RT-MAT 98-11

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Abril 1998

TITLE

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SHORT RUNNING TITLE

EXPONENTIAL SOLUTION FOR INTEGRAL EQUATION

EXPONENTIAL SOLUTION FOR INFINITE DIMENSIONAL VOLTERRA-STIELTJES LINEAR INTEGRAL EQUATION OF TYPE (K)

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ABSTRACT

Here we are dealing with the linear Volterra-Stieltjes integral equation on B-spaces of type (K) with kernel K and resolvent R, which encloses a large class of evolutive systems as the PDE, NFDE and impulsive action equations. If the uniform semivariation of K is 0,5 at most, then there exists an operator B, with $R = e^{B}$.

It is exhaustively well known the importance in to get the operator-solution of an evolutive system having the exponential form.

In the frame of the linear integral equations of type (K) we have, until now, results yielding the resolvent in the exponential form only in the case in wich special kernels appear, e.g., when the equation (K) is a Stieltjes equation - (see remark 2.3 below). Here we will be giving more general conditions enlarging in this way our options.

In the following section 1 we will point general results on the theory of the linear integral equations of type (K). In the section 2 we will give the results concerning the exponential expression of the resolvent R.

A.M.S. classification: 45N05, 45D05, 45L99, 47B38
Keywords and phrases: Integral equations of type (K), solution-operator in the exponential form, semivariation.

1. Volterra-Stieltjes integral equations of type (K).

The Volterra-Stieltjes linear integral equation of type (K) deals with the forcing and state variables being regulated mappings and is considered in many works. The main events in the development of the theory are due to D.B. Hinton, who originated it in 1966, and C.S. Hönig and S. Schwabik (1974), and others. For historical remarks and references, see Hönig [6]. Here we work in the context by Hönig.

This type of equation encompasses very general classes of evolutive systems, as the linear ODE, PDE, Neutral Functional Differential Equations, controlled equations with impulsive action and the Stieltjes equations

$$y(t) - x + \int_a^t dA(s)y(s) = f(t), \tag{1.1}$$

as one can see in [5, pp. 82-94], [11], and in [7] and in [1] - [3].

Given $[a, b] \subset \mathbb{R}$ and a Banach space X, we define the *semi-variation* of $g : [a, b] \to L(X)$ as

$$SV[g] = \sup_{d \in D} \sup \left\{ \| \sum_{i=1}^{|d|} [g(t_i) - g(t_{i-1})]x_i \| \in X, \| x_i \| \leq 1 \right\},$$

where D is the set of all finite partitions of the interval, $[a, b]$, $d : t_0 = a < t_1 < ... < t_n = b$, and $|d| = n$. Sometimes we will denote $SV[g]$ by $SV_{[a,b]}[g]$. If $SV[g] < \infty$, we say that g is of bounded semivariation, and we will declare this fact by writing $g \in SV([a, b], L(X))$. Note that SV is a seminorm.

The following properties on the semi-variation of α will be useful:

Proposition 1.1 [4; I.1.2, I.3.1, I.3.3.] If $\alpha \in SV([a, b], L(X))$, then:

(i) if $[c, d] \subset [a, b]$, we have $\alpha \in SV([c, d], L(X))$ and $SV_{[c,d]}[\alpha] \leq SV_{[a,b]}[\alpha]$

(ii) the function $t \in [a, b] \to SV_{[a,t]}[\alpha]$ is increasing,

(iii) if $c \in (a, b)$ then $SV_{[a,b]}[\alpha] \leq SV_{[a,c]}[\alpha] + SV_{[c,b]}[\alpha]$,

(iv) α is bounded and $\| \alpha(t) \| \leq \| \alpha(a) \| + SV_{[a,t]}[\alpha]$.

We say that $f : [a,b] \to X$ is *regulated*, and write $f \in G([a,b],X)$ if f has only discontinuities of the first kind. $G([a,b],X)$ is a Banach space when endowed with the *sup* norm.

For α, $\beta \in SV([a,b],L(X))$ and $f \in G([a,b],X)$, there exists the *interior* (or *Dushnik-type*) *integral*

$$F_\alpha(f) = \int_a^b \cdot d\alpha(t) \cdot f(t) = \lim_{d \in D} \sum_{i=1}^{|d|} [\alpha(t_i) - \alpha(t_{i-1})] f(s_i) \in X$$

where $s_i \in (t_{i-1}, t_i)$, $0 \le i \le |d|$ and $d = \{t_0, t_1, ..., t_{|d|}\} \in D$ (see [6, Theorem 1.11]). In a connected way we define

$$\left[\int_a^b \cdot d\alpha(t) \circ \beta(t)\right](x) = \int_a^b \cdot d\alpha(t) \cdot [\beta(t)x] \ .$$

Actually the interior integral is an extension of the usual Riemann-Stieltjes integral.

Given I_X the identity mapping of $L(X)$ and the set

$$Q = \{(t,s) \in [a,b] \times [a,b]; a \le s \le t \le b\} \subset \mathbb{R}^2 \ ,$$

and a mapping $T : Q \to L(X)$, with $T^t(s) = T_s(t) = T(t,s)$, we write $T \in G^\sigma_\Delta \cdot SV^u(Q, L(X))$ or shortly $T \in G^\sigma_\Delta \cdot SV^u$ if T satisfies all the three properties:

(Δ) : the mapping $t \to T(t,t)$ is regulated ;

(G^σ) : $T_s x \in G([a,b], L(X))$,

where $T_s x(t) = T(t,s)x$, for every $t \in [a,b]$ and $x \in X$, and

(SV^u) : $SV^u_{[a,b]}[T] = SV^u[T] = \sup_{a \le t \le b} SV[T^t] < \infty$.

If, moreover, $T \in G^\sigma_\Delta \cdot SV^u$ has the property

$$(\Delta^0): \qquad T(t,t) = 0,$$

then we write $T \in G^\sigma_0 \cdot SV^u(Q, L(X))$ or shortly $T \in G^\sigma_0 \cdot SV^u$.

If, instead of $(\Delta^0), T \in G^\sigma_\Delta \cdot SV^u$ satisfies

$$(\Delta^I): \qquad T(t,t) = I_X$$

we write $T \in G^\sigma_I \cdot SV^u(Q, L(X))$, or shortly $T \in G^\sigma_I \cdot SV^u$. Note that SV^u is a seminorm, and that (see e.g. [7]):

$$\| \int_a^b \cdot d_s T(t,s) f(s) \| \leq SV^u[T] \cdot \| f \| \text{ for every } f \in G([a,b], X) \tag{1.2}$$

The operators in $G^\sigma_0 \cdot SV^u$ represent in the sense of the classical Riesz representation theorem - using by now the interior integral (see [6, Th. 2.10]) - exactly the non-antecipative [or causal] operators acting on the left continuous elements of $G([a,b], X)$. The equation (K) which we will be dealing with is

$$(K): \quad x(t) - x(a) + \int_a^t \cdot d_s K(t,s) \cdot x(s) = u(t) - u(a) \quad (a \leq t \leq b) \tag{1.3}$$

with $x, u \in G([a,b], X)$ and $K \in G^\sigma_0 \cdot SV^u$.

In the following proposition we will define the resolvent R, associated to K:

Proposition 1.2 ([6, Th. 3.4]) Suppose (1.3) and that there exists one and only one mapping $R \in G^\sigma_I \cdot SV^u(Q, L(X))$ satisfying

$$R(t,s)x - x + \int_s^t \cdot d_\tau K(t,\tau) \cdot R(\tau, s)x = 0 \tag{1.4}$$

for every $x \in X$ and $a \leq s \leq t \leq b$. Then the solution of (1.3) forced by u, is given by

$$x(t) = u(t) + R(t,a)[x(a) - u(a)] - \int_a^t \cdot d_s R(t,s) \cdot u(s).$$

In the next proposition we will give necessary and sufficient conditions on K, with the sake of to have the existence and unicity of such R. We will need the following definitions before:

Definition 1.3 For $K \in G_0^\sigma \cdot SV^u$ and $d \in D$ let

$$c(K,d) = \sup_{1 \leq i \leq |d|} \sup\{SV_{[s_{i-1},t]}[K^t]; s_{i-1} \leq t \leq s_i\}.$$

Definition 1.4 If $K \in G_0^\sigma \cdot SV^u$ we define $K^- \in G_0^\sigma \cdot SV^u$ as

$$K^-(t,s) = K(t-,s) = \lim_{\tau \downarrow t} K(\tau,s) \qquad (a < s \leq t \leq b).$$

Note that the existence of such operator is released in a straightforward way by the Banach-Steinhaus theorem.

A result allowing the existence and unicity of the resolvent R is done by the

Proposition 1.5 [7; Th. 3.8] Let be $K \in G_0^\sigma \cdot SV^u$, satisfying: there exists a division $d \in D$ with $c(K^-,d) < 1$.

Then there exists an unique R fulfilling (1.4) if and only if for every $t \in [a,b)$, we have $[I_X - K(t+,t)]^{-1} \in L(X)$. Moreover the resolvent R associated to K it is done by the *Neumann series*:

$$R(t,s) = I_X + \sum_{n=1}^{\infty} (-1)^n K^{(n)}(t,s) \tag{1.5}$$

with $K^{(n)}(n \geq 1)$ being $K^{(1)} = K$ and

$$K^{(n+1)}(t,s) = \int_s^t \cdot d_\sigma K^{(n)}(t,\sigma) \circ K(\sigma,s)$$

for every $s,t \in [a,b]$, $s \leq t$.

After those preliminary results, we present in the next section the main result in this work.

2. Exponential representation of the resolvent

The next theorem will be pointing about the possibility in to have $R = e^B$ for some operator B, on Q. Before, however, we need a result done in the next proposition.

Proposition 2.1 Let be $K \in G_0^\sigma \cdot SV^u(Q, L(X))$. Then

$$\sup_{\|x\|\leq 1} \| \int_s^t \cdot d_\sigma K(t,\sigma) \cdot [K(\sigma,s)x] \| \leq (SV^u_{[a,b]}[K])^2 \; .$$

for every $s,t \in [a,b]$.

Proof: According the inequality (1.2) we have for every $x \in X$,

$$\| \int_s^t \cdot d_\sigma K(t,\sigma) \cdot [K^\sigma(s)x] \| \leq \; SV^u_{[s,t]}[K] \cdot \| K_s(\sigma)x \| \leq$$

$$\leq SV^u_{[a,b]}[K] \cdot \; \| K_s(\sigma)x \| \leq \; SV^u_{[a,b]}[K] \cdot \sup_{\sigma\in[a,b]} \| K_s(\sigma)x \| \; . \tag{2.2}$$

with $K_\sigma \in G([s,t], L(X)) \subset G([a,b], L(X))$. The second inequality in (2.2) can be achieved with the use of the Proposition 1.1 (i)

But, for a fixed $\sigma \in [a,b]$ we have

$$\sup_{\|x\|\leq 1} \| K(\sigma,s)x \| \leq \; SV_{[s,\sigma]}[K^\sigma] \; , \tag{2.3}$$

and then taking the supremum on $\sigma \in [a,b]$ in both the sides in (2.3), we have:

$$\sup_{\|x\|\leq 1}\sup_{\sigma\in[a,b]} \| K(\sigma,s)x \| \leq SV^u_{[a,b]}[K] \ . \tag{2.4}$$

Comparing (2.2) and (2.4) we get the result. □

Theorem 2.2 Suppose $K \in G_0^\sigma \cdot SV^u$. If $SV^u[K] < \frac{1}{2}$ then there exists an operator $B \in G_0^\sigma \cdot SV^u$ such that for every $(t,s) \in Q$:

$$R(t,s) = e^{B(t,s)} \tag{2.5}$$

Proof: First of all we prove the existence of the resolvent R. Following the definitions we have $c(K^-, d) < SV^u[K^-] < SV^u[K] < \frac{1}{2}$.

This implies (Prop. 1.5), that R can be done by the Neumann series (1.5).

According a result by Nagumo within Banach algebra ([9, Th. I.4.12]) and using straightforward arguments (see for example [10; Lemma]) we get (2.5) if, for instance, we have for all nonnegative real r:

$$[R(t,s) + rI_X]^{-1} \in L(X) \ . \tag{2.6}$$

Using (1.4), the expression in (2.6) is true provide

$$\left[\lambda I_X - \int_s^t \cdot d_\sigma K(t,\sigma) \circ R(\sigma,s)\right]^{-1} \in L(X) \ , \tag{2.7}$$

for all real $\lambda \geq 1$.

Otherwise we have (see [8]): if

$$\| \int_s^t \cdot d_\sigma K(t,\sigma) \circ R(\sigma,s) \| < 1 \tag{2.8}$$

then (2.7) is true.

Using (1.4) and the Prop. 1.5 again, we get

$$\sum_{n=1}^{\infty} (-1)^{n+1} K^{(n)}(t,s) = \int_s^t \cdot d_\sigma K(t,\sigma) \circ R(\sigma,s) \ . \tag{2.9}$$

Gathering the result in (2.9) with the one in the Prop. 2.1, we obtain:

$$\| \int_s^t \cdot K(t,\sigma) \circ R(\sigma,s) \| \leq \sum_{n=1}^{\infty} (SV^u[K])^n \ .$$

Because $SV^u[K] \leq 1/2$ we get (2.8), and so the theorem. □

The next remark shows the unique result in the present framework telling actually about exponential representation of the resolvent. It will help us in to look at the results of this section from an exterior point of view.

Remark 2.3 Consider the Stieltjes equation (1.1). If $A \in SV([a,b], L(X))$ we get

$$R(t,s) = e^{[A(t)-A(s)]}$$

if and only if A is continuous ([7; Remark p. 37]). The result, essentially, is obtained because in this case we have at (1.1) $(R(t,0))_{t \geq a}$ satisfying the, semigroup properties. But this situation is no longer true for general (K) - in (1.3) - at all.

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