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SPATIALLY PERIODIC EQUILIBRIA FOR A NON LOCAL EVOLUTION EQUATION

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Abstract. In this work we prove the existence of a global attractor for the non local evolution equation $\frac{\partial m(r,t)}{\partial t} = -m(r,t) + \tanh(\beta J * m(r,t))$ in the space of τ-periodic functions, for τ sufficiently large. We also show the existence of non constant (unstable) equilibria in these spaces.

1. Introduction.

We consider here the non local evolution equation

$$\frac{\partial m(r,t)}{\partial t} = -m(r,t) + \tanh(\beta J * m(r,t)) \tag{1.1}$$

where $m(r,t)$ is a real function on $\mathbf{R} \times \mathbf{R}_+$; $\beta > 1, J \in C^1(\mathbf{R})$ is a non negative even function with integral equal to 1, and whose support is the interval $[-1,1]$. The $*$ product denotes convolution, namely:

$$(J * m)(x) = \int_{\mathbf{R}} J(x-y)m(y)\,dy \quad .$$

This equation arises as a continuum limit of one-dimensional Ising spin systems with Glauber dynamics and Kac potentials [6]; m represents then a magnetization density and β^{-1} the temperature of the system.

Equation (1.1) clearly has the spatially homogeneous equilibria 0 and $\pm m_\beta$; m_β being the positive solution of the equation

$$m_\beta = \tanh \beta m_\beta \quad . \tag{1.2}$$

It also has been proved in [5] that, in the space of continuous bounded functions in $\mathbf{R}$, there exists a exponentially stable stationary solution whose asymptotic values at $\pm\infty$ are $\pm m_\beta$ (the instanton).

We consider here the same equation restricted to the subspace $\mathcal{P}_{2\tau}$ of functions periodic in space with a given period 2τ, $\tau > 1$. As we will see below this leads naturally to the consideration of a flow in $L^2(S^1)$, where S^1 denotes the one dimensional unit sphere. In this space, one can show existence of a global compact attractor and the existence of stationary solutions in addition to the ones mentioned above (see [7] for a similar approach in the case of a semilinear parabolic equation).

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2. **The flow in $L^2(S^1)$.** We start by observing that, using uniqueness of solutions, it is easy to show that $\mathcal{P}_{2\tau}$ is invariant under the flow defined by (1.1). Now, if $\tau > 1$ is a given positive number, we define J^τ as the 2τ periodic extension of the restriction of J to $[-\tau,\ \tau]$.

Lemma 2.1. *If $u \in \mathcal{P}_{2\tau}$, then*

$$(J * u)(x) = \int_{-\tau}^{\tau} J^\tau(x-y)u(y)\,d\,y.$$

Proof: If $u \in \mathcal{P}_{2\tau}$, then

$$\begin{aligned}(J * u)(x) &= \int_{\mathbf{R}} J(x-y)u(y)\,d\,y \\ &= \int_{x-\tau}^{x+\tau} J(x-y)u(y)\,d\,y \\ &= \int_{x-\tau}^{x+\tau} J^\tau(x-y)u(y)\,d\,y \\ &= \int_{-\tau}^{\tau} J^\tau(x-y)u(y)\,d\,y.\end{aligned}$$

In view of this lemma, the problem (1.1), restricted to $\mathcal{P}_{2\tau}$, with $\tau > 1$, can be written as

$$\frac{\partial m(x,t)}{\partial t} = -m(x,t) + \tanh\left(\beta \int_{-\tau}^{\tau} J^\tau(x-y)m(y)\,d\,y\right)$$

Now, define $\varphi : [-\tau, \tau] \to S^1$ (the exponential map), by

$$\varphi(x) = e^{i\frac{\pi}{\tau}x}$$

and, for any $u \in \mathcal{P}_{2\tau}$, $v : S^1 \to \mathbf{R}$ by

$$v(\varphi(x)) = u(x).$$

In particular, we write $\tilde{J}(\varphi(x)) = J^\tau(x)$. Then, a simple computation shows that $u = u(x,t)$ is a 2τ-periodic solution of (1.1) if and only if $v(w,t) = u(\varphi^{-1}(w), t)$ is a solution of

$$\frac{\partial m(w,t)}{\partial t} = -m(w,t) + \tanh\left(\beta \tilde{J} * m(w,t)\right) \tag{2.1}$$

where now $*$ denotes convolution in S^1, that is

$$\left(\tilde{J} * m\right)(w) = \int_{S^1} \tilde{J}(w \cdot z^{-1})m(z)\,d\,z$$

and $d\,z = \frac{\tau}{\pi}\,d\,\theta$ where $d\,\theta$ denotes integration with respect to arclength. This will be the measure adopted in S^1 in the sequel. From now on, we drop the ˜ sign in J for simplicity.

Equation (2.1) generates a C^1 flow $T(t)$ in $X = L^2(S^1)$ since its right-hand side is a Lipschitz continuous function in this space.

3. **Existence of the global attractor.** In this section we prove the existence of a global maximal invariant compact set $A \subset X$ for the flow T, which attracts the bounded sets of X (the global attractor) (see [1] or [8]).

We recall that a set $\mathbf{B} \subset X$ is an *absorbing set* for the flow T if, for any bounded set C in X, there is a $t_1 > 0$ such that $T(t)C \subset B$ for any $t \geq t_1$ (see [8]).

Lemma 3.1. *For any $\varepsilon > 0$, the ball of radius $\frac{\sqrt{2\tau}}{1-\varepsilon}$ is an absorbing set for the flow $T(t)$.*

Proof: For $u \in X$, we denote the norm of u by $||u||$. Let $u(t) = T(t)u$. Then, while $||u|| \geq \frac{\sqrt{2\tau}}{1-\varepsilon}$, we have

$$
\begin{aligned}
\frac{d}{dt}\int_{S^1} |u|^2\, dz &= -2\left(\int_{S^1} u^2\, dz - \int_{S^1} u \tanh(\beta J * u)\, dz\right)\\
&\leq -2\int_{S^1} u^2\, dz + 2\left(\int_{S^1} u^2\, dz\right)^{\frac{1}{2}}\left(\int_{S^1} (\tanh(\beta J * u))^2\, dz\right)^{\frac{1}{2}}\\
&\leq -2\int_{S^1} u^2\, dz + 2\sqrt{2\tau}\left(\int_{S^1} u^2\, dz\right)^{\frac{1}{2}}\\
&\leq -2(||u||^2 - \sqrt{2\tau}||u||)\\
&\leq -2||u||^2(1 - \frac{\sqrt{2\tau}}{||u||})\\
&\leq -2\varepsilon||u||^2
\end{aligned}
$$

Therefore, while $||u|| \geq \frac{\sqrt{2\tau}}{1-\varepsilon}$, we have

$$\frac{d}{dt}||u(t)||^2 \leq -2\varepsilon||u||^2$$

Thus

$$||u(t)|| \leq e^{-\varepsilon(t-t_0)}||u(t_0)||. \tag{3.1}$$

and the result follows immediately.

Remark 3.2. *The estimate (3.1) above actually shows uniform exponential decay with rate ε to the ball of radius $\frac{\sqrt{2\tau}}{1-\varepsilon}$.*

Theorem 3.3. *There exists a global attractor A for the flow $T(t)$ generated by 2.1 in X, which is contained in the ball of radius $\sqrt{2\tau}$.*

Proof: If $u(w,t)$ is a solution of (2.1), we have by the variation of constants formula

$$u(w,t) = e^{-t}u(w,0) + \int_0^t e^{s-t}\tanh\{\beta(J * u)(w,s)\}\, ds \quad . \tag{3.2}$$

Write $T_1(t)u(w) = e^{-t}u(w,0)$, $T_2(t)u(w) = \int_0^t e^{s-t}\tanh\{\beta(J * u)(w,s)\}\, ds$ and suppose $u(\cdot,0) \in C$, where C is a bounded set in X contained, say, in a ball of radius R. Then $||T_1(t)u|| \to 0$ as $t \to \infty$, uniformly in u. Also, using estimate (3.1) we can see that $||u(t)|| \leq K$, for $t \geq 0$ where $K = \max\{R, \frac{\sqrt{2\tau}}{1-\varepsilon}\ \}$.

Therefore, for $t \geq 0$

$$
\begin{aligned}
\frac{\partial}{\partial w}T_2 u(w) &= \int_0^t e^{s-t}\frac{\partial}{\partial w}\tanh\{\beta(J*u)(w,s)\}\,d\,s \\
&= \beta\int_0^t e^{s-t}\operatorname{sech}^2(\beta J*u(w,s))\cdot(J'*u)(w,s)\,d\,s.
\end{aligned}
$$

Thus, we obtain

$$
\begin{aligned}
|\frac{\partial}{\partial w}T_2 u(w)| &\leq \beta\int_0^t e^{s-t}|J'*u(w,s)|d\,s \\
&\leq \beta\int_0^t e^{s-t}\int_{S^1}|J'(w\cdot z^{-1})u(z,s)|\,d\,z\,d\,s \\
&\leq \beta\int_0^t e^{s-t}||J'||\,||u(\cdot,s)||\,d\,s \\
&\leq K\beta||J'||\int_0^t e^{s-t}\,d\,s \\
&\leq K\beta||J'||
\end{aligned}
$$

It follows that, for $t \geq 0$ and any $u \in C$ the value of $||\frac{\partial}{\partial w}T_2 u||$ is bounded by a constant (independent of t and u). Thus $\bigcup_{t\geq 0} T_2(t)C$ is relatively compact in X by Sobolev's imbedding theorem.

The existence of the attractor follows immediately from Theorem 1.1 of [8]. The estimate on its size is an easy consequence of Lemma (3.1).

Now, once estimates in L^2 for solutions in the global attractor have been proved, one can use a bootstrap argument to obtain more regularity for them.

Theorem 3.4. *The global attractor A is bounded in $\mathcal{C}^k$, for any integer $k \geq 0$.*

Proof: If $u(w,t)$ is a solution of (1.1) in A, we have by the variation of constants formula

$$
u(w,t) = e^{-(t-t_0)}u(x,t_0) + \int_{t_0}^t e^{s-t}\tanh\{\beta(J*u)(w,s)\}\,d\,s \tag{3.3}
$$

Since $u(w,t_0)$ is bounded by $\sqrt{2\tau}$ for any choice of t_0, letting $t_0 \to -\infty$, we obtain

$$
u(w,t) = \int_{-\infty}^t e^{s-t}\tanh\{\beta(J*u)(w,s)\}\,d\,s \tag{3.4}
$$

(equality in L^2)

From this formula we obtain

$$
\begin{aligned}
|u(w,t)| &\leq |\int_{-\infty}^t e^{s-t}\tanh\{\beta(J*u)(w,s)\}\,d\,s| \\
|u(w,t)| &< \int_{-\infty}^t e^{s-t}\,d\,s \\
|u(w,t)| &< 1
\end{aligned} \tag{3.5}
$$

Now, proceeding as in the proof of Theorem 3.3, and using the bound of the attractor obtained there, we get

$$\begin{aligned}
|\frac{\partial}{\partial w}u(w,t)| &\leq |\int_{-\infty}^{t} e^{s-t}\frac{\partial}{\partial w}\tanh\{\beta(J*u)(w,s)\}\,ds| \\
&\leq \sqrt{2\tau}\beta\int_{-\infty}^{t} e^{s-t}||J'||\,ds \\
&\leq \sqrt{2\tau}\beta||J'||
\end{aligned}$$

Differentiating once more, we obtain

$\frac{\partial^2}{\partial w^2}u(w,t) =$

$\int_{-\infty}^{t} e^{s-t}\left\{2\beta^2\text{sech}(\beta J*u(w,s))\cdot\text{sech}\tanh(\beta J*u(w,s))\cdot(J'*u(w,s))^2+\right.$
$\left.\beta\text{sech}^2(\beta J*u(w,s))(J'*u'(w,s))\right\}\,ds$

and so

$$\begin{aligned}
|\frac{\partial^2}{\partial w^2}u(w,t)| &\leq \beta\int_{-\infty}^{t} e^{s-t}2\beta(J'*u(w,s))^2+J'*u'(w,s)\,ds \\
&\leq \beta\int_{-\infty}^{t} e^{s-t}2\beta||J'||^2||u(\cdot,s)||^2+||J'||||u'(\cdot,s)||\,ds \\
&\leq \beta^2 2\tau||J'||^2+\beta\sqrt{2\tau}||J'||
\end{aligned}$$

In the same way, we can obtain bounds for the derivatives of u of any order, in terms of J, J' and derivatives of lower order of u, concluding the proof.

4. **Stationary solutions.** The following functional $\mathcal{F}(m)$ is used in [5] to prove the existence of the instanton.

$$\mathcal{F}(m)=\int[f(m(x))-f(m_\beta)]\,dx+\frac{1}{4}\int\int J(x-y)[m(x)-m(y)]^2\,dx\,dy$$

where $f(m)$ (the free energy density) is given by

$$f(m)=-\frac{1}{2}m^2-\beta^{-1}i(m)$$

and $i(m)$ is the entropy density

$$i(m)=-\frac{1+m}{2}log\{\frac{1+m}{2}\}-\frac{1-m}{2}log\{\frac{1-m}{2}\}$$

A difficulty encountered with this functional in the space of continuous bounded functions in $\mathbf{R}$ is that $\mathcal{F}$ is not defined in the whole space. In fact, $\mathcal{F}(m)<\infty$ if, and only if $m(x)$ is close -in a certain sense- to $\pm m_\beta$ in a neighborhood of the infinity (see [5] for details).

In our setting, however, we have a similar functional defined in the whole phase space, as follows

$$\mathcal{F}(u) = \int_{S^1}[f(u(w))-f(m_\beta)]\,dw+\frac{1}{4}\int_{S^1}\int_{S^1}J(w\cdot z^{-1})[u(w)-u(z)]^2\,dw\,dz \quad .$$

We also define, for any $u\in X$

$$\dot{\mathcal{F}}(u) = \limsup_{t\to 0} \frac{1}{t}\left\{\mathcal{F}(T(t)u) - \mathcal{F}(u)\right\}$$

The following result is an adaptation of proposition 2.8 of [5] to our context.

Lemma 4.1. *Let $u(\cdot,t)$ be a solution of (2.1) with $u(\cdot,0)\leq 1$. Then $(\mathcal{F}(u(.,t))$ is well defined for all $t\geq 0$, it is differentiable with respect to t for $t>0$ and*

$$\dot{\mathcal{F}}(u(t)) = \frac{d}{dt}\mathcal{F}(u(\cdot,t)) = -I(u(\cdot,t)) \leq 0$$

where, for any $h\in L^2(S^1)$ with $||h||_\infty \leq 1$,

$$I(h(\cdot)) = \int_{S^1}[(J*h)(w) - \beta^{-1}arctanh(h(w))][tanh\beta(J*h)(w) - h(w)]\,d\,w$$

We are now ready to establish the existence of non trivial periodic equilibria for equation (1.1). For any $n\in\mathbf{N}^*$, we define the subspace A_n of X by

$$A_n = \left\{v\in X|v(\varphi(\frac{\tau}{n}+y)) = -v(\varphi(y))\right\}$$

It is easy to show, using uniqueness, that these subspaces are invariant under $T(t)$. Our result is given by

Theorem 4.2. *For any $n_0\in\mathbf{N}$, there is a $\tau(n_0)$ such that, if $\tau\geq\tau(n_0)$, there exists a nontrivial stationary solution of (2.1) in A_n, for any $n\leq n_0$.*

Proof: Consider the function in A_n defined by $l(\varphi(x)) = m_\beta$ for $0\leq x<\frac{\tau}{n}$, and $l(x,t)$ the solution of (1.1), with initial condition $l(\cdot,0)=l$. We then have, if $\tau>n$

$$\begin{aligned}
\mathcal{F}(l) &= \frac{1}{4}\int_{-\tau}^{\tau}\int_{-\tau}^{\tau} J^\tau(x-y)(l(x)-l(y))^2\,d\,y\,d\,x\\
&= \frac{1}{4}\sum_{j=-n}^{n-1}\int_{j\frac{\tau}{n}}^{j\frac{\tau}{n}+1}\int_{x-1}^{j\frac{\tau}{n}} J^\tau(x-y)4m_\beta^2\,d\,y\,d\,x\\
&+ \frac{1}{4}\sum_{j=-(n-1)}^{n}\int_{j\frac{\tau}{n}-1}^{j\frac{\tau}{n}}\int_{j\frac{\tau}{n}}^{x+1} J^\tau(x-y)4m_\beta^2\,d\,y\,d\,x \quad .
\end{aligned}$$

In Figure 1 we show the triangular regions (in the (x,y)-plane) where we have non-zero contributions to the above double integral. These are the regions which lie inside the support of $J^\tau(x-y)$ and where $l(x)$ and $l(y)$ are different. The first sum in the formula above corresponds to the triangles just bellow the diagonal, the second one to the triangles just above the diagonal. The single triangles in the corners are due to the periodicity of $l(y)$ and of J^τ, they lead to the first term in the first summation and to the last term in the second. Using the decomposition above we get:

$$\begin{aligned}
\mathcal{F}(l) &= 2nm_\beta^2 \int_0^1 \int_{x-1}^0 J^\tau(x-y)\,d\,y\,d\,x + 2nm_\beta^2 \int_{-1}^0 \int_0^{x+1} J^\tau(x-y)\,d\,y\,d\,x \\
&\le 4nm_\beta^2 \int_0^1 \int_{x-1}^x J^\tau(x-y)\,d\,y\,d\,x \\
&= 4nm_\beta^2 \int_0^1 \int_{-1}^0 J^\tau(z)\,d\,z\,d\,x \\
&= 2nm_\beta^2
\end{aligned}$$

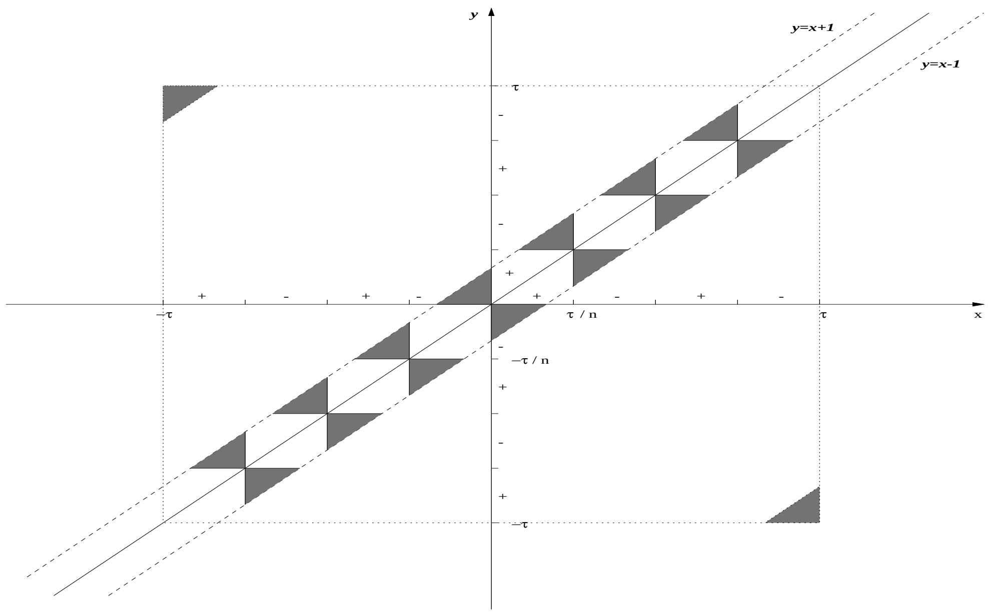

FIGURE 1. Triangular regions which give non zero contributions to the double integral in the proof of Theorem (4.2). We have taken $n = 4$ in the figure.

On the other hand

$$\mathcal{F}(0) = \int_{-\tau}^{\tau} (f(0) - f(m_\beta))\,d\,x = 2\tau(f(0) - f(m_\beta)).$$

Therefore, if $\frac{\tau}{n} > \frac{m_\beta^2}{(f(0)-f(m_\beta))}$, $\mathcal{F}(0) > \mathcal{F}(l)$ and the ω-limit of l does not contain the null stationary solution.

Now, the existence of a global compact attractor implies precompacity of the orbits of $T(t)$. It follows then by La Salle's invariance principle (see [2]) that $l(x,t) \to M$, where M is the maximal invariant subset of $E = \{u \in L^2(S^1) | \dot{\mathcal{F}}(u) = 0\}$. Observe now that, if $\dot{\mathcal{F}}(u) = 0$, then

$$\frac{1}{\beta}[\beta(J * u)(w) - arctanh(u(w))][\tanh \beta(J * u)(w) - u(w)] \equiv 0.$$

Since arctanh is an increasing function, we must have

$$\tanh \beta(J * u)(w) - u(w) \equiv 0.$$

that is, u is a stationary point. Since any point in E is a stationary point of (2.1), the result follows immediately.

Remark 4.3. *Since $m_\beta \to 0$ as $\beta \to 1$ and $\mathcal{F}(0) = \tau(1 - \frac{1}{\beta})m_\beta^2 + O(m_\beta^3)$ as $m_\beta \to 0$, it follows that $\mathcal{F}(0) < \mathcal{F}(l)$ if β is close to 1. Therefore, for any $n \geq 1$ and τ* fixed *the argument above cannot be used to show existence of solutio ns in A_n.*

We now want to show that the solutions in A_n are unstable. To achieve this goal we first need some preliminary definitions and auxiliary results from the theory of positive operators which were taken mainly from [4] and [3].

Let E be a Banach space. A closed convex subset $K \subset E$ is called a *cone* if $t \cdot K \subset K$ for every $t \geq 0$ and $K \bigcap (-K) = \{0\}$. We say that a cone K is *reproducing* if any $x \in E$ can be written as $x = u - v$ with $u, v \in K$.

If a cone exists in E, we can define an ordering in it by $x \leq y$ iff $y - x \in K$. A linear operator A on E is *positive* if $AK \subset K$. A is *strongly positive* if it is positive and, for every $x \in K - \{0\}$ there is an integer $m \geq 1$ such that $A^m(x) \in IntK$. If $e \in K - \{0\}$, A is *e-positive* if A is positive and, for every $x \in K - \{0\}$, there is an integer $m \geq 1$ and positive numbers α, β such that

$$\alpha e \leq A^m x \leq \beta e.$$

If A is a strongly positive operator, it can be proved (see for example [3, p. 59]) that A is also e-positive for any $e \in IntK$.

The result stated below is one version of *Krein-Rutman's Theorem*. A proof can be found in [4, Theorem 6.1] and [3, Theorem 2.10].

Theorem 4.4. *Let E be a Banach space, K a reproducing cone in E, and A a positive compact operator in E with a point of its spectrum different from 0. Then A has a positive eigenvalue ρ, not less in modulus than any other eigenvalue associated to an eigenvector in K. If A is e-positive, then ρ is also simple.*

We will apply Krein-Rutman's theorem to show the instability of solutions in A_n. For this, we need the following result:

Proposition 4.5. *Let $E = \mathcal{C}(S^1)$ be the space of real-valued continuous functions on S^1 with the* sup*-norm, K the cone of positive functions and T the operator in E defined by*

$$T(u)(w) = \theta(w) \int_{S^1} J(w \cdot z^{-1}) u(z) \, d\, z,$$

where θ is a strictly positive continuous function on S^1. Then, T is strictly positive.

Proof: We first observe that the support of J in S^1 is contained in an arc around $1 = e^{i0}$ with length $\frac{2\pi}{\tau}$, that is

$$\text{supp } J \subset \{ e^{i\frac{\pi}{\tau}\theta} \mid -1 \leq \theta \leq 1 \ \}$$

Now, for $u \in K - \{0\}$ we define

$$M_j = \{ w \in S^1 \ | T^j u(w) > 0 \ \}, \text{ for } 0, 1, 2, \cdot,$$

Note that $M_0 \neq \emptyset$. For all $w \in S^1$ such that $w \cdot z^{-1} \in$ supp J for some $z \in M_0$ (that is, if w is in some arc of length $\frac{2\pi}{\tau}$ centered around some point of M_0) then

$T(u)(w) > 0$, since the integrand is a non vanishing positive continuous function. It follows that M_1 contains an arc of length $\frac{2\pi}{\tau}$. If now, w is an end point of this arc and wz^{-1} is in supp J we conclude in the same way that $T^2u(w) = T(T(w) > 0$ and, therefore M_2 contains an arc with length $\frac{4\pi}{\tau}$, if $\frac{4\pi}{\tau} \leq 2\pi$.

Proceeding inductively, we conclude that $M_n = S^1$ if $\frac{n}{\tau} > 1$ and, thus $T^n > 0$, which means that T is strictly positive.

We are now ready to prove instability of solutions in A_n.

Theorem 4.6. *The solutions in A_n, obtained in Theorem (4.2), are all unstable.*

Proof: Let $\bar{m}(x)$ be a nontrivial equilibrium in A_n. The linearization of the evolution equation (2.1) around $\bar{m}$ is

$$\partial_t v = -v + (1 - \bar{m}^2)\beta J * v \equiv \mathcal{L}v$$

Since (by (3.5)), $(1-\bar{m}^2) > 0$, it follows from Proposition (4.5) that the operator $Tv = (1 - \bar{m}^2)\beta J * v$ is strictly positive as an operator in $\mathcal{C}(S^1)$. It is also compact by Arzela-Ascoli's theorem. Now, since $\mathcal{L}(\bar{m}') = 0$, it follows that $T(\bar{m}') = \bar{m}'$ and therefore T has a continuous eigenfunction associated to the eigenvalue 1. It follows from Krein-Rutman's Theorem (theorem (4.4)) that T must have an eigenfunction in the positive cone associated with a positive eigenvalue $\rho > 1$

Therefore $\mathcal{L} = -I + T$ has a positive eigenvalue and $\bar{m}$ is unstable.

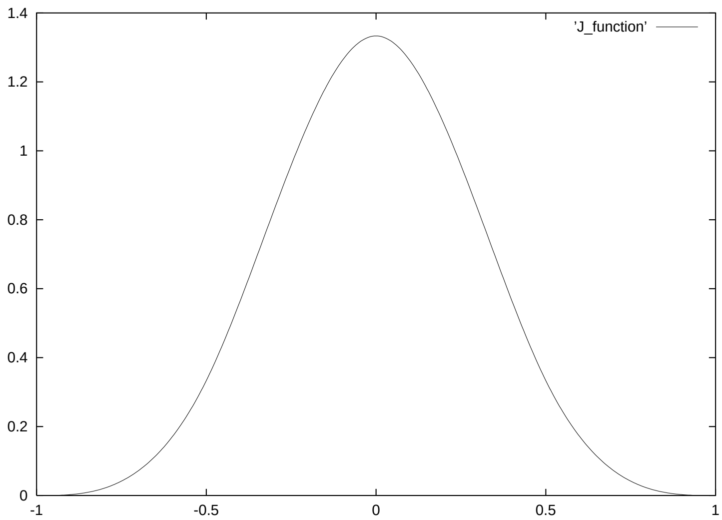

FIGURE 2. Graphic of the spline function $J(x)$ used in the numerical experiments. It is symmetric and assumes the values $J(x) = 4/3 - 8x^2(1 - x)$ in $[0, 0.5]$ and $J(x) = 8(1 - x)^3/3$ in $[0.5, 1]$.

5. **Numerical Simulations.** We have implemented a numerical scheme for solving equation (1.1) for 2τ-periodic solutions, based on a second order predictor-corrector method for the discretization of the differential equation. The convolution is computed through a multiple Simpson-rule, while the J function (shown in

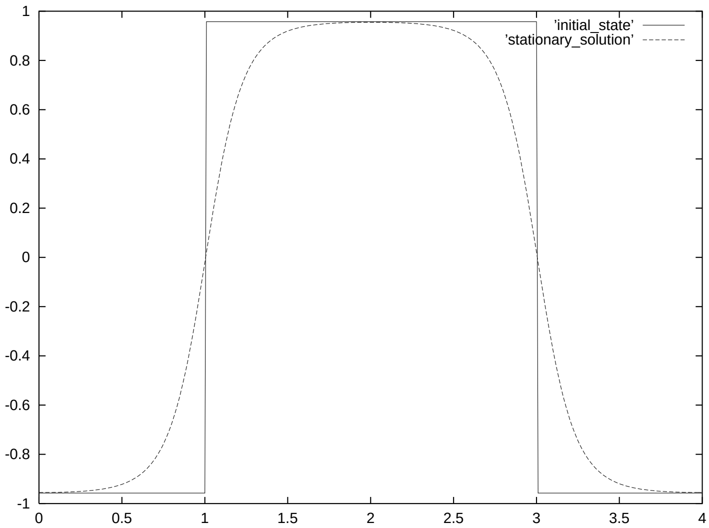

FIGURE 3. Stationary solution: $\tau = 2$ and $n = 1$

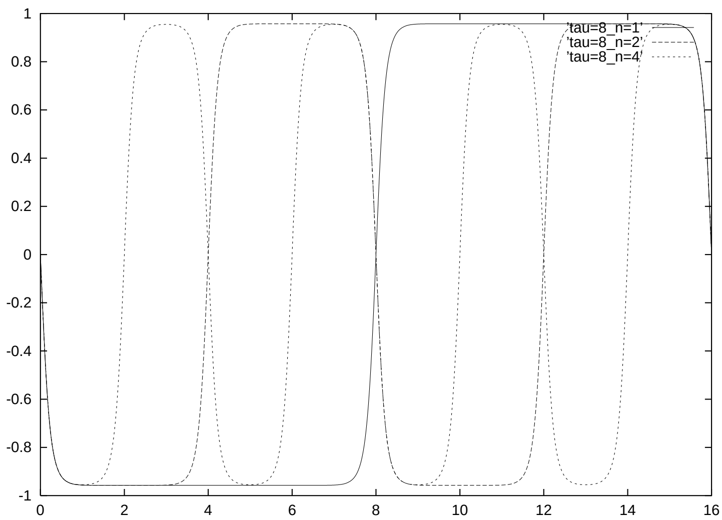

FIGURE 4. Stationary solutions: $\tau = 8$ and $n = 4$, $n = 2$, $n = 1$.

Figure 2) is chosen as a cubic spline, giving it the further property of being twice differentiable.

Numerical simulations helped on gathering evidence of the existence of the stationary solutions that we proved to exist. On the other hand, numerical experiments also provide means for illustrating the behaviour of the solutions in certain cases. In Figure 3 we plot the stationary solution obtained as in Theorem 4.2 with $\tau = 2$ and $n = 1$, after the temporal evolution of the initial state. We used the value $\beta = 2$ in the simulations.

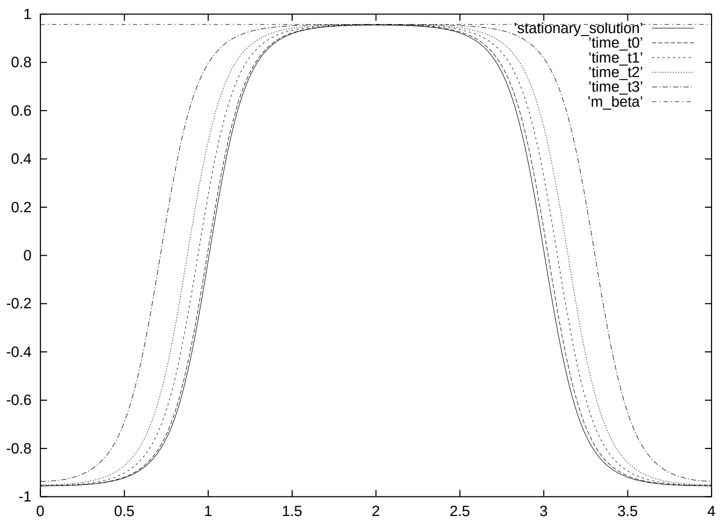

FIGURE 5. Unstable behaviour of perturbed stationary solution: $\tau = 2$ and $n = 1$. The solution departs from a small perturbation of the stationary solution of Figure 3 (time t0), until it gets to the constant stationary state equal to m_β.

In Figure 4 we display how the shape of these stationary solutions vary with n, for a fixed value of τ. In this case we employ $\tau = 8$ and show in the same graph the cases $n = 4$, $n = 2$ and $n = 1$. We point out that the stationary function for $\tau = 8$ and $n = 4$ is actually the same (up to translation invariance) as the one for $\tau = 2$ and $n = 1$, presented in Figure 3 (they are both 4-periodic). All these stationary solutions present a similar shape, having almost flat parts close to $+m_\beta$ or $-m_\beta$ and transition zones between them. We also observed in the experiments that for $n = 8$ (in the case $\tau = 8$) we don't obtain a new stationary solution (the conditions of Theorem 4.2 are not satisfied in this case).

In Figure 5 we present results of the use of the numerical method as an illustration of the unstable behaviour of the stationary solutions. In order to observe the instability, we perturbed the stationary solution ($\tau = 2$, $n = 1$, $\beta = 2$) by one percent of the modulus of its derivative. The choice of this perturbation was based on the fact that $T(\bar{v}) \geq \bar{v}$ (for $\bar{v} = |\bar{m}'|$) and this inequality is strict over an interval. For the perturbed initial state we computed the time evolution. The solution remains close to the stationary one for a long time, departing slowly from it, until reaching a point when it converges very fast to the stable stationary point given by the constant solution equal to m_β. This evolution is shown in Figure 5, where we plot the solution at several times during integration.

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