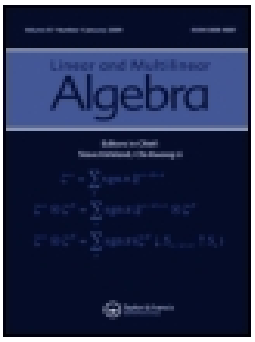

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Degree-inverting involution on full square and triangular matrices

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ABSTRACT

In this short note, we investigate the degree-inverting involutions on the full square matrices, assuming the base field is algebraically closed of characteristic not 2. We obtain a partial result if the grading group is arbitrary, and a complete answer for abelian groups. We also classify the degree-inverting involutions on the triangular matrices, for arbitrary groups, and arbitrary fields of characteristic not 2.

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1. Introduction

Graded rings appear naturally in several branches of Mathematics and Physics. For instance, one can construct a grading from a filtered algebra, a local valuation ring, a polynomial ring, an automorphism of finite order of an algebra, a finite-dimensional Lie algebra over an algebraically closed field of characteristic zero, etc.

Recall that a G-grading on an algebra $\mathcal{A}$ is a vector space decomposition $\mathcal{A} = \bigoplus_{g\in G} \mathcal{A}_g$ such that $\mathcal{A}_g\mathcal{A}_h \subseteq \mathcal{A}_{gh}$, for all $g, h \in G$. Some graded algebras are endowed also with a graded involution, in the following sense: ψ is an involution of $\mathcal{A}$, and $\psi(\mathcal{A}_g) \subseteq \mathcal{A}_{g^{-1}}$, for any $g \in G$. Examples include: the usual transposition of square matrices with any good grading, the usual involution on Leavitt path algebras endowed with their usual grading, etc. It is worth mentioning that, in a recent work [1], the authors proved that the graded involution enriches the structure of the Graded Grothendieck group of a graded ring. Thus, understanding gradings and graded involution on a given algebra seem to be an interesting problem. From now on, we will refer the graded involution as *degree-inverting involution*.

On the other hand, in [2], the authors call a *graded involution* an involution satisfying $\psi(\mathcal{A}_g) \subseteq \mathcal{A}_g$, for all $g \in G$. They proved that the degree-preserving involutions are fundamental to give a description of group gradings on some simple Lie algebras, a question raised by Patera and Zassenhaus [3]. After the contribution of several authors, the classification of the degree-preserving involutions on matrix algebras, and the gradings on simple

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Lie algebras are essentially complete [2,4–7] (among others), see also the monograph [8]. Thus, degree-preserving involution is an essential tool as well.

In this paper, using the ideas of the degree-preserving case [4,9] (see also [8]), we classify degree-inverting involutions on matrix algebras and on upper triangular matrices, improving the results of [10].

This paper is divided as follows: we include a little preliminary theory in Section 2. Then, we provide partial results for degree-inverting involution on graded division algebras (Section 3). Next, we develop a similar theory as presented in [8, Section 2.4] (see also the paper by Elduque [9]) to study the matrix algebra case in Section 4. Finally, in Section 5, we obtain results for the upper triangular matrices case.

Conventions. In the whole paper, we assume that G is an arbitrary group. We explicitly mention the results in the special case of an abelian group. We assume that $\mathbb{F}$ is a field of characteristic not 2. Except in Section 5, we assume that $\mathbb{F}$ is algebraically closed.

2. Preliminaries

2.1. Graded algebras

We shall work with graded algebras rather than graded rings, as follows. Let G be any group. We say that an algebra $\mathcal{A}$ is G-graded if there exists a vector-space decomposition $\mathcal{A} = \bigoplus_{g\in G} \mathcal{A}_g$ such that $\mathcal{A}_g\mathcal{A}_h \subseteq \mathcal{A}_{gh}$, for all $g, h \in G$. The subspace $\mathcal{A}_g$ is called *homogeneous component of degree g*. A nonzero element $x \in \mathcal{A}_g$ is called a homogeneous element of degree g. We denote $\deg x = g$.

A map $f : \mathcal{A} \to \mathcal{B}$ between two G-graded algebras is called a *graded homomorphism* if f is an algebra homomorphism, and $f(\mathcal{A}_g) \subseteq \mathcal{B}_g$ for all $g \in G$. If, moreover, f is an isomorphism, then f is called a graded isomorphism; in this case, $\mathcal{A}$ and $\mathcal{B}$ are said to be isomorphic.

A *graded division algebra* is an associative algebra $\mathcal{D}$ with 1, where each nonzero homogeneous element $x \in \mathcal{D}$ is invertible.

Now let $\mathcal{R} = M_n(\mathbb{F})$ be a matrix algebra endowed with a G-grading. Then the graded version of the Density Theorem (see, for instance, [8, Theorem 2.5 (Section 2.1)]) tells us that we can find a graded division algebra $\mathcal{D}$, $\dim \mathcal{D} = \ell^2$, and a sequence $(g_1, \ldots, g_m) \in G^m$, such that $\mathcal{R} \cong M_m(\mathbb{F}) \otimes \mathcal{D}$, where the grading is given by

$$\deg e_{ij} \otimes d = g_i \deg(d) g_j^{-1}, \quad d \in \mathcal{D} \text{ homogeneous.} \tag{1}$$

Let $\mathcal{A} = \bigoplus_{g\in G} \mathcal{A}_g$ be a G-graded algebra. We say that V is a *G-graded right $\mathcal{A}$-module* if V is a right $\mathcal{A}$-module and there exists a vector space decomposition $V = \bigoplus_{g\in G} V_g$ such that $V_h\mathcal{A}_g \subseteq V_{hg}$, for all $g, h \in G$. Similarly we define the notion of graded left modules.

Given two G-graded right $\mathcal{A}$-modules V and W, we say that $f : V \to W$ is a *graded map of degree g* if f is an $\mathcal{A}$-homomorphism, and $f(V_h) \subseteq W_{gh}$, for all $h \in G$. The graded maps of degree 1 are also known as graded $\mathcal{A}$-homomorphisms. We denote $\mathrm{Hom}_g(V, W)$ the set of all graded maps of degree g; and let $\mathrm{Hom}_{\mathrm{gr}}(V, W) = \bigoplus_{g\in G} \mathrm{Hom}_g(V, W)$. If V and W are finite-dimensional, then we have $\mathrm{Hom}(V, W) = \mathrm{Hom}_{\mathrm{gr}}(V, W)$, so $\mathrm{Hom}(V, W)$ gets a vector-space grading (see, for instance, [8, p. 10]).

Now, let $\mathcal{D}$ be a finite-dimensional graded division algebra, and let V be a finite-dimensional G-graded right $\mathcal{D}$-module. Then $\mathcal{R} = \mathrm{End}_{\mathcal{D}}(V) = \mathrm{Hom}(V, V)$ is a G-graded algebra isomorphic to a matrix algebra. Moreover, V is a G-graded left $\mathcal{R}$-module.

Finally, we provide a precise definition of the following:

Definition 2.1: Let $\mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g$ be a G-graded algebra. An involution ψ on $\mathcal{A}$ is a *degree-inverting involution* if $\psi(\mathcal{A}_g) \subseteq \mathcal{A}_{g^{-1}}$, for all $g \in G$.

In this paper, *involution* will mean a first kind involution, that is, a $C(\mathcal{A})$-linear map, where $C(\mathcal{A})$ is the centre of $\mathcal{A}$.

2.2. Factor sets

Let T be a finite group. A map $\sigma : T \times T \to \mathbb{F}^{\times}$, where $\mathbb{F}^{\times}$ is the set of invertible elements of $\mathbb{F}$, is called a *2-cocycle* or a *factor set* if

$$\sigma(u, v)\sigma(uv, w) = \sigma(u, vw)\sigma(v, w), \quad \forall u, v, w \in T.$$

These objects are interesting and intensively studied in the context of cohomology of groups, Galois Theory, Brauer Groups (equivalently, central simple algebras and division algebras), and so on (see, for instance, [11,12]). However, we do not need such generalities, and we limit ourselves within the theory we will need.

Denote by $Z^2(T, \mathbb{F}^{\times})$ the set of all factor sets. Since $\mathbb{F}^{\times}$ is commutative with respect to the product, the $Z^2(T, \mathbb{F}^{\times})$ acquires a natural structure of an abelian group, by point-wise product.

We can construct algebras from factor sets. Given an arbitrary map $\sigma : T \times T \to \mathbb{F}^{\times}$ denote by $\mathbb{F}^{\sigma}T$ the following algebra: $\mathbb{F}^{\sigma}T$ has a basis $\{X_u \mid u \in T\}$, and the product is defined by $X_u X_v = \sigma(u, v)X_{uv}$. Note that $\mathbb{F}^{\sigma}T$ is associative if and only if $\sigma \in Z^2(T, \mathbb{F}^{\times})$. For instance, if $\sigma = 1$ (the constant function), then $\mathbb{F}^{\sigma}T$ is the group algebra of T. Next, we investigate the isomorphism classes of algebras given by factor sets.

For any arbitrary map $\lambda : T \to \mathbb{F}^{\times}$, we obtain a factor set $\delta\lambda$ by the formula

$$\delta\lambda(u, v) := \frac{\lambda(u)\lambda(v)}{\lambda(uv)}.$$

Since $\delta(\lambda_1\lambda_2) = \delta\lambda_1\delta\lambda_2$, $B^2(T, \mathbb{F}^{\times}) := \{\delta\lambda \mid \lambda : T \to \mathbb{F}^{\times}\}$ is a subgroup of $Z^2(T, \mathbb{F}^{\times})$. We denote the quotient by $H^2(T, \mathbb{F}^{\times}) = Z^2(T, \mathbb{F}^{\times})/B^2(T, \mathbb{F}^{\times})$, and call it the *second cohomology group of T*. Given $\sigma \in Z^2(T, \mathbb{F}^{\times})$, we denote by $[\sigma]$ the element $\sigma B^2(T, \mathbb{F}^{\times})$ in $H^2(T, \mathbb{F}^{\times})$.

Lemma 2.1 ([11, Chapter 2, Lemma 1.1]): *Let $\sigma_1, \sigma_2 \in Z^2(T, \mathbb{F}^{\times})$. Then $\mathbb{F}^{\sigma_1}T \cong \mathbb{F}^{\sigma_2}T$ if and only if $[\sigma_1] = [\sigma_2]$.*

The following is an easy manipulation:

Lemma 2.2: *Let $[\sigma] \in H^2(T, \mathbb{F}^{\times})$. Then, there exists $\sigma' \in [\sigma]$ such that $\sigma'(u, 1) = \sigma'(1, u) = 1$, for all $u \in T$.*

Hence, combining the two previous results, given $\mathbb{F}^\sigma T$, we can assume that $\sigma(u,1) = \sigma(1,u) = 1$, for all $u \in T$.

Finally, it is worth mentioning that, if char $\mathbb{F}$ does not divide $|T|$, then $\mathbb{F}^\sigma T$ is semiprimitive (that is, its Jacobson radical is zero).

2.3. Graded division algebras

Finite-dimensional graded division algebras have a nice description when the base field is algebraically closed. We briefly recall the structure here (for more details, refer to [8, Section 2.1]). Assume that $\mathbb{F}$ is algebraically closed and let $\mathcal{D} = \bigoplus_{g\in G}\mathcal{D}_g$ be a finite-dimensional graded division algebra over $\mathbb{F}$. Let $T = \{g \in G \mid \mathcal{D}_g \neq 0\}$ be its support. Then it is easy to see that T is a subgroup of G. We use multiplicative notation for the product of T, and denote by 1 its neutral element.

Moreover, $\mathcal{D}_1 \supseteq \mathbb{F}$ is a division algebra. So $\mathcal{D}_1 = \mathbb{F}$, since $\mathbb{F}$ is algebraically closed and $\dim_{\mathbb{F}} \mathcal{D}_1 < \infty$. This also implies $\dim \mathcal{D}_g = 1$, for all $g \in T$. Let $\{X_u \mid u \in T\}$ be a homogeneous basis of $\mathcal{D}$. Then $X_uX_v = \sigma(u,v)X_{uv}$, for some $\sigma(u,v) \in \mathbb{F}^\times$. Since $\mathcal{D}$ is associative, from $(X_uX_v)X_w = X_u(X_vX_w)$, we derive that σ is a 2-cocycle. Hence, $\mathcal{D} \cong \mathbb{F}^\sigma T$, the twisted group algebra of T by σ. Conversely, for any finite group T and any $\sigma \in Z^2(T,\mathbb{F}^\times)$, the natural T-grading on $\mathbb{F}^\sigma T$ turns it into a graded division algebra.

Now, assume that T is abelian. Recall that an *alternating bicharacter* is a map $\zeta : T \times T \to \mathbb{F}^\times$ satisfying, $\forall u, v, w \in T$:

$$\begin{aligned}\zeta(uv,w) &= \zeta(u,w)\zeta(v,w),\\ \zeta(u,vw) &= \zeta(u,v)\zeta(u,w),\\ \zeta(u,u) &= 1.\end{aligned}$$

Let $\beta : T \times T \to \mathbb{F}^\times$ be defined by $\beta(u,v) = \sigma(u,v)\sigma(v,u)^{-1}$. A direct computation shows that β is an alternating bicharacter; moreover, $\mathcal{D}$ is central if and only if β is nondegenerate. Finally, Theorem 2.15 of [8] tells that the pair (T,β) uniquely determines an isomorphism class of finite-dimensional central graded division algebras over $\mathbb{F}$ with commutative support. Hence, if T is abelian, the pairs (T,β) are in bijection with the elements of the second cohomology group $H^2(T,\mathbb{F}^\times)$.

2.4. Realization of graded division algebras with commutative support

Let ε be a primitive n-root of unity in $\mathbb{F}$. Consider the elements

$$X = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0\\ 0 & 0 & 1 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 0 & 1\\ 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} \varepsilon^{n-1} & & & & 0\\ & \varepsilon^{n-2} & & & \\ & & \ddots & & \\ & & & \varepsilon & \\ 0 & & & & 1 \end{pmatrix}. \tag{2}$$

Note that $\varepsilon XY = YX$ and $X^n = Y^n = 1$. Moreover, $\{X^iY^j \mid i,j = 0,1,\ldots,n\}$ is a vector space basis of $M_n(\mathbb{F})$. Also, $\mathcal{A}_{(i,j)} = \mathrm{Span}\{X^iY^j\}$ constitute a $\mathbb{Z}_n \times \mathbb{Z}_n$-grading on $M_n(\mathbb{F})$. This grading is called an ε-grading, and it is a division grading.

Now, if $M_n(\mathbb{F})$ is endowed with a division grading, then, as mentioned in the previous section, the support T of the grading is a group, and the product is determined by a non-degenerate alternating bicharacter $\beta : T \times T \to \mathbb{F}^\times$. Thus, we obtain a decomposition $T = H_1^2 \times H_2^2 \times \cdots \times H_s^2$, where each H_i is β-invariant and $H_i \cong \mathbb{Z}_{m_i}$. Moreover, we obtain

$$M_n \cong M_{m_1} \otimes M_{m_2} \otimes \cdots \otimes M_{m_s}, \tag{3}$$

where Supp $M_{m_i} = H_i^2$ and M_{m_i} has an ε_i-grading (see [8, Section 2.2] for more details).

Thus, if $\mathcal{D}$ is a central finite-dimensional graded division algebra over an algebraically closed field $\mathbb{F}$, then we can realize $\mathcal{D}$ as a matrix algebra. Such realization is made after a choice of Kronecker product identification as in (3), and, for each M_{m_i}, a choice of a basis as in (2).

3. Degree-inverting involution on graded division algebras

As mentioned above, over an algebraically closed field $\mathbb{F}$, a finite-dimensional G-graded-division algebra assumes the form $\mathbb{F}^\sigma T$, where $T \subseteq G$ is a finite subgroup, and $\sigma : T \times T \to \mathbb{F}^\times$ is a 2-cocycle.

Lemma 3.1: *Given $\sigma \in Z^2(T, \mathbb{F}^\times)$, let $\bar{\sigma} : T \times T \to \mathbb{F}^\times$ be defined by $\bar{\sigma}(u, v) = \sigma(v^{-1}, u^{-1})$. Then $[\bar{\sigma}] = [\sigma^{-1}]$.*

Proof: We have

$$\begin{aligned}\sigma(u, v)\bar{\sigma}(u, v) &= \sigma(u, v)\sigma(v^{-1}, u^{-1})\\ &= \sigma(uv, v^{-1})^{-1}\sigma(v, v^{-1})\sigma(u, vv^{-1})\sigma(v^{-1}, u^{-1}).\end{aligned}$$

Also,

$$\sigma(uv, v^{-1}) = \sigma(uvv^{-1}, u^{-1})^{-1}\sigma(v^{-1}, u^{-1})\sigma(uv, v^{-1}u^{-1}).$$

Thus, continuing from the first equation,

$$\begin{aligned}\sigma(u, v)\bar{\sigma}(u, v) &= \sigma(u, u^{-1})\sigma(v, v^{-1})\sigma(uv, (uv)^{-1})^{-1}\\ &= \delta\lambda(u, v),\end{aligned}$$

where $\lambda(u) := \sigma(u, u^{-1})$. ■

We fix a $\sigma \in Z^2(T, \mathbb{F}^\times)$, and a homogeneous basis $\{X_u \mid u \in T\}$ of $\mathbb{F}^\sigma T$.

Proposition 3.2: *$\mathbb{F}^\sigma T$ admits a degree-inverting involution if and only if $[\sigma]^2 = 1$.*

Proof: Assume that ρ is a degree-inverting involution on $\mathbb{F}^\sigma T$. Let $\mu : T \to \mathbb{F}^\times$ be such that $\rho(X_u) = \mu(u)X_{u^{-1}}$, for all $u \in T$. Note that, for any $u, v \in T$,

$$\begin{aligned}\rho(X_uX_v) &= \rho(X_v)\rho(X_u) = \mu(u)\mu(v)\sigma(v^{-1}, u^{-1})X_{v^{-1}u^{-1}},\\ \rho(X_uX_v) &= \sigma(u, v)\rho(X_{uv}) = \sigma(u, v)\mu(uv)X_{(uv)^{-1}}.\end{aligned}$$

Thus $\sigma = (\delta\mu)\bar{\sigma}$, which implies $[\sigma]^2 = 1$, by Lemma 3.1.

Conversely, if $[\sigma] = [\sigma^{-1}] = [\bar{\sigma}]$, let $\mu : T \to \mathbb{F}^\times$ be such that $\sigma = (\delta\mu)\bar{\sigma}$. We claim that $\rho : \mathbb{F}^\sigma T \to \mathbb{F}^\sigma T$ defined by $\rho(X_u) = \mu(u)X_{u^{-1}}$ is a degree-inverting involution. By definition, ρ inverts the degrees, so we only need to show that it is an involution. We have

$$\rho(X_u X_v) = \sigma(u, v)\mu(uv)X_{(uv)^{-1}},$$
$$\rho(X_v)\rho(X_u) = \mu(v)\mu(u)\sigma(v^{-1}, u^{-1})X_{v^{-1}u^{-1}},$$

and both coincide by the choice of μ. Finally,

$$\rho\rho(X_u) = \mu(u)\mu(u^{-1})X_u.$$

So, we need to show that $\mu(u)\mu(u^{-1}) = 1$, for all $u \in T$. However, we note that, for any $u, v \in T$, we have

$$\frac{\mu(u)\mu(v)}{\mu(uv)} = \frac{\sigma(u, v)}{\bar{\sigma}(u, v)}.$$

In particular, $\mu(u)\mu(u^{-1}) = \sigma(u, u^{-1})\bar{\sigma}(u, u^{-1})^{-1}\mu(uu^{-1}) = \mu(1)$, for any $u \in T$. Taking $u = 1$, we obtain $\mu(1) = 1$. Hence, $\mu(u)\mu(u^{-1}) = 1$, for any $u \in T$, and we are done. ■

Lemma 3.3: *There exists an isomorphism* $\mathrm{Aut}_G(\mathbb{F}^\sigma T) \cong \mathrm{Hom}(T, \mathbb{F}^\times)$.

Proof: Given $\psi \in \mathrm{Aut}_G(\mathbb{F}^\sigma T)$, we have $\psi(X_u) = \chi(u)X_u$, for some $\chi : T \to \mathbb{F}^\times$, for all $u \in T$. It is easy to check that χ is a group homomorphism. Conversely, given $\chi : T \to \mathbb{F}^\times$, the map ψ defined by $\psi(X_u) = \chi(u)X_u$ is a G-graded automorphism of $\mathbb{F}^\sigma T$. So, we obtain a bijection $\psi \mapsto \chi$.

Finally, note that, if $\psi_i \mapsto \chi_i$, for $i = 1, 2$, then $\psi_1\psi_2 \mapsto \chi_1\chi_2$. So, the bijection is a group isomorphism. ■

Denote $\hat{T} = \mathrm{Hom}(T, \mathbb{F}^\times)$. As a consequence of the previous lemma, $\mathrm{Aut}_G(\mathbb{F}^\sigma T) \cong \hat{T}$ is an abelian group.

Lemma 3.4: *Let ρ be a degree-inverting involution on $\mathbb{F}^\sigma T$. Then, for any $\psi \in \mathrm{Aut}_G(\mathbb{F}^\sigma T)$, $\rho \circ \psi$ is a degree-inverting involution on $\mathbb{F}^\sigma T$. Every degree-inverting involution is obtained by such way.*

Proof: Using that $\psi(X_u) = \chi(u)X_u$, for all $u \in T$, we obtain that $\rho \circ \psi$ is a degree-inverting involution by direct computation. If ρ' is another degree-inverting involution, then $\rho\rho'$ is a graded automorphism, thus $\rho\rho' = \psi$, for some $\psi \in \mathrm{Aut}_G(\mathbb{F}^\sigma T)$. Thus, $\rho' = \rho \circ \psi$. ■

Given a group H, we denote $S(H) = \{h^2 \mid h \in H\}$. Notice that, if H is abelian, then $S(H)$ is a subgroup of H.

Lemma 3.5: *ρ and $\rho \circ \psi$ are equivalent if and only if $\psi \in S(\mathrm{Aut}_G(\mathbb{F}^\sigma T))$.*

Proof: For any $\psi \in \mathrm{Aut}_G(\mathbb{F}^\sigma T)$, note that $\rho \circ \psi = \psi^{-1} \circ \rho$.

So, if $\psi = \varphi^2$, for some $\varphi \in \mathrm{Aut}_G(\mathbb{F}^\sigma T)$, then

$$\rho\psi = \rho\varphi\varphi = \varphi^{-1}\rho\varphi,$$

which shows that $\rho\psi \sim \rho$. Conversely, assume that $\rho\psi = \varphi^{-1}\rho\varphi$, for some φ. Then we obtain $\rho\psi = \rho\varphi^2$, which implies $\psi = \varphi^2 \in S(\mathrm{Aut}_G(\mathbb{F}^\sigma T))$. ■

We summarize the results.

Theorem 3.6: *Let $\mathbb{F}$ be a field, T a finite group, and $\sigma : T \times T \to \mathbb{F}^\times$ a 2-cocycle. Then $\mathbb{F}^\sigma T$ admits a degree-inverting involution if and only if $[\sigma]^2 = 1$. In this case, there exist $|\hat{T}/S(\hat{T})|$ non-equivalent classes of degree-inverting involution on $\mathbb{F}^\sigma T$.*

Now, we are interested in the case where we have simultaneously $\mathbb{F}^\sigma T$ isomorphic to a matrix algebra, and $[\sigma]$ of order 2. The last one can be achieved if we compute the Schur multiplier $M(T)$. The former one is equivalent to: (a) $|T| = n^2$, for some n, and (b) T admits an irreducible (projective) σ-representation of degree n.

Although some works were dedicated to either answer the first question, or to compute the Schur multiplier (see, for instance, [11,12]), we were not able to find a single example of a non-abelian group satisfying both conditions. So we leave the following question.

Question 3.7: *Find a non-abelian finite group T of order n^2, for some $n \in \mathbb{N}$, and a 2-cocyle $\sigma : T \times T \to \mathbb{F}^\times$ such that $[\sigma]^2 = 1$, and $\mathbb{F}^\sigma T \cong M_n(\mathbb{F})$.*

3.1. Abelian case

Things become easier if we assume a priori the grading group abelian.

The following was essentially proved in [10]:

Lemma 3.8: *Let $\psi_0 : \mathcal{D} \to \mathcal{D}$ be a degree-inverting anti-automorphism, where $\mathcal{D}$ is a central finite-dimensional graded division algebra with support T, where T is an abelian group. Then T is an elementary 2-group.*

Proof: As mentioned in Subsection 2.4, $\mathrm{Supp}\,\mathcal{D} = H_1^2 \times H_2^2 \times \cdots \times H_s^2$, where each $H_i \cong \mathbb{Z}_{n_i}$, and $\mathcal{D} \cong M_{n_1} \otimes M_{n_2} \otimes \cdots \otimes M_{n_s}$, where each M_{n_i} is endowed with an ε_i-grading.

Since every nonzero homogeneous component of $\mathcal{D}$ has dimension 1, we see that each $1 \otimes \cdots 1 \otimes M_{n_i} \otimes 1 \cdots \otimes 1$ is invariant under the anti-automorphism, with support $1 \times \cdots 1 \times H_i^2 \times 1 \cdots \times 1$. From Lemma 4.6 of [10], we obtain $n_i = 2$ and $H_i \cong \mathbb{Z}_2$. ■

So, an immediate consequence is the following fact:

Corollary 3.9: *Let $\mathcal{D}$ be a central finite-dimensional graded division algebra over an algebraically closed field $\mathbb{F}$, and assume that* $\mathrm{Supp}\,\mathcal{D}$ *is commutative. Then an involution on $\mathcal{D}$ is a degree-preserving involution if and only if it is a degree-inverting involution.*

4. Degree-inverting involution on matrix algebras

In this section we investigate degree-inverting involutions on matrix algebras over an algebraically closed field of characteristic not 2. The arguments in this section are a copy of the ordinary case [8, Section 2.4] (see also the original paper by Elduque [9]). If a matrix algebra is endowed with a grading and a degree-inverting involution, then its support does not need to be commutative. This is a contrast with the degree-preserving involution case (see, for instance, [8, Proposition 2.49]).

We fix an algebraically closed field $\mathbb{F}$ and an arbitrary group G. Let $\mathcal{D}$ be a finite-dimensional G-graded division algebra, and let T be its support (then $T \subseteq G$ is a finite subgroup). Let V be a finite-dimensional G-graded right $\mathcal{D}$-module. We define

$$V^* = \{f : V \to \mathcal{D}, f \text{ is a graded } \mathcal{D}\text{-linear map}\}.$$

Thus, V^* has a natural G-grading. For homogeneous $f \in V^*$ and $v \in V$, we denote $\langle f, v\rangle = f(v)$ to emphasize the duality between V and V^*. Moreover, one has

$$\deg\langle f, v\rangle = \deg f \deg v.$$

Let $\mathcal{R} = \mathrm{End}_{\mathcal{D}}(V)$. Then $\mathcal{R}$ is a matrix algebra endowed with a G-grading. The natural action of $\mathcal{R}$ on V turns V a graded left $\mathcal{R}$-module. Also, V^* has a structure of graded right $\mathcal{R}$-module given by

$$\langle fr, v\rangle = \langle f, rv\rangle, \quad r \in \mathcal{R}, \quad f \in V^*, \quad v \in V.$$

Assume that $\mathcal{R}$ has a degree-inverting anti-automorphism ψ. Then V^* becomes a left $\mathcal{R}$-module by

$$r \cdot f := f\psi(r), \quad r \in \mathcal{R}, \quad f \in V^*. \tag{4}$$

Lemma 4.1: *With* (4), *V^* is an inverted-graded left $\mathcal{R}$-module, that is, V^* is a left $\mathcal{R}$-module and*

$$\mathcal{R}_g \cdot V_t^* \subseteq V_{tg^{-1}}^*, \quad \forall g, t \in G.$$

Proof: Let $r \in \mathcal{R}_g, f \in V_t^*, v \in V_h$. Then

$$\deg(r \cdot f)h = \deg\langle r \cdot f, v\rangle = \deg\langle f\psi(r), v\rangle = \deg\langle f, \psi(r)v\rangle = tg^{-1}h,$$

thus, $\mathcal{R}_g V_t^* \subseteq V_{tg^{-1}}^*$. ■

For any G-graded vector space $W = \bigoplus_{g\in G} W_g$, we define $W^{[-]} = \bigoplus_{g\in G} W_g^{[-]}$, where $W_g^{[-]} = W_{g^{-1}}$. These are known as *Veronese modules* (see [13, Example 1.2.7], for a more general construction).

Lemma 4.2: *V is an inverted-graded left $\mathcal{R}$-module if and only if $V^{[-]}$ is a graded left $\mathcal{R}$-module.*

Proof: Assume that $V^{[-]}$ is a graded left $\mathcal{R}$-module. Then

$$\mathcal{R}_g V_t = \mathcal{R}_g V^{[-]}_{t^{-1}} \subseteq V^{[-]}_{gt^{-1}} = V_{tg^{-1}}.$$

Conversely, if V is an inverted-graded left $\mathcal{R}$-module, then

$$\mathcal{R}_g V^{[-]}_t = \mathcal{R}_g V_{t^{-1}} \subseteq V_{t^{-1}g^{-1}} = V^{[-]}_{gt}.$$

■

Lemma 4.3: *There exists a degree-inverting $\mathcal{R}$-isomorphism $\varphi_1 : V^{[g_0]} \to V^*$, for some $g_0 \in G$. Equivalently, $\varphi_1 : V^{[g_0]} \to V^{*[-]}$ is a G-graded $\mathcal{R}$-isomorphism.*

Proof: It follows from Lemma 4.2 and Lemma 2.7 of [8]. ■

From now on, we fix $g_0 \in G$ and $\varphi_1 : V^{[g_0]} \to V^*$, as in Lemma 4.3.

Lemma 4.4: *There exists a homogeneous anti-automorphism $\psi_0 : \mathcal{D} \to \mathcal{D}$ such that*

$$\varphi_1(vd) = \psi_0(d)\varphi_1(v), \tag{5}$$

for all $v \in V, d \in \mathcal{D}$. Moreover, $\deg \psi_0(d) = g_0^{-1}(\deg d)^{-1}g_0$, *for any nonzero homogeneous $d \in D$.*

Proof: For any homogeneous $d \in \mathcal{D}$, let $R_d : V \to V$ be the right multiplication by d, and $L_d : V^* \to V^*$ the left multiplication. We will prove that the following sets coincide:

$$S_1 = \{\varphi : V^{[g]} \to V^* \text{degree-inverting } \mathcal{R}\text{-isomorphism, for some } g \in G\},$$
$$S_2 = \{\varphi_1 \circ R_d \mid d \in \mathcal{D}^\times \text{ homogeneous}\},$$
$$S_3 = \{L_d \circ \varphi_1 \mid d \in \mathcal{D}^\times \text{ homogeneous}\}.$$

It is clear that $S_2, S_3 \subseteq S_1$. Given $\varphi \in S_1$, we have $\varphi_1^{-1} \circ \varphi \in \operatorname{End}_{\mathcal{R}}(V) \cong \mathcal{D}$. Thus, for some nonzero homogeneous $d \in \mathcal{D}$, we have $\varphi_1^{-1} \circ \varphi = R_d$; which implies $\varphi = \varphi_1 \circ R_d \in S_2$. Similarly, $\varphi \circ \varphi_1^{-1} \in \operatorname{End}_{\mathcal{R}}(V^*) \cong \mathcal{D}$, so we can find a nonzero homogeneous $d \in \mathcal{D}$ such that $\varphi \circ \varphi_1^{-1} = L_d$. Hence, $\varphi = L_d \circ \varphi_1 \in S_3$.

Now, since $S_2 = S_3$, given a nonzero homogeneous $d \in \mathcal{D}$, we can find a nonzero homogeneous $d' \in \mathcal{D}$ such that $L_{d'} \circ \varphi_1 = \varphi_1 \circ R_d$. Define $\psi_0 : \mathcal{D} \to \mathcal{D}$ linearly, such that $\psi(d) = d'$. By construction, ψ_0 is a linear isomorphism, and it is an anti-homomorphism. Also, $L_{\psi(d)} \circ \varphi_1 = \varphi_1 \circ R_d$ is equivalent to $\psi(d)\varphi_1(v) = \varphi_1(vd)$, for all $v \in V$. Moreover, from this relation, we derive the following:

$$\deg \psi(d) \left((\deg v)g_0\right)^{-1} = \left((\deg v)(\deg d)g_0\right)^{-1}.$$

Or, equivalently, $\deg \psi(d) = g_0^{-1}(\deg d)^{-1}g_0$. ■

Remark 4.1: If it happens that $g_0 \in \operatorname{Supp} \mathcal{D}$, then, by the proof of Lemma 4.4, we can replace φ_1 by $\varphi_1 \circ R_{d_0}$, where $d_0 \in \mathcal{D}$ is homogeneous with $\deg d_0 = g_0$. Thus, $\deg \psi_0(d) =$

$(\deg d)^{-1}$ for all homogeneous d, so that the new $\psi_0 : \mathcal{D} \to \mathcal{D}$ is a degree-inverting involution on $\mathcal{D}$.

Now, we have a non-degenerate $\mathbb{F}$-bilinear form $B : V \times V \to \mathcal{D}$ given by

$$B(v, w) = \langle \varphi_1(v), w \rangle.$$

This form satisfies the following properties:

(i) $\deg B(v, w) = g_0^{-1}(\deg v)^{-1} \deg w$, for all homogeneous $v, w \in V$,
(ii) B is ψ_0-sesquilinear, that is, $B(vd, w) = \psi_0(d)B(v, w)$, $B(v, wd) = B(v, w)d$, $v, w \in V$, $d \in \mathcal{D}$,
(iii) $B(rv, w) = B(v, \psi(r)w)$, $v, w \in V$, $r \in \mathcal{R}$.

Conversely, a pair (B, ψ_0) satisfying (i)–(iii) determines uniquely ψ, that is, we can recover ψ from the pair (B, ψ_0). Indeed, let $\{w_1, \dots, w_n\}$ be a homogeneous $\mathcal{D}$-basis of V. Let $\Phi = (x_{ij})$, where $x_{ij} = B(w_i, w_j)$, be the matrix of B. Given $r \in \mathcal{R}$, let $R = (r_{ij})$ be its matrix form, and $\psi(R) = (r'_{ij})$ the matrix form of $\psi(r)$. Then, we have

$$B(rw_k, w_\ell) = B\left(\sum_{i=1}^{n} w_i r_{ik}, w_\ell\right) = \sum_{i=1}^{n} \psi_0(r_{ik}) x_{i\ell}$$

$$B(w_k, \psi(r) w_\ell) = B\left(w_k, \sum_{i=1}^{n} w_i r'_{i\ell}\right) = \sum_{i=1}^{n} x_{ki} r'_{i\ell}$$

So, we obtain the equation $\psi_0(R)^t \Phi = \Phi R$. Hence,

$$\psi : X \in \mathcal{R} \mapsto \Phi^{-1} \psi_0(X^t) \Phi \in \mathcal{R}, \tag{6}$$

where we identify, via Kronecker product, $\mathcal{R} = M_n(\mathcal{D})$, $\psi_0(X)$ means that we are applying ψ_0 in the entries of X, and ^{t} is the usual transpose involution of the $n \times n$ matrices $M_n(\mathcal{D})$. Assuming further that ψ_0 is the usual matrix transpose, then we can get rid of ψ_0 in equation (6); where the transposition should be understood as the usual transposition of $M_{n'}(\mathbb{F})$ (where $M_{n'}(\mathbb{F}) = M_n(\mathbb{F}) \otimes \mathcal{D}$).

We summarize the results obtained so far:

Proposition 4.5 (cf. [8, Theorem 2.57]): *Let G be any group, $\mathcal{D}$ a graded division algebra, V a finite-dimensional graded right $\mathcal{D}$-module and $\mathcal{R} = \mathrm{End}_{\mathcal{D}}(V)$. Assume that ψ is a degree-inverting anti-automorphism of $\mathcal{R}$. Then there exist $g_0 \in G$, an anti-automorphism ψ_0 on $\mathcal{D}$ satisfying* $\deg \psi_0(d) = g_0^{-1}(\deg d)^{-1} g_0$ *for all homogeneous $d \in \mathcal{D}$, and a non-degenerate form $B : V \times V \to \mathcal{D}$ satisfying* (i)–(iii). *If (ψ_0', B') is another such pair, then there exists a nonzero homogeneous $d \in \mathcal{D}$ such that $B' = dB$ and $\psi_0'(x) = d\psi_0(x)$, $\forall x \in \mathcal{D}$.*

Conversely, given a pair (ψ_0, B) satisfying (i)–(iii), *there exists a degree-inverting anti-automorphism on $\mathcal{R}$.*

From now on, we assume that ψ is a degree-inverting *involution*.

Lemma 4.6: *If ψ is an involution, then*

$$B(w,v) = \varepsilon_B \psi_0(B(v,w)), \quad \forall v,w \in V,$$

where $\varepsilon_B \in \{1,-1\}$.

Proof: Define $\bar{B}(v,w) = \psi_0(B(w,v))$. Then $\bar{B}$ is a non-degenerate ψ_0-sesquilinear form satisfying (ii). Thus, we can find an invertible $\mathcal{D}$-linear $Q : V \to V$ such that $\bar{B}(v,w) = B(Qv,w)$, for all $v,w \in V$. Hence, for any $r \in \mathcal{R}$, $v,w \in V$,

$$\begin{aligned} B(v,rw) &= B(\psi(r)v,w) = \psi_0\bar{B}(w,\psi(r)v) = \psi_0 B(Qw,\psi(r)v) = \psi_0 B(rQw,v) = \\ &= \bar{B}(v,rQw) = B(Qv,rQw). \end{aligned}$$

Taking $r = 1$, we see that $B(v,w) = B(Qv,Qw)$ for all $v,w \in V$. Hence, we have

$$B(v,rw) = B(Qv,rQw) = B(v,Q^{-1}rQw).$$

So $r = Q^{-1}rQ$, for all $r \in \mathcal{R}$. This gives $Q = \lambda \in \mathbb{F}$. Moreover, $B(v,w) = \lambda^2 B(v,w)$, for all $v,w \in V$, which implies $\lambda \in \{1,-1\}$. Thus, $\psi_0 B(w,v) = \bar{B}(v,w) = \varepsilon_B B(v,w)$, where $\varepsilon_B = \lambda$. ■

As a result, B is *balanced*, that is, $B(v,w) = 0$ if and only if $B(w,v) = 0$.

Given any $\mathcal{D}$-subspace $U \subseteq V$, we define

$$U^\perp = \{x \in V \mid B(x,U) = 0\} = \{x \in V \mid B(U,x) = 0\}.$$

The following result is standard:

Lemma 4.7: *Let $B : V \times V \to \mathcal{D}$ be a non-degenerate balanced $\mathbb{F}$-bilinear form. Given a $\mathcal{D}$-subspace $U \subseteq V$, we have $V = U \oplus U^\perp$ if and only if $B|_U$ is non-degenerate.*

Now, using Lemma 4.7, we can construct a homogeneous $\mathcal{D}$-basis of V

$$\{v_1,\ldots,v_m,v'_{m+1},v''_{m+1},\ldots,v'_s,v''_s\}, \tag{7}$$

satisfying

(a) $B(v_i,v_i) \neq 0$, $i = 1,2,\ldots,m$,
(b) $B(v'_j,v''_j) = 1$, $j > m$,
(c) all the remaining $B(v,w) = 0$.

Let $g_i = \deg v_i$, $g'_j = \deg v'_j$, $g''_j = \deg v''_j$. If $m > 0$, then $T \ni \deg B(v_1,v_1) = g_0^{-1}$. Also,

$$1 = \deg B(v'_j,v''_j) = g_0^{-1}g_j'^{-1}g''_j,$$

so $g''_j = g'_j g_0$, for all $j > m$. Moreover, we have

Lemma 4.8: *If $s > m$, then $g_0^2 = 1$.*

Proof: Since

$$B(v_s'', v_s') = \varepsilon_B \psi_0(B(v_s', v_s'')) = \varepsilon_B 1,$$

we obtain $1 = g_0^{-1}(g'')^{-1}g' = g_0^{-2}$. Thus, $g_0^2 = 1$. ∎

Now, if $\varepsilon_B = 1$ then we call ψ orthogonal, and otherwise, ψ is symplectic. We note that $\varepsilon_B = -1$ implies $m = 0$ in the previous notations. Using (6), we can construct the matrix of Φ, and determine ψ in matrix form. It will be convenient to use the basis $\{v_1, \dots, v_m, v'_{m+1}, \dots, v'_s, v''_{m+1}, \dots, v''_s\}$. We summarize the results

Theorem 4.9: *Let G be any group, and $\mathbb{F}$ an algebraically closed field,* $\operatorname{char}\mathbb{F} \neq 2$. *Let $\mathcal{R} = M_n(\mathcal{D})$ be a matrix algebra endowed with a G-grading parametrized by $(\mathcal{D}, \gamma)$. Then $\mathcal{R}$ admits a degree-inverting involution ψ if and only if there exists $g_0 \in G$, the graded division algebra $\mathcal{D}$ admits an involution ψ_0 satisfying* $\deg \psi_0(d) = g_0^{-1}(\deg d)^{-1}g_0$, $\forall d \in \mathcal{D}$ *homogeneous, and*

$$\gamma = (g_1, \dots, g_m, g'_{m+1}, \dots, g'_s, g''_{m+1}, \dots, g''_s)$$

where $g''_j = g'_j g_0$, for all $j > m$. Moreover, if $g_0 \notin T$ then $m = 0$; if $g_0 \in T$, then we can assume ψ_0 a degree-inverting involution; and if $s > m$ then $g_0^2 = 1$.

Let $\{X_u \mid u \in T\}$ be a homogeneous basis of $\mathcal{D}$. In any case, $\psi(e_{ij} \otimes X) = \Phi^{-1}e_{ji} \otimes \psi_0(X)\Phi$, for $e_{ij} \otimes X \in \mathcal{R}$, where Φ is given by:

(i) *if ψ is orthogonal,*

$$\Phi = \begin{pmatrix} I_m \otimes X_{g_0} & & \\ & 0 & I_s \otimes X_1 \\ & I_s \otimes X_1 & 0 \end{pmatrix},$$

(ii) *if ψ is symplectic, then*

$$\Phi = \begin{pmatrix} 0 & I_s \\ -I_s & 0 \end{pmatrix} \otimes X_1.$$

Remark 4.2: It is worth mentioning that, if G is assumed to be abelian, then we obtain a complete description of degree-inverting involutions on $M_n(\mathcal{D})$: the involution ψ_0 on $\mathcal{D}$ will be degree-inverting, and we apply Corollary 3.9.

5. Degree-inverting involution on upper triangular matrices

In this section we shall classify degree-inverting involutions on the algebra of upper triangular matrices. The final result is similar to the degree-preserving involution case [14]. However, in the degree-inverting case, the support of the grading does not need to be commutative. We shall improve the result obtained in [10], since we only impose the restriction $\operatorname{char}\mathbb{F} \neq 2$.

Let $\mathbb{F}$ be an arbitrary field of characteristic not 2, and G any group. It is known that every group grading on UT_n is elementary [15], that is, every grading admits an isomorphic structure where each matrix unit e_{ij} is homogeneous. Moreover, an isomorphism class of G-gradings on UT_n is uniquely determined by a sequence $\eta = (g_1, \ldots, g_{n-1}) \in G^{n-1}$, where $\deg e_{i,i+1} = g_i$, for $i = 1, 2, \ldots, n-1$ (see [16, Theorem 2.3]).

From now on, we fix a G-grading on UT_n, given by $\eta = (g_1, g_2, \ldots, g_{n-1})$. Let $J = J(UT_n)$ be the Jacobson radical, which is clearly a graded ideal. We denote by τ the canonical involution of UT_n, that is, $\tau(e_{ij}) = e_{n-j+1,n-i+1}$. Note that τ is the flip along the secondary diagonal of M_n.

Let ρ be a degree-inverting involution of UT_n. Since $\rho(J^m) = J^m$, for every $m \geq 1$, we have that ρ induces a degree-inverting involution on J/J^2 (which we denote by ρ as well). Moreover, we know that every automorphism of UT_n is inner (see, for instance, [17]); hence, $\rho = \mathrm{Int}(u) \circ \tau$, for some inner automorphism $\mathrm{Int}(u)$ (where $u \in UT_n$ is invertible). Thus, $\rho(e_{i,i+1} + J^2) = e_{n-i,n-i+1} + J^2$; that is, $\deg e_{i,i+1} = (\deg e_{n-i,n-i+1})^{-1}$. This proves

Lemma 5.1: *(UT_n, η) admits a degree-inverting involution if, and only if, $g_i = g_{n-i+1}^{-1}$ for each $i = 1, 2, \ldots, \lceil n/2 \rceil$.*

Proof: The argument above proves the 'only if' part. The 'if' part is obvious, since τ will invert degree, under this condition. ■

Remark 5.1: Note that, in contrast with the graded-involution case, the existence of a degree-inverting involution does not imply that the support of the grading is commutative.

Now, assume from now on that η satisfies the condition of Lemma 5.1. It is clear that τ is a degree-inverting involution in this case. Since we wrote $\rho = \mathrm{Int}(u) \circ \tau$, we note that $\mathrm{Int}(u)$ is a graded automorphism of UT_n. Thus, u is homogeneous of degree 1. Moreover, since $\rho^2 = 1$, one has $\tau(u) = \pm u$. We note that $\tau(u) = -u$ happens only if n is even. Indeed, if $n = 2m + 1$, then $\tau(e_{m+1,m+1}) = e_{m+1,m+1}$. Since u is invertible, the entry $(m + 1, m + 1)$ of u must be nonzero; and at the same time, it should coincide with its opposite, a contradiction.

Suppose $n = 2m$, and let $D = \mathrm{diag}(1, \ldots, 1, -1, \ldots, -1)$. The involution $s(x) = D\tau(x)D$ is called the *symplectic involution* of UT_n.

Finally, if $n = 2m + 1$, then we can multiply u by some scalar (note that, $\mathrm{Int}(u) = \mathrm{Int}(\lambda u)$), in such a way that its $m + 1$ entry is 1 (this is an important step in the proof of the next lemma, see [14, Lemma 2.4]). Also, if $\tau(u) = -u$, then

$$\rho(x) = u\tau(x)u^{-1} = uDD\tau(x)DDu^{-1} = \mathrm{Int}(uD)(s(x)).$$

In this case, $s(uD) = uD$. So, we can replace u by uD to obtain $s(u) = u$. Hence, in any case, we always obtain the equation

$$\rho = \mathrm{Int}(u) \circ \rho_0,$$

with $\rho_0(u) = u$, where ρ_0 is either τ or s.

Lemma 5.2: *Assume* $\mathrm{char}\,\mathbb{F} \neq 2$. *Let u be an invertible homogeneous element of degree* 1. *Let ρ_0 be either τ or s, in such a way that $\rho_0(u) = u$; and if $n = 2m + 1$, assume that the entry $(m + 1, m + 1)$ of u is 1. Then there exists a homogeneous invertible element $v \in UT_n$, of degree* 1, *such that $u = v\rho_0(v)$.*

Proof: The proof is exactly the construction of the proof of Lemma 2.4 of [14] (see also [10, Lemma 6.9]). As an example, we include here the case $n = 2m$, and $\rho_0 = \tau$. Write

$$u = \begin{pmatrix} X & Z \\ 0 & Y \end{pmatrix},$$

where $X, Y \in UT_m$ are invertible, and $Z \in M_m$. Then

$$v = \begin{pmatrix} \mathrm{I}_m & \frac{1}{2}Z \\ 0 & Y \end{pmatrix}$$

satisfies $u = v\tau(v)$. Moreover, let $\mathcal{U}$ be the set of pairs (i,j) such that $u = \sum_{(i,j)\in\mathcal{U}} \alpha_{ij}e_{ij}$, for $\alpha_{ij} \neq 0$. Since u is homogeneous of degree 1, and every matrix unit is homogeneous; $\deg e_{ij} = 1$, for all $(i,j) \in \mathcal{U}$. Now, by construction, $v = \sum_{(i,j)\in\mathcal{U}'} \beta_{ij}e_{ij}$, for some $\mathcal{U}' \subseteq \mathcal{U}$. In particular, v is a linear combination of homogeneous elements of degree 1. This implies v is homogeneous of degree 1.

The proof is similar for the other cases. ■

As a conclusion, $\rho = \mathrm{Int}(u) \circ \rho_0 = \mathrm{Int}(v) \circ \mathrm{Int}(\rho_0(v)) \circ \rho_0$, where ρ_0 is either τ or s, and $\rho_0(u) = u$. A straightforward argument shows that, in this case, ρ is equivalent to ρ_0. Indeed, we need to find a graded automorphism φ such that $\varphi(\rho_0(x)) = \rho(\varphi(x))$. Taking $\varphi = \mathrm{Int}(v)$, we have

$$\rho(\varphi(x)) = \rho(vxv^{-1}) = \mathrm{Int}(v)\mathrm{Int}(\rho_0(v))\rho_0(vxv^{-1}) = \mathrm{Int}(v)(\rho_0(x)) = \varphi(\rho_0(x)).$$

We summarize our main result of this section:

Theorem 5.3: *Let $\mathbb{F}$ be a field of characteristic not 2, and G any group. Let (UT_n, η) be G-graded, where $\eta = (g_1, g_2, \ldots, g_{n-1})$. Then (UT_n, η) admits a degree-inverting involution if, and only if, $g_i = g_{n-i+1}^{-1}$, for all $i = 1, 2, \ldots, n-1$. In this case, every degree-inverting involution is equivalent either to τ or to s; where s can occur if, and only if, n is even.*

Our definition of elementary grading on UT_n is not the standard one. Usually one defines an elementary grading on UT_n as we did for matrix algebras, that is, a sequence $\gamma = (h_1, h_2, \ldots, h_n) \in G^n$ defines a G-grading on UT_n by $\deg e_{ij} = h_i h_j^{-1}$. However, we cannot find a friendly way to write the condition of existence of a degree-inverting involution on UT_n in the standard notation. Nonetheless, if the grading group is abelian then the condition is nicely written, and we reobtain a result of [10]:

Corollary 5.4 ([10, Corollary 5.11]): *Let $\mathbb{F}$ be a field of characteristic not 2, and G be an abelian group. Let UT_n be endowed with an elementary G-grading given by $\gamma = (h_1, \ldots, h_n)$. Then UT_n admits a degree-inverting involution if and only if $h_1h_n^{-1} = h_2h_{n-1}^{-1} = \cdots = h_nh_1^{-1}$. In this case, every degree-inverting involution is equivalent either to τ or to s; where s can occur if, and only if, n is even.*

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