linear dynamic equations on time scales, whose functions are Perron Δ-integrable and we prove the existence and uniqueness of their solutions (see Theorem 4.13). Still in Sec. 4.2, we give a relation between the solutions of our dynamic equations and the solutions of our Volterra-Stieltljes integral equation (see Theorem 4.14), and we present a variation-of-constant formula for nonhomogeneous dynamic equations on time scales (see Theorem 4.22). The main purpose of Sec. 5 is to investigate necessary and sufficient conditions for a dynamic equation on time scales defined in a Banach space with Perron Δ-integrable functions to be approximately controllable and strictly controllable (see Theorem 5.2). Yet in this section, we prove that Theorem 5.2 generalizes some other results on controllability for dynamic equations on time scales defined in $\mathbb{R}^n$ with rd-continuous and regressive functions. In Sec. 6, we provide examples of strictly controllable dynamic systems on time scales taking values in $\mathbb{R}^2$ and in an arbitrary Banach space.

2. Vector Integrals

In this section, we present a brief overview of the theory of nonabsolute integration, due to Jaroslav Kurzweil and Ralph Henstock, for integrands taking values in Banach spaces.

Throughout this section, X, Y and Z are Banach spaces and $L(U,V)$ is the Banach space of continuous linear mapping $T : U \to V$ where $U, V \in \{X, Y, Z\}$. When $U = V$, we write simply $L(U)$ instead of $L(U,V)$.

For $-\infty < a < b < +\infty$, $[a,b]$ denotes the corresponding compact interval of the real line. A finite set $d = \{s_0, s_1, \ldots, s_{|d|}\} \subset [a,b]$ is called a *division* of $[a,b]$, whenever $a = s_0 < s_1 < \cdots < s_{|d|} = b$. The set of all divisions of $[a,b]$ is denoted by $\mathcal{D}[a,b]$.

We recall that a function $f : [a,b] \to X$ is called *step function*, if there exist a division $d = (t_i) \in \mathcal{D}[a,b]$ and elements $c_i \in X$, $i = 1,\ldots,|d|$, such that $f(t) = c_i$ for all $t \in (t_{i-1}, t_i)$, $i = 1,\ldots,|d|$. We denote by $E([a,b],X)$ the set of all step functions $f : [a,b] \to X$.

Definition 2.1. Let $f : [a,b] \to X$ be a function. We define the *variation* of f on $[a,b]$ by

$$\mathrm{var}_a^b f = \sup_{d \in \mathcal{D}[a,b]} \sum_{i=1}^{|d|} \|f(s_i) - f(s_{i-1})\|_X.$$

Moreover, if $\mathrm{var}_a^b f < \infty$, we say that f is a *function of bounded variation on* $[a,b]$ and we denote by $\mathrm{BV}([a,b],X)$ the space of all functions $f : [a,b] \to X$ of bounded variation on $[a,b]$.

In the sequel, we present a more general set of functions, described by Bourbaki (1949–1951) and Dieudonné (1960). See [7, 10].

Definition 2.2. A function $f : [a,b] \to X$ is called *regulated*, if at any point $t \in [a,b]$, it possesses one-sided limits, that is, the limit $\lim_{s\to t^-} f(s) = f(t^-) \in X$ exists for every $t \in (a,b]$ and the limit $\lim_{s \to t^+} f(s) = f(t^+) \in X$ exists for every $t \in [a,b)$. In this case, we write $f \in G([a,b],X)$.

It is well known that $\mathrm{BV}([a,b],X)$ and $G([a,b],X)$ equipped with the variation norm $\|f\|_{\mathrm{BV}} := \|f(a)\| + \mathrm{var}_a^b f$ and with the usual supremum norm $\|f\|_\infty = \sup_{t\in[a,b]} \|f(t)\|$, respectively, are Banach spaces. See [19] for a proof of these facts.

It is not difficult to prove the following inclusions. See, e.g. [19].

Remark 2.3. It is well known that

$$E([a,b],X) \subset G([a,b],X) \quad \text{and} \quad \mathrm{BV}([a,b],X) \subset G([a,b],X).$$

Furthermore, we can extend the previous sets to unbounded intervals as follows.

Definition 2.4. Let $[t_0,+\infty) \subset \mathbb{R}$ be an arbitrary interval and $f : [t_0,+\infty) \to X$ be a function. Then,

(i) f is *locally of bounded variation* on $[t_0,+\infty)$, if $f \in \mathrm{BV}([a,b],X)$ for all $[a,b] \subset [t_0,+\infty)$;
(ii) f belongs to $G([t_0,+\infty),X)$, if $f \in G([a,b],X)$ for all $[a,b] \subset [t_0,+\infty)$;
(iii) f belongs to $G_0([t_0,+\infty),X)$, if $f \in G([t_0,+\infty),X)$ and $\sup_{s\in[t_0,+\infty)} e^{-(s-t_0)} \|f(t)\| < \infty$.

In [18], the reader can find a proof of the fact that $G_0([t_0,+\infty),X)$, endowed with the norm

$$\|f\|_{[t_0,+\infty)} := \sup_{s\in[t_0,+\infty)} e^{-(s-t_0)}\|f(t)\|, \quad f \in G([t_0,+\infty),X)$$

is a Banach space.

The next two results bring important properties of the space $G([a,b],X)$ and they can be found in [19, 15].

Theorem 2.5. *For every function $f \in G([a,b],X)$ (in particular, $f \in \mathrm{BV}([a,b],X)$), there exists a sequence of step functions $\{\varphi_n\}$ which converges uniformly to f, that is,*

$$\sup_{s\in[a,b]} \|f(s)-\varphi_n(s)\| = \|f-\varphi_n\|_\infty \overset{n\to\infty}{\to} 0.$$

Proposition 2.6. *Every regulated function is bounded on compact intervals.*

In what follows, we show how to construct a function in $G_0([t_0,+\infty),X)$ from a given function in $G([a,b],X)$. This fact will be used several times later.

Remark 2.7. Let $[a,b] \subset [t_0,+\infty)$ and $f \in G([a,b],X)$. Define $\widehat{f} : [t_0,+\infty) \to X$ by

$$\widehat{f}(t) = \begin{cases} f(a), & t \in [t_0,a], \\ f(t), & t \in [a,b], \\ f(b), & t \in [b,+\infty). \end{cases}$$

Then, it is not difficult to verify that $\|\widehat{f}\|_{[t_0,+\infty)} \leq \|f\|_\infty$ and by Proposition 2.6, $\widehat{f} \in G_0([t_0,+\infty),X)$. Furthermore, since $\mathrm{BV}([a,b],X) \subset G([a,b],X)$, the same statement holds for functions in $\mathrm{BV}([a,b],X)$.

In the sequel, we state a Helly's choice principle for Banach space-valued functions as presented in [19].

Theorem 2.8. *If $\{f_n\}$ is a uniformly bounded sequence of functions defined from $[a,b]$ into X, then there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ converging pointwisely to a function $f \in \mathrm{BV}([a,b],X)$, that is,*

$$\lim_{n\to\infty} f_{n_k}(t) = f(t),$$

for all $t \in [a,b]$.

In the next lines, we recall the concept of δ-fine tagged division of $[a,b] \subset \mathbb{R}$. See [28].

Definition 2.9. A *tagged division* of $[a,b]$ is any finite collection of point-interval pairs $(\xi_i, [t_{i-1}, t_i])$ with division points $a = t_0 \leq t_1 \leq \cdots \leq t_{|d|} = b$ and *tags* $\xi_i \in [t_{i-1}, t_i]$, $i = 1, 2, \ldots, |d|$. We denote such a tagged division simply by $d = (\xi_i, [t_{i-1}, t_i])$.

Given a positive function $\delta : [a,b] \to (0, +\infty)$, called a *gauge* on $[a,b]$, a tagged division $d = (\xi_i, [t_{i-1}, t_i])$ of $[a,b]$ is called *δ-fine*, whenever

$$[t_{i-1}, t_i] \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i)), \quad i = 1, 2, \ldots, |d|.$$

The following result deals with the existence of at least one δ-fine tagged division. A proof of this fact can be found in [28, Lemma 1.4].

Lemma 2.10 (Cousin lemma). *Given a gauge δ on $[a,b]$, there is a δ-fine tagged division of $[a,b]$.*

Before presenting the definition of Perron-Stieltjes integrable functions, we recall the concept of bilinear triples.

Definition 2.11. A triple of Banach spaces X, Y and Z is *bilinear*, if there exists a bilinear mapping $B : X \times Y \to Z$ such that

$$\|B(x,y)\|_Z \leq \|x\|_X \|y\|_Y \quad \text{for all } (x,y) \in X \times Y.$$

We denote by $\mathcal{B}(X,Y,Z)$ the set of all bilinear triples.

It is not difficult to see that $\mathcal{B} = (L(X,Y), X, Y)$ is a bilinear triple with $B : L(X,Y) \times X \to Y$ defined by $B(F,x) = Fx \in Y$. Another usual bilinear triple is $\mathcal{B} = (L(X,Y), L(Y,Z), L(X,Z))$ with $B : L(X,Y) \times L(Y,Z) \to L(X,Z)$ given by the composition $G \circ F \in L(X,Z)$ of the operators $F \in L(X,Y)$ and $G \in L(Y,Z)$.

In the sequel, we give a definition of the abstract Perron-Stieltjes integral. See, e.g. [17, 22, 23, 29, 28].

Definition 2.12. Let $\mathcal{B} = (X,Y,Z)$ be a bilinear triple. For given functions $f : [a,b] \to X$ and $g : [a,b] \to Y$ and a tagged division $d = (\xi_i, [t_{i-1}, t_i])$ of $[a,b]$, we define Riemannian-type sums

$$S(f, dg, d) = \sum_{i=1}^{|d|} f(\xi_i)(g(t_i) - g(t_{i-1})) \quad \text{and}$$

$$S(df, g, d) = \sum_{i=1}^{|d|} [f(t_i) - f(t_{i-1})]g(\xi_i).$$

We say that $I \in Z$ is the *Perron-Stieltjes integral of f with respect to g*, if for every $\varepsilon > 0$, there exists a gauge δ on $[a, b]$ such that

$$\|S(f, dg, d) - I\| < \varepsilon$$

for all δ-fine tagged divisions $d = (\xi_i, [t_{i-1}, t_i])$ of $[a, b]$. Similarly, $J \in Z$ is the *Perron-Stieltjes integral of g with respect to f*, if for every $\varepsilon > 0$, there exists a gauge δ on $[a, b]$ such that

$$\|S(df, g, d) - J\| < \varepsilon$$

for all δ-fine tagged divisions $d = (\xi_i, [t_{i-1}, t_i])$ of $[a, b]$. In these cases, we write

$$I = \int_a^b f(s)dg(s) \quad \text{and} \quad J = \int_a^b d[f(s)]g(s).$$

Moreover, we use the conventions $\int_a^b f(s)dg(s) = -\int_b^a f(s)dg(s)$ and $\int_a^b d[f(s)] g(s) = -\int_b^a d[f(s)]g(s)$, whenever $a \leq b$. In addition, when $Y = \mathbb{R}$ and g is the identity function, the Perron-Stieltjes integral of f with respect to g is called Perron integral.

Remark 2.13. The integrals defined above are linear and additive with respect to intervals. For more details, see [29].

In what follows, we exhibit the uniform convergence theorem for Perron-Stieltjes integrals. The reader may consult [29, Theorem 11] for a proof.

Theorem 2.14. *Let $\mathcal{B} = (X, Y, Z)$ be a bilinear triple and $f : [a, b] \to X$ be a function. If $x : [a, b] \to Y$ is the uniform limit of a sequence $\{\varphi_n\}$ of step functions, $\varphi_n : [a, b] \to Y$, such that the Perron-Stieljtes integral $\int_a^b d[f(s)]\varphi_n(s)$ exists for all $n \in \mathbb{N}$, then the integral Perron-Stieljtes integral $\int_a^b d[f(s)]x(s)$ exists and*

$$\int_a^b d[f(s)]x(s) = \lim_{n\to\infty} \int_a^b d[f(s)]\varphi_n(s).$$

The next result shows that the indefinite Perron-Stieltjes integral is not continuous in general. See [29, Theorem 17 and Remark 18] for a proof.

Theorem 2.15. *Assume that $\mathcal{B} = (X, Y, Z)$ is a bilinear triple and the functions $f : [a, b] \to X$ and $g : [a, b] \to Y$ are such that the integral Perron-Stieljtes integral*

$\int_a^b d[f(s)]g(s)$ exists. If $c \in [a,b]$, then

$$\lim_{s\to c^-}\left(\int_a^s d[f(s)]g(s) - f(s)g(c) + f(c)g(c)\right) = \int_a^c d[f(s)]g(s)$$

and

$$\lim_{s\to c^+}\left(\int_s^b d[f(s)]g(s) + f(s)g(c) - f(c)g(c)\right) = \int_c^b d[f(s)]g(s).$$

In the sequel, we present some properties of the indefinite Perron-Stieltjes integral. Their proofs follow as in [28, Theorem 1.14] with obvious adaptations for the case of Banach space-valued functions.

Theorem 2.16. *Consider functions $f : [a,b] \to X$ and $g \in G([a,b],\mathbb{R})$ such that the Perron-Stieltjes integral $\int_a^b f(s)dg(s)$ exists. Then, the functions*

$$h(t) = \int_a^t f(s)dg(s), \quad t \in [a,b] \quad \text{and} \quad k(t) = \int_t^b f(s)dg(s), \quad t \in [a,b]$$

are regulated on $[a,b]$, and satisfies

$$h(t^+) = h(t) + f(t)\Delta^+ g(t), \quad k(t^+) = k(t) - f(t)\Delta^+ g(t), \quad t \in [a,b),$$

$$h(t^-) = h(t) - f(t)\Delta^- g(t), \quad k(t^-) = k(t) + f(t)\Delta^- g(t), \quad t \in (a,b],$$

where $\Delta^+ g(t) = g(t^+) - g(t)$ and $\Delta^- g(t) = g(t) - g(t^-)$.

The next three results were borrowed from [29].

Proposition 2.17. *If $f \in G([a,b],X)$ and $g : [a,b] \to \mathbb{R}$ is nondecreasing, then the Perron-Stieltjes integral $\int_a^b f(s)dg(s)$ exists and*

$$\left\|\int_a^b f(s)dg(s)\right\| \le \|f\|_\infty [g(b) - g(a)],$$

where $\|f\|_\infty = \sup_{s\in[a,b]} \|f(s)\|$.

Proposition 2.18. *Assume that $\mathcal{B} = (X,Y,Z)$ is a bilinear triple and $f : [a,b] \to X$ is such that $\mathrm{var}_a^b f < \infty$. If $g \in G([a,b],Y)$, then the Perron-Stieltjes integral $\int_a^b d[f(s)]g(s)$ exists.*

Proposition 2.19. *Assume that $\mathcal{B} = (X,Y,Z)$ is a bilinear triple. If the functions $f : [a,b] \to X$ and $g \in G([a,b],Y)$ (in particular, $g \in \mathrm{BV}([a,b],Y)$) are such that the Perron-Stieltjes integral $\int_a^b d[f(s)]g(s)$ exists and $\mathrm{var}_a^b f < \infty$, then*

$$\left\|\int_a^b d[f(s)]g(s)\right\| \le \mathrm{var}_a^b f \|g\|_\infty,$$

where $\|g\|_\infty = \sup_{s\in[a,b]} \|g(s)\|$.

We end this section by presenting an integration-by-parts theorem for Perron-Stieltjes integrals. Its proof is analogous to that of [20, Theorem 1.15] and therefore, we omit it here.

Theorem 2.20. *Let $f : [a,b] \to X$ and $g : [a,b] \to \mathbb{R}$ be such that the Perron-Stieltjes integral $\int_a^b f(s)dg(s)$ exists. If $\widetilde{f} : [a,b] \to X$ is given by $\widetilde{f}(t) = \int_a^t f(s)dg(s)$ and $\alpha \in \mathrm{BV}([a,b], L(X))$, then the Perron-Stieltjes integral $\int_a^b d[\alpha(s)]\widetilde{f}(s)$ exists and*

$$\int_a^b d[\alpha(s)]\widetilde{f}(s) = \alpha(b)\widetilde{f}(b) - \alpha(a)\widetilde{f}(a) - \int_a^b \alpha(s)f(s)dg(s).$$

3. The Time Scales Calculus

In this section, we present fundamental results of the theory of Δ-integral. We refer the interested reader to [3, 4] for basic concepts of the theory of time scale calculus. At first, we recall the definition of a special division of $[a,b]_{\mathbb{T}}$. See [26].

Definition 3.1. Let $\mathbb{T}$ be a time scale. A *division* of $[a,b]_{\mathbb{T}}$ is a finite sequence of points $d_{\mathbb{T}} = \{s_0, s_1, \ldots, s_{|d|}\} \subset [a,b]_{\mathbb{T}}$, where $a = s_0 < s_1 < \cdots < s_{|d|} = b$. We say that $d_{\mathbb{T}}$ is a *tagged division* of $[a,b]_{\mathbb{T}}$, whenever

$$a = s_0 \leq \tau_1 \leq s_1 \leq \cdots \leq s_{|d|-1} \leq \tau_{|d|} \leq s_{|d|} = b$$

with $s_i > s_{i-1}$, $s_i \in \mathbb{T}$ and $\tau_i \in [s_{i-1}, s_i]_{\mathbb{T}}$, for $1 \leq i \leq |d|$. We denote such tagged division by $d_{\mathbb{T}} = (\tau_i, [s_{i-1}, s_i]_{\mathbb{T}})$, where τ_i is the associated tag point in $[s_{i-1}, s_i]_{\mathbb{T}}$.

We say that $\delta = (\delta_L, \delta_R)$ is a Δ-gauge of $[a,b]_{\mathbb{T}}$, provided $\delta_L(t) > 0$ on $(a,b]_{\mathbb{T}}$, $\delta_R(t) > 0$ on $[a,b)_{\mathbb{T}}$, $\delta_L(a) \geq 0$, $\delta_R(b) \geq 0$ and $\delta_R(t) \geq \mu(t)$ for all $t \in [a,b)_{\mathbb{T}}$.

If δ is a Δ-gauge on $[a,b]_{\mathbb{T}}$, then a tagged division $d_{\mathbb{T}} = (\tau_i, [s_{i-1}, s_i]_{\mathbb{T}})$ is called δ-*fine*, whenever

$$\tau_i - \delta_L(\tau_i) \leq s_{i-1} < s_i \leq \tau_i + \delta_R(\tau_i), \quad 1 \leq i \leq |d|.$$

Note that similarly as in real line case, that is, when $\mathbb{T} = \mathbb{R}$, we can ensure the existence of at least one δ-fine tagged divisions $d_{\mathbb{T}}$ of $[a,b]_{\mathbb{T}}$. As a matter of fact, this result is a generalization of the Cousin lemma (see Lemma 2.10) for a Δ-gauge of a time scale interval and it can be found in [26, Lemma 1.9].

In what follows, we present a definition of Δ-integral of a function $f : [a,b]_{\mathbb{T}} \to X$ by means of δ-fine tagged divisions. We refer to this integral as Perron Δ-integral. See also [26].

Definition 3.2. Let $\mathbb{T}$ be a time scale. We say that $f : [a,b]_{\mathbb{T}} \to X$ is *Perron Δ-integrable* on $[a,b]_{\mathbb{T}}$, if there is an element $I \in X$ such that given $\varepsilon > 0$, there exists a Δ-gauge δ on $[a,b]_{\mathbb{T}}$ for which

$$\left\| \sum_{i=1}^{|d|} f(\tau_i)(s_i - s_{i-1}) - I \right\| < \varepsilon,$$

for all δ-fine tagged division $d_{\mathbb{T}} = (\tau_i, [s_{i-1}, s_i]_{\mathbb{T}})$ of $[a,b]_{\mathbb{T}}$. In this case, we write $I = \int_a^b f(t)\Delta t$ which is called the *Perron Δ-integral* of f. Moreover, a function

$f : \mathbb{T} \to X$ is said to be *locally Perron* Δ*-integrable*, if the Perron Δ-integral $\int_a^b f(t)\Delta t$ exists for all $[a,b]_{\mathbb{T}} \subset \mathbb{T}$.

As in [14], in order to present a relation between Perron-Stieltjes integrals and Perron Δ-integrals, we define an *extension* of a given time scale $\mathbb{T}$ by

$$\mathbb{T}^* = \begin{cases} (-\infty, \sup \mathbb{T}] & \text{if } \sup \mathbb{T} < \infty, \\ (-\infty, +\infty) & \text{otherwise.} \end{cases}$$

Moreover, given a function $f : \mathbb{T} \to X$, an extension $f^* : \mathbb{T}^* \to X$ is defined by

$$f^*(t) = f(t^*) \quad \text{for all } t \in \mathbb{T}^*,$$

where t is a real number such that $t \leq \sup \mathbb{T}$ and $t^* = \inf\{s \in \mathbb{T}; s \geq t\}$. Since $\mathbb{T}$ is a closed set, it is clear that $t^* \in \mathbb{T}$ and therefore, f^* is well defined.

In what follows, we present a result which shows how f^* inherits some properties of f. Its proof, when $X = \mathbb{R}^n$, can be found in [27, Lemma 4] and for arbitrary Banach space-valued functions, see [6, Lemma 3.20].

Lemma 3.3. *Let $\mathbb{T}$ be a time scale, $f : \mathbb{T} \to X$ be a function and $f^* : \mathbb{T}^* \to X$ be the extension of f. Then, the following statements are true*:

(i) *If f is a regulated function on $\mathbb{T}$, then f^* is also regulated on $\mathbb{T}^*$.*
(ii) *If f is left-continuous on $\mathbb{T}$, then f^* is left-continuous on $\mathbb{T}^*$.*
(iii) *If f is right-continuous on $\mathbb{T}$, then f^* is right-continuous at right-dense points of $\mathbb{T}$.*

The next result shows that the existence of Perron Δ-integrals and Perron-Stieltjes integrals are related. For a proof of this fact, see [6, Theorem 3.24] and for the case when $X = \mathbb{R}^n$, see [14, Theorem 4.2].

Theorem 3.4. *Let $\mathbb{T}$ be a time scale and $f : [a,b]_{\mathbb{T}} \to X$ be a function. Define $g(t) = t^*$ for all $t \in [a,b]$. Then, the Perron Δ-integral $\int_a^b f(t)\Delta t$ exists if and only if the Perron-Stieltjes integral $\int_a^b f^*(t)dg(t)$ exists. In this case, both integrals have the same value.*

Remark 3.5. By Remark 2.13 and Theorem 3.4, we conclude that the Perron Δ-integral is linear and additive with respect to adjacent intervals. Moreover, if $f : \mathbb{T} \to X$ is a Perron Δ-integrable function on a time scale $\mathbb{T}$, then $\int_a^b f(s)\Delta s = -\int_b^a f(s)\Delta s$, whenever $a, b \in \mathbb{T}$ and $a \leq b$.

The next statement will be crucial to our proposes. Its proof can be found in [14, Theorem 4.1], for the case where $X = \mathbb{R}^n$, and in [6, Theorem 3.15] for Banach space-valued functions.

Theorem 3.6. *Let $\mathbb{T}$ be a time scale and $f : \mathbb{T} \to X$ be a function such that the Perron Δ-integral $\int_a^b f(t)\Delta t$ exists for all $a, b \in \mathbb{T}$, $a < b$. Choose an arbitrary $a \in \mathbb{T}$*

and define

$$F_1(t) = \int_a^t f(s)\Delta(s), \quad t \in \mathbb{T},$$

$$F_2(t) = \int_a^t f^*(s)dg(s), \quad t \in \mathbb{T}^*,$$

where $g(s) = s^*$ *for every* $s \in \mathbb{T}^*$. *Then,* $F_2 = F_1^*$.

We end this section by presenting a result which gives a sufficient condition for the existence of a Perron Δ-integral.

Proposition 3.7. *Let* $\mathbb{T}$ *be a time scale and* $f : \mathbb{T} \to X$ *be a regulated function. Then, the Perron* Δ*-integral* $\int_a^b f(s)\Delta s$ *exists for all* $a, b \in \mathbb{T}$ *and*

$$\left\| \int_a^b f(s)\Delta s \right\| \leq \|f\|_{[a,b]_{\mathbb{T}}}[g(b) - g(a)], \tag{3.1}$$

where $g(s) = s^*$ *for all* $s \in \mathbb{T}^*$ *and* $\|f\|_{[a,b]_{\mathbb{T}}} = \sup_{s\in[a,b]_{\mathbb{T}}} \|f(s)\|$.

Proof. Let $f : \mathbb{T} \to X$ be a regulated function and define $g(s) = s^*$ for all $s \in \mathbb{T}^*$. Then, it is clear that g is nondecreasing and by Lemma 3.3, $f : \mathbb{T}^* \to X$ is also regulated. Moreover, by Proposition 2.17, the Perron-Stieltjes integral $\int_a^b f^*(s)dg(s)$ exists for all $a, b \in \mathbb{T}$ and by Theorem 3.4, the Perron Δ-integral $\int_a^b f(s)\Delta s$ exists for all $a, b \in \mathbb{T}$.

Let us prove (3.1). Since $\int_a^b f(s)\Delta s = -\int_b^a f(s)\Delta s$ for all $a, b \in \mathbb{T}$ (see Remark 3.5), we may assume, without loss of generality, that $a < b$. Let ε be given. Once the Perron Δ-integral $\int_a^b f(s)\Delta s$ exists, there is a Δ-gauge δ on $[a, b]_{\mathbb{T}}$ such that

$$\left\| \sum_{i=1}^{|d|} f(\tau_i)(s_i - s_{i-1}) - \int_a^b f(s)\Delta s \right\| < \varepsilon,$$

provided $d_{\mathbb{T}} = (\tau_i, [s_{i-1}, s_i]_{\mathbb{T}})$ is a δ-fine tagged division of $[a, b]_{\mathbb{T}}$. Then,

$$\begin{aligned} \left\| \int_a^b f(s)\Delta s \right\| &\leq \left\| \int_a^b f(s)\Delta s - \sum_{i=1}^{|d|} f(\tau_i)(s_i - s_{i-1}) \right\| + \left\| \sum_{i=1}^{|d|} f(\tau_i)(s_i - s_{i-1}) \right\| \\ &\leq \varepsilon + \left\| \sum_{i=1}^{|d|} f(\tau_i)(s_i - s_{i-1}) \right\|. \end{aligned} \tag{3.2}$$

Furthermore, since f^* is regulated, by Proposition 2.6, we have

$$f(s) \leq \sup_{s\in[a,b]_{\mathbb{T}}} \|f(s)\| \leq \sup_{s\in[a,b]} \|f^*(s)\| < \infty \quad \text{for all } s \in [a, b]_{\mathbb{T}}. \tag{3.3}$$

Equation (3.3) together with the fact that $g|_{\mathbb{T}}$ is the identity function yield

$$\begin{aligned}\left\|\sum_{i=1}^{|d|} f(\tau_i)(s_i - s_{i-1})\right\| &\leq \sum_{i=1}^{|d|} \|f(\tau_i)(s_i - s_{i-1})\| \\ &\leq \sup_{s\in[a,b]_{\mathbb{T}}} \|f(s)\| \sum_{i=1}^{|d|} |(s_i - s_{i-1})| \\ &\overset{s_i > s_{i-1}}{=} \sup_{s\in[a,b]_{\mathbb{T}}} \|f(s)\| \sum_{i=1}^{|d|} g(s_i) - g(s_{i-1}) \\ &= \|f\|_{[a,b]_{\mathbb{T}}}[g(b) - g(a)]. \end{aligned} \tag{3.4}$$

Finally, by (3.2) and (3.4), we obtain

$$\left\|\int_a^b f(s)\Delta s\right\| \leq \varepsilon + \|f\|_{[a,b]_{\mathbb{T}}}[g(b) - g(a)]$$

and the statement is proved once ε can be made arbitrarily small. □

4. Main Results

In this section, we are interested in proving existence and uniqueness of solutions of homogeneous and nonhomogeneous dynamic equations on time scales, whose integral forms contain Perron Δ-integrals. To this end, we organize this section into two subsections. In the first subsection, we prove the existence and uniqueness of solutions of a linear Volterra-Stieltjes integral equation of the second kind and in the second subsection, we provide conditions under which homogeneous and nonhomogeneous dynamic equations on time scales possess solutions. Furthermore, we prove that the solutions of our dynamic equations and the solutions of our Volterra-Stieltjes integral equation are related, and we state a version of the variation-of-constant formula for nonhomogeneous dynamic equations on time scales. The results presented here generalize those presented in [3].

4.1. *Volterra-Stieltjes integral equations*

In this subsection, we consider X and Y Banach spaces, $L(X,Y)$ the Banach space of continuous linear mappings $T : X \to Y$, $J \subset \mathbb{R}$ an interval and we investigate conditions under which the linear Volterra-Stieltjes integral equation of the second kind

$$y(t) = \int_{t_0}^{t} d[A(s)]y(s) + h(t), \quad t \in [t_0, v], \tag{4.1}$$

admits solutions, where $[t_0, v] \subset J$, $A : J \to L(X,Y)$, $h : J \to X$ and the integral in (4.1) is in the sense of Perron-Stieltjes.

In what follows, we prove that if A is locally of bounded variation, then the Perron-Stieltjes integral of y with respect to A exists, whenever y is a regulated function defined on a closed interval.

Theorem 4.1. *Let $v \in J$ and $A : J \to L(X, Y)$ be locally of bounded variation, that is, $\mathrm{var}_a^b A < \infty$ for all $[a, b] \subset J$. If $y \in G([t_0, v], X)$, then the Perron-Stieltjes integral*

$$\int_{t_0}^{v} d[A(s)]y(s)$$

exists.

Proof. Consider $\widehat{X} = L(X, Y)$, $\widehat{Y} = X$ and $\widehat{Z} = Y$. Then, $\widehat{X}, \widehat{Y}$ and $\widehat{Z}$ are Banach spaces and $B : \widehat{X} \times \widehat{Y} \to \widehat{Z}$, defined by $B(F, x) = Fx$ is a bilinear triple. Moreover, by hypothesis, $\mathrm{var}_{t_0}^v A < \infty$. Therefore, by Proposition 2.18, the Perron-Stieltjes integral $\int_{t_0}^{v} d[A(s)]y(s)$ exists for all $y \in G([t_0, v], X)$. □

In order to prove the existence of solutions of the linear Volterra-Stieltjes integral equation (4.1), we prove the following auxiliary result.

Theorem 4.2. *Let $v \in J$ and $A : J \to L(X, Y)$ be locally of bounded variation. Then, the mapping $T : \mathrm{BV}([t_0, v], X) \to \mathrm{BV}([t_0, v], X)$ given by*

$$Ty(t) = \int_{t_0}^{t} d[A(s)]y(s), \quad t \in [t_0, v]$$

is well-defined and it is a bounded compact linear operator on $\mathrm{BV}([t_0, v], X)$.

Proof. At first, we notice that if $v \in J$ and $y \in \mathrm{BV}([t_0, v], X)$, then the Perron-Stieltjes integral $\int_{t_0}^{t} d[A(s)]y(s)$ exists for all $t \in [t_0, v]$ (see Remark 2.3 and Theorem 4.1). Let us prove that $Ty \in \mathrm{BV}([t_0, v], X)$. Consider $t_1, t_2 \in [t_0, v]$. By Proposition 2.19, we have

$$\begin{aligned} \|Ty(t_2) - Ty(t_1)\| &= \left\| \int_{t_0}^{t_2} d[A(s)]y(s) - \int_{t_0}^{t_1} d[A(s)]y(s) \right\| \\ &= \left\| \int_{t_1}^{t_2} d[A(s)]y(s) \right\| \leq \mathrm{var}_{t_1}^{t_2} A \|y\|_\infty. \end{aligned}$$

Thus, for every division $d = (t_i)$ of $[t_0, v]$, we have

$$\sum_{i=1}^{|d|} \|Ty(t_i) - Ty(t_{i-1})\| \leq \mathrm{var}_{t_0}^v A \|y\|_\infty,$$

which implies that

$$\mathrm{var}_{t_0}^v Ty \leq \mathrm{var}_{t_0}^v A \|y\|_\infty \leq \mathrm{var}_{t_0}^v A(\|y(t_0)\| + \mathrm{var}_{t_0}^v y) = \mathrm{var}_{t_0}^v A \|y\|_{\mathrm{BV}} < \infty, \tag{4.2}$$

once

$$\|y(s)\| \leq \|y(s) - y(t_0)\| + \|y(t_0)\| \leq \mathrm{var}_{t_0}^{s} y + \|y(t_0)\|$$
$$\leq \mathrm{var}_{t_0}^{v} y + \|y(t_0)\| \quad \text{for all } s \in [t_0, v].$$

Therefore, $Ty : [t_0, v] \to X$ is of bounded variation on $[t_0, v]$ for all $y \in \mathrm{BV}([t_0, v], X)$ and consequently, T is well defined. Moreover, by the linearity of the Perron-Stieltjes integral (see Remark 2.13), T is linear and it is bounded since, $\|Ty(t_0)\| = 0$ (see Remark 3.5) and by (4.2),

$$\|Ty\|_{\mathrm{BV}} = \|Ty(t_0)\| + \mathrm{var}_{t_0}^{v} Ty < \mathrm{var}_{t_0}^{v} A \|y\|_{\mathrm{BV}}.$$

It remains to show that T is compact. Let $\{y_k\}$ be a uniformly bounded sequence in $\mathrm{BV}([t_0, v], X)$, that is, there exists $C > 0$ such that $\|y_k\| \leq C$ for all $k \in \mathbb{N}$. By Helly's choice theorem (see Theorem 2.8), the sequence $\{y_k\}$ contains a subsequence $\{y_{k_l}\}$ which converges pointwisely to a function $y_0 \in \mathrm{BV}([t_0, v], X)$. To simplify the notation, we denote the subsequence $\{y_{k_l}\}$ by $\{y_k\}$. For all $k \in \mathbb{N}$, define $z_k : [t_0, v] \to X$ by

$$z_k(s) = y_k(s) - y_0(s), \quad s \in [t_0, v].$$

It is clear that $z_k \in \mathrm{BV}([t_0, v], X)$ and

$$\lim_{k \to \infty} z_k(s) = 0, \quad s \in [t_0, v]. \tag{4.3}$$

Furthermore, define

$$z(t) = \int_{t_0}^{t} d[A(s)] y_0(s), \quad s \in [t_0, v].$$

Then, evidently $z \in \mathrm{BV}([t_0, v], X)$ and for all $t_1, t_2 \in [t_0, v]$, we have

$$\begin{aligned}
&\|Ty_k(t_2) - z(t_2) - [Ty_k(t_1) - z(t_1)]\| \\
&= \left\| \int_{t_1}^{t_2} d[A(s)] y_k(s) - \int_{t_1}^{t_2} d[A(s)] y_0(s) \right\| \\
&= \left\| \int_{t_1}^{t_2} d[A(s)] (y_k(s) - y_0(s)) \right\| \\
&= \left\| \int_{t_1}^{t_2} d[A(s)] z_k(s) \right\| \\
&\overset{\text{Proposition } 2.19}{\leq} \mathrm{var}_{t_1}^{t_2} A \|z_k\|_\infty,
\end{aligned}$$

which implies that

$$\mathrm{var}_{t_0}^{v} (Ty_k - z) \leq \mathrm{var}_{t_0}^{v} A \|z_k\|_\infty. \tag{4.4}$$

By Eqs. (4.3) and (4.4), we can conclude

$$\lim_{k\to\infty} \|Ty_k - z\|_{\mathrm{BV}} = \lim_{k\to\infty} \|Ty_k(t_0) - z(t_0)\| + \lim_{k\to\infty} \mathrm{var}_{t_0}^{v}(Ty_k - z)$$
$$= \lim_{k\to\infty} \mathrm{var}_{t_0}^{v}(Ty_k - z) = 0.$$

Therefore, $\{Ty_k\}$ converges to z in $\mathrm{BV}([t_0, v], X)$ and the operator T is compact.

□

The next result is known as the Fredholm alternative for linear operators defined in Banach spaces and will be essential in the proof of the existence of solutions of the linear Volterra-Stieltjes integral equation (4.1). For a proof of it, see [12, p. 609].

Theorem 4.3 (Fredholm alternative). *Let X be a Banach space, $T \in L(X)$, $x \in X$ and $\lambda \in \mathbb{R}$ with $\lambda \neq 0$. If T is compact, then the equation $Tx - \lambda x = y$ has a solution for every $y \in X$ if and only if the equation $Tx = \lambda x$ has only the trivial solution.*

In the sequel, we present the main result of this subsection.

Theorem 4.4. *Let $v \in J$ and $A : J \to L(X, Y)$ be locally of bounded variation on J and left-continuous. Then, the linear Volterra-Stieltjes integral equation* (4.1) *admits a unique solution in* $\mathrm{BV}([t_0, v], X)$ *for each given* $h \in \mathrm{BV}([t_0, v], X)$.

Proof. At first, we prove that the homogeneous linear Volterra-Stieltjes integral equation

$$y(t) = \int_{t_0}^{t} d[A(s)]y(s), \quad t \in [t_0, v] \tag{4.5}$$

admits only the trivial solution in $\mathrm{BV}([t_0, v], X)$.

Let $x : [t_0, v] \to X$ be a solution of (4.5) and define

$$C = \{t \in [t_0, v]; \|x(t)\| = 0\}.$$

Then, C is a nonempty set, since $t_0 \in C$ (see Remark 3.5), and it is upper bounded. Let $c = \sup C$. Note that $\|x(c)\| = 0$. Indeed, by Theorem 2.15, we have

$$\|x(c)\| = \left\| \int_{t_0}^{c} d[A(s)]x(s) \right\| \leq \lim_{\tau\to c^-} \left\| \int_{t_0}^{\tau} d[A(s)]x(s) \right\| + \|A(c) - A(c^-)\| \|x(c)\| = 0,$$

once $\|x(t)\| = 0$ for all $t \in [t_0, c)$ and A is left-continuous.

We claim that $c = v$. Assume that $c < v$ and let us consider two cases.

Case 1. A is right-continuous at c.

In this case, define $V : [t_0, v] \to X$ by

$$V(t) = \mathrm{var}_{t_0}^{t} A \quad \text{for all } t \in [t_0, v].$$

Thus, by hypothesis, V is well defined. It is also nondecreasing and satisfies

$$V(t) - V(s) = \mathrm{var}_{s}^{t} A \quad \text{for all } s, \quad t \in [t_0, v], \quad s \le t.$$

Since A is continuous at c, V is also continuous at c and hence, there exists $\overline{t} \in (c, v]$ such that

$$V(\overline{t}) - V(c) = \mathrm{var}_{c}^{\overline{t}} A < \frac{1}{2}. \tag{4.6}$$

By Proposition 2.19, for all $s_1, s_2 \in [c, \overline{t}]$, $s_1 < s_2$, we have

$$\|x(s_2) - x(s_1)\| = \left\| \int_{s_1}^{s_2} d[A(s)]x(s) \right\| \le \mathrm{var}_{s_1}^{s_2} A \sup_{s \in [s_1, s_2]} \|x(s)\|. \tag{4.7}$$

On the other hand,

$$\|x(s)\| \le \|x(s) - x(c)\| + \|x(c)\| \le \mathrm{var}_{c}^{\overline{t}} x \quad \text{for all } s \in [s_1, s_2] \quad \text{and}$$

$$\mathrm{var}_{s_1}^{s_2} A = \mathrm{var}_{c}^{\overline{t}} A - \mathrm{var}_{s_1}^{c} A - \mathrm{var}_{s_2}^{\overline{t}} A < \frac{1}{2}.$$

Then, (4.7) becomes

$$\|x(s_2) - x(s_1)\| \le \frac{1}{2} \|x\|_{\mathrm{BV}([c,\overline{t}],X)},$$

which implies

$$\|x\|_{\mathrm{BV}([c,\overline{t}],X)} \le \frac{1}{2} \|x\|_{\mathrm{BV}([c,\overline{t}],X)} \quad \text{and} \quad \|x\|_{\mathrm{BV}([c,\overline{t}],X)} = 0.$$

Consequently, $\|x(t)\| = 0$ for all $t \in [c, \overline{t}]$, which contradicts the fact that $c = \sup C$. Therefore, $c = v$ and $x(t) = 0$ for all $t \in [t_0, v]$.

Case 2. A is not right-continuous at c.

Define $\widetilde{A} : [t_0, v] \to X$ by

$$\widetilde{A}(t) = \begin{cases} A(t), & t \in [t_0, c], \\ A(t) - A(c^+) + A(c), & t \in (c, v]. \end{cases}$$

Then, $\widetilde{A}$ is continuous at c. By Theorem 2.15 and by the fact that $\|x(c)\| = 0$ for all $t \ge c$, we have

$$\begin{aligned} \left\| \int_c^t d[A(s)]x(s) \right\| &\le \left\| \lim_{\tau \to c^+} \int_\tau^t d[A(s)]x(s) \right\| + \|A(c^+) - A(c)\| \|x(c)\| \\ &\le \left\| \lim_{\tau \to c^+} \int_\tau^t d[A(s) - A(c^+) + A(c)]x(s) \right\| \\ &\quad + \left\| \lim_{\tau \to c^+} \int_\tau^t d[A(c^+) - A(c)]x(s) \right\| \\ &= \left\| \lim_{\tau \to c^+} \int_\tau^t d[\widetilde{A}(s)]x(s) \right\| = \left\| \int_c^t d[\widetilde{A}(s)]x(s) \right\|. \end{aligned} \tag{4.8}$$

Define $\widetilde{V} : [t_0, v] \to X$ by

$$\widetilde{V}(t) = \mathrm{var}_{t_0}^t \widetilde{A} \quad \text{for all } t \in [t_0, v].$$

Notice that

$$\mathrm{var}_c^t \widetilde{A} \leq \mathrm{var}_c^t A \quad \text{for all } t \geq c$$

and hence, $\widetilde{V}$ is well defined. Moreover, $\widetilde{V}$ is nondecreasing and satisfies

$$\widetilde{V}(t) - \widetilde{V}(s) = \mathrm{var}_s^t \widetilde{A} \quad \text{for all } s, \quad t \in [t_0, v] s \leq t.$$

Once $\widetilde{A}$ is continuous at c, $\widetilde{V}$ is also continuous at c. Then, there exists $\overline{t} \in (c, v]$ such that

$$\widetilde{V}(\overline{t}) - \widetilde{V}(c) = \mathrm{var}_c^{\overline{t}} \widetilde{A} < \frac{1}{2}. \tag{4.9}$$

By Proposition 2.19 and Eq. (4.8), for all $s_1, s_2 \in [c, \overline{t}]$, $s_1 < s_2$, we have

$$\|x(s_2) - x(s_1)\| = \left\| \int_{s_1}^{s_2} d[A(s)]x(s) \right\| \leq \left\| \int_{s_1}^{s_2} d[\widetilde{A}(s)]x(s) \right\| \leq \mathrm{var}_{s_1}^{s_2} \widetilde{A} \sup_{s \in [s_1, s_2]} \|x(s)\|$$

and analogously to Case 1, we conclude that $\|x(t)\| = 0$ for all $t \in [c, v]$. Therefore, $c = v$.

From Cases 1 and 2, the only solution of (4.5) is the trivial one.

Now, since Eqs. (4.1) and (4.5) are equivalent to

$$y - Ty = h \quad \text{and} \quad y - Ty = 0, \tag{4.10}$$

respectively, and T is a compact operator (see Theorem 4.2), the existence of the solutions of the Volterra-Stieltjes integral equation (4.1) is guaranteed by Theorem 4.3 with $\lambda = 1$. In order to prove the uniqueness, assume that there exist two solutions $y_1 : [t_0, v] \to X$ and $y_2 : [t_0, v] \to X$ of (4.1) and define $z(t) = y_1(t) - y_2(t)$ for all $t \in [t_0, v]$. Then,

$$\begin{aligned} z(t) &= \int_{t_0}^t d[A(s)]y_1(s) - \int_{t_0}^t d[A(s)]y_2(s) = \int_{t_0}^t d[A(s)][y_1(s) - y_2(s)] \\ &= \int_{t_0}^t d[A(s)]z(s) \end{aligned}$$

and hence, z is solution of the Volterra-Stieltjes integral equation (4.5) and $z(t) = 0$ for all $t \in [t_0, v]$. □

We end this subsection by presenting a version of Theorem 4.4 for finite dimensional Banach spaces, but instead of requiring h is of bounded variation, we consider the larger space of regulated functions. This is possible due to an Arzelà–Ascoli-type theorem for regulated functions taking values in $\mathbb{R}^n$, by Dana Fraňková. See [16].

Theorem 4.5. *Let $v \in J$ and $A : J \to L(\mathbb{R}^n, Y)$ be locally of bounded variation on J and left-continuous. Then, the linear Volterra-Stieltjes integral equation* (4.1) *admits a unique solution in $G([t_0, v], \mathbb{R}^n)$ for each given $h \in G([t_0, v], \mathbb{R}^n)$.*

Proof. Let $v \in J$ and define $T : G([t_0, v], \mathbb{R}^n) \to G([t_0, v], \mathbb{R}^n)$ by

$$Ty(t) = \int_{t_0}^{t} d[A(s)]y(s), \quad t \in [t_0, v]. \tag{4.11}$$

By Theorem 4.1, the integral in (4.11) exists and by Theorem 2.15, $Ty \in G([t_0, v], \mathbb{R}^n)$. Therefore, T is well-defined. Moreover, T is clearly linear and by Proposition 2.19, we have

$$\|Ty\|_\infty \leq \text{var}_{t_0}^{v} A \|y\|_\infty$$

for all $y \in G([t_0, v], \mathbb{R}^n)$. Therefore, T is a bounded linear operator on $G([t_0, v], \mathbb{R}^n)$. By the Arzelà–Ascoli-type theorem for regulated functions taking values in $\mathbb{R}^n$ (see, e.g. [6, Corollary 1.19] or [16]), every bounded sequence $\{y_k\}$ in $G([t_0, v], \mathbb{R}^n)$ contains a subsequence $\{y_{k_l}\}$ which converges to a function $y_0 \in G([t_0, v], \mathbb{R}^n)$. Using the same ideas as in the proof of Theorem 4.2, we can conclude that T is compact.

The existence and uniqueness of a solution of (4.1) for each given $h \in G([t_0, v], \mathbb{R}^n)$ is guaranteed by the Fredholm alternative (Theorem 4.3), since the proof of the fact that the homogeneous linear Volterra-Stieltjes integral equation

$$y(t) = \int_{t_0}^{t} d[A(s)]y(s) \tag{4.12}$$

admits only the trivial solution in $G([t_0, v], \mathbb{R}^n)$ is analogous to the proof of the corresponding fact in Theorem 4.4. Indeed, following the ideas of Theorem 4.4, if $x : [t_0, v] \to \mathbb{R}^n$ is a solution of (4.12) and $c = \sup\{t \in [t_0, v]; \|x(t)\| = 0\}$, then the left-continuity of A on J implies $\|x(c)\| = 0$. Assuming that $c < v$ and A is continuous at c, we obtain the existence of $\overline{t} \in (c, v]$ such that $\text{var}_c^{\overline{t}} A < \frac{1}{2}$. Moreover, for all $s \in [c, \overline{t}]$, we have

$$\begin{aligned} \|x(s)\| &= \left\| \int_{t_0}^{s} d[A(s)]x(s) \right\| \\ &\leq \left\| \int_{t_0}^{c} d[A(s)]x(s) \right\| + \left\| \int_{c}^{s} d[A(s)]x(s) \right\| \\ &\leq \|x(c)\| + \text{var}_c^s A \sup_{\theta \in [c,s]} \|x(\theta)\| \\ &\leq \frac{1}{2} \sup_{\theta \in [c,\overline{t}]} \|x(\theta)\|. \end{aligned}$$

Thus,

$$\sup_{\theta \in [c,\overline{t}]} \|x(\theta)\| \leq \frac{1}{2} \sup_{\theta \in [c,\overline{t}]} \|x(\theta)\|$$

and consequently,

$$\|x(t)\| = 0 \quad \text{for all } t \in [c, \overline{t}], \tag{4.13}$$

which contradicts the fact that $c = \sup\{t \in [t_0, v]; \|x(t)\| = 0\}$. On the other hand, if A is not continuous at c, then the proof of (4.13) follows as Case 2 from Theorem 4.4, with obvious adaptations. Therefore, $c = v$ and the proof is complete. □

4.2. *Dynamic equations*

In this subsection, we prove existence and uniqueness of a solution for homogeneous and nonhomogeneous dynamic equations on time scales, whose integral forms contain Perron Δ-integrals. Furthermore, we prove a version of the variation-of-constant formula for nonhomogeneous dynamic equations on time scales. The results obtained here generalize those presented in [3].

Let $\mathbb{T}$ be a time scale, X be a Banach space and $L(X)$ be the Banach space of continuous linear mappings $T : X \to X$. Given $t_0 \in \mathbb{T}$, we define $\mathbb{T}_0 = [t_0, +\infty) \cap \mathbb{T}$ and we denote by $G_0(\mathbb{T}_0, X)$ the vector space of all regulated functions $x : \mathbb{T}_0 \to X$ such that

$$\|x\|_{\mathbb{T}_0} := \sup_{s \in \mathbb{T}_0} e^{-(s-t_0)} \|x(s)\| < \infty.$$

Consider the nonhomogeneous linear dynamic equation

$$x^{\Delta} = a(t)x + f(t) \tag{4.14}$$

and its corresponding homogeneous equation

$$x^{\Delta} = a(t)x \tag{4.15}$$

on a time scale $\mathbb{T}$, where both functions $a : \mathbb{T} \to L(X)$ and $f : \mathbb{T} \to X$ satisfy the following conditions:

(A1) The Perron Δ-integrals

$$\int_{t_1}^{t_2} f(s)\Delta s \quad \text{and} \quad \int_{t_1}^{t_2} a(s)y(s)\Delta s$$

exist for all $t_1, t_2 \in \mathbb{T}_0$, whenever $y : \mathbb{T}_0 \to X$ is regulated.

(A2) There is a locally Perron Δ-integrable function $L : \mathbb{T}_0 \to \mathbb{R}$ such that

$$\left\| \int_{t_1}^{t_2} a(s)[z(s) - y(s)]\Delta s \right\| \leq \|z - y\|_{\mathbb{T}_0} \int_{t_1}^{t_2} L(s)\Delta s$$

for all $z, y \in G_0(\mathbb{T}_0, X)$ and all $t_1, t_2 \in \mathbb{T}_0$.

(A3) There is a locally Perron Δ-integrable function $K : \mathbb{T}_0 \to \mathbb{R}$ such that

$$\left\| \int_{t_1}^{t_2} f(s)\Delta s \right\| \leq \int_{t_1}^{t_2} K(s)\Delta s$$

for all $t_1, t_2 \in \mathbb{T}_0$.

In what follows, we present a definition of a solution for the dynamic equations (4.14) and (4.15).

Definition 4.6. Let $t_0 \in \mathbb{T}$ and $\mathbb{T}_0 = [t_0, +\infty) \cap \mathbb{T}$. We say that a function $\overline{x} : \mathbb{T}_0 \to X$ is a *solution of the nonhomogeneous dynamic equation* (4.14) with initial condition $\overline{x}(t_0) = x_0 \in X$, if it is satisfies

$$\overline{x}(t) = x_0 + \int_{t_0}^{t} a(s)\overline{x}(s)\Delta s + \int_{t_0}^{t} f(s)\Delta s \quad \text{for all } t \in \mathbb{T}_0.$$

Moreover, a function $x : \mathbb{T}_0 \to X$ is said to be a *solution of the homogeneous dynamic equation* (4.15) with initial condition $x(t_0) = x_0 \in X$, if the equality

$$x(t) = x_0 + \int_{t_0}^{t} a(s)x(s)\Delta s$$

holds for all $t \in \mathbb{T}_0$.

The next statement gives an interesting property of the solutions of the dynamic equations (4.14) and (4.15).

Theorem 4.7. *The solutions of the dynamic equations* (4.14) *and* (4.15) *are rd-continuous.*

Proof. Let $\overline{x} : \mathbb{T}_0 \to X$ be a solution of the nonhomogeneous dynamic equation (4.14). Then, by the definition of solution, for all $t \in \mathbb{T}_0$, we have

$$\overline{x}(t) = \overline{x}(t_0) + \int_{t_0}^{t} a(s)\overline{x}(s)\Delta s + \int_{t_0}^{t} f(s)\Delta s. \tag{4.16}$$

By Theorem 3.4, the Perron-Stieltjes integrals $\int_{t_0}^{t} a^*(s)\overline{x}^*(s)dg(s)$ and $\int_{t_0}^{t} f^*(s)$ $dg(s)$ exist for all $t \in \mathbb{T}_0$, where $g(t) = t^*$ for all $t \in \mathbb{T}_0^*$. Moreover,

$$\int_{t_0}^{t} a^*(s)\overline{x}^*(s)dg(s) = \int_{t_0}^{t} a(s)\overline{x}(s)\Delta s \quad \text{and} \quad \int_{t_0}^{t} f^*(s)dg(s) = \int_{t_0}^{t} f(s)\Delta s \tag{4.17}$$

for all $t \in \mathbb{T}_0$.

On the other hand, since $g|_{\mathbb{T}_0}$ is the identity function, $g|_{\mathbb{T}_0}$ is right-continuous and by Lemma 3.3, g is rd-continuous on $\mathbb{T}_0^*$. Then, by Theorem 2.16, the functions $t \ni \mathbb{T}_0^* \mapsto \int_{t_0}^{t} a^*(s)\overline{x}^*(s)dg(s)$ and $t \ni \mathbb{T}_0^* \mapsto \int_{t_0}^{t} f^*(s)dg(s)$ are rd-continuous on $\mathbb{T}_0^*$. This fact together with Eqs. (4.16) and (4.17) show that $\overline{x}$ is rd-continuous The same applies for the solutions of the homogeneous dynamic equation (4.15). □

Our goal is to prove that the solutions of the dynamic equations (4.14) and (4.15) are related to solutions of linear Volterra-Stieltjes integral equations of the second kind. At first, we present a relation between $G_0(\mathbb{T}_0, X)$ and $G_0(\mathbb{T}_0^*, X)$, where $G_0(\mathbb{T}_0^*, X)$ is given in Definition 2.4(iii).

Remark 4.8. Let $y \in G_0(\mathbb{T}_0^*, X)$. It is clear that $\mathbb{T}_0 \subset \mathbb{T}_0^*$ and hence, we have

$$\|y\|_{\mathbb{T}_0} = \sup_{s \in \mathbb{T}_0} e^{-(s-t_0)}\|y(s)\| \le \sup_{s \in \mathbb{T}_0^*} e^{-(s-t_0)}\|y(s)\| = \|y\|_{\mathbb{T}_0^*} < \infty,$$

which implies $y|_{\mathbb{T}_0} \in G_0(\mathbb{T}_0, X)$.

Define an auxiliary operator $A : \mathbb{T}_0^* \to L(G_0(\mathbb{T}_0^*, X), X)$ by

$$A(t)y = \int_{t_0}^{t} a^*(s)y(s)dg(s) \tag{4.18}$$

for all $t \in \mathbb{T}_0^*$ and all $y \in G_0(\mathbb{T}_0^*, X)$, where $g(s) = s^*$ for every $s \in \mathbb{T}_0^*$, the integral in (4.18) is in the sense of Perron-Stieltjes and $L(G_0(\mathbb{T}_0^*, X), X)$ denotes the space of linear continuous mappings defined from $G_0(\mathbb{T}_0^*, X)$ into X.

We assert that A is well defined, that is, the integral on the right-hand side of (4.18) exists for all $t \in \mathbb{T}_0^*$ and all $y \in G_0(\mathbb{T}_0^*, X)$. Indeed, by Remark 4.8, $z = y|_{\mathbb{T}_0}$ belongs to $G_0(\mathbb{T}_0, X)$ for all $y \in G_0(\mathbb{T}_0^*, X)$ and by condition (A1), the Perron Δ-integral $\int_{t_0}^{t} a(s)z(s)\Delta s$ exists for all $t \in \mathbb{T}_0$. By Theorem 3.4, the Perron-Stieltjes integral $\int_{t_0}^{t} a^*(s)z^*(s)dg(s) = \int_{t_0}^{t} a^*(s)y(s)dg(s)$ also exists for all $t \in \mathbb{T}_0^*$.

The next result shows that $A : \mathbb{T}_0^* \to L(G_0(\mathbb{T}_0^*, X), X)$, defined by (4.18), is locally of bounded variation on $\mathbb{T}_0^*$.

Lemma 4.9. *Let* $A : \mathbb{T}_0^* \to L(G_0(\mathbb{T}_0^*, X), X)$ *be defined by* (4.18) *and assume that condition* (A2) *holds. Then,*

$$\operatorname{var}_a^b A < \infty \quad \textit{for all } [a, b] \subset \mathbb{T}_0^*. \tag{4.19}$$

Proof. Note that by the definition of $G_0(\mathbb{T}_0^*, X)$, $\|y\|_{\mathbb{T}_0^*} < \infty$ for all $y \in G_0(\mathbb{T}_0^*, X)$ (see Remark 4.8). Then, by condition (A2), for every $a, b \in \mathbb{T}_0^*$ and every division $d = (s_i)$ of $[a, b]$, we have

$$\begin{aligned}
&\sum_{i=1}^{|d|} \|A(s_i)y - A(s_{i-1})y\| \\
&\quad = \sum_{i=1}^{|d|} \left\| \int_{t_0}^{s_i} a^*(s)y(s)dg(s) - \int_{t_0}^{s_{i-1}} a^*(s)y(s)dg(s) \right\| \\
&\quad \overset{\text{Theorem 3.6}}{=} \sum_{i=1}^{|d|} \left\| \int_{t_0}^{s_i^*} a(s)y(s)\Delta s - \int_{t_0}^{s_{i-1}^*} a(s)y(s)\Delta s \right\| \\
&\quad = \sum_{i=1}^{|d|} \left\| \int_{s_{i-1}^*}^{s_i^*} a(s)y(s)\Delta s \right\| \\
&\quad \overset{(A2)}{\leq} \sum_{i=1}^{|d|} \left(\|y\|_{\mathbb{T}_0} \int_{s_{i-1}^*}^{s_i^*} L(s)\Delta s \right) \\
&\quad = \|y\|_{\mathbb{T}_0} \left(\sum_{i=1}^{|d|} \int_{s_{i-1}^*}^{s_i^*} L(s)\Delta s \right)
\end{aligned}$$

$$\overset{\text{Remark 4.8}}{\leq} \|y\|_{\mathbb{T}_0^*} \left(\sum_{i=1}^{|d|} \int_{s_{i-1}^*}^{s_i^*} L(s)\Delta s \right)$$

$$= \|y\|_{\mathbb{T}_0^*} \int_{a^*}^{b^*} L(s)\Delta s < \infty.$$

Taking the supremum over all divisions $d = (s_i)$ of $[a, b]$, we obtain $\operatorname{var}_a^b A < \infty$ and since the statement holds for all $a, b \in \mathbb{T}_0^*$, we conclude (4.19). □

In what follows, we prove that the Perron-Stieltjes integral of y with respect to A exists, whenever y is a regulated function defined on a closed interval.

Theorem 4.10. *Let $v \in \mathbb{T}_0^*$ and $A : \mathbb{T}_0^* \to L(G_0(\mathbb{T}_0^*, X), X)$ be given by (4.18). If $y \in G([t_0, v], X)$, then the Perron-Stieltjes integral*

$$\int_{t_0}^{v} d[A(s)]y(s)$$

exists.

Proof. Consider $\widehat{X} = L(G_0(\mathbb{T}_0^*, X), X)$ and $\widehat{Y} = X = \widehat{Z}$. Then, $\widehat{X}, \widehat{Y}$ and $\widehat{Z}$ are Banach spaces once $G_0(\mathbb{T}_0^*, X)$ and X are Banach spaces (see the comment following Definition 2.4). Define $B : \widehat{X} \times \widehat{Y} \to \widehat{Z}$ by

$$B(F, x) = Fz_x,$$

where $z_x : \mathbb{T}_0^* \to X$ is given by $z_x(t) = x$ for all $t \in \mathbb{T}_0^*$. It is clear that $z_x \in G_0(\mathbb{T}_0^*, X)$ and $\mathcal{B} = (\widehat{X}, \widehat{Y}, \widehat{Z})$ is a bilinear triple. Moreover, by Lemma 4.9, $\operatorname{var}_{t_0}^v A < \infty$ and by Proposition 2.18, the Perron-Stieltjes integral $\int_{t_0}^v d[A(s)]y(s)$ exists for all $y \in G([t_0, v], X)$. □

In the sequel, we present a relation between the Perron-Stieltjes integral $\int d[A(s)]y(s)$ and the Perron-Stieltjes Δ integral $\int a(s)y(s)\Delta s$. Its proof follows the same ideas as in [13, Theorem 4.7].

Theorem 4.11. *Let $A : \mathbb{T}_0^* \to L(G_0(\mathbb{T}_0^*, X), X)$ be given by (4.18). If $v \in \mathbb{T}_0^*$, then*

$$\int_{t_0}^{t} d[A(s)]y(s) = \int_{t_0}^{t^*} a(s)y(s)\Delta s$$

for all $t \in [t_0, v]$ and all $y \in G([t_0, v], X)$.

Proof. We start by proving that if $v \in \mathbb{T}_0^*$ and $\varphi : [t_0, t] \to X$ is a step function, then

$$\int_{t_0}^{t} d[A(s)]\varphi(s) = \int_{t_0}^{t^*} a(s)\varphi(s)\Delta s$$

for all $t \in [t_0, v]$. By Remark 2.3, φ is regulated on $[t_0, v]$ and by Theorem 2.18, the Perron-Stieltjes integral $\int_{t_0}^t d[A(s)]\varphi(s)$ exists for all $t \in [t_0, v]$. Let $t \in [t_0, v]$

be fixed. Since φ is a step function, there exists a division $d = (s_i)$ of $[t_0, t]$ and $c_1, \ldots, c_{|d|} \in X$ such that

$$\varphi(s) = c_i \quad \text{for every } s \in (s_{i-1}, s_i).$$

Thus, by the definition of A, if $i \in \{1, \ldots, |d|\}$ and $s_{i-1} < t_1 < t_2 < s_i$, we have

$$\begin{aligned}\int_{t_1}^{t_2} d[A(s)]\varphi(s) &= \int_{t_1}^{t_2} d[A(s)]c_i = A(t_2)c_i - A(t_1)c_i = \int_{t_1}^{t_2} a^*(s)c_i dg(s) \\ &= \int_{t_1}^{t_2} a^*(s)\varphi(s)dg(s). \end{aligned} \tag{4.20}$$

On the other hand, Theorem 2.16 ensures

$$\lim_{\xi \to s_{i-1}^+} \int_{s_{i-1}}^{\xi} a^*(s)\varphi(s_{i-1})dg(s) = a^*(s_{i-1})\varphi(s_{i-1})\Delta^+ g(s_{i-1}) \tag{4.21}$$

and

$$\begin{aligned}&\lim_{\xi \to s_{i-1}^+} \int_{\xi}^{\tau} a^*(s)\varphi(s)dg(s) \\ &\quad = \int_{s_{i-1}}^{\tau} a^*(s)\varphi(s)dg(s) - a^*(s_{i-1})\varphi(s_{i-1})\Delta^+ g(s_{i-1}), \quad \tau \in (s_{i-1}, s_i). \end{aligned} \tag{4.22}$$

Combining (4.21) and (4.22), we obtain

$$\begin{aligned}&\lim_{\xi \to s_{i-1}^+} \left(\int_{\xi}^{\tau} a^*(s)\varphi(s)dg(s) + \int_{s_{i-1}}^{\xi} a^*(s)\varphi(s_{i-1})dg(s) \right) \\ &\quad = \int_{s_{i-1}}^{\tau} a^*(s)\varphi(s)dg(s), \quad \tau \in (s_{i-1}, s_i). \end{aligned} \tag{4.23}$$

Moreover, by Theorem 2.15, if $\tau \in (s_{i-1}, s_i)$, then

$$\begin{aligned}\int_{s_{i-1}}^{\tau} d[A(s)]\varphi(s) &= \lim_{\xi \to s_{i-1}^+} \left(\int_{\xi}^{\tau} d[A(s)]\varphi(s) + A(\xi)\varphi(s_{i-1}) - A(s_{i-1})\varphi(s_{i-1}) \right) \\ &= \lim_{\xi \to s_{i-1}^+} \left(\int_{\xi}^{\tau} d[A(s)]\varphi(s) + \int_{s_{i-1}}^{\xi} a^*(s)\varphi(s_{i-1})dg(s) \right) \\ &\overset{(4.20)}{=} \lim_{\xi \to s_{i-1}^+} \left(\int_{\xi}^{\tau} a^*(s)\varphi(s)dg(s) + \int_{s_{i-1}}^{\xi} a^*(s)\varphi(s_{i-1})dg(s) \right) \\ &\overset{(4.23)}{=} \int_{s_{i-1}}^{\tau} a^*(s)\varphi(s)dg(s). \end{aligned}$$

Therefore,

$$\int_{s_{i-1}}^{\tau} d[A(s)]\varphi(s) = \int_{s_{i-1}}^{\tau} a^*(s)\varphi(s_{i-1})dg(s) \tag{4.24}$$

for all $i \in \{1, \ldots, |d|\}$ and all $\tau \in (s_{i-1}, s_i)$. Analogously,

$$\int_{\tau}^{s_i} d[A(s)]\varphi(s) = \int_{\tau}^{s_i} a^*(s)\varphi(s_{i-1})dg(s) \tag{4.25}$$

for all $i \in \{1, \ldots, |d|\}$ and all $\tau \in (s_{i-1}, s_i)$. According to (4.24) and (4.25), we obtain

$$\int_{s_{i-1}}^{s_i} d[A(s)]\varphi(s) = \int_{s_{i-1}}^{s_i} a^*(s)\varphi(s)dg(s) \quad \text{for all } i \in \{1, \ldots, |d|\}. \tag{4.26}$$

Therefore, by (4.26) and Theorem 3.6, we conclude

$$\begin{aligned}\int_{t_0}^{t} d[A(s)]\varphi(s) &= \sum_{i=1}^{|d|} \int_{s_{i-1}}^{s_i} d[A(s)]\varphi(s) = \sum_{i=1}^{|d|} \int_{s_{i-1}}^{s_i} a^*(s)\varphi(s)dg(s) \\ &= \int_{t_0}^{t} a^*(s)\varphi(s)dg(s) = \int_{t_0}^{t^*} a(s)\varphi(s)\Delta s. \end{aligned} \tag{4.27}$$

In order to prove the assertion to regulated functions, we consider $y \in G([t_0, v], X)$ and a sequence of step functions $\varphi_k : [t_0, v] \to X$, $k \in \mathbb{N}$ such that

$$\lim_{k\to\infty} \|\varphi_k(s) - y(s)\|_\infty = 0, \tag{4.28}$$

where the existence of this sequence is guaranteed by Theorem 2.5. By (4.27), we obtain

$$\int_{t_0}^{t} d[A(s)]\varphi_k(s) = \int_{t_0}^{t^*} a(s)\varphi_k(s)\Delta s \quad \text{for all } k \in \mathbb{N} \tag{4.29}$$

and by Theorem 2.14, we have

$$\lim_{k\to\infty} \int_{t_0}^{t} d[A(s)]\varphi_k(s) = \int_{t_0}^{t} d[A(s)]y(s). \tag{4.30}$$

Combining (4.29) with (4.30), we get

$$\lim_{k\to\infty} \int_{t_0}^{t^*} a(s)\varphi_k(s)\Delta s = \lim_{k\to\infty} \int_{t_0}^{t} d[A(s)]\varphi_k(s) = \int_{t_0}^{t} d[A(s)]y(s).$$

We conclude the proof by showing

$$\lim_{k\to\infty} \int_{t_0}^{t^*} a(s)\varphi_k(s)\Delta s = \int_{t_0}^{t^*} a(s)y(s)\Delta s. \tag{4.31}$$

Indeed, by Remarks 2.7 and 4.8, the functions $\widehat{\varphi}_k, \widehat{y} : \mathbb{T}_0^* \to X$ defined by

$$\widehat{\varphi}_k(t) = \begin{cases} \varphi_k(t), & t \in [t_0, v], \\ \varphi_k(v) & t \in \mathbb{T}_0^* \backslash [t_0, v] \end{cases}$$

and

$$\widehat{y}(t) = \begin{cases} y(t), & t \in [t_0, v], \\ y(v) & t \in \mathbb{T}_0^* \backslash [t_0, v] \end{cases}$$

belong to $G_0(\mathbb{T}_0^*, X)$ and

$$\|\widehat{\varphi}_k - \widehat{y}\|_{\mathbb{T}_0} \leq \|\widehat{\varphi}_k - \widehat{y}\|_{\mathbb{T}_0^*} \leq \|\widehat{\varphi}_k - \widehat{y}\|_\infty = \|\varphi_k - y\|_\infty.$$

Moreover, condition (A2) implies

$$\begin{aligned} \left\| \int_{t_0}^{t^*} a(s)\varphi_k(s)\Delta s - \int_{t_0}^{t^*} a(s)y(s)\Delta s \right\| &= \left\| \int_{t_0}^{t^*} a(s)[\widehat{\varphi}_k(s) - \widehat{y}(s)]\Delta s \right\| \\ &\leq \|\widehat{\varphi}_k - \widehat{y}\|_{\mathbb{T}_0} \int_{t_0}^{t^*} L(s)\Delta s \\ &\leq \|\varphi_k - y\|_\infty \int_{t_0}^{t^*} L(s)\Delta s \qquad (4.32) \end{aligned}$$

and hence, (4.31) follows by (4.28) and (4.32). □

Now, we are able to prove the existence and uniqueness of a solution of a linear Volterra-Stieltjes integral equation with which the dynamic equations (4.14) and (4.15) will be related to.

Theorem 4.12. *Let $v \in \mathbb{T}_0^*$ and $A : \mathbb{T}_0^* \to L(G_0(\mathbb{T}_0^*, X), X)$ be given by* (4.18). *Then, the linear Volterra-Stieltjes integral equation*

$$y(t) = \int_{t_0}^{t} d[A(s)]y(s) + h(t), \quad t \in [t_0, v], \qquad (4.33)$$

admits a unique solution in $\mathrm{BV}([t_0, v], X)$, *for any* $h \in \mathrm{BV}([t_0, v], X)$.

Proof. By Theorem 4.4 and Lemma 4.27, it is enough to show that $A : \mathbb{T}_0^* \to L(G_0(\mathbb{T}_0^*, X), X)$, defined by (4.18), is left-continuous. Indeed, since $g|_{\mathbb{T}_0}$ is the identity function, Lemma 3.3 ensures that g is left-continuous on $\mathbb{T}_0^*$. Therefore, by Theorem 2.16, A is left-continuous. □

In what follows, we prove the existence and uniqueness of a solution for the dynamic equations (4.14) and (4.15). Note that we do not require the rd-continuity of the functions involved. Therefore, this result is more general than the ones found in the literature. See [1, 3, 4, 21], for example, and the comment after Lemma 4.24.

Theorem 4.13. *Assume that* $a : \mathbb{T} \to L(X)$ *and* $f : \mathbb{T} \to X$ *satisfy conditions* (A1)–(A3). *Then, the dynamic equations* (4.14) *and* (4.15) *admit unique solutions.*

Proof. We start by proving the existence and uniqueness of a solution of the nonhomogeneous dynamic equation (4.1). Define $\widetilde{h} : \mathbb{T}_0^* \to X$ by

$$\widetilde{h}(t) = \int_{t_0}^{t} f^*(s)dg(s) \quad \text{for all } t \in \mathbb{T}_0^*,$$

where $g(s) = s^*$ for all $s \in \mathbb{T}_0^*$. By condition (A1) and Theorem 3.4, the Perron-Stieltjes integral $\int_{t_0}^{t} f^*(s)dg(s)$ exists for all $t \in \mathbb{T}_0^*$ and consequently, $\widetilde{h}$ is well defined. On the other hand, by Theorem 3.6 and condition (A3), we have

$$\begin{aligned}
|\widetilde{h}(t_2) - \widetilde{h}(t_1)| &= \left\| \int_{t_0}^{t_2} f^*(s)dg(s) - \int_{t_0}^{t_1} f^*(s)dg(s) \right\| \\
&\overset{\text{Theorem 3.6}}{=} \left\| \int_{t_0}^{t_2^*} f(s)\Delta s - \int_{t_0}^{t_1^*} f(s)\Delta s \right\| \\
&= \left\| \int_{t_0}^{t_1^*} f(s)\Delta s + \int_{t_1^*}^{t_2^*} f(s)\Delta s - \int_{t_0}^{t_1^*} f(s)\Delta s \right\| \\
&= \left\| \int_{t_1^*}^{t_2^*} f(s)\Delta s \right\| \\
&\overset{\text{Condition }(A3)}{\leq} \int_{t_1^*}^{t_2^*} K(s)\Delta s < \infty \qquad (4.34)
\end{aligned}$$

for all $t_1, t_2 \in \mathbb{T}_0^*$, which implies $\widetilde{h}$ is locally of bounded variation on $\mathbb{T}_0^*$, that is, $\widetilde{h} \in \mathrm{BV}([a, b], X)$ for all $[a, b] \subset \mathbb{T}_0^*$. Consequently, if $x_0 \in X$, then $h : \mathbb{T}_0^* \to X$, defined by $h(t) = \widetilde{h}(t) + x_0$, is locally of bounded variation on $\mathbb{T}_0^*$.

Let $v \in \mathbb{T}_0^*$ be arbitrary. By Theorem 4.12, there exists a unique solution $y : [t_0, v] \to X$ of the Volterra-Stieltjes integral equation (4.33) in $\mathrm{BV}([t_0, v], X)$, that is,

$$y(t) = x_0 + \int_{t_0}^{t} d[A(s)]y(s) + \int_{t_0}^{t} f^*(s)dg(s) \quad \text{for all } t \in [t_0, v], \qquad (4.35)$$

where $A : \mathbb{T}_0^* \to L(G_0(\mathbb{T}_0^*, X), X)$ is given by (4.18). By Theorems 3.6 and 4.11, we have

$$\int_{t_0}^{t} d[A(s)]y(s) = \int_{t_0}^{t^*} a(s)y(s)\Delta s \quad \text{for all } t \in [t_0, v] \quad \text{and}$$

$$\int_{t_0}^{t} f^*(s)dg(s) = \int_{t_0}^{t^*} f(s)\Delta s \quad \text{for all } t \in [t_0, v].$$

Thus,

$$y(t) = x_0 + \int_{t_0}^{t^*} a(s)y(s)\Delta s + \int_{t_0}^{t^*} f(s)\Delta s \quad \text{for all } t \in [t_0, v]. \tag{4.36}$$

Consider the set

$$S := \{y : [t_0, v_y] \to X;\ v_y \in \mathbb{T}_0^* \text{ and } y \text{ is a solution of (4.33) in } \mathrm{BV}([t_0, v_y], X) \\ \text{with } y(t_0) = x_0\}.$$

Since (4.35) holds for every $v \in \mathbb{T}_0^*$, the set S is nonempty and $\mathbb{T}_0^* = \bigcup_{y \in S}[t_0, v_y]$. Moreover, if $y_1, y_2 \in S$, then either $[t_0, v_{y_1}] \subset [t_0, v_{y_2}]$ or $[t_0, v_{y_2}] \subset [t_0, v_{y_1}]$. Assume, without loss of generality, that $[t_0, v_{y_1}] \subset [t_0, v_{y_2}]$. It is clear that $y_2|_{[t_0, v_{y_1}]} \in \mathrm{BV}([t_0, v_{y_1}], X)$ and by the uniqueness of a solution (see Theorem 4.4), $y_2|_{[t_0, v_{y_1}]} = y_1$. Therefore,

$$y_1(t) = y_2(t) \quad \text{for all } t \in [t_0, v_{y_1}] \cap [t_0, v_{y_2}] \quad \text{and all } y_1, y_2 \in S. \tag{4.37}$$

Define $z : \mathbb{T}_0^* \to X$ by $z(t) = y(t)$, whenever $y \in S$ and $t \in [t_0, v_y]$. By (4.37), z is well defined and by (4.36), we have

$$z(t) = x_0 + \int_{t_0}^{t^*} a(s)z(s)\Delta s + \int_{t_0}^{t^*} f(s)\Delta s \quad \text{for all } t \in \mathbb{T}_0^*.$$

On the other hand, since $t^* = t$ for all $t \in \mathbb{T}_0$, we have

$$z(t) = x_0 + \int_{t_0}^{t} a(s)z(s)\Delta s + \int_{t_0}^{t} f(s)\Delta s \quad \text{for all } t \in \mathbb{T}_0.$$

Consequently, $x : \mathbb{T}_0 \to X$, defined by $x = z|_{\mathbb{T}_0}$, is a solution of the dynamic equation (4.14) with initial condition $x(t_0) = x_0$. The uniqueness of x follows directly from the uniqueness of a solution of (4.33).

The proof for the homogeneous dynamic equation (4.15) is analogous (with $h(t) = x_0$ for all $t \in \mathbb{T}_0^*$) and therefore, we omit it here. □

The next result is a direct consequence of the proof of Theorem 4.13.

Theorem 4.14. *Let $A : \mathbb{T}_0^* \to L(G_0(\mathbb{T}_0^*, X), X)$ be given by* (4.18) *and $x_0 \in X$. Assume that $a : \mathbb{T} \to L(X)$ and $f : \mathbb{T} \to X$ satisfy conditions* (A1)–(A3)*. Then, the* *following statements hold:*

(i) *If* $y : \mathbb{T}_0^* \to X$ *is a function such that the equality*

$$y(t) = x_0 + \int_{t_0}^{t} d[A(s)]y(s) + \int_{t_0}^{t} f^*(s)dg(s) \tag{4.38}$$

holds for all $t \in \mathbb{T}_0$, *then* $x := y|_{\mathbb{T}_0}$ *is the solution of the nonhomogeneous dynamic equation* (4.14) *with initial condition* x_0. *Conversely, if* $x : \mathbb{T}_0 \to X$ *is the solution of the nonhomogeneous dynamic equation* (4.14) *with initial condition* x_0, *then* $x = y|_{\mathbb{T}_0}$, *where* y *is given by* (4.38).

(ii) *If the function* $y : \mathbb{T}_0^* \to X$ *satisfies*

$$y(t) = x_0 + \int_{t_0}^{t} d[A(s)]y(s) \tag{4.39}$$

for all $t \in \mathbb{T}$, *then* $x := y|_{\mathbb{T}_0}$ *is the solution of the homogeneous dynamic equation* (4.15) *with initial condition* x_0. *Reciprocally, if* $x : \mathbb{T}_0 \to X$ *is the solution of the homogeneous dynamic equation* (4.15) *with initial condition* x_0, *then* $x = y|_{\mathbb{T}_0}$, *where* y *is given by* (4.39).

In the sequel, we specify Theorem 4.13 for $X = \mathbb{R}^n$, in which case condition (A3) is not required.

Theorem 4.15. *Assume that* $X = \mathbb{R}^n$ *and let* $A : \mathbb{T}_0^* \to L(G_0(\mathbb{T}_0^*, X), X)$ *be given by* (4.18) *and* $x_0 \in X$. *If* $a : \mathbb{T} \to L(X)$ *and* $f : \mathbb{T} \to X$ *satisfy conditions* (A1) *and* (A2), *then there exists a unique solution of the nonhomogeneous dynamic equation* (4.14).

Proof. By condition (A1) and Theorems 2.16 and 3.6, for all $v \in \mathbb{T}_0^*$, the function $h : [t_0, v] \to X$, given by

$$h(t) = \int_{t_0}^{t} f^*(s)dg(s), \quad t \in [t_0, v],$$

where $g(s) = s^*$ for all $s \in \mathbb{T}_0^*$ is well-defined and $h \in G([t_0, v], X)$. Then, by Theorem 4.5, Lemma 4.9 and by the fact that $A : \mathbb{T}_0^* \to L(G_0(\mathbb{T}_0^*, X), X)$, defined by (4.18), is left-continuous, we conclude that for all $x_0 \in X$, there exists a unique function $y : [t_0, v] \to X$ for which

$$y(t) = x_0 + \int_{t_0}^{t} d[A(s)]y(s) + h(t), \quad t \in [t_0, v].$$

By Theorems 3.6 and 4.11, we have

$$y(t) = \int_{t_0}^{t^*} a(s)y(s)\Delta s + \int_{t_0}^{t^*} f(s)\Delta s, \quad t \in [t_0, v].$$

Using the same ideas as in the proof of Theorem 4.13, we obtain a unique function $z : \mathbb{T}_0^* \to X$ such that

$$z(t) = \int_{t_0}^{t^*} a(s)z(s)\Delta s + \int_{t_0}^{t^*} f(s)\Delta s, \quad t \in \mathbb{T}_0^*$$

and consequently, $x = z|_{\mathbb{T}_0}$ is a solution of the nonhomogeneous dynamic equation (4.14) with initial condition $x(t_0) = x_0$. □

In order to prove the existence of a fundamental operator associated to the linear dynamic equation (4.15), we need some auxiliary results.

Theorem 4.16. *Let $v \in \mathbb{T}_0^*$ and $A : \mathbb{T}_0^* \to L(G_0(\mathbb{T}_0^*, X), X)$ be given by* (4.18)*. If $\Phi \in \mathrm{BV}([t_0, v], L(X))$, then the Perron-Stieltjes integral*

$$\int_{t_0}^{v} d[A(s)]\Phi(s)$$

exists.

Proof. Consider $\widehat{X} = L(G_0(\mathbb{T}_0^*, X), X)$ and $\widehat{Y} = L(X) = \widehat{Z}$. Since $G_0(\mathbb{T}_0^*, X)$ and X are Banach spaces, $\widehat{X}, \widehat{Y}$ and $\widehat{Z}$ are also Banach spaces. Define $B : \widehat{X} \times \widehat{Y} \to \widehat{Z}$ by

$$B(F, G) = G \circ F.$$

It is clear that $\mathcal{B} = (\widehat{X}, \widehat{Y}, \widehat{Z})$ is a bilinear triple and by Lemma 4.9, $\mathrm{var}_{t_0}^{v} A < \infty$. Therefore, by Proposition 2.18, the Perron-Stieltjes integral $\int_{t_0}^{v} d[A(s)]\Phi(s)$ exists for all $\Phi \in G([t_0, v], L(X))$, in particular, $\int_{t_0}^{v} d[A(s)]\Phi(s)$ exists for all $\Phi \in \mathrm{BV}([t_0, v], L(X))$. □

From now on, we denote by $I \in L(X)$ the identity operator. The proof of the next result is analogous to the proof of Theorem 4.4 and therefore, we omit it here.

Theorem 4.17. *Let $v \in \mathbb{T}_0^*$ and $A : \mathbb{T}_0^* \to L(G_0(\mathbb{T}_0^*, X), X)$ be given by* (4.18)*. Then, for a given $s \in [t_0, v]$, the equation*

$$\Phi(t) = I + \int_{s}^{t} d[A(s)]\Phi(s), \quad t \in [t_0, v], \tag{4.40}$$

admits at least one nontrivial solution in $\mathrm{BV}([t_0, v], L(X))$*.*

The next result guarantees the existence of a fundamental operator associated to the dynamic equation (4.15).

Theorem 4.18. *There exists a unique operator $U : \mathbb{T}_0 \times \mathbb{T}_0 \to L(X)$, called fundamental operator of the homogeneous linear dynamic equation* (4.15)*, such that $U(t, t) = I$ for all $t \in \mathbb{T}_0$, and if $x : \mathbb{T}_0 \to X$ is a solution of the homogeneous dynamic equation* (4.15) *with initial condition $x(t_0) = x_0 \in X$, then*

$$x(t) = U(t, t_0)x_0$$

for all $t \in \mathbb{T}_0$.

Proof. Consider the set

$$S := \{\Phi_i : [t_0, v_i] \to L(X);\ v_i \in \mathbb{T}_0^* \text{ and } \Phi_i \text{ is a solution of (4.40) in } \mathrm{BV}([t_0, v_i], L(X))\}.$$

By Theorem 4.17, S is nonempty. Define $\Phi : \mathbb{T}_0^* \to L(X)$ by $\Phi(t) = \Phi_i(t)$, where $\Phi_i \in S$ and $t \in [t_0, v_i]$. Notice that Φ is well defined because $\mathbb{T}_0^* = \bigcup_{\Phi_i \in S} [t_0, v_i]$ and by the uniqueness of a solution of (4.40), if $\Phi_i, \Phi_j \in S$ with $i \neq j$, then $\Phi_i(t) = \Phi_j(t)$ for all $t \in [t_0, v_i] \cap [t_0, v_j]$. Then, it is clear that for all $t \in \mathbb{T}_0^*$, we have

$$\Phi(t) = I + \int_{t_0}^{t} d[A(s)]\Phi(s) \tag{4.41}$$

and Φ is locally of bounded variation on $\mathbb{T}_0^*$. Define $V : \mathbb{T}_0^* \times \mathbb{T}_0^* \to L(X)$ by

$$V(t,\tau) = I + \int_{\tau}^{t} d[A(s)]\Phi(s) \quad \text{for all } t, \quad \tau \in \mathbb{T}_0^*. \tag{4.42}$$

Thus, $V(t, t_0) = \Phi(t)$ and $V(t,t) = I$ for all $t \in \mathbb{T}_0^*$. Let $y : \mathbb{T}_0^* \to X$ be defined by $y(t) = V(t, t_0)x_0$ for all $t \in \mathbb{T}_0^*$. Then, for all $t \in \mathbb{T}_0^*$, we have

$$\begin{aligned}
\int_{t_0}^{t} d[A(s)]y(s) &= \int_{t_0}^{t} d[A(s)]V(s,t_0)x_0 \\
&= \left(\int_{t_0}^{t} d[A(s)]\Phi(s)\right) x_0 \\
&= (\Phi(t) - I)x_0 = (V(t,t_0) - I)x_0 \\
&= y(t) - x_0.
\end{aligned}$$

By Theorem 4.14(i), $x := y|_{\mathbb{T}_0}$ is the solution of the homogeneous dynamic equation (4.15) with initial condition x_0 and $U := V|_{\mathbb{T}_0 \times \mathbb{T}_0}$. □

The next result presents some properties of the operator $V : \mathbb{T}_0^* \times \mathbb{T}_0^* \to L(X)$ defined by (4.42). Its proof can be found in [8, Theorem 4.3].

Theorem 4.19. *Let $s \in \mathbb{T}_0^*$ and $V(\cdot, s) : \mathbb{T}_0^* \to L(X)$ be defined by*

$$V(t,s) = I + \int_{s}^{t} d[A(r)]\Phi(r) \quad \textit{for all } t \in \mathbb{T}_0^*,$$

where Φ is given by (4.41). *Then, V satisfies the following properties*:

(i) *For all $t, \tau \in \mathbb{T}_0^*$, we have*

$$V(t,\tau) = I + \int_{\tau}^{t} d[A(r)]V(r,\tau).$$

(ii) $V(t,t) = I$ *for all* $t \in \mathbb{T}_0^*$.
(iii) $V(t,\tau) = V(t,r)V(r,\tau)$ *for all* $t, \tau, r \in \mathbb{T}_0^*$.
(iv) *There exists* $[V(t,\tau)]^{-1} \in L(X)$ *and* $[V(t,\tau)]^{-1} = V(\tau, t)$ *for all* $t, \tau \in \mathbb{T}_0^*$.

(v) $V(\cdot,\tau)$ *and* $V(\tau,\cdot)$ *belongs to* $\mathrm{BV}([a,b],L(X))$ *for all* $[a,b]\subset\mathbb{T}_0^*$ *and all* $\tau\in\mathbb{T}_0^*$.
(vi) *For every compact set* $[a,b]\subset\mathbb{T}_0^*$, *there exists a constant* $M>0$ *such that*

$$\|V(t,\tau)\|\leq M\quad \text{for all } t,\quad \tau\in[a,b].$$

Using similar ideas as in the proof of Theorem 4.11 and by Theorem 4.19, we obtain the next result.

Theorem 4.20. *The fundamental operator of the homogeneous linear dynamic equation* (4.15) $U:\mathbb{T}_0\times\mathbb{T}_0\to L(X)$ *given in Theorem* 4.18, *satisfies the following properties*:

(i) *For all* $s,r\in\mathbb{T}_0$, *we have*

$$U(s,r)=I+\int_r^s a(\tau)U(\tau,r)\Delta\tau.$$

(ii) $U(t,t)=I$ *for all* $t\in\mathbb{T}_0$.
(iii) $U(t,\tau)=U(t,r)U(r,\tau)$ *for all* $t,\tau,r\in\mathbb{T}_0$.
(iv) *There exists* $[U(t,\tau)]^{-1}\in L(X)$ *and* $[U(t,\tau)]^{-1}=U(\tau,t)$ *for all* $t,\tau\in\mathbb{T}_0$.
(v) $U(\cdot,\tau)$ *and* $U(\tau,\cdot)$ *are regulated on* $[a,b]_{\mathbb{T}}$ *for all* $[a,b]_{\mathbb{T}}\subset\mathbb{T}_0$ *and all* $\tau\in\mathbb{T}_0$.
(vi) *For every compact set* $[a,b]_{\mathbb{T}}\subset\mathbb{T}_0$, *there exists a constant* $M>0$ *such that*

$$\|U(t,\tau)\|\leq M\quad \text{for all } t,\quad \tau\in[a,b]_{\mathbb{T}}.$$

In what follows, we introduce a variation-of-constant formula for linear Volterra-Stieltjes integral equation of the second kind. Its proof is similar to the proof of [8, Theorem 4.10] and therefore, we omit it here.

Theorem 4.21. *Let* $A:\mathbb{T}_0^*\to L(G_0(\mathbb{T}_0^*,X),X)$ *be given by* (4.18) *and* $h:\mathbb{T}_0^*\to X$ *be a function such that* $h\in\mathrm{BV}([a,b],X)$ *for all* $[a,b]\subset\mathbb{T}_0^*$. *If* $y:\mathbb{T}_0^*\to X$ *is a function such that the equality*

$$y(t)=y(t_0)+\int_{t_0}^t d[A(s)]y(s)+h(t)$$

hold for all $t\in\mathbb{T}_0^*$, *then*

$$y(t)=V(t,t_0)y(t_0)+h(t)-h(t_0)-\int_{t_0}^t d_s[V(t,s)](h(s)-h(t_0)),\quad t\in\mathbb{T}_0^*,$$

where $V:\mathbb{T}_0^*\times\mathbb{T}_0^*\to L(X)$ *is given by*

$$V(t,\tau)=I+\int_\tau^t d[A(s)]V(s,\tau),\quad t,\tau\in\mathbb{T}_0^*.$$

Using Theorem 4.21 and the relation between the solution of the nonhomogeneous dynamic equation (4.14) and the solution of a linear Volterra-Stieltjes integral equation, given in Theorem 4.14, we obtain a variation-of-constant formula for the dynamic equation (4.14).

Theorem 4.22. *Let $A : \mathbb{T}_0^* \to L(G_0(\mathbb{T}_0^*, X), X)$ be given by* (4.18) *and $x_0 \in X$. Assume that $a : \mathbb{T} \to L(X)$ and $f : \mathbb{T} \to X$ satisfy conditions* (A1)–(A3)*. Then, the solution of the dynamic equation* (4.15)*, with initial condition x_0, is given by*

$$x(t) = U(t, t_0)x_0 + \int_{t_0}^{t} U(t, s)f(s)\Delta s, \quad s \in \mathbb{T}_0,$$

where $U : \mathbb{T}_0 \times \mathbb{T}_0 \to L(X)$ is given in Theorem 4.18*.*

Proof. Let $x_0 \in X$. By Theorem 4.14(i), if $x : \mathbb{T}_0 \to X$ is the unique solution of the dynamic equation (4.15) with initial condition x_0, then $x = y|_{\mathbb{T}_0}$, where $y : \mathbb{T}_0^* \to X$ is given by

$$y(t) = x_0 + \int_{t_0}^{t} d[A(s)]y(s) + \int_{t_0}^{t} f^*(s)dg(s), \quad t \in \mathbb{T}_0^*.$$

For all $t \in \mathbb{T}_0^*$, consider $h(t) = \int_{t_0}^{t} f^*(s)dg(s)$, where $g(s) = s^*$ for all $s \in \mathbb{T}_0^*$. Then, by Theorem 3.6 and condition (A3), h is locally of bounded variation on $\mathbb{T}_0^*$ (see (4.34)) and by Theorem 4.21,

$$y(t) = V(t, t_0)x_0 + h(t) - h(t_0) - \int_{t_0}^{t} d_s[V(t, s)](h(s) - h(t_0)), \quad t \in \mathbb{T}_0^*, \qquad (4.43)$$

where $V : \mathbb{T}_0^* \times \mathbb{T}_0^* \to L(X)$ is given by

$$V(t, \tau) = I + \int_{\tau}^{t} d[A(s)]V(s, \tau), \quad t, \tau \in \mathbb{T}_0^*.$$

By Theorems 2.20 and 4.19, for all $t \in \mathbb{T}_0^*$, we have

$$\begin{aligned}\int_{t_0}^{t} d_s[V(t, s)](h(s) - h(t_0)) &= \int_{t_0}^{t} d_s[V(t, s)]h(s) \\ &\overset{\text{Theorem 2.20}}{=} V(t, t)h(t) - V(t, t_0)h(t_0) \\ &\quad - \int_{t_0}^{t} V(t, s)f^*(s)dg(s) \\ &\overset{\text{Theorem 4.19}}{=} h(t) - \int_{t_0}^{t} V(t, s)f^*(s)dg(s). \qquad (4.44)\end{aligned}$$

Then, by (4.43), (4.44) and Theorem 3.6, we obtain

$$y(t) = V(t, t_0)x_0 + \int_{t_0}^{t^*} V(t, s)f(s)\Delta(s) \quad \text{for all } t \in \mathbb{T}_0^*.$$

Therefore,

$$x(t) = U(t, t_0)x_0 + \int_{t_0}^{t} U(t, s)f(s)\Delta(s) \quad \text{for all } t \in \mathbb{T}_0,$$

since $x = y|_{\mathbb{T}_0}$, $U = V|_{\mathbb{T}_0 \times \mathbb{T}_0}$ and $t^* = t$ for all $t \in \mathbb{T}_0$. ◻

We point out that the result on the existence and uniqueness of a solution of the dynamic equation

$$x^{\Delta} = A(t)x + \overline{f}(t), \tag{4.45}$$

on a time scale $\mathbb{T}$, as well as the variation-of-constant formula, both presented in [3, 4, 1], require that $A : \mathbb{T} \to \mathbb{R}^{n\times n}$ belongs to $\mathcal{R}(\mathbb{T}, \mathbb{R}^{n\times n})$, where the set $\mathcal{R}(\mathbb{T}, \mathbb{R}^{n\times n})$ is described in the next definition, and $\overline{f} : \mathbb{T} \to \mathbb{R}^n$ is rd-continuous. Moreover, the authors of [3, 4] proved that if $t_0 \in \mathbb{T}$ and $y_0 \in \mathbb{R}^n$, then the solution of the dynamic equation (4.45) with initial condition y_0 is given by

$$y(t) = e_A(t, t_0)y_0 + \int_{t_0}^{t} e_A(t, \sigma(\tau))\overline{f}(\tau)\Delta\tau \quad \text{for all } t \in \mathbb{T}_0,$$

where $\mathbb{T}_0 = [t_0, +\infty) \cap \mathbb{T}$ and e_A is the fundamental operator of the homogeneous linear dynamic equation

$$x^{\Delta} = A(t)x.$$

Definition 4.23. Let $\mathbb{T}$ be a time scale. An $n \times n$-matrix-valued function $A : \mathbb{T} \to \mathbb{R}^{n\times n}$ is called *regressive* (with respect to $\mathbb{T}$) if $I + \mu(t)A(t)$ is invertible for all $t \in \mathbb{T}^k$. Moreover, the set of all regressive $n \times n$-matrix-valued function which are rd-continuous is denoted by $\mathcal{R}(\mathbb{T}, \mathbb{R}^{n\times n}) = \mathcal{R}(\mathbb{T})$.

Our next goal is to show that our main results, namely Theorems 4.13 and 4.22, generalize [3, Theorems 5.8 and 5.24; 1, Theorem 3.1; 4, Theorem 2.1]. To this end, we prove, in the next result, that if $A : \mathbb{T} \to \mathbb{R}^{n\times n}$ and $\overline{f} : \mathbb{T} \to \mathbb{R}^n$ are rd-continuous, then A and $\overline{f}$ satisfy conditions (A1)–(A3).

Lemma 4.24. *Let $\mathbb{T}$ be a time scale and $\mathbb{T}_0 = [t_0, +\infty) \cap \mathbb{T}$, with $t_0 \in \mathbb{T}$. If $\overline{f} : \mathbb{T} \to \mathbb{R}^n$ and $A : \mathbb{T} \to \mathbb{R}^{n\times n}$ are rd-continuous, then conditions* (A1)–(A3) *are satisfied.*

Proof. Since A is rd-continuous, for all regulated function $y : \mathbb{T}_0 \to \mathbb{R}^n$, the product $Ay : \mathbb{T}_0 \to \mathbb{R}^n$, given by $A(s)y(s)$ for all $s \in \mathbb{T}_0$, is also a regulated function. Moreover, every rd-continuous function is a regulated function, then $\overline{f}$ is also regulated. Therefore, by Proposition 3.7, condition (A1) is satisfied.

Before proving condition (A2), we notice that the function $t \ni\mapsto \|A(t)\|e^{t-t_0}$ is regulated. Thus, by Proposition 3.7, the Perron Δ-integral $\int_{t_1}^{t_2} \|A(s)\|e^{s-t_0}\Delta s$ exists for all $t_1, t_2 \in \mathbb{T}_0$. Let $t_1, t_2 \in \mathbb{T}_0$ be fixed and $y, z \in G_0(\mathbb{T}_0, X)$. Once $\int_{t_1}^{t_2} A(s)[y(s) - z(s)]\Delta s = -\int_{t_2}^{t_1} A(s)[y(s) - z(s)]\Delta s$, we may assume that $t_1 < t_2$. By the definition of the Perron Δ-integral, for all $\varepsilon > 0$, there exists a Δ-gauge δ on $[t_1, t_2]_{\mathbb{T}}$ such that

$$\left\| \sum_{i=1}^{|d|} A(\tau_i)[y(\tau_i) - z(\tau_i)](s_i - s_{i-1}) - \int_{t_1}^{t_2} A(s)[y(s) - z(s)]\Delta s \right\| < \varepsilon \quad \text{and}$$

$$\sum_{i=1}^{|d|} \|A(\tau_i)\| e^{\tau_i - t_0}(s_i - s_{i-1})$$
$$< \varepsilon + \int_{t_1}^{t_2} \|A(s)\| e^{s-t_0} \Delta s,$$

provided $d_{\mathbb{T}} = (\tau_i, [s_{i-1}, s_i]_{\mathbb{T}})$ is a δ-fine tagged division of $[t_1, t_2]_{\mathbb{T}}$. Then,

$$\begin{aligned}
&\left\| \int_{t_1}^{t_2} A(s)[y(s) - z(s)]\Delta s \right\| \\
&\quad \leq \left\| \int_{t_1}^{t_2} A(s)[y(s) - z(s)]\Delta s - \sum_{i=1}^{|d|} A(\tau_i)[y(\tau_i) - z(\tau_i)](s_i - s_{i-1}) \right\| \\
&\qquad + \left\| \sum_{i=1}^{|d|} A(\tau_i)[y(\tau_i) - z(\tau_i)](s_i - s_{i-1}) \right\| \\
&\quad \overset{s_i > s_{i-1}}{\leq} \varepsilon + \sum_{i=1}^{|d|} \|A(\tau_i)\| \|y(\tau_i) - z(\tau_i)\| (s_i - s_{i-1}) \\
&\quad = \varepsilon + \sum_{i=1}^{|d|} \|A(\tau_i)\| \|y(\tau_i) - z(\tau_i)\| e^{-(\tau_i - t_0)} e^{(\tau_i - t_0)} (s_i - s_{i-1}) \\
&\quad \leq \varepsilon + \|y - z\|_{\mathbb{T}_0} \sum_{i=1}^{|d|} \|A(\tau_i)\| e^{(\tau_i - t_0)} (s_i - s_{i-1}) \\
&\quad \leq \varepsilon(1 + \|y - z\|_{\mathbb{T}_0}) + \|y - z\|_{\mathbb{T}_0} \int_{t_1}^{t_2} \|A(s)\| e^{s-t_0} \Delta s.
\end{aligned}$$

Therefore, condition (A2) holds by taking ε sufficiently small and $L(s) = \|A(s)\| e^{s-t_0}$ for all $s \in \mathbb{T}_0$. Similarly, since $\overline{f}$ is regulated, $\|\overline{f}\|_{[t_1,t_2]_{\mathbb{T}}} = \sup_{t \in [t_1,t_2]_{\mathbb{T}}} \|\overline{f}(t)\| < \infty$ for all $t_1, t_2 \in \mathbb{T}_0$. Therefore, the function $K : \mathbb{T}_0 \to \mathbb{R}$ defined by

$$K(s) = \|\overline{f}(s)\| \quad \text{for all } s \in \mathbb{T}_0$$

is locally Perron Δ-integrable. Moreover, given $\varepsilon > 0$, there exists a Δ-gauge $\overline{\delta}$ on $[t_1, t_2]_{\mathbb{T}}$ such that

$$\left\| \sum_{i=1}^{|\overline{d}|} \overline{f}(\tau_i)(s_i - s_{i-1}) - \int_{t_1}^{t_2} \overline{f}(s)\Delta s \right\| < \varepsilon \quad \text{and}$$

$$\sum_{i=1}^{|\overline{d}|} \|\overline{f}(\tau_i)\| (s_i - s_{i-1}) < \varepsilon + \int_{t_1}^{t_2} K(s)\Delta s,$$

provided $\overline{d}_{\mathbb{T}} = (\tau_i, [s_{i-1}, s_i]_{\mathbb{T}})$ is a $\overline{\delta}$-fine tagged division of $[t_1, t_2]_{\mathbb{T}}$. Then,

$$\begin{aligned}\left\|\int_{t_1}^{t_2} \overline{f}(s)\Delta s\right\| &\leq \left\|\sum_{i=1}^{|\overline{d}|} \overline{f}(\tau_i)(s_i - s_{i-1}) - \int_{t_1}^{t_2} \overline{f}(s)\Delta s\right\| + \sum_{i=1}^{|\overline{d}|} \|\overline{f}(\tau_i)\|(s_i - s_{i-1}) \\ &\leq 2\varepsilon + \int_{t_1}^{t_2} K(s)\Delta s,\end{aligned}$$

which proves condition (A3). □

Therefore, the result appearing in [3, Theorems 5.8; 1, Theorem 3.1] is a particular case of Theorem 4.13. Moreover, since Theorem 4.13 does not require any continuity for the functions a and f and neither that a is a regressive matrix, we notice that Theorem 4.13 supports a larger class of functions than the class of rd-continuous functions. In particular, if a and f are regulated, by Proposition 3.7, a and f satisfy all the hypotheses of Theorem 4.13.

In order to prove that Theorem 4.22 generalizes [3, Theorems 5.24; 4, Theorem 2.1], we need the following auxiliary result.

Theorem 4.25. *If $f : \mathbb{T}_0 \to \mathbb{R}^n$ and $a : \mathbb{T} \to \mathbb{R}^{n\times n}$ are rd-continuous, then*

$$\int_{t_0}^{t} U(t,s)f(s)\Delta(s) = \int_{t_0}^{t} U(t,\sigma(s))f(s)\Delta s \quad \textit{for all } t \in \mathbb{T}_0, \tag{4.46}$$

where U is the fundamental operator of the homogeneous linear dynamic equation (4.15) *given in Theorem* 4.18.

Proof. At first, we notice that by Lemma 4.24, a and f satisfies all the hypotheses of Theorems 4.13, 4.20 and 4.22. Then, by Theorem 4.20, we have

$$U(s,r) = I + \int_r^s a(\tau)U(\tau,r)\Delta\tau \tag{4.47}$$

for all $r, s \in \mathbb{T}_0$. Moreover,

$$U(t,r)U(r,s) = U(t,s) \quad \text{for all } t, r, s \in \mathbb{T}_0. \tag{4.48}$$

By Eqs. (4.47) and (4.48), we obtain

$$\begin{aligned}U(t,t_0) &= U(t,r)U(r,t_0) \\ &= \left(I + \int_r^t a(\tau)U(\tau,r)\Delta\tau\right)U(r,t_0) \\ &= U(r,t_0) + \int_r^t a(\tau)U(\tau,t_0)\Delta\tau\end{aligned} \tag{4.49}$$

for all $t, r \in \mathbb{T}_0$. Denote $U(t,t_0)$ by $\beta(t)$ for all $t \in \mathbb{T}_0$. By (4.49), we conclude

$$\beta(t) - \beta(r) = \int_r^t a(\tau)\beta(\tau)\Delta\tau \quad \text{for all } t, \quad r \in \mathbb{T}_0. \tag{4.50}$$

Therefore, β is a solution of the dynamic equation

$$\beta^{\Delta}(s) = a(s)\beta(s). \tag{4.51}$$

On the other hand, by Theorem 4.22, the function $x : \mathbb{T}_0 \to \mathbb{R}^n$, given by

$$x(t) = U(t, t_0)x_0 + \int_{t_0}^{t} U(t, s)f(s)\Delta s, \quad t \in \mathbb{T}_0$$

is the unique solution of the dynamic equation

$$x^{\Delta} = a(t)x + f(t), \tag{4.52}$$

with initial condition $x(t_0) = x_0$. Let $y : \mathbb{T}_0 \to \mathbb{R}^n$ be defined by

$$y(t) = U(t, t_0)x_0 + \int_{t_0}^{t} U(t, \sigma(s))f(s)\Delta s, \quad t \in \mathbb{T}_0.$$

By Eq. (4.48), we can rewrite y by

$$y(t) = \beta(t)\alpha(t), \quad t \in \mathbb{T}_0, \tag{4.53}$$

where

$$\alpha(t) = x_0 + \int_{t_0}^{t} U(t_0, \sigma(s))f(s)\Delta s, \quad t \in \mathbb{T}_0.$$

By Theorem [3, Theorem 1.20], we obtain

$$\begin{aligned}
y^{\Delta}(t) &= \beta^{\Delta}(t)\alpha(t) + \beta(\sigma(t))\alpha^{\Delta}(t) \\
&\overset{(4.51)}{=} a(t)\beta(t)\alpha(t) + \beta(\sigma(t))U(t_0, \sigma(t))f(t) \\
&\overset{(4.53)}{=} a(t)y(t) + U(\sigma(t), t_0)U(t_0, \sigma(t))f(t) \\
&\overset{(4.48)}{=} a(t)y(t) + U(\sigma(t), \sigma(t))f(t) \\
&\overset{(4.47)}{=} a(t)y(t) + If(t) \\
&= a(t)y(t) + f(t).
\end{aligned}$$

Therefore, y is a solution of the dynamic equation (4.52) with initial condition $y(t_0) = x_0$ and by the uniqueness of a solution (see Theorem 4.13), we have $y(t) = x(t)$ for all $t \in \mathbb{T}_0$. Consequently,

$$\int_{t_0}^{t} U(t, s)f(s)\Delta s = \int_{t_0}^{t} U(t, \sigma(s))f(s)\Delta s$$

for all $t \in \mathbb{T}_0$. □

We highlight that, by Theorems 4.22 and 4.25, the result in [3, Theorems 5.24; 4, Theorem 2.1] is a particular case of Theorem 4.22. Furthermore, in Theorem 4.22, the functions involved in the dynamic equation (4.14), are not necessarily rd-continuous and they are defined in an arbitrary Banach space instead in $\mathbb{R}^n$.

5. Controllability

This section is devoted to the study of controllability for dynamic equations on time scales.

Let X and $\mathbb{U}$ be a Banach spaces and $\mathbb{T}_0 = [t_0, +\infty) \cap \mathbb{T}$, where $\mathbb{T}$ is a time scale and $t_0 \in \mathbb{T}$. Consider the control system on the time scale $\mathbb{T}$ described by

$$x^{\Delta} = a(t)x + B(t)u(t), \tag{5.1}$$

where $B : \mathbb{T} \to L(\mathbb{U}, X)$, $u : \mathbb{T} \to \mathbb{U}$ and $a : \mathbb{T} \to L(X)$. Assume that the following conditions are satisfied:

(C1) The Perron Δ-integrals

$$\int_{t_1}^{t_2} B(s)u(s)\Delta s \quad \text{and} \quad \int_{t_1}^{t_2} a(s)y(s)\Delta s$$

exist for all $t_1, t_2 \in \mathbb{T}$, whenever $y : \mathbb{T} \to X$ is regulated.

(C2) There is a locally Perron Δ-integrable function $L : \mathbb{T} \to \mathbb{R}$ such that

$$\left\| \int_{t_1}^{t_2} a(s)[z(s) - y(s)]\Delta s \right\| \leq \|z - y\|_{\mathbb{T}_0} \int_{t_1}^{t_2} L(s)\Delta s$$

for all $z, y \in G_0(\mathbb{T}_0, X)$ and all $t_1, t_2 \in \mathbb{T}_0$.

(C3) There is a locally Perron Δ-integrable function $K : \mathbb{T}_0 \to \mathbb{R}$ such that

$$\left\| \int_{t_1}^{t_2} B(s)u(s)\Delta s \right\| \leq \int_{t_1}^{t_2} K(s)\Delta s$$

for all $t_1, t_2 \in \mathbb{T}_0$.

We denote by $\mathcal{U}$ the space of all control functions, $u : \mathbb{T} \to \mathbb{U}$, such that conditions (C1) and (C3) are fulfilled.

By Theorems 4.13 and 4.22, the dynamic equation (5.1), with initial condition $\widetilde{x} \in X$ and control $u \in \mathcal{U}$, admits a unique solution described by

$$x(t) = U(t, t_0)\widetilde{x} + \int_{t_0}^{t} U(t, s)B(s)u(s)\Delta s, \quad s \in \mathbb{T}_0,$$

where $U : \mathbb{T}_0 \times \mathbb{T}_0 \to L(X)$ is given in Theorem 4.18. Moreover, we denote $x(\cdot)$ by $x(\cdot, \widetilde{x}, u)$.

Let us now define a concept of controllability for the dynamic system (5.1).

Definition 5.1. Let $T \in \mathbb{T}_0$ be fixed and $S \subseteq X$ be such that $0 \in S$, where 0 denotes the neutral element of X. The state $d \in S$ is said to be:

(i) *approximately controllable* at time T to a point $\widetilde{x} \in X$, if there exists a sequence $\{u_n\}$ in $\mathcal{U}$ such that $x(T, d, u_n) \to \widetilde{x}$ as $n \to \infty$;

(ii) *strictly controllable at time* T to a point $\widetilde{x} \in X$, if there exists $u \in \mathcal{U}$ such that $x(T, d, u) = \widetilde{x}$.

The dynamic system (5.1) is *approximately controllable* (*strictly controllable*) *at time* T, if all points of S are approximately controllable (strictly controllable) at time T to all points of X.

For all $t \in \mathbb{T}_0$ and all $d \in S$, define $G(t) : \mathcal{U} \to X$ by

$$G(t)u = \int_{t_0}^{t} U(t,s)B(s)u(s)\Delta s, \tag{5.2}$$

where $U : \mathbb{T}_0 \times \mathbb{T}_0 \to L(X)$ is given in Theorem 4.18.

The next result gives necessary and sufficient conditions for the system (5.1) to be approximately controllable (strictly controllable).

Theorem 5.2. *Let* $T \in \mathbb{T}_0$ *and* $G(T) : \mathcal{U} \to X$ *be given by* (5.2). *Then, the following assertions hold*:

(i) *The dynamic system* (5.1) *is approximately controllable at time* T *if and only if the range of* $G(T)$ *is everywhere dense in* X.
(ii) *The dynamic system* (5.1) *is strictly controllable at time* T *if and only if the mapping* $G(T)$ *is onto.*

Proof. We start by proving item (i). Let $\widetilde{x} \in X$ be arbitrary. Since the dynamic system (5.1) is approximately controllable at time T, Definition 5.1 yields all points in S are approximately controllable at time T to all points of X. In particular, $d = 0$ is approximately controllable at time T to $\widetilde{x}$. Therefore, there exists a sequence $\{u_n\}$ in $\mathcal{U}$ such that $x(T, 0, u_n) \to \widetilde{x}$ as $n \to \infty$, that is, $G(T)u_n \to \widetilde{x}$, as $n \to \infty$. Hence, the range of $G(T)$ is everywhere dense in X. Conversely, for arbitrary $d \in S$ and $\widetilde{x} \in X$, $\widetilde{x} - U(T, t_0)d \in X$ and if the range of $G(T)$ is everywhere dense in X, then there exists a sequence $\{u_n\}$ in $\mathcal{U}$ such that $G(T)u_n \to \widetilde{x} - U(T, t_0)d$ as $n \to \infty$, that is, $x(T, d, u_n) \to \widetilde{x}$, as $n \to \infty$, that is, the dynamic system (5.1) is approximately controllable at time T.

As in item (i), in order to prove item (ii), we take $\widetilde{x}$ arbitrarily and $d = 0$. Once the dynamic system (5.1) is controllable at time T, there exists $u \in \mathcal{U}$ such that $x(T, 0, u) = \widetilde{x}$, that is, $G(T)u = \widetilde{x}$. Therefore, $G(T)$ is onto. On the other hand, if $G(T)$ is onto, then for all $d \in S$ and all $\widetilde{x} \in X$, there exists $u \in \mathcal{U}$ such that $G(T)u = \widetilde{x} - U(T, t_0)d$, which implies that $x(T, d, u) = \widetilde{x}$. Therefore, the dynamic system (5.1) is strictly controllable at time T. □

In the following theorem, we consider the particular case where $X = \mathbb{R}^n$, $\mathbb{U} = \mathbb{R}^p$, $p < n$, and use the notation M' to denote the transpose of a given matrix M.

Theorem 5.3. *Let* $\mathbb{T}$ *be a time scale and* $\mathbb{T}_0 = [t_0, +\infty) \cap \mathbb{T}_0$, *with* $t_0 \in \mathbb{T}$. *Assume that* $f : \mathbb{T} \to \mathbb{R}^n$ *and* $a : \mathbb{T} \to \mathbb{R}^{n \times n}$ *are rd-continuous. Then,* $G(T)$ *is onto if and only if the rows of the matrix* $U(t_0, \sigma(T))B(T)$ *are linearly independent, where* $T \in \mathbb{T}_0$, $G(T) : \mathcal{U} \to X$ *is defined by* (5.2) *and* $U : \mathbb{T}_0 \times \mathbb{T}_0 \to L(X)$ *is given in Theorem* 4.18.

Proof. Let $T \in \mathbb{T}_0$ and suppose that the rows of the matrix $U(t_0, \sigma(T))B(T)$ are linearly independent. Then, the matrix

$$\mathcal{C}(t_0, T) = \int_{t_0}^{T} U(T, \sigma(s))B(T)B'(s)U'(t_0, s)\Delta s$$

is positive definite. Let $\widetilde{x} \in X$ be arbitrary and define

$$u(s) = B'(s)U'(t_0, s)\mathcal{C}^{-1}(t_0, T)\widetilde{x} \quad \text{for all } s \in [t_0, T]_{\mathbb{T}}.$$

By (5.2) and Theorem 4.25, we obtain

$$G(T)u = \int_{t_0}^{T} U(T, \sigma(s))B(s)u(s)\Delta s = \widetilde{x},$$

which shows that $G(T)$ is onto.

Reciprocally, if the rows of the matrix $U(t_0, \sigma(T))B(T)$ are linearly dependent, then there exists $\widetilde{x} \in X$, $\widetilde{x} \neq 0$, for which

$$\widetilde{x}'U(t_0, \sigma(s))B(s) \equiv 0 \quad \text{for all } s \in [t_0, T]_{\mathbb{T}}. \tag{5.3}$$

Since $G(T)$ is onto, there exists u such that $G(T)u = \widetilde{x}$. Therefore, by Theorem 4.25, we have

$$G(T)u = \int_{t_0}^{T} U(T, \sigma(s))B(s)u(s)\Delta s = \widetilde{x}.$$

Multiplying the above equation by $\widetilde{x}'U(t_0, \sigma(T))$, we obtain, by (5.3),

$$\widetilde{x}'U(t_0, \sigma(T))\widetilde{x} = \int_{t_0}^{T} \widetilde{x}'U(t_0, \sigma(s))B(s)u(s)\Delta s = 0,$$

which contradicts the fact that U is the fundamental operator of the dynamic equation (5.1). □

The next result is a straightforward consequence of Theorems 5.2 and 5.3. Unlike the proofs of the results presented in [5, Theorem 4.3; 9, Theorem 2.2; 25, Theorem 1] which deals with the rank of some matrices, our result provides another characterization of strict controllability and its proof uses linear dependence of a much simpler matrix.

Corollary 5.4. *Let $\mathbb{T}$ be a time scale and $\mathbb{T}_0 = [t_0, +\infty) \cap \mathbb{T}_0$, with $t_0 \in \mathbb{T}$. Assume that $f : \mathbb{T} \to \mathbb{R}^n$ and $a : \mathbb{T} \to \mathbb{R}^{n \times n}$ are rd-continuous. Then, the dynamic system* (5.1) *is strictly controllable at time $T \in \mathbb{T}_0$ if and only if the rows of the matrix $U(t_0, \sigma(T))B(T)$ are linearly independent, where $U : \mathbb{T}_0 \times \mathbb{T}_0 \to L(X)$ is given in Theorem* 4.18.

6. Examples

In this section, we give some examples of strictly controllable dynamic systems on time scales defined in $\mathbb{R}^n$ and in arbitrary Banach spaces.

The next remark was borrowed from [3, Examples 5.9 and 5.19]. It is important because it describes fundamental operator of a certain linear dynamic equation on two different time scales.

Remark 6.1. Let I be the identity $n \times n$-matrix and $U(\cdot,\cdot)$ be the fundamental operator given in Theorem 4.18.

(i) If $\mathbb{T} = \mathbb{R}$, then $\sigma(t) = t$ for all $t \in \mathbb{T}$. Moreover, any $n \times n$-matrix-valued function a on $\mathbb{T}$ is such that $I + \mu(t)a(t)$ is invertible for all $t \in \mathbb{T}^k$ and hence, it is regressive. In this case, a matrix-valued function a is rd-continuous if and only if it is continuous. Then, the initial value problem

$$x^\Delta = a(t)x, \quad x(t_0) = x_0$$

has a unique solution provided a is continuous. Moreover, if a is a constant $n \times n$-matrix, then $U(t,t_0) = e^{a(t-t_0)}$ for all $t \in \mathbb{R}$.

(ii) If $\mathbb{T} = \mathbb{Z}$, then $\sigma(t) = t + 1$ and any $n \times n$-matrix-valued function a on $\mathbb{T}$ is rd-continuous. Moreover, in order for a matrix-valued function a on $\mathbb{T}$ to be regressive, the matrix $I + a(t)$ needs to be invertible for each $t \in \mathbb{Z}$. Furthermore, if a is a constant $n \times n$-matrix, then $U(t,t_0) = (I + a)^{(t-t_0)}$ for all $t \in \mathbb{Z}$.

Example 6.2. Let $\mathbb{T}$ be a time scale, $t_0 \in \mathbb{T}$, $\mathbb{T}_0 = [t_0, +\infty) \cap \mathbb{T}$ and $\mathcal{U}$ be the set of all regulated functions $u : \mathbb{T} \to \mathbb{R}$ such that $\|u\|_{\mathbb{T}_0} = \sup_{s\in\mathbb{T}_0} |u(s)| < c$ for some constant $c > 0$.

Consider the following control dynamic system:

$$x^\Delta = ax + Bu(t) \tag{6.1}$$

on the time scale $\mathbb{T}$, where $u \in \mathcal{U}$,

$$a = \begin{pmatrix} -\dfrac{8}{45} & \dfrac{1}{30} \\ -\dfrac{1}{45} & -\dfrac{1}{30} \end{pmatrix} \quad \text{and}$$

$$B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

At first, we notice that, since a and B are real constant matrices, a and B are rd-continuous. Then, by Lemma 4.24, conditions (C1) and (C2), from Sec. 5, are satisfied. Moreover, by Proposition 3.7, we have

$$\left\| \int_{t_1}^{t_2} Bu(s)\Delta s \right\| \leq \|B\| \|u\|_{[t_1,t_2]_\mathbb{T}} [g(t_2) - g(t_1)] \leq \|B\| c[g(t_2) - g(t_1)] \tag{6.2}$$

for all $t_1, t_2 \in \mathbb{T}_0$ and all $u \in \mathcal{U}$, where $g(t) = t^*$ for all $t \in \mathbb{T}_0^*$. Since $g|_{\mathbb{T}_0}$ is the identity function, g is delta differentiable. Then, the function $K : \mathbb{T}_0 \to \mathbb{R}$ given by

$$K(t) = \|B\|cg^{\Delta}(t), \quad t \in \mathbb{T}_0$$

is well-defined and

$$\int_{t_1}^{t_2} K(s)\Delta s = \|B\|c[g(t_2) - g(t_1)] \tag{6.3}$$

for all $t_1, t_2 \in \mathbb{T}_0$. By Eqs. (6.2) and (6.3), for all $t_1, t_2 \in \mathbb{T}_0$, we have

$$\left\|\int_{t_1}^{t_2} Bu(s)\Delta s\right\| \leq \int_{t_1}^{t_2} K(s)\Delta s,$$

which proves condition (C3).

Let us consider two cases for the time scale $\mathbb{T}$.

(a) $\mathbb{T} = \mathbb{R}$.

Note that a is continuous, since it is a real constant 2×2-matrix. Then, by Remark 6.1(i), there exists a solution of the dynamic equation (6.1) and $U(t, t_0) = e^{a(t-t_0)}$. On the other hand, it is clear that $U(t_0, \sigma(t))$ is invertible, for all $t \in \mathbb{T}_0$ and consequently, the rows of $U(t_0, \sigma(t)) = U(t_0, \sigma(t))B(t)$ are linearly independent, for all $t \in \mathbb{T}_0$. Therefore, Corollary 5.4 guarantees that the dynamic system (6.1) is strictly controllable.

(b) $\mathbb{T} = \mathbb{Z}$. Note that

$$I + a = \begin{pmatrix} \frac{37}{45} & \frac{1}{30} \\ -\frac{1}{45} & \frac{29}{30} \end{pmatrix}$$

is invertible, since the determinant of $I + a$ is not zero. Therefore, by Remark 6.1(ii), a is regressive and $U(t, t_0) = (I + a)^{(t-t_0)}$ for all $t \in \mathbb{Z}$. On the other hand, once $(I + a)$ is invertible, $(I + a)^{(t-t_0)}$ is also invertible for all $t \in \mathbb{T}_0$ and consequently, $U(\sigma(t), t_0) = U(t + 1, t_0)$ is invertible, for all $t \in \mathbb{T}_0$, which implies that the rows of $U(\sigma(t), t_0) = U(\sigma(t), t_0)B(t)$ are linearly independent, for all $t \in \mathbb{T}_0$. Then, by Corollary 5.4, we conclude that the dynamic equation (6.1) is strictly controllable.

Example 6.3. Let X be a Banach space. Consider the time scale $\mathbb{T} = h\mathbb{Z} = \{hk : k \in \mathbb{Z}\}$ with $h > 0$. Let $t_0 = h$ and consider the homogeneous linear dynamic equation

$$x^{\Delta} = a(t)x \tag{6.4}$$

on the time scale $\mathbb{T}_0 = [t_0, +\infty) \cap \mathbb{T}$, where $a : \mathbb{T} \to L(X)$ is given by

$$a(t)x = \begin{cases} 0, & (t, x) \in \mathbb{T}\backslash\mathbb{T}_0 \times X, \\ \dfrac{x}{t}, & (t, x) \in \mathbb{T}_0 \times X. \end{cases} \tag{6.5}$$

Since a is regulated, for all regulated functions $y : \mathbb{T}_0 \to X$, the product $ay : \mathbb{T}_0 \to X$, given by $a(s)y(s)$ for all $s \in \mathbb{T}_0$, is also a regulated function. Then, by Proposition 3.7, the Perron Δ-integral $\int_{t_1}^{t_2} a(s)y(s)\Delta s$ exists for all $t_1, t_2 \in \mathbb{T}_0$ and all regulated functions $y : \mathbb{T}_0 \to X$. This leads to condition (C1).

Let us prove condition (C2). Consider $y, z \in G_0(\mathbb{T}_0, X)$ and define $L : \mathbb{T}_0 \to \mathbb{R}$ by

$$L(s) = \frac{e^{s-t_0}}{s} \quad \text{for all } s \in \mathbb{T}_0.$$

Let $t_1, t_2 \in \mathbb{T}_0$ and assume that $t_1 < t_2$. Then, by [3, Theorem 1.79] (with obvious adaptation to Banach space-valued functions), we get

$$\begin{aligned}
\left\| \int_{t_1}^{t_2} a(s)[y(s) - z(s)]\Delta s \right\| &= \left\| \sum_{k=\frac{t_1}{h}}^{\frac{t_2}{h}-1} a(kh)[y(kh) - z(kh)]h \right\| \\
&= \left\| \sum_{k=\frac{t_1}{h}}^{\frac{t_2}{h}-1} \frac{y(kh) - z(kh)}{kh} h \right\| \\
&\leq \sum_{k=\frac{t_1}{h}}^{\frac{t_2}{h}-1} \frac{\|y(kh) - z(hk)\| e^{-(kh-t_0)} e^{kh-t_0}}{k} \\
&\leq \|y - z\|_{\mathbb{T}_0} \sum_{k=\frac{t_1}{h}}^{\frac{t_2}{h}-1} \frac{e^{kh-t_0}}{k} \\
&= \|y\|_{[t_1,t_2]_{\mathbb{T}_0}} \sum_{k=\frac{t_1}{h}}^{\frac{t_2}{h}-1} \frac{e^{kh-t_0}}{kh} h \\
&= \|y\|_{[t_1,t_2]_{\mathbb{T}_0}} \int_{t_1}^{t_2} L(s)\Delta s
\end{aligned}$$

and hence, condition (C2) is fulfilled.

On the other hand, if $x : \mathbb{T}_0 \to X$ is the solution of the dynamic equation (6.4), with initial condition $x(t_0) = x_0$, then, for all $t \in \mathbb{T}_0$, we have

$$x(t) = x_0 + \sum_{k=\frac{t_0}{h}}^{\frac{t}{h}-1} \frac{x(kh)h}{kh} = x_0 + \sum_{k=\frac{t_0}{h}}^{\frac{t}{h}-1} \frac{x(kh)}{k} = x_0 + \sum_{k=t_0}^{t-h} \frac{x(k)h}{k}.$$

Thus, a straightforward calculation shows that $U(t, t_0) = (I + h)^{\frac{t-t_0}{h}}$, where $I \in L(X)$ is the identity operator.

Now, let $\mathbb{U}$ be a Banach space, with $\mathbb{U} \subset X$ and $\mathcal{U}$ be the set of all regulated functions $u : \mathbb{T} \to \mathbb{U}$ such that for every interval $[a, b]_{\mathbb{T}} \subset \mathbb{T}_0$, there exists a constant

$M > 0$ for which $\|u\|_{[a,b]_{\mathbb{T}}} < M$. Consider the nonhomogeneous dynamic system

$$x^{\Delta} = a(t)x + B(t)u(t), \tag{6.6}$$

where $B : \mathbb{T} \to L(\mathbb{U}, X)$ is defined by $B(t)y = y$ for all $y \in \mathbb{U}$ and all $t \in \mathbb{T}$, $a : \mathbb{T} \to L(X)$ is given by (6.5) and $u \in \mathcal{U}$. Then, by Proposition 3.7, the Perron Δ-integral $\int_{t_1}^{t_2} u(s)\Delta s = \int_{t_1}^{t_2} B(s)u(s)\Delta s$ exists for all $t_1, t_2 \in \mathbb{T}_0$ and

$$\left\| \int_{t_1}^{t_2} u(s)\Delta s \right\| \leq \|u\|_{[t_1,t_2]_{\mathbb{T}}}[g(t_2) - g(t_1)] \leq M[g(t_2) - g(t_1)] = \int_{t_1}^{t_2} K(s)\Delta s,$$

where $K(s) = Mg^{\Delta}(s)$ for all $s \in \mathbb{T}_0$ and $g(s) = s^*$ for all $s \in \mathbb{T}_0^*$. Therefore, condition (C3) holds.

In what follows, we aim to prove that the dynamic system (6.6) is strictly controllable.

Let $t \in \mathbb{T}_0$ be fixed and $G(t) : \mathcal{U} \to X$ be given by

$$G(t)u = \int_{t_0}^{t} (I + h)^{\frac{t-s}{h}} u(s)\Delta s.$$

Note that if $x \in X$ and $u : \mathbb{T} \to \mathbb{U}$ is given by

$$u(s) = \begin{cases} \dfrac{U(t_0, t)x}{t - t_0} = \dfrac{(I + h)^{\frac{t_0 - t}{h}} x}{t - t_0} & \text{if } s \in \mathbb{T}\backslash\mathbb{T}_0, \\ \dfrac{U(s, t)x}{t - t_0} = \dfrac{(I + h)^{\frac{s - t}{h}} x}{t - t_0} & \text{if } s \in \mathbb{T}_0, \end{cases}$$

then, by Theorem 4.20, u is well defined and $u \in \mathcal{U}$. Moreover,

$$G(t)u = \int_{t_0}^{t} (I + h)^{\frac{t-s}{h}} \frac{(I + h)^{\frac{s-t}{h}}}{t - t_0} x\Delta s = \int_{t_0}^{t} \frac{x}{t - t_0}\Delta s = x$$

which implies that $G(t)$ is onto and by Theorem 5.2, the dynamic system (6.6) is strictly controllable at time $t \in \mathbb{T}_0$. Since t is arbitrary, we conclude that the dynamic system (6.6) is strictly controllable.

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