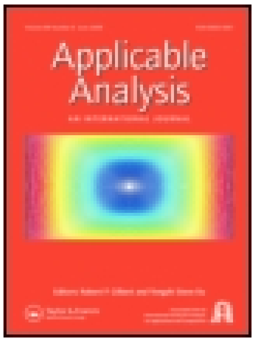

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Roughness-induced effects on the convection–diffusion–reaction problem in a thin domain

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ABSTRACT

In this paper, we investigate a convection–diffusion–reaction problem in a thin domain endowed with the Robin-type boundary condition describing the reaction catalyzed by the upper wall. Motivated by the microfluidic applications, we allow the oscillating behavior of the upper boundary and analyze the resonant case where the amplitude and period of the oscillation have the same small order as the domain's thickness. Depending on the magnitude of the reaction mechanism, we rigorously derive three different asymptotic models via the unfolding operator method. In particular, we identify the critical case in which the effects of the domain's geometry and all physically relevant processes become balanced.

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1. Introduction

The flow problems posed in thin domains (domains whose longitudinal dimension is much larger than the transverse one) are of great interest due to their practical importance. In real-life applications, the boundary of such domains are usually not perfectly smooth, i.e. they usually have some small rugosities, dents, etc. In solid mechanics, the typical examples of such structures would be thin rods, plates or shells. Lubrication devices and blood circulatory system are the obvious examples associated to fluid mechanics. No matter the context is, introducing the small parameter as the perturbation quantity in the domain boundary makes the analysis very challenging from the mathematical point of view.

Motivated by the numerous applications in which the effective flow is significantly affected by the irregular wall roughness, we suppose that the upper boundary of our thin domain has an oscillating behavior. Namely, the considered domain reads

$$R^\varepsilon = \left\{(x,y) \in \mathbb{R}^2 : 0 < x < 1, 0 < y < \varepsilon h\left(\frac{x}{\varepsilon}\right)\right\}, \quad 0 < \varepsilon \ll 1. \tag{1}$$

In the sequel, we address the following elliptic boundary-value problem:

$$\begin{cases} -\kappa\,\Delta u^\varepsilon + Q^\varepsilon(y)\partial_x u^\varepsilon + cu^\varepsilon = f^\varepsilon & \text{in } R^\varepsilon, \\ \kappa\dfrac{\partial u^\varepsilon}{\partial \nu^\varepsilon} = \varepsilon^\alpha(g(x) - u^\varepsilon) & \text{on } \Gamma^\varepsilon = \left\{(x,y) \in \mathbb{R}^2 : 0 < x < 1, y = \varepsilon h\left(\dfrac{x}{\varepsilon}\right)\right\}, \\ \dfrac{\partial u^\varepsilon}{\partial \nu^\varepsilon} = 0 & \text{on } \partial R^\varepsilon \backslash \Gamma^\varepsilon. \end{cases} \tag{2}$$

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Here $\kappa, c = \text{const.} > 0$, the vector $\nu^\epsilon = (\nu_1^\epsilon, \nu_2^\epsilon)$ is the unit outward normal to ∂R^ϵ and $\frac{\partial}{\partial \nu^\epsilon}$ is the outside normal derivative. For the function Q^ε, we assume

$$Q^\varepsilon(y) = Q\left(\frac{y}{\varepsilon}\right),$$

where $Q \in L^\infty(0, h_1)$ is a non-negative function, $h_1 = \max_{x\in\mathbb{R}} h(x)$. This assumption is reasonable from the point of view of the applications, since we are tackling the process in a thin domain and Q^ε can be interpreted as the entering (unidirectional) velocity in e.g. the solute transport problem (see [1,2]). The boundary perturbation function h satisfies the usual assumptions listed in ($\mathbf{H_h}$), see Section 2.1. Finally, we suppose $g \in L^2(0,1)$. As you can see, the governing equation is endowed with the Robin-type boundary condition which models the reaction catalyzed by the upper wall. By taking the reaction coefficient in the form ε^α, $\alpha \in \mathbb{R}$ (see $(2)_2$), our aim is to address different order of magnitudes of the prescribed reaction mechanism. Such type of elliptic boundary-value problems describes many processes naturally arising in chemical engineering, in particular related to microfluidic applications (see e.g. [3]). Our goal is to study the asymptotic behavior of the described problem, as $\varepsilon \to 0$.

To achieve our goal, we employ the homogenization technique based on the unfolding method proposed in [4,5] (see also [6]). Due to its ability to elegantly treat the surface integrals, the unfolding method has been extensively used for derivation of lower-dimensional approximations in the last period. We refer the reader e.g. to [7–10]. In this work, we adapt the variant of this method introduced by Arrieta and Villanueva-Pesqueira [11,12] for thin domains. As a result, we obtain three different asymptotic models, depending on the value of the coefficient α. More precisely, for $\alpha < 1$, the process turns out to be dominated by the function g from the Robin boundary condition, with $g \in H^1(0,1)$ (see Theorem 3.4). For $\alpha > 1$, the effective model does not depend on g (see Theorem 3.5), meaning that the reaction mechanism does not affect the process. Between those two cases, we identify the critical (and the most interesting) case $\alpha = 1$ capturing the effects of the domain's geometry and all the physical processes relevant to the problem as well (see Theorem 3.1). We firmly believe that the results presented here could prove useful in numerical simulations of the convection–diffusion–reaction problems in thin domains with irregularities.

To conclude the Introduction, let us provide more bibliographic remarks on the subject. In [13], the Neumann problem for the Laplace equation posed in a domain (of thickness $\mathcal{O}(1)$) with highly oscillating boundary has been considered via asymptotic expansion method. Using rigorous analysis in appropriate functional setting, a thin-domain situation has been addressed in [14]. It should be emphasized that, in both papers, a homogeneous Neumann boundary condition has been imposed and that the transition to a Robin-type boundary condition cannot be considered straightforward whatsoever. The present work can be viewed as the continuation of our recent work [15] in which a thin domain without boundary oscillations has been studied. Notice that introducing boundary irregularities to the problem forced us to completely change the approach.

2. Preliminary results

In this section, we introduce some notations and the functional setting. First we notice that the variational formulation of (2) reads

$$\int_{R^\varepsilon} \kappa \nabla u^\varepsilon \nabla \varphi + Q^\varepsilon(y)\partial_x u^\varepsilon \varphi + c u^\varepsilon \varphi \mathrm{d}x\mathrm{d}y + \varepsilon^\alpha \int_{\Gamma^\varepsilon} u^\varepsilon \varphi \mathrm{d}S$$
$$= \int_{R^\varepsilon} f^\varepsilon \varphi \mathrm{d}x\mathrm{d}y + \varepsilon^\alpha \int_{\Gamma^\varepsilon} g\varphi \mathrm{d}S, \quad \forall \varphi \in H^1(R^\varepsilon). \tag{3}$$

Remark 2.1: The existence and uniqueness in $H^1(R^\varepsilon)$ follows as in [15, Lemma 3.4] under condition

$$c > ||Q||^2_{L^\infty(0,h_1)}/4\kappa. \tag{4}$$

In fact, it can be proved that the bilinear form $a_\varepsilon : H^1(R^\varepsilon) \times H^1(R^\varepsilon) \mapsto \mathbb{R}$ set by

$$a_\varepsilon(u,v) = \int_{R^\varepsilon} \kappa \nabla u \nabla v + Q^\varepsilon(y)\partial_x uv + cuv\mathrm{d}x\mathrm{d}y$$

is continuous and uniformly coercive.

Thus, we have a family of solutions $\{u^\varepsilon\}_{\varepsilon>0}$ given by (3) and we are concerned here about the asymptotic behavior at $\varepsilon = 0$.

2.1. The unfolding operator

In order to study the convergence of the solutions u^ε, we apply the unfolding method firstly introduced in [4,5] (see also [6]) for oscillating coefficients and perforated domains. Here, we just give some notations and recall the main results concerning this method in the thin domain situation. The proofs and all the details can be found in [11,12].

The function h used to set the thin domain (1) satisfies the following hypothesis:

($\mathbf{H_h}$) $h : \mathbb{R} \to \mathbb{R}$ is a strictly positive, Lipschitz and L-periodic with $h' \in L^\infty(\mathbb{R})$. Also, if $h_0 = \min_{x\in\mathbb{R}} h(x)$ and $h_1 = \max_{x\in\mathbb{R}} h(x)$, we have $0 < h_0 \le h(x) \le h_1$ for all $x \in \mathbb{R}$.

Throughout this paper, we use the following notations. We call Y^* the representative cell of the thin domain R^ε which is given by

$$Y^* = \left\{(y_1, y_2) \in \mathbb{R}^2 : 0 < y_1 < L \quad \text{and} \quad 0 < y_2 < h(y_1)\right\}. \tag{5}$$

The average of $\varphi \in L^1_{loc}(\mathbb{R}^2)$ on a measure set $\mathcal{O} \subset \mathbb{R}^2$ is denoted by $\langle\varphi\rangle_{\mathcal{O}} := \frac{1}{|\mathcal{O}|}\int_{\mathcal{O}} \varphi(x)\,\mathrm{d}x$, where $|\mathcal{O}|$ sets the Lebesgue measure of any measure set $\mathcal{O}$.

We will also need the following functional spaces which are defined by periodic functions in the variable $y_1 \in (0, L)$

$$L^2_\#(Y^*) = \{\varphi \in L^2(Y^*) \ : \ \varphi(y_1, y_2) \text{ is } L-\text{periodic in} y_1\},$$
$$L^2_\#\left((0,1)\times Y^*\right) = \{\varphi \in L^2((0,1)\times Y^*) \ : \ \varphi(x, y_1, y_2) \text{ is } L-\text{periodic in } y_1\},$$
$$H^1_\#(Y^*) = \{\varphi \in H^1(Y^*) \ : \ \varphi|_{\partial_{left} Y^*} = \varphi|_{\partial_{right} Y^*}\}.$$

If we denote by $[a]_L$ the unique integer number such that $a = [a]_L L + \{a\}_L$ where $\{a\}_L \in [0, L)$, then for each $\varepsilon > 0$ and any $x \in \mathbb{R}$, we have

$$x = \varepsilon\left[\frac{x}{\varepsilon}\right]_L L + \varepsilon\left\{\frac{x}{\varepsilon}\right\}_L \quad \text{where } \left\{\frac{x}{\varepsilon}\right\}_L \in [0, L).$$

Let us also denote

$$I_\varepsilon = \ \text{Int}\left(\bigcup_{k=0}^{N_\varepsilon} [kL\varepsilon, (k+1)L\varepsilon]\right),$$

where N_ε is the largest integer such that $\varepsilon L(N_\varepsilon + 1) \le 1$. We also set

$$\Lambda_\varepsilon = (0,1)\backslash I_\varepsilon = [\varepsilon L(N_\varepsilon+1), 1), \quad R_0^\varepsilon = \left\{(x,y)\in\mathbb{R}^2 : x \in I_\varepsilon, 0 < y < \varepsilon h\left(\frac{x}{\varepsilon}\right)\right\}$$
$$\text{and} \quad R_1^\varepsilon = \left\{(x,y)\in\mathbb{R}^2 : x \in \Lambda_\varepsilon, 0 < y < \varepsilon h\left(\frac{x}{\varepsilon}\right)\right\}.$$

Notice that we have $\Lambda_\varepsilon = \emptyset$ if $\varepsilon L(N_\varepsilon + 1) = 1$. In this case $R_0^\varepsilon = R^\varepsilon$ and $R_1^\varepsilon = \emptyset$.

Definition 2.2: Let φ be a Lebesgue-measurable function in R^ε. The unfolding operator $\mathcal{T}_\varepsilon$ acting on φ is defined as the following function in $(0,1)\times Y^*$:

$$\mathcal{T}_\varepsilon\varphi(x,y_1,y_2) = \begin{cases} \varphi\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_L L+\varepsilon y_1, \varepsilon y_2\right) \text{ a.e. } (x,y_1,y_2)\in I_\varepsilon\times Y^*, \\ 0 \text{ a.e. } (x,y_1,y_2)\in\Lambda_\varepsilon\times Y^*. \end{cases}$$

Proposition 2.3: *The unfolding operator satisfies the following properties:*

(1) $\mathcal{T}_\varepsilon$ *is linear and satisfies* $\mathcal{T}_\varepsilon(\varphi\psi)=\mathcal{T}_\varepsilon(\varphi)\mathcal{T}_\varepsilon(\psi)$.
(2) *Let* φ *a Lebesgue function in* Y^* *extended periodically in the first variable. Then,* $\varphi^\varepsilon(x,y)=\varphi(\frac{x}{\varepsilon},\frac{y}{\varepsilon})$ *is measurable in* R^ε *and* $\mathcal{T}_\varepsilon(\varphi^\varepsilon)(x,y_1,y_2)=\varphi(y_1,y_2)$*. Moreover, if* $\varphi\in L^2(Y^*)$*, then* $\varphi^\varepsilon\in L^2(R^\varepsilon)$.
(3) *For all* $\varphi^\varepsilon\in L^1(R^\varepsilon)$*, we have*

$$\frac{1}{L}\int_{(0,1)\times Y^*}\mathcal{T}_\varepsilon(\varphi)(x,y_1,y_2)\mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2=\frac{1}{\varepsilon}\int_{R^\varepsilon}\varphi(x,y)\mathrm{d}x\mathrm{d}y-\frac{1}{\varepsilon}\int_{R_1^\varepsilon}\varphi(x,y)\mathrm{d}x\mathrm{d}y.$$

(4) $T_\varepsilon(\varphi)\in L^2((0,1)\times Y^*)$ *for all* $\varphi\in L^2(R^\varepsilon)$ *with*

$$||\mathcal{T}_\varepsilon(\varphi)||_{L^2((0,1)\times Y^*)}=\left(\frac{L}{\varepsilon}\right)^{\frac{1}{2}}||\varphi||_{L^2(R_0^\varepsilon)}\le\left(\frac{L}{\varepsilon}\right)^{\frac{1}{2}}||\varphi||_{L^2(R^\varepsilon)}.$$

(5) $\partial_{y_1}T_\varepsilon(\varphi)=\varepsilon T_\varepsilon(\partial_x\varphi)$ *and* $\partial_{y_2}T_\varepsilon(\varphi)=\varepsilon T_\varepsilon(\partial_y\varphi)$ *a.e.* $(0,1)\times Y^*$ *for all* $H^1(R^\varepsilon)$.
(6) *If* $\varphi\in H^1(R^\varepsilon)$*, then* $\mathcal{T}_\varepsilon(\varphi)\in L^2((0,1);H^1(Y^*))$ *with*

$$\left|\left|\partial_{y_1}\mathcal{T}_\varepsilon(\varphi)\right|\right|_{L^2((0,1)\times Y^*)}=\varepsilon\left(\frac{L}{\varepsilon}\right)^{\frac{1}{2}}||\partial_x\varphi||_{L^2(R_0^\varepsilon)}\le\varepsilon\left(\frac{L}{\varepsilon}\right)^{\frac{1}{2}}||\partial_x\varphi||_{L^2(R^\varepsilon)},$$

$$\left|\left|\partial_{y_2}\mathcal{T}_\varepsilon(\varphi)\right|\right|_{L^2((0,1)\times Y^*)}=\varepsilon\left(\frac{L}{\varepsilon}\right)^{\frac{1}{2}}\left|\left|\partial_y\varphi\right|\right|_{L^2(R_0^\varepsilon)}\le\varepsilon\left(\frac{L}{\varepsilon}\right)^{\frac{1}{2}}\left|\left|\partial_y\varphi\right|\right|_{L^2(R^\varepsilon)}.$$

From now on, we will use the following rescaled norms in the thin open sets:

$$|||\varphi|||_{L^2(R^\varepsilon)}=\varepsilon^{-1/2}||\varphi||_{L^2(R^\varepsilon)},\quad\forall\varphi\in L^2(R^\varepsilon),$$
$$|||\varphi|||_{H^1(R^\varepsilon)}=\varepsilon^{-1/2}||\varphi||_{H^1(R^\varepsilon)},\quad\forall\varphi\in H^1(R^\varepsilon).$$

Notice that Proposition 2.3 is essential to pass to the limit since allows us to transform an integral over R^ε into one over the fixed set $(0,1)\times Y^*$.

Hence, the unfolding criterion for integrals (u.c.i.) plays an important role.

Definition 2.4: A sequence (φ_ε) satisfies the unfolding criterion for integrals (u.c.i) if

$$\frac{1}{\varepsilon}\int_{R_1^\varepsilon}|\varphi_\varepsilon|\mathrm{d}x\mathrm{d}y\to 0.$$

It is known that any sequence $(\varphi_\varepsilon)\subset L^2(R^\varepsilon)$ with norm $|||\cdot|||_{L^2(R^\varepsilon)}$ uniformly bounded satisfies the (u.c.i). Moreover, if we have (ψ_ε) set as

$$\psi_\varepsilon(x,y)=\psi\left(\frac{x}{\varepsilon},\frac{y}{\varepsilon}\right)$$

for any $\psi\in L^2(Y^*)$, then $(\varphi_\varepsilon\psi_\varepsilon)$ also satisfies (u.c.i).

In the sequel, we recall some convergence properties of the unfolding operator.

Proposition 2.5: (1) *Let $\varphi \in L^2(0,1)$. Then, $\mathcal{T}_\varepsilon\varphi \to \varphi$ strongly in $L^2((0,1)\times Y^*)$.*
(2) *Let (φ_ε) be a sequence in $L^2(0,1)$ such that $\varphi_\varepsilon \to \varphi$ strongly in $L^2(0,1)$. Then, $\mathcal{T}_\varepsilon\varphi_\varepsilon \to \varphi$ strongly in $L^2((0,1)\times Y^*)$.*

Next, we give a suitable decomposition in order to introduce other convergence results. We write $\varphi_\varepsilon(x,y) = V_\varepsilon(x) + \varphi_r(x,y)$ where V is set as

$$V_\varepsilon(x) := \frac{1}{\varepsilon h_0}\int_0^{\varepsilon h_0} \varphi_\varepsilon(x,s)\,\mathrm{d}s \quad \text{a.e. } x \in (0,1). \tag{6}$$

Proposition 2.6: *Let $\varphi_\varepsilon \in H^1(R^\varepsilon)$ with $|||\varphi_\varepsilon|||_{H^1(R^\varepsilon)}$ uniformly bounded and $V_\varepsilon(x)$ defined as in (6). Then, there exists a function $\varphi \in H^1(0,1)$ such that, up to subsequences*

$$\begin{aligned}
&V_\varepsilon \rightharpoonup \varphi \quad \textit{weakly in } H^1(0,1) \textit{ and strongly in } L^2(0,1),\\
&\mathcal{T}_\varepsilon V_\varepsilon \to \varphi \quad \textit{strongly in } L^2\left((0,1)\times Y^*\right),\\
&|||\varphi_\varepsilon - V_\varepsilon|||_{L^2(R^\varepsilon)} \to 0,\\
&|||\varphi_\varepsilon - \varphi|||_{L^2(R^\varepsilon)} \to 0,\\
&\mathcal{T}_\varepsilon\varphi_\varepsilon \to \varphi \quad \textit{strongly in } L^2\left((0,1);H^1(Y^*)\right).
\end{aligned}$$

Finally, we recall a compactness result proved in [12, Theorem 3.1] which allows us to pass to the limit in the gradient sequences.

Theorem 2.7: *Let $\varphi_\varepsilon \in H^1(R^\varepsilon)$ with $|||\varphi_\varepsilon|||_{H^1(R^\varepsilon)}$ uniformly bounded.*
Then, there exist $\varphi \in H^1(0,1)$ and $\varphi_1 \in L^2((0,1);H^1_\#(Y^*))$ such that (up to a subsequence)

$$\begin{aligned}
&\mathcal{T}_\varepsilon\varphi_\varepsilon \to \varphi \quad \textit{strongly in } L^2\left((0,1);H^1(Y^*)\right),\\
&\mathcal{T}_\varepsilon\partial_x\varphi_\varepsilon \rightharpoonup \partial_x\varphi + \partial_{y_1}\varphi_1 \quad \textit{weakly in } L^2\left((0,1)\times Y^*\right),\\
&\mathcal{T}_\varepsilon\partial_y\varphi_\varepsilon \rightharpoonup \partial_{y_2}\varphi_1 \quad \textit{weakly in } L^2\left((0,1)\times Y^*\right).
\end{aligned}$$

2.2. Boundary unfolding

In this section, we set the unfolding operator on the oscillating upper boundary of R^ε. For this sake, we adapt the one introduced in [4,5] yielding the appropriated results to our case. Notice that under assumptions ($\mathbf{H_h}$) we have that Γ^ε is a Lipschitz border.

Definition 2.8: Let ϕ be a measurable function on Γ_ε. The boundary unfolding operator $\mathcal{T}^b_\varepsilon$ is defined by

$$\mathcal{T}^b_\varepsilon\phi(x,y) = \begin{cases} \phi\left(\varepsilon\left[\dfrac{x}{\varepsilon}\right]L + \varepsilon y\right) & \text{a.e. } I_\varepsilon \times \partial_u Y^*,\\ 0 \quad \text{a.e. } \Lambda_\varepsilon \times \partial_u Y^*, \end{cases}$$

where $\partial_u Y^*$ is the upper boundary of the representative cell Y^* given by

$$\partial_u Y^* = \{(y_1,y_2)\in\mathbb{R}^2 \;:\; y_1 \in (0,L), \text{ and } y_2 = h(y_1)\}.$$

Proposition 2.9: *The boundary unfolding satisfies the following properties:*

(1) $\mathcal{T}_\varepsilon^b$ *is linear.*
(2) $\mathcal{T}_\varepsilon^b(\varphi\psi) = \mathcal{T}_\varepsilon^b(\varphi)\mathcal{T}_\varepsilon^b(\psi)$, *for all* φ, ψ *Lebesgue measurable in* Γ^ε.
(3) *For any* $\varphi \in L^1(R^\varepsilon)$,

$$\frac{1}{L}\int_{(0,1)\times\partial_u Y^*} \mathcal{T}_\varepsilon^b\varphi(x,y)\mathrm{d}x\mathrm{d}\sigma(y) = \int_{\Gamma_0^\varepsilon}\varphi\mathrm{d}S = \int_{\Gamma^\varepsilon}\varphi\mathrm{d}S - \int_{\Gamma_1^\varepsilon}\varphi\mathrm{d}S, \tag{7}$$

where Γ_i^ε *is the upper boundary of* R_i^ε *for* $i = 0,1$.
(4) *Suppose that* $\varphi \in L^2(\Gamma^\varepsilon)$. *Then,*

$$||\mathcal{T}_\varepsilon^b\varphi||_{L^2((0,1)\times\partial_u Y^*)} \le \frac{1}{L}||\varphi||_{L^2(\Gamma^\varepsilon)}.$$

(5) *(Unfolding criterion for integrals) Suppose that* $\varphi_\varepsilon \in L^2(\Gamma_\varepsilon)$ *is such that* $||\varphi_\varepsilon||_{L^2(\Gamma^\varepsilon)} \le c$, *with* c *independent on* ε. *Then,*

$$\int_{\Gamma_1^\varepsilon}|\varphi|\mathrm{d}S \to 0.$$

(6) *Let* $\psi_\varepsilon \in H^1(R^\varepsilon)$ *such that* $\mathcal{T}_\varepsilon\psi_\varepsilon \rightharpoonup \widehat{\psi}$ *in* $L^2((0,1);H^1(Y^*))$ *with* $\widehat{\psi} \in H^1(0,1)$. *Then,*

$$\mathcal{T}_\varepsilon^b\psi_\varepsilon \rightharpoonup \widehat{\psi} \quad in \quad L^2((0,1);H^{\frac{1}{2}}\left(\partial_u Y^*\right)).$$

Proof: It is not difficult to see that Properties 1, 2, 4 and 6 follow from the definition of the boundary unfolding operator. We discuss the remaining ones.

3.: Indeed,

$$\begin{aligned}\int_{(0,1)\times\partial_u Y^*}\mathcal{T}_\varepsilon^b\varphi(x,y)\mathrm{d}x\mathrm{d}S &= \sum_{k=0}^{N_\varepsilon-1}\int_{kL\varepsilon}^{(k+1)L\varepsilon}\int_{\partial_u Y^*}\varphi\left(\varepsilon\left[\frac{x}{\varepsilon}\right]L + \varepsilon y\right)\mathrm{d}\sigma(y)\mathrm{d}x \\ &= \sum_{k=0}^{N_\varepsilon-1} L\varepsilon\int_{\partial_u Y^*}\varphi\left(\varepsilon kL + \varepsilon y\right)\mathrm{d}\sigma(y)\mathrm{d}x.\end{aligned} \tag{8}$$

From the change of variables $s = \varepsilon kL + \varepsilon y$, we get $\mathrm{d}\sigma(s) = \varepsilon\mathrm{d}\sigma(y)$. Also, denoting by

$$\Gamma^{\varepsilon,k} = \left\{(x,y) : kL\varepsilon \le x \le (k+1)L\varepsilon,\ y = \varepsilon h\left(\frac{x}{\varepsilon}\right)\right\} \quad \text{for } k \ge 0$$

it is clear that

$$\bigcup_{k=0}^{N_\varepsilon-1}\Gamma^{\varepsilon,k} = \Gamma_0^\varepsilon.$$

Then, by (8) we get

$$\sum_{k=0}^{N_\varepsilon-1} L\int_{\Gamma^{\varepsilon,k}}\varphi(s)\mathrm{d}\sigma(s) = \int_{\Gamma_0^\varepsilon}\varphi\mathrm{d}S. \tag{9}$$

5.: Let $\varphi_\varepsilon \in L^2(\Gamma^\varepsilon)$ such that $||\varphi_\varepsilon||_{L^2(\Gamma^\varepsilon)} \le C$ with C independent on ε. Then

$$\int_{\Gamma_1^\varepsilon}|\varphi|\mathrm{d}S \le |\Gamma_1^\varepsilon|^{1/2}||\varphi||_{L^2(\Gamma^\varepsilon)} \le C|\Gamma_1^\varepsilon|^{1/2} \to 0. \qquad \blacksquare$$

3. Main results

Here, we prove our main results. As emphasized, the asymptotic behavior of the considered problem greatly depends on the value of the coefficient α appearing in the Robin boundary condition $(2)_2$. First we analyze the critical case $\alpha = 1$. After that, we address two remaining characteristic cases $\alpha < 1$ and $\alpha > 1$, respectively.

3.1. Case $\alpha = 1$

Theorem 3.1: *Let u^ε be the solution of the problem (2) with $\alpha = 1$, $f^\varepsilon \in L^2(R^\varepsilon)$ and $|||f^\varepsilon|||_{L^2(R^\varepsilon)}$ uniformly bounded. Also, assume there exists $\hat{f} \in L^2((0,1) \times Y^*)$ such that*

$$\mathcal{T}_\varepsilon f^\varepsilon \rightharpoonup \hat{f} \quad \text{weakly in } L^2\left((0,1) \times Y^*\right).$$

Then, there exist $u \in H^1(0,1)$ and $u_1 \in L^2((0,1); H^1_\#(Y^*))$ such that

$$\begin{aligned} &\mathcal{T}_\varepsilon u_\varepsilon \to u \quad \text{strongly in } L^2\left((0,1); H^1(Y^*)\right), \\ &\mathcal{T}_\varepsilon \partial_x u_\varepsilon \rightharpoonup \partial_x u + \partial_{y_1} u_1 \quad \text{weakly in } L^2\left((0,1) \times Y^*\right), \\ &\mathcal{T}_\varepsilon \partial_y u_\varepsilon \rightharpoonup \partial_{y_2} u_1 \quad \text{weakly in } L^2\left((0,1) \times Y^*\right). \end{aligned}$$

Moreover, we have that u is the solution of

$$\begin{cases} -\kappa q u_{xx} + p u_x + \left(c + \dfrac{|\partial_u Y^*|}{|Y^*|}\right) u = \bar{f} + \dfrac{|\partial_u Y^*|}{|Y^*|} g & \text{in } (0,1), \\ u_x(0) = u_x(1) = 0, \end{cases}$$

where the homogenized coefficients q and p are given by

$$q = \frac{1}{|Y^*|} \int_{Y^*} \left(1 - \partial_{y_1} X\right) \mathrm{d}y_1 \mathrm{d}y_2 \quad \text{and} \quad p = \frac{1}{|Y^*|} \int_{Y^*} Q\left(1 - \partial_{y_1} X\right) \mathrm{d}y_1 \mathrm{d}y_2$$

and $X \in H^1_\#(Y^*)$ with $\int_{Y^*} X \mathrm{d}y_1 \mathrm{d}y_2 = 0$ is the unique solution of

$$\int_{Y^*} \nabla X \nabla \varphi \mathrm{d}y_1 \mathrm{d}y_2 = \int_{Y^*} \partial_{y_1} \varphi \mathrm{d}y_1 \mathrm{d}y_2 \quad \forall \varphi \in H^1_\#(Y^*). \tag{10}$$

Also, the forcing term $\bar{f}$ is given by

$$\bar{f} = \frac{1}{|Y^*|} \int_{Y^*} \hat{f} \mathrm{d}y_1 \mathrm{d}y_2.$$

Proof: (a) Uniform bounds.

Take $\varphi = u^\varepsilon$ as a test function in (3). Then

$$\begin{aligned} &\frac{1}{\varepsilon} \int_{R^\varepsilon} \kappa |\nabla u^\varepsilon|^2 + Q^\varepsilon(y) \partial_x u^\varepsilon \nabla u^\varepsilon + c(u^\varepsilon)^2 \mathrm{d}x\mathrm{d}y + \int_{\Gamma^\varepsilon} (u^\varepsilon)^2 \mathrm{d}S \\ &\quad = \frac{1}{\varepsilon} \int_{R^\varepsilon} f^\varepsilon u^\varepsilon \mathrm{d}x\mathrm{d}y + \int_{\Gamma^\varepsilon} g u^\varepsilon \mathrm{d}S, \quad \forall \varphi \in H^1(R^\varepsilon). \end{aligned}$$

Hence, by Remark 2.1, assumption (4) and definition of the norm $|||\cdot|||$, one gets using the coercive constant $m > 0$ independent of ε that

$$m|||u^\varepsilon|||^2_{H^1(R^\varepsilon)} + ||u^\varepsilon||^2_{L^2(\Gamma^\varepsilon)} \le |||f^\varepsilon|||_{L^2(R^\varepsilon)} |||u^\varepsilon|||_{L^2(R^\varepsilon)} + ||g||_{L^2(\Gamma^\varepsilon)} ||u^\varepsilon||_{L^2(\Gamma^\varepsilon)}. \tag{11}$$

Thus, since we have from [16–18] that there exists $C_0 > 0$ independent of ε such that

$$||\varphi||_{L^2(\Gamma^\varepsilon)} \leq C_0 \varepsilon^{-1/2} \|\varphi\|_{H^1(R^\varepsilon)}, \tag{12}$$

we obtain that

$$m|||u^\varepsilon|||^2_{H^1(R^\varepsilon)} \leq |||u^\varepsilon|||_{H^1(R^\varepsilon)} \left(|||f^\varepsilon|||_{L^2(R^\varepsilon)} + C_0||g||_{L^2(\Gamma^\varepsilon)}\right).$$

By hypotheses $|||f^\varepsilon|||_{L^2(R^\varepsilon)}$ is uniformly bounded. Also,

$$||g||^2_{L^2(\Gamma^\varepsilon)} \leq \max_{x\in[0,1]} \sqrt{1 + \|h'\|^2_{L^\infty(\mathbb{R})}} \|g\|^2_{L^2(0,1)}.$$

Therefore, there exists $C > 0$ independent of $\varepsilon > 0$ such that

$$|||u^\varepsilon|||_{H^1(R^\varepsilon)} \leq C$$

which implies that u^ε is uniformly bounded in $||| \cdot |||_{H^1}$.

(b) Limiting problem.

Let us apply Propositions 2.3 and 2.9 in (3). Then,

$$\begin{aligned}
&\int_{(0,1)\times Y^*} k\mathcal{T}_\varepsilon \nabla u^\varepsilon \mathcal{T}_\varepsilon \nabla\varphi + \mathcal{T}_\varepsilon Q^\varepsilon \mathcal{T}_\varepsilon \partial_x u^\varepsilon \mathcal{T}_\varepsilon \varphi + c\mathcal{T}_\varepsilon u^\varepsilon \varphi \mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2 + \int_{(0,1)\times\partial_u Y^*} \mathcal{T}^b_\varepsilon u^\varepsilon \mathcal{T}^b_\varepsilon \varphi \mathrm{d}\sigma(y)\\
&\qquad + \frac{L}{\varepsilon}\int_{R^\varepsilon_1} \kappa\nabla u^\varepsilon \nabla\varphi + Q^\varepsilon(y)\partial_x u^\varepsilon\varphi + cu^\varepsilon\varphi \mathrm{d}x\mathrm{d}y + L\int_{\Gamma^\varepsilon_1} u^\varepsilon\varphi \mathrm{d}S\\
&\quad = \int_{(0,1)\times Y^*} \mathcal{T}_\varepsilon f^\varepsilon \mathcal{T}_\varepsilon\varphi \mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2 + \int_{(0,1)\times\partial_u Y^*} \mathcal{T}^b_\varepsilon g \mathcal{T}^b_\varepsilon \varphi \mathrm{d}x\mathrm{d}\sigma(y)\\
&\qquad + \frac{L}{\varepsilon}\int_{R^\varepsilon_1} f^\varepsilon\varphi \mathrm{d}x\mathrm{d}y + L\int_{\Gamma^\varepsilon_1} g\varphi \mathrm{d}S,
\end{aligned} \tag{13}$$

for any $\varphi \in H^1(R^\varepsilon)$.

Since we have uniform bounds for the solutions of (2) in the $|||.|||_{H^1(R^\varepsilon)}$ norm, we can apply Theorem 2.7. Thus, there exist $u \in H^1(0,1)$ and $u_1 \in L^2((0,1); H^1_\#(Y^*))$ such that

$$\begin{aligned}
&\mathcal{T}_\varepsilon u_\varepsilon \to u \quad \text{strongly in } L^2\left((0,1); H^1(Y^*)\right),\\
&\mathcal{T}_\varepsilon \partial_x u_\varepsilon \rightharpoonup \partial_x u + \partial_{y_1} u_1 \quad \text{weakly in } L^2\left((0,1)\times Y^*\right),\\
&\mathcal{T}_\varepsilon \partial_y u_\varepsilon \rightharpoonup \partial_{y_2} u_1 \quad \text{weakly in } L^2\left((0,1)\times Y^*\right).
\end{aligned} \tag{14}$$

Also, by Proposition 2.9, one gets

$$\mathcal{T}^b_\varepsilon u_\varepsilon \to u \quad \text{in} L^2((0,1); H^{\frac{1}{2}}\left(\partial_u Y^*\right)). \tag{15}$$

By (14) and (15), for test functions $\varphi(x,y) = \varphi(x)$, we can pass to the limit in (13) yielding

$$\begin{aligned}
&\int_{(0,1)\times Y^*} \kappa\left(\partial_x u + \partial_{y_1} u_1\right)\partial_x\varphi + Q\left(\partial_x u + \partial_{y_1} u_1\right)\varphi + cu\varphi \mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2\\
&\quad + \int_{(0,1)\times\partial_u Y^*} u\varphi \mathrm{d}x\mathrm{d}\sigma(y) = \int_{(0,1)\times Y^*} \hat{f}\varphi \mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2 + \int_{(0,1)\times\partial_u Y^*} g\varphi \mathrm{d}x\mathrm{d}\sigma(y).
\end{aligned} \tag{16}$$

Now we obtain the relation between u_1 and the solution X of the auxiliary problem (10). For this sake, consider the sequence

$$v^\varepsilon(x,y) = \varepsilon\phi(x)\psi\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right), \quad (x,y) \in R^\varepsilon, \tag{17}$$

where $\phi \in C_0^\infty(0,1)$ and $\psi \in H^1_\#(Y^*)$. Then, Propositions 2.3 and 2.5 provide

$$\begin{aligned} &\mathcal{T}_\varepsilon v^\varepsilon \to 0 \quad \text{strongly in } L^2\left((0,1)\times Y^*\right),\\ &\mathcal{T}_\varepsilon \partial_x v^\varepsilon \to \phi\partial_{y_1}\psi \quad \text{strongly in } L^2\left((0,1)\times Y^*\right),\\ &\mathcal{T}_\varepsilon \partial_y v^\varepsilon \to \phi\partial_{y_2}\psi \quad \text{strongly in } L^2\left((0,1)\times Y^*\right). \end{aligned} \tag{18}$$

Now, we take v^ε as a test function in (13). Passing to the limit as $\varepsilon \to 0$, we get

$$\int_{(0,1)\times Y^*} \left(\partial_x u + \partial_{y_1} u_1, \partial_{y_2} u_1\right)\phi\nabla_y\psi \,\mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2 = 0.$$

From the density of tensor product $C_0^\infty(0,1)\otimes H^1_\#(Y^*)$ in $L^2((0,1);H^1_\#(Y^*))$, we can rewrite the above equation as

$$\int_{(0,1)\times Y^*} \left(\partial_x u + \partial_{y_1} u_1, \partial_{y_2} u_1\right)\nabla_y\psi \,\mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2 = 0, \quad \forall\psi \in L^2((0,1);H^1_\#(Y^*)). \tag{19}$$

It is not difficult to check that (19) has a unique solution in the Hilbert space $H^1(0,1)\times L^2((0,1);H^1_\#(Y^*)/\mathbb{R})$. We refer the reader to [12] for details.

Since X is the unique L-periodic solution of the problem (10) and u is independent of y_1 and y_2, we have that $-\partial_x u(x)X(y_1,y_2)$ satisfies

$$\int_{Y^*} -\partial_x u\nabla X\nabla\psi \,\mathrm{d}y_1\mathrm{d}y_2 = \int_{Y^*} -\partial_x u\partial_{y_1}\psi \,\mathrm{d}y_1\mathrm{d}y_2, \quad \forall\psi \in L^2((0,1);H^1_\#(Y^*)). \tag{20}$$

Consequently, it follows from (19) that

$$u_1(x,y_1,y_2) = -\partial_x u(x)X(y_1,y_2). \tag{21}$$

Now we are in position to rewrite (16) as

$$\begin{aligned} &\int_{(0,1)\times Y^*} \kappa\left(\partial_x u - \partial_x u\partial_{y_1}X\right)\partial_x\varphi + Q\left(\partial_x u - \partial_x u\partial_{y_1}X\right)\varphi + cu\varphi \,\mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2\\ &\quad + \int_{(0,1)\times\partial_u Y^*} u\varphi \,\mathrm{d}x\mathrm{d}\sigma(y) = \int_{(0,1)\times Y^*} \hat{f}\varphi \,\mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2 + \int_{(0,1)\times\partial_u Y^*} g\varphi \,\mathrm{d}x\mathrm{d}\sigma(y). \end{aligned}$$

Hence, since u and φ are independent on y_1 and y_2, we get

$$\begin{aligned} &\int_0^1 \kappa\partial_x u\left[\int_{Y^*}\left(1-\partial_{y_1}X\right)\mathrm{d}y_1\mathrm{d}y_2\right]\partial_x\varphi + \left[\int_{Y^*} Q\left(1-\partial_{y_1}X\right)\mathrm{d}y_1\mathrm{d}y_2\right]\partial_x u\varphi\,\mathrm{d}x + |Y^*|c\int_0^1 u\varphi\,\mathrm{d}x\\ &\quad + |\partial_u Y^*|\int_0^1 u\varphi\,\mathrm{d}x = \int_0^1\left[\int_{Y^*}\hat{f}\,\mathrm{d}y_1\mathrm{d}y_2\right]\varphi\,\mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2 + |\partial_u Y^*|\int_0^1 g\varphi\,\mathrm{d}x. \end{aligned}$$

Dividing both sides by $|Y^*|$ gives

$$\int_0^1 \left[\kappa\, q\,\partial_x u\partial_x\varphi + p\,\partial_x u\varphi + c\,u\varphi\right]\mathrm{d}x + \frac{|\partial_u Y^*|}{|Y^*|}\int_0^1 u\varphi\,\mathrm{d}x = \int_0^1 \bar{f}\varphi\,\mathrm{d}x + \frac{|\partial_u Y^*|}{|Y^*|}\int_0^1 g\varphi\,\mathrm{d}x,$$

for all $\varphi \in H^1(0,1)$. It follows from [19] that the coefficients q and p are strictly positive, and then, the limit equation is a well posed problem. Thus, u^ε is a convergent sequence which leads us to the end of the proof. ∎

Remark 3.2: Notice the effect of the oscillating behavior on the homogenized coefficients q and p given by the auxiliary function X. Also, we emphasize the effect of the Lebesgue measure of the sets Y^* and $\partial_u Y^*$ at the limit equation, as well as, the flux condition on the border sets by g. All these ingredients can be seen at the homogenized equation.

Remark 3.3: See [12] for a discussing on some properties of the homogenized coefficient q. In particular, they show that $0 < q < 1$.

3.2. Case $\alpha < 1$

Theorem 3.4: *Let u^ε be the solution of the problem (2) with $\alpha < 1$, $f^\varepsilon \in L^2(R^\varepsilon)$ and $|||f^\varepsilon|||_{L^2(R^\varepsilon)}$ uniformly bounded. Also, assume $g \in H^1(0,1)$. Then, as $\varepsilon \to 0$,*

$$\mathcal{T}_\varepsilon u^\varepsilon \to g \quad \text{strongly in } L^2\left((0,1); H^1(Y^*)\right)$$

in such way that

$$|||u^\varepsilon - g|||_{L^2(R^\varepsilon)} \to 0. \tag{22}$$

Proof: Let

$$w^\varepsilon(x,y) = u^\varepsilon(x,y) - g(x) \quad \forall (x,y) \in R^\varepsilon, \tag{23}$$

where g is extended trivially to R^ε.

One can rewrite (3) as follows:

$$\int_{R^\varepsilon} \kappa \nabla w^\varepsilon \nabla \varphi + Q^\varepsilon(y) \partial_x w^\varepsilon \varphi + c w^\varepsilon \varphi \mathrm{d}x\mathrm{d}y + \varepsilon^\alpha \int_{\Gamma^\varepsilon} w^\varepsilon \varphi \mathrm{d}S$$
$$+ \int_{R^\varepsilon} \kappa \nabla g \nabla \varphi + Q^\varepsilon(y) \partial_x g \varphi + c g \varphi \mathrm{d}x\mathrm{d}y = \int_{R^\varepsilon} f^\varepsilon \varphi \mathrm{d}x\mathrm{d}y, \tag{24}$$

for all $\varphi \in H^1(R^\varepsilon)$.

(a) Uniform bounds.

We can proceed as in (11). It follows from Remark 2.1 and condition (4) taking $\varphi = w^\varepsilon$ in (24) that

$$m|||w^\varepsilon|||^2_{H^1(R^\varepsilon)} \le m|||w^\varepsilon|||^2_{H^1(R^\varepsilon)} + \varepsilon^{\alpha-1}||w^\varepsilon||^2_{L^2(\Gamma^\varepsilon)}$$
$$\le C|||w^\varepsilon|||_{H^1(R^\varepsilon)},$$

where C is a positive constant given by

$$C = \max\left\{|||f^\varepsilon|||_{L^2(R^\varepsilon)}, \kappa|||\nabla g|||_{L^2(R^\varepsilon)}, \|Q\|_{L^\infty(0,h_1)}|||\nabla g|||_{L^2(R^\varepsilon)}, c|||g|||_{L^2(R^\varepsilon)}\right\}.$$

Notice that C does not depend on ε. Indeed, it follows from the assumption $|||f^\varepsilon|||_{L^2(R^\varepsilon)}$ uniformly bounded and estimates

$$|||\nabla g|||^2_{L^2(R^\varepsilon)} = \frac{1}{\varepsilon}\int_0^1 \int_0^{\varepsilon h(x/\varepsilon)} g'(x)^2 \mathrm{d}x\mathrm{d}y \le \frac{1}{\varepsilon}\int_0^1 g'(x)^2 \varepsilon h(x/\varepsilon)\,\mathrm{d}x$$
$$\le h_1 \|g'\|^2_{L^2(0,1)}$$

and

$$|||g|||^2_{L^2(R^\varepsilon)} \le h_1 \|g\|^2_{L^2(0,1)}.$$

Thus, there exists positive constants C_1 and C_2, independent of ε, such that

$$|||w^\varepsilon|||_{H^1(R^\varepsilon)} \le C_1 \quad \text{and} \quad \varepsilon^{\alpha-1}||w^\varepsilon||^2_{L^2(\Gamma^\varepsilon)} \le C_2. \tag{25}$$

(b) Limits of w^ε.

By the uniform bound of w^ε in the $|||.|||_{H^1}$ norm, it follows from Theorem 2.7 that there exist $w \in H^1(0,1)$ and $w_1 \in L^2((0,1); H^1_\#(Y^*))$ such that

$$\begin{aligned} &\mathcal{T}_\varepsilon w^\varepsilon \to w \quad \text{strongly in } L^2\left((0,1); H^1(Y^*)\right), \\ &\mathcal{T}_\varepsilon \partial_x w^\varepsilon \rightharpoonup \partial_x w + \partial_{y_1} w_1 \quad \text{weakly in } L^2\left((0,1) \times Y^*\right), \\ &\mathcal{T}_\varepsilon \partial_y w^\varepsilon \rightharpoonup \partial_{y_2} w_1 \quad \text{weakly in } L^2\left((0,1) \times Y^*\right). \end{aligned} \tag{26}$$

Also, by Proposition 2.9, one gets

$$\mathcal{T}^b_\varepsilon w^\varepsilon \to w \quad \text{in} L^2((0,1); H^{\frac{1}{2}}\left(\partial_u Y^*\right)). \tag{27}$$

Now, due to Proposition 2.9 and (25), there exists $C > 0$ independent of ε such that

$$\begin{aligned} ||w||_{L^2\left((0,1);H^{\frac{1}{2}}(\partial_u Y^*)\right)} &\leq ||w - \mathcal{T}^b_\varepsilon w^\varepsilon||_{L^2\left((0,1);H^{\frac{1}{2}}(\partial_u Y^*)\right)} + ||\mathcal{T}^b_\varepsilon w^\varepsilon||_{L^2\left((0,1);H^{\frac{1}{2}}(\partial_u Y^*)\right)} \\ &\leq ||w - \mathcal{T}^b_\varepsilon w^\varepsilon||_{L^2\left((0,1);H^{\frac{1}{2}}(\partial_u Y^*)\right)} + C||w^\varepsilon||_{L^2(\Gamma^\varepsilon)} \\ &\leq ||w - \mathcal{T}^b_\varepsilon w^\varepsilon||_{L^2\left((0,1);H^{\frac{1}{2}}(\partial_u Y^*)\right)} + C\varepsilon^{1-\alpha}. \end{aligned} \tag{28}$$

Therefore, since $\alpha < 1$ and w depends just on x-variable, we can pass to the limit in the inequality (28) as $\varepsilon \to 0$ obtaining that

$$w = 0 \quad \text{in } (0,1).$$

Hence, due to (23) and (26), one can conclude that

$$\mathcal{T}_\varepsilon u^\varepsilon \to g \quad \text{strongly in } L^2\left((0,1); H^1(Y^*)\right).$$

It follows from Proposition 2.6 that

$$|||u^\varepsilon - g|||_{L^2(R^\varepsilon)} \to 0$$

concluding the proof of the theorem. ■

3.3. Case $\alpha > 1$

Theorem 3.5: *Let u^ε be the solution of the problem (2) with $\alpha > 1$, $f^\varepsilon \in L^2(R^\varepsilon)$ and $|||f^\varepsilon|||_{L^2(R^\varepsilon)}$ uniformly bounded. Assume that there exists $\hat{f} \in L^2((0,1) \times Y^*)$ such that*

$$\mathcal{T}_\varepsilon f^\varepsilon \rightharpoonup \hat{f} \quad \textit{weakly in } L^2\left((0,1) \times Y^*\right).$$

Then, there exist $u \in H^1(0,1)$ and $u_1 \in L^2((0,1); H^1_\#(Y^*))$ such that

$$\begin{aligned} &\mathcal{T}_\varepsilon u_\varepsilon \to u \quad \textit{strongly in } L^2\left((0,1); H^1(Y^*)\right), \\ &\mathcal{T}_\varepsilon \partial_x u_\varepsilon \rightharpoonup \partial_x u + \partial_{y_1} u_1 \quad \textit{weakly in } L^2\left((0,1) \times Y^*\right), \\ &\mathcal{T}_\varepsilon \partial_y u_\varepsilon \rightharpoonup \partial_{y_2} u_1 \quad \textit{weakly in } L^2\left((0,1) \times Y^*\right). \end{aligned}$$

Moreover, u is the solution of

$$\begin{cases} -\kappa q u_{xx} + p u_x + c u = \bar{f} & \textit{in } (0,1) \\ u_x(0) = u_x(1) = 0, \end{cases}$$

where the homogenized coefficients q and p are given by

$$q = \frac{1}{|Y^*|}\int_{Y^*}\left(1-\partial_{y_1}X\right)\mathrm{d}y_1\mathrm{d}y_2 \quad \text{and} \quad p = \frac{1}{|Y^*|}\int_{Y^*}Q\left(1-\partial_{y_1}X\right)\mathrm{d}y_1\mathrm{d}y_2$$

and $X \in H^1_\#(Y^*)$ *with* $\int_{Y^*} X\mathrm{d}y_1\mathrm{d}y_2 = 0$ *is the unique solution of*

$$\int_{Y^*}\nabla X\nabla\varphi\mathrm{d}y_1\mathrm{d}y_2 = \int_{Y^*}\partial_{y_1}\varphi\mathrm{d}y_1\mathrm{d}y_2 \quad \forall\varphi \in H^1_\#(Y^*).$$

Also, the forcing term $\bar{f}$ *is given by*

$$\bar{f} = \frac{1}{|Y^*|}\int_{Y^*}\hat{f}\mathrm{d}y_1\mathrm{d}y_2.$$

Proof: The proof follows the same arguments as in that one of Theorem 3.1.

(a) Uniform bounds.

Take $\varphi = u^\varepsilon$ as a test function in (3). Then, arguing again as in (11), we get from Remark 2.1 and the condition (4) that

$$m|||u^\varepsilon|||^2_{H^1(R^\varepsilon)} + \varepsilon^{\alpha-1}||u^\varepsilon||^2_{L^2(\Gamma^\varepsilon)} \leq |||f^\varepsilon|||_{L^2(R^\varepsilon)}|||u^\varepsilon|||_{L^2(R^\varepsilon)} + \varepsilon^{\alpha-1}||g||_{L^2(\Gamma^\varepsilon)}||u^\varepsilon||_{L^2(\Gamma^\varepsilon)}.$$

Thus, it follows from (12) that there exists $C > 0$, independent of $\varepsilon > 0$, such that

$$|||u^\varepsilon|||_{H^1(R^\varepsilon)} \leq |||f^\varepsilon|||_{L^2(R^\varepsilon)}|||u^\varepsilon|||_{L^2(R^\varepsilon)} + C\varepsilon^{\alpha-1}||g||_{L^2(0,1)}|||u^\varepsilon|||_{H^1(R^\varepsilon)}.$$

Consequently, $|||f^\varepsilon|||_{L^2(R^\varepsilon)}$ uniformly bounded, $\alpha > 1$ and $\varepsilon < 1$ guarantee u^ε uniformly bounded in $|||\cdot|||_{H^1}$.

(b) Limiting problem

Let us rewrite (3) using Proposition 2.3. We have

$$\begin{aligned}
&\int_{(0,1)\times Y^*} k\mathcal{T}_\varepsilon\nabla u^\varepsilon\mathcal{T}_\varepsilon\nabla\varphi + \mathcal{T}_\varepsilon Q^\varepsilon\mathcal{T}_\varepsilon\partial_x u^\varepsilon\mathcal{T}_\varepsilon\varphi + c\mathcal{T}_\varepsilon u^\varepsilon\varphi\mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2\\
&\quad + \varepsilon^{\alpha-1}\int_{(0,1)\times\partial_u Y^*}\mathcal{T}^b_\varepsilon u^\varepsilon\mathcal{T}^b_\varepsilon\varphi\mathrm{d}\sigma(y)\\
&\quad + \frac{L}{\varepsilon}\int_{R^\varepsilon_1}\kappa\nabla u^\varepsilon\nabla\varphi + Q^\varepsilon(y)\partial_x u^\varepsilon\varphi + cu^\varepsilon\varphi\mathrm{d}x\mathrm{d}y + L\varepsilon^{\alpha-1}\int_{\Gamma^\varepsilon_1}u^\varepsilon\varphi\mathrm{d}S\\
&= \int_{(0,1)\times Y^*}\mathcal{T}_\varepsilon f^\varepsilon\mathcal{T}_\varepsilon\varphi\mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2 + \varepsilon^{\alpha-1}\int_{(0,1)\times\partial_u Y^*}\mathcal{T}^b_\varepsilon g\mathcal{T}^b_\varepsilon\varphi\mathrm{d}x d\sigma(y)\\
&\quad + \frac{L}{\varepsilon}\int_{R^\varepsilon_1}f^\varepsilon\varphi\mathrm{d}x\mathrm{d}y + L\varepsilon^{\alpha-1}\int_{\Gamma^\varepsilon_1}g\varphi\mathrm{d}S. \qquad (29)
\end{aligned}$$

Hence, since we have uniform bounds for the solutions in the $|||.|||_{H^1(R^\varepsilon)}$ norm, we can apply Theorem 2.7. Thus, there exist $u \in H^1(0,1)$ and $u_1 \in L^2((0,1); H^1_\#(Y^*))$ such that

$$\begin{aligned}
&\mathcal{T}_\varepsilon u_\varepsilon \to u \quad \text{strongly in } L^2\left((0,1); H^1(Y^*)\right),\\
&\mathcal{T}_\varepsilon\partial_x u_\varepsilon \rightharpoonup \partial_x u + \partial_{y_1}u_1 \quad \text{weakly in } L^2\left((0,1)\times Y^*\right),\\
&\mathcal{T}_\varepsilon\partial_y u_\varepsilon \rightharpoonup \partial_{y_2}u_1 \quad \text{weakly in } L^2\left((0,1)\times Y^*\right). \qquad (30)
\end{aligned}$$

Also, by Proposition 2.9, one gets

$$\mathcal{T}^b_\varepsilon u_\varepsilon \to u \quad \text{in} L^2((0,1); H^{\frac{1}{2}}\left(\partial_u Y^*\right)). \qquad (31)$$

By (30), (15) and $\alpha > 1$, for test functions $\varphi(x, y) = \varphi(x)$, we can pass to the limit (29) getting

$$\int_{(0,1)\times Y^*} \kappa \left(\partial_x u + \partial_{y_1} u_1\right) \partial_x \varphi + Q\left(\partial_x u + \partial_{y_1} u_1\right) \varphi + cu\varphi \mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2$$
$$= \int_{(0,1)\times Y^*} \hat{f}\varphi \mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2 \quad \forall \varphi \in H^1(0,1). \tag{32}$$

Now, one can proceed as in the proof of Theorem 3.1 to obtain

$$\int_0^1 \left[\kappa\, q\, \partial_x u \partial_x \varphi + p\, \partial_x u \varphi + c\, u\varphi\right] \mathrm{d}x = \int_0^1 \bar{f}\varphi \mathrm{d}x, \quad \forall \varphi \in H^1(0,1)$$

which completes the proof. ■

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