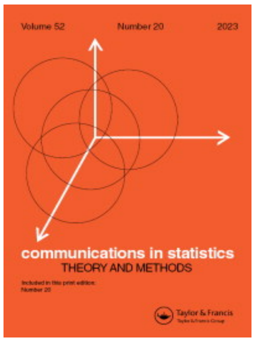

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Some asymptotic inferential aspects of the Kumaraswamy distribution

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ABSTRACT

The Kumaraswamy distribution is doubly limited, continuous, very flexible, and is widely applied in hydrology and related areas. Recently, several families of distributions based on this distribution have emerged. To make a contribution regarding some asymptotic aspects related to the inferential analysis, we derived an analytic expression of order $n^{-1/2}$, where n is the sample size, for the skewness coefficient of the distribution of the maximum likelihood estimators of the parameters of the Kumaraswamy distribution. A simulation study and an application are presented to illustrate that, when the sample size is small, the likelihood inferences may not be reliable. We also obtain Bartlett correction factors for the likelihood ratio statistic as well as the results of the *bootstrap* likelihood ratio test and *bootstrap* Bartlett correction and present a Monte Carlo simulation study to compare the rejection rates of the tests in question.

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1. Introduction

The Kumaraswamy distribution Kumaraswamy (1980) is doubly limited, continuous, and very flexible. It has been extensively studied because of its similarities with the beta distribution, but overcoming it with some computational advantages. Recent contributions in the literature include Andrade (2020), Bai et al. (2021), Cordeiro et al. (2018), and Pinto et al. (2020), among others. Jones (2009) presents a detailed study of properties, advantages, and disadvantages of the Kumaraswamy distribution.

It is well known that the maximum likelihood estimators (MLE) are asymptotically normally distributed under standard regularity conditions. Inferences regarding this asymptotic distribution, when the sample size is moderate or small, may not be adequate. One way to check, in finite samples, whether the MLE distribution is close to the normal distribution, is to calculate its skewness coefficient. A value far from zero gives indications that the MLE distribution is far from the normal distribution.

There are several measures of asymmetry, but the most used by practitioners is Pearson's standardized third cumulant, defined by $A = \kappa_3/\kappa_2^{3/2}$, where κ_r is the rth cumulant of the distribution. In general, κ_3 does not have a closed-form. This issue was overcome by Bowman and Shenton (1998), who expanded the third cumulant of the distribution of the maximum

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likelihood estimator up to the order n^{-2}. Based on this expansion, they obtained the skewness coefficient up to the order $n^{-1/2}$, γ_1. This coefficient can be seen as a more accurate estimate of the exact asymmetry coefficient of the MLE distribution.

The result of Bowman and Shenton (1998) is very general, so the statistical literature contains many expressions of the skewness coefficient for specific models. For example, Cordeiro and Cordeiro (2001) derived the specific expression of γ_1 for generalized linear models and Cavalcanti et al. (2009) extended this result to the exponential family of non linear models. Magalhães, Botter, and Sandoval (2013) and Magalhães et al. (2019) derived γ_1 for the beta regression model with fixed and varying dispersion, respectively, and Magalhães, Gallardo, and Gómez (2019) obtained γ_1 for Weibull censored data. The idea of obtaining the skewness coefficient has been well received in the literature and to contribute to this research area, our main goal is to derive the specific skewness coefficient of order $n^{-1/2}$ for the MLE of the parameters of the Kumaraswamy distribution.

The likelihood ratio statistic Wilks (1938) is among the most widely used in statistical tests. It is known that, under the null hypothesis and under general conditions of regularity, this statistic has an asymptotic chi-squared distribution, with an approximation error of $\mathcal{O}(n^{-1})$. When n is small or even moderate the use of this approximation may not be adequate. An alternative to circumvent this problem and to improve the quality of the approximation of the likelihood ratio statistic (S_{LR}) distribution to the chi-squared distribution was proposed by Bartlett (1937). He suggested multiplying S_{LR} by a factor C_B, known as Bartlett's correction factor, thus obtaining a modified likelihood ratio statistic, S^*_{LR}, whose rate of convergence to the chi-squared distribution is faster. Recent works related to this topic include Araújo, Cysneiros, and Montenegro (2020), Melo et al. (2022), and Magalhães and Gallardo (2020). Given the importance of this topic for statistical inference, here we also propose correcting the likelihood ratio statistic in order to study and compare the performance of the likelihood ratio (LR) tests with and without correction, considering the Kumaraswamy distribution.

The work is organized as follows. In Section 2, we introduce the Kumaraswamy distribution. In Section 3, we obtain an analytical expression for the skewness coefficients of order $n^{-1/2}$ of the distribution of the MLE of the Kumaraswamy distribution parameters. In Section 3.1, we perform a simulation study to compare the performance of these coefficients with the sample skewness and in Section 3.2, we illustrate the results with an application to a real dataset. In Section 4, we obtain the Bartlett correction factor for the LR test of the parameters of the Kumaraswamy distribution. In addition, we include the bootstrap test and the Bartlett bootstrap correction factor for the S_{LR} and in Section 4.1 we develop a simulation study to compare their performances. This article concludes with a brief discussion in Section 5.

2. Kumaraswamy distribution

A random variable Y has a Kumaraswamy distribution with parameters $\alpha > 0$ and $\beta > 0$, $Y \sim Kum(\alpha, \beta)$, if the cumulative distribution function (CDF) and probability density function (PDF) are given, respectively, by

$$F(y;\alpha,\beta) = 1 - (1 - y^{\alpha})^{\beta} \text{ and } f(y;\alpha,\beta) = \alpha\beta y^{\alpha-1}(1 - y^{\alpha})^{\beta-1},$$

for $y \in (0, 1)$. The general function of the q-order quantile is easily obtained by applying $y_q = \left[1 - (1 - q)^{1/\beta}\right]^{1/\alpha}, 0 < q < 1$.

Let $Y_1, \dots, Y_n$ be a random sample of size n from the $Kum(\alpha, \beta)$ distribution. The log-likelihood function for $\boldsymbol{\theta} = (\alpha, \beta)^\top$ and an observed sample $\boldsymbol{y} = (y_1, \dots, y_n)^\top$ is given by

$$\ell(\boldsymbol{\theta}; \boldsymbol{y}) = \ell(\boldsymbol{\theta}) = n \log(\alpha\beta) + (\alpha - 1) \sum_{i=1}^{n} \log(y_i) + (\beta - 1) \sum_{i=1}^{n} \log(1 - y_i^\alpha). \tag{1}$$

The score vector obtained by differentiating (1) with respect to $\boldsymbol{\theta}$ is given by $\boldsymbol{U_\theta} = \boldsymbol{U_\theta}(\boldsymbol{\theta}) = (U_\alpha, U_\beta)^\top = (\partial\ell/\partial\alpha, \partial\ell/\partial\beta)^\top$, with

$$\frac{\partial \ell}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} \log(y_i) + (1 - \beta) \sum_{i=1}^{n} \frac{y_i^\alpha \log(y_i)}{1 - y_i^\alpha} \quad \text{and} \quad \frac{\partial \ell}{\partial \beta} = \frac{n}{\beta} + \sum_{i=1}^{n} \log(1 - y_i^\alpha). \tag{2}$$

The MLE of $\boldsymbol{\theta}$ is obtained by solving $\boldsymbol{U_\theta} = \boldsymbol{0}$, which has no analytical expression. For more details, we suggest Jones (2009) and Lemonte (2011).

In this section, the following notation is adopted for log-likelihood derivatives, where r, s, t,... denote components of the vector $\boldsymbol{\theta}$: $U_r = \partial\ell/\partial\theta_r$, $U_{rs} = \partial^2\ell/\partial\theta_r\partial_s$, and so on. Also, the standard notation for the cumulative log-likelihood derivatives is: $\kappa_{rs} = E(U_{rs})$, $\kappa_{r,s} = E(U_r U_s)$, $\kappa_{rs,t} = E(U_{rs} U_t)$, and so on, where all κ's correspond to a total over the sample and are generally of order n.

The Fisher information matrix for $\boldsymbol{\theta}$ has the form

$$\boldsymbol{K_\theta} = \boldsymbol{K_\theta}(\boldsymbol{\theta}) = \begin{bmatrix} k_{\alpha,\alpha} & k_{\alpha,\beta} \\ k_{\beta,\alpha} & k_{\beta,\beta} \end{bmatrix}, \tag{3}$$

where $k_{\alpha,\alpha} = n\overline{A}/\alpha^2$, $k_{\alpha,\beta} = k_{\beta,\alpha} = n\overline{B}/\alpha$, $k_{\beta,\beta} = n/\beta^2$. For sake of brevity, the quantities $\overline{A}$ and $\overline{B}$ are presented in Appendix A. Note that the matrix $\boldsymbol{K_\theta}$ is not diagonal. This indicates that the parameters α and β are not orthogonal and therefore their MLE estimators $\hat{\alpha}$ and $\hat{\beta}$ are not asymptotically independent.

Under the usual regularity conditions for maximum likelihood estimation, when the sample size is large, $\hat{\boldsymbol{\theta}} \sim N_2(\boldsymbol{\theta}, \boldsymbol{K_\theta}^{-1})$, approximately, where $\boldsymbol{K_\theta}^{-1}$ is the inverse of Fisher's information matrix, presented in Appendix A.

3. The skewness coefficient

As discussed in the introductory section, calculating the skewness coefficient is simple way to verify whether the approximation to normality is adequate. Since there is no closed-form of this coefficient for the distribution of the MLE of the Kumaraswamy's parameters, the approximation of order $\mathcal{O}(n^{-2})$ for the third cumulant of a general MLE $\hat{\theta}$, $\kappa_3(\hat{\theta}) = E\left[\hat{\theta} - E\left(\hat{\theta}\right)\right]^3$, derived by Bowman and Shenton (1998) is very useful.

From the general result of Bowman and Shenton (1998), we derived, after some algebraic manipulations, the specific third cumulant of $\hat{\alpha}$ and $\hat{\beta}$ in the Kumaraswamy distribution. They can be, respectively, expressed as

$$\kappa_3(\hat{\alpha}) = \frac{\alpha^3}{n^2(\overline{A} - \beta^2\overline{B}^2)^3} \left(a_0 + \frac{a_1}{\beta - 1} + \frac{a_2}{\beta - 2} + \frac{a_3}{\beta - 3} \right), \tag{4}$$

$$\kappa_3(\hat{\beta}) = \frac{\beta^3}{n^2(\overline{A} - \beta^2\overline{B}^2)^3} \left(b_0 + \frac{b_1}{\beta - 1} + \frac{b_2}{\beta - 2} + \frac{b_3}{\beta - 3} \right),$$

where the quantities a_0 to a_3 and b_0 to b_3 are presented in Appendix B.

By (4) and the inverse of the Fisher information matrix, given in (A.1), the skewness coefficient of the distribution of $\hat{\alpha}$ and $\hat{\beta}$ in the Kumaraswamy distribution up to order $n^{-1/2}$ is given by

$$\gamma_1(\hat{\alpha}) = \frac{1}{\sqrt{n\left(\overline{A} - \beta^2\overline{B}^2\right)^3}}\left(a_0 + \frac{a_1}{\beta - 1} + \frac{a_2}{\beta - 2} + \frac{a_3}{\beta - 3}\right), \ \beta \notin \{1, 2\}, \tag{5}$$

$$\gamma_1(\hat{\beta}) = \frac{1}{\sqrt{n\overline{A}^3\left(\overline{A} - \beta^2\overline{B}^2\right)^3}}\left(b_0 + \frac{b_1}{\beta - 1} + \frac{b_2}{\beta - 2} + \frac{b_3}{\beta - 3}\right), \ \beta \notin \{1, 2, 3\}.$$

Note that the skewness coefficient of the $\hat{\alpha}$ distribution does not depend on the α parameter, instead depending on the β parameter. On the other hand, the skewness coefficient of the $\hat{\beta}$ distribution depends only on the β parameter. As n increases, the values of $\gamma_1(\hat{\alpha})$ and $\gamma_1(\hat{\beta})$ decrease at a rate of $\mathcal{O}(n^{-1/2})$, as expected. The amounts $\left(a_0 + \frac{a_1}{\beta-1} + \frac{a_2}{\beta-2} + \frac{a_3}{\beta-3}\right)$ and $\left(b_0 + \frac{b_1}{\beta-1} + \frac{b_2}{\beta-2} + \frac{b_3}{\beta-3}\right)$ determine the sign of the asymmetry of the $\hat{\alpha}$ and $\hat{\beta}$ distributions, respectively.

Equation (5) is our first main result. The expansion proposed by Bowman and Shenton (1998) is very general and there is no guarantee that it will have closed-form for all models. For instance, although Simas, Cordeiro, and Rocha (2010) found the skewness coefficient for the dispersion models, it has no closed-form for the simplex distribution. As can be seen in (5), we are able to express γ_1 of the parameters of a Kumaraswamy distribution in a closed-form. Additionally, it has a very nice form and can be easily implemented in statistical softwares such as R (R Core Team 2020).

3.1. Simulation study

In this section, a Monte Carlo simulation study is performed to evaluate the skewness coefficient of the MLE distributions of the parameters α and β of the Kumaraswamy distribution, defined in (5), and to compare them with the sample skewness coefficients, defined by the ratio of sample moments.

The sample sizes considered in the simulation are $n = 15$, 25, 40, and 60, and the true values of the parameters are fixed at $\alpha = 0.5$ and $\beta = 0.5$, 5 and 7.5, since different values of α lead to the same results. The number of Monte Carlo replications is set at $R = 5{,}000$. For each replication, we find the MLE $\hat{\boldsymbol{\theta}} = (\hat{\alpha}, \hat{\beta})^\top$ by maximizing the log-likelihood function using the quasi-Newton method BFGS. All simulations were performed using the R software.

The sample skewness coefficient of the rth component of $\hat{\boldsymbol{\theta}}$, is calculated as

$$g_1 = m_3/m_2^{3/2}, \ m_s = \frac{1}{5,000}\sum_{i=1}^{5,000}\left(\hat{\theta}_{ri} - \bar{\hat{\boldsymbol{\theta}}}_r\right)^s,$$

where m_s is the sample moment of order s and $\bar{\hat{\boldsymbol{\theta}}}_r = \sum_{i=1}^{5,000}\hat{\theta}_{ri}/5,000$, with $r = 1$ and 2. For each of the 5,000 Monte Carlo replications, we calculate the expressions in (5). The γ_1 coefficient is the arithmetic mean of the 5,000 calculated values. The results obtained are shown in Table 1.

Table 1. Sample (g_1) and $n^{-1/2}$ skewness (γ_1) coefficients.

		$\hat{\alpha}$		$\hat{\beta}$	
β	n	g_1	γ_1	g_1	γ_1
0.5	15	2.5520	1.4909	2.2657	1.3920
	25	1.4601	1.1694	1.4767	1.0669
	40	1.1568	0.9282	0.9928	0.8399
	60	0.7792	0.7614	0.7865	0.6835
5.0	15	1.0046	0.8873	4.2904	2.2581
	25	0.7508	0.6881	3.895	1.7131
	40	0.5259	0.5441	1.7564	1.3417
	60	0.4588	0.4447	1.2966	1.0883
7.5	15	0.8854	0.8680	3.3256	2.4531
	25	0.6779	0.6720	2.9803	1.8588
	40	0.5898	0.5310	2.1690	1.4541
	60	0.4503	0.4336	1.4217	1.1792

We point out that the $n^{-1/2}$ skewness coefficient of the distribution of the MLE of α is generally close to the sample skewness coefficient. For β, this pattern only occurs when $\beta = 0.5$. For other values of β, they are reasonably close when $n = 60$. In addition, when the sample size increases, the sample and the $n^{-1/2}$ skewness coefficient decrease, as expected. However, even for $n = 60$, they are far from zero, mainly for the MLE distribution of β. This result indicates that the normal distribution approach for the MLE of the Kumaraswamy distribution parameters can be misleading with small and moderate sample sizes. This result shows that the distance of the distribution of $\hat{\beta}$ from normality is strongest when the values of β are greater than 0.5. In contrast, the $\hat{\alpha}$ distribution appears to be closer to normal as the values of β increase. Our results are related to those of Lemonte (2011), whose simulations show that the MLE of α and β are severely biased. We also note that, in almost all cases, the γ_1 coefficient has values lower than the g_1 coefficient, indicating that the skewness coefficient of order $n^{-1/2}$ underestimates the sample asymmetry.

To enable visualization of some of our results, we created three figures containing estimated MLE densities of $\hat{\alpha}$ and $\hat{\beta}$, in order to compare them with the density of the normal distribution with increasing values of n. The results in Table 1 are presented in Figures 1–3. We chose the sample sizes $n = 15, 25,$ and 60, $\alpha = 0.5$ and $\beta = 0.5, 5$ and 7.5 (chosen cases to cover $\alpha < 1$ and $\beta < 1$, $\alpha < 1$ and $\beta > 1$). These figures illustrate the behavior that we described earlier in the comments on the tables. Since the formulas for the skewness coefficients of α and β do not depend on α, we only vary parameter β.

3.2. Real-world data analysis

Here we consider data on food expenditure, income, and number of people in each household from a random sample of 38 households in a large city in the USA (Griffiths et al. 1993). The variables are defined as (x_1): household food expenses, (x_2): household income and (x_3): number of people in the household. Our interest is to model the proportion of income spent on food, that is, $Y = x_1/x_2$.

When Y follows a Kum(α, β) distribution, Table 2 shows the maximum likelihood estimates of α and β, the respective standard errors and the estimated values of the asymmetry coefficients, obtained from the expressions in (5). The estimated values of γ_1 suggest that a

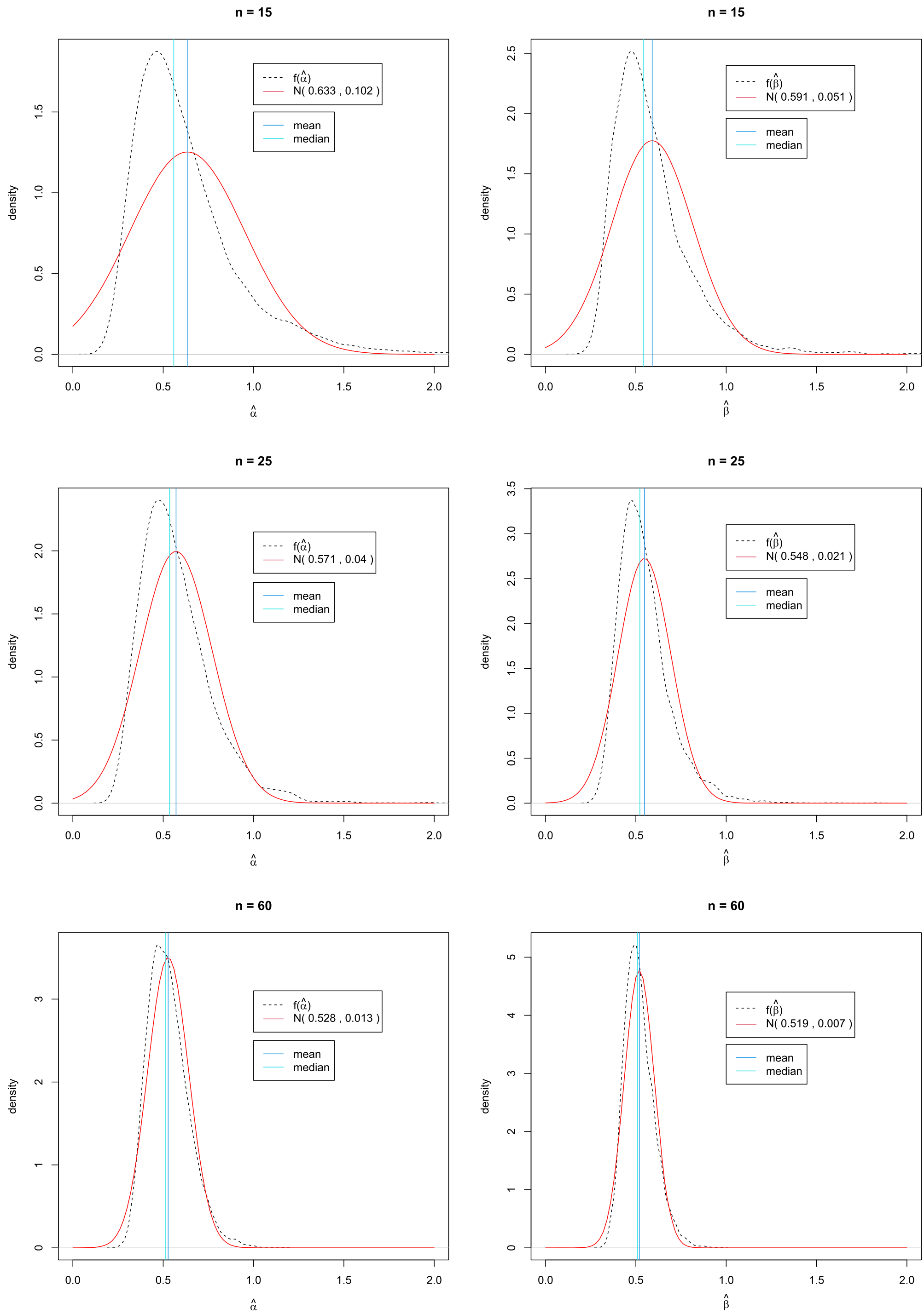

Figure 1. Estimated density of $\hat{\alpha}$ and $\hat{\beta}$ for Kum(0.5, 0.5) and normal density.

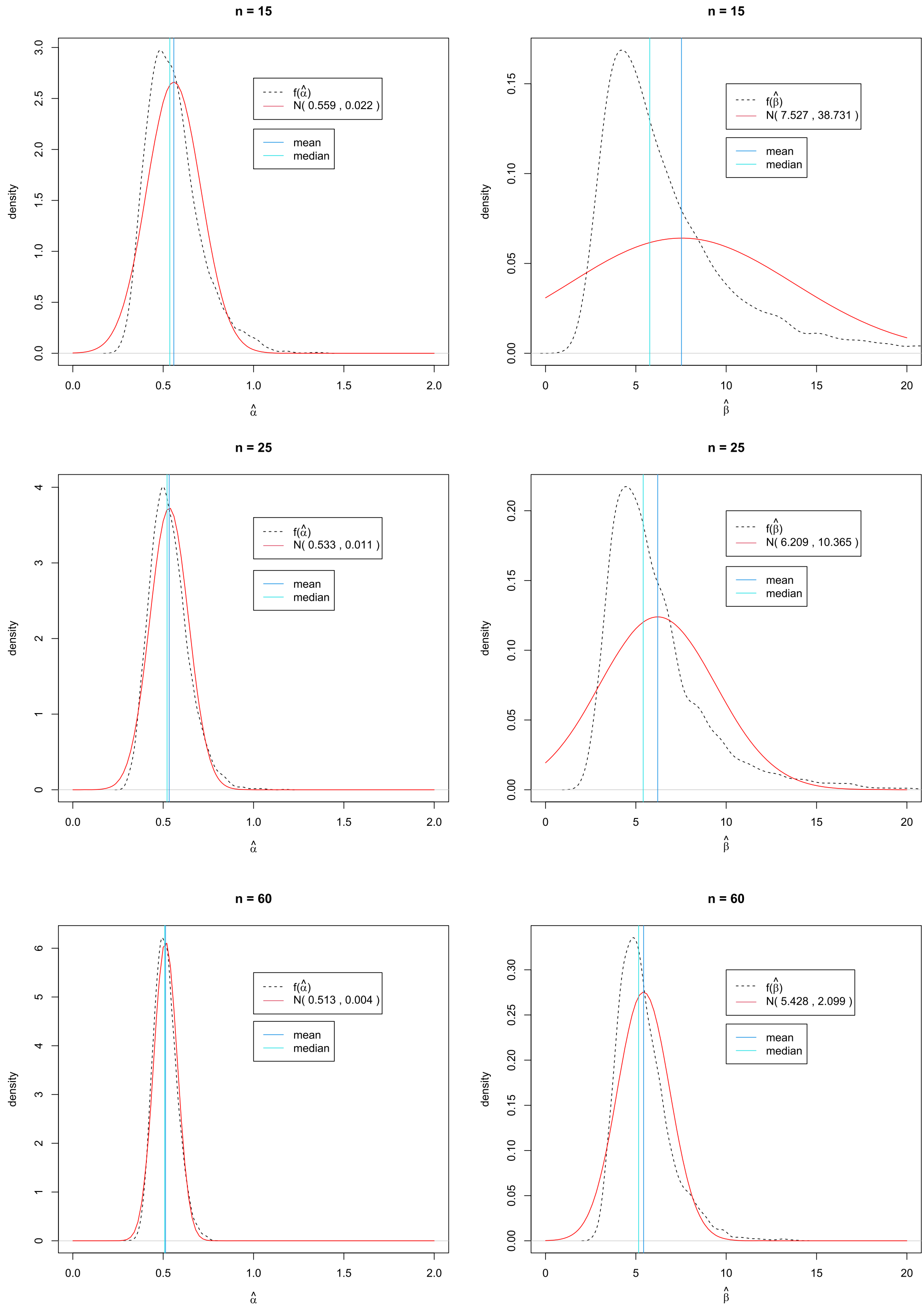

Figure 2. Estimated density of $\hat{\alpha}$ and $\hat{\beta}$ for Kum(0.5, 5) and normal density.

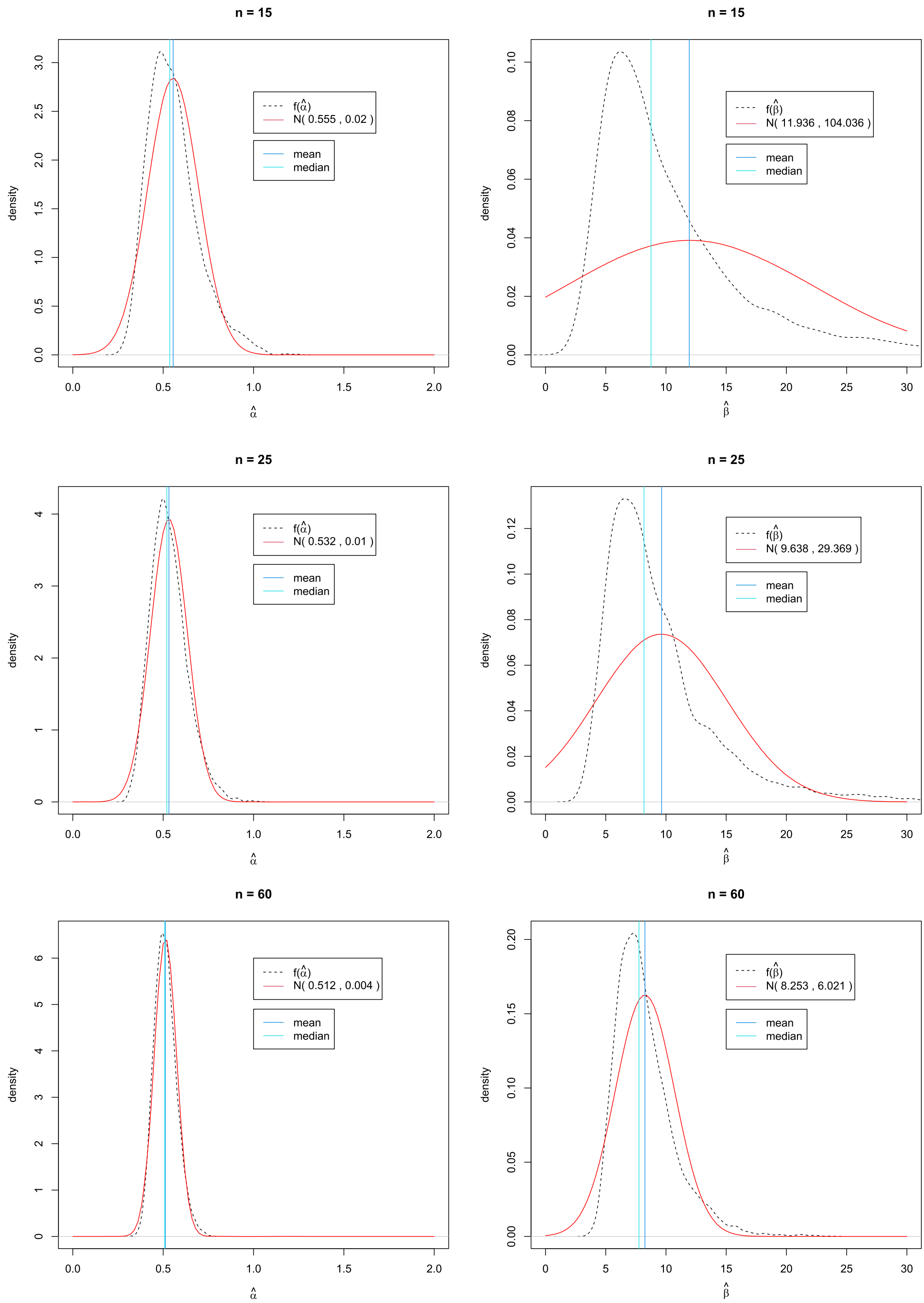

Figure 3. Estimated density of $\hat{\alpha}$ and $\hat{\beta}$ for Kum(0.5, 7.5) and normal density.

Table 2. Estimates of α, β and γ_1

Parameter	Estimate	Standard error	γ_1
α	2.9545	0.3905	−0.3068
β	26.9649	11.7863	1.0404

normal approximation would be less appropriate for the distribution of $\hat{\beta}$ than for $\hat{\alpha}$. This result is consistent with what was observed in the simulation study.

4. Corrected likelihood ratio test

In this section, we obtain the Bartlett correction factor for the LR test of the parameters of the Kumaraswamy distribution, as well as some of its variations presented in the literature. In addition, we include the bootstrap test and the Bartlett bootstrap correction factor for the likelihood ratio statistic, S_{LR}.

Let $\boldsymbol{Y} = (Y_1, \ldots, Y_n)^\top$ denote a random sample of size n from the Kumaraswamy distribution indexed by the parameter $\boldsymbol{\theta} = (\alpha, \beta)^\top$. The associated log-likelihood function is given in (1). Suppose the interest is to test the following hypotheses

$$H_0 : \alpha = \alpha_0, \beta = \beta_0 \quad \text{against} \quad H_1 : \alpha \neq \alpha_0 \text{ or } \beta \neq \beta_0 \tag{6}$$

or

$$H_0 : \alpha = \alpha_0 \quad \text{against} \quad H_1 : \alpha \neq \alpha_0 \tag{7}$$

or

$$H_0 : \beta = \beta_0 \quad \text{against} \quad H_1 : \beta \neq \beta_0. \tag{8}$$

The LR statistic to test the null hypothesis (6), (7), or (8) is given by

$$S_{LR} = 2\left\{\ell(\hat{\boldsymbol{\theta}}; \boldsymbol{y}) - \ell(\tilde{\boldsymbol{\theta}}; \boldsymbol{y})\right\},$$

where $\tilde{\boldsymbol{\theta}} = (\alpha_0, \beta_0)^\top$, for the null hypothesis given in (6), $\tilde{\boldsymbol{\theta}} = (\alpha_0, \tilde{\beta})^\top$ or $\tilde{\boldsymbol{\theta}} = (\tilde{\alpha}, \beta_0)^\top$, with $\tilde{\beta}$ and $\tilde{\alpha}$ denoting the maximum likelihood estimators of β and α, restricted to hypotheses (7) and (8), respectively, and $\hat{\boldsymbol{\theta}} = (\hat{\alpha}, \hat{\beta})^\top$ the unrestricted estimator.

Under each of the null hypotheses $S_{LR} \sim \chi^2_q$, where q is the number of restrictions imposed on the parameters by H_0 (in this case $q = 2$ or 1, depending on the null hypothesis considered).

The LR statistic corrected by the Bartlett correction factor, S^*_{LR} is given by

$$S^*_{LR} = S_{LR} \cdot C_B, \tag{9}$$

where $C_B^{-1} = (1 + d)$, with $d = (\epsilon_2 - \epsilon_{2-q})/q$ being a constant of order n^{-1} and

$$\epsilon_2 = \sum_{\alpha,\beta} \{\ell_{rstu} - \ell_{rstuvw}\}, \tag{10}$$

where $\ell_{rstu} = \kappa^{rs}\kappa^{tu}\left\{\frac{1}{4}\kappa_{rstu} - \kappa^{(u)}_{rst} + \kappa^{(su)}_{rt}\right\}$ and

$$\ell_{rstuvw} = \kappa^{rs}\kappa^{tu}\kappa^{vw} \times \left\{\kappa_{rtv}(\kappa_{suw}/6 - \kappa^{(u)}_{sw}) + \kappa_{rtu}(\kappa_{svw}/4 - \kappa^{(v)}_{sw}) + \kappa^{(v)}_{rt}\kappa^{(u)}_{sw} + \kappa^{(u)}_{rt}\kappa^{(v)}_{sw}\right\}.$$

Note that the indices r, s, t, u, v, and w in (10) vary over all the two parameters of the vector $\boldsymbol{\theta}$. The calculation of ϵ_{2-q} comes from (10), and the sum is calculated over the perturbation parameters (if any). Thus, the modified statistic, S^*_{LR}, has an approximate distribution of χ^2_q, with order approximation error $\mathcal{O}(n^{-2})$.

The modified statistic is usually given by (9). However, there are variations of this test statistic in the literature:

$$S^*_{LR1} = S_{LR} \cdot (1+d)^{-1}, \quad S^*_{LR2} = S_{LR} \cdot \exp(-d), \quad S^*_{LR3} = S_{LR} \cdot (1-d). \tag{11}$$

All statistics in (11) are equivalent to the order $\mathcal{O}(n^{-1})$ (Lemonte, Ferrari, and Cribari-Neto 2010). Note that the S^*_{LR2} statistic has the advantage of never having negative values.

For the simple null hypothesis (6), we have $d = \frac{\epsilon_2}{2}$ with

$$\epsilon_2 = \sum_{\alpha,\beta} \{\ell_{rstu} - \ell_{rstuvw}\}, \tag{12}$$

where $\ell_{rstu} = \kappa^{rs}\kappa^{tu}\left\{\frac{1}{4}\kappa_{rstu} - \kappa^{(u)}_{rst} + \kappa^{(su)}_{rt}\right\}$ and

$$\ell_{rstuvw} = \kappa^{rs}\kappa^{tu}\kappa^{vw} \times \left\{\kappa_{rtv}(\kappa_{suw}/6 - \kappa^{(u)}_{sw}) + \kappa_{rtu}(\kappa_{svw}/4 - \kappa^{(v)}_{sw}) + \kappa^{(v)}_{rt}\kappa^{(u)}_{sw} + \kappa^{(u)}_{rt}\kappa^{(v)}_{sw}\right\}.$$

For the composite null hypotheses (7) and (8), we have that $d = \epsilon_2 - \epsilon_1$, with ϵ_2 given in (12) and $\epsilon_1 = \ell_{\beta\beta\beta\beta} - \ell_{\beta\beta\beta\beta\beta\beta}$ and $\epsilon_1 = \ell_{\alpha\alpha\alpha\alpha} - \ell_{\alpha\alpha\alpha\alpha\alpha\alpha}$, respectively. All fourth-order cumulants of the Kumaraswamy distribution are given in Appendix C. Now we describe the bootstrap test and the Bartlett bootstrap correction factor for the LR statistic.

Under the null hypothesis H_0, we generate *Bbootstrap* samples $\left(\boldsymbol{y}^{*1}, \ldots, \boldsymbol{y}^{*B}\right)$ of size n using the parametric *bootstrap* method described by Tibshirani and Efron (1993). For each *bootstrap* sample, the value of the LR statistic is calculated as S^{*b}_{LR}, $b = 1, \ldots, B$, and for the B values obtained, the quantile of order $(1-\alpha)$, denoted by $\hat{q}_{(1-\alpha)}$, is found, where α is the fixed significance level. From the original sample $\boldsymbol{y} = (y_1, \ldots, y_n)^\top$, we calculate the value of S_{LR}, and we reject the null hypothesis if $S_{LR} > \hat{q}_{(1-\alpha)}$. In the simulation study, we use the notation $S_{LR_{Boot}}$ to represent this test.

An alternative method to calculate the Bartlett correction factor for S_{LR} was introduced by Rocke (1989) and is also based on the parametric *bootstrap* method. For the original sample $\boldsymbol{y}$, we calculate S_{LR}. Then, we generate, under H_0, the *Bbootstrap* samples $\left(\boldsymbol{y}^{*1}, \ldots, \boldsymbol{y}^{*B}\right)$ of size n and we calculate S^{*b}_{LR}, $b = 1, \ldots, B$. Finally, the LR statistic corrected by the Bartlett *bootstrap* method is calculated as:

$$S^*_{LR_{Boot}} = \frac{S_{LR}}{\overline{S^{*b}_{LR}}} q,$$

where q represents the number of constraints under the null hypothesis and $\overline{S^{*b}_{LR}} = \sum_{b=1}^{B} S^{*b}_{LR}/B$. This statistic has an asymptotic distribution χ^2_q. For more details on the Bartlett corrections, including the bootstrap Bartlett adjustment, see Cordeiro and Cribari-Neto (2014).

4.1. Simulation

Here we present a simulation study to compare the finite sample performance of the tests based on S_{LR}, S^*_{LR1}, S^*_{LR2}, S^*_{LR3}, $S_{LR_{Boot}}$, and $S^*_{LR_{Boot}}$, considering the Kumaraswamy distribution with parameters α and β.

The simulation results were obtained from the R software (R Core Team 2020). For these simulations, the MLE values were obtained from the *optim* function, by selecting the quasi-Newton method, BFGS. The values set for the parameters α, β, n and R were the same defined in the previous simulation (see Section 3.1). We still consider $B=500$ bootstrap replications. Tables 3–6 present the rejection rates (in percentage) of the six tests of the simple null hypothesis $H_0 : \alpha = \alpha_0; \beta = \beta_0$, considering the nominal levels 1%, 5%, and 10%.

Table 3. Rejection rates for $n = 15$ and $H_0 : \alpha = \alpha_0, \beta = \beta_0$.

		Nominal level 1%					
α	β	S_{LR}	S^*_{LR1}	S^*_{LR2}	S^*_{LR3}	$S_{LR_{Boot}}$	$S^*_{LR_{Boot}}$
0.5	0.5	1.14	0.88	0.88	0.88	0.96	0.98
	5.0	1.46	1.08	1.08	1.08	1.38	1.16
	7.5	1.32	0.96	0.92	0.92	1.10	0.92
		Nominal level 5%					
α	β	S_{LR}	S^*_{LR1}	S^*_{LR2}	S^*_{LR3}	$S_{LR_{Boot}}$	$S^*_{LR_{Boot}}$
0.5	0.5	6.34	5.26	5.22	5.18	5.58	5.32
	5.0	6.22	5.14	5.12	5.06	5.38	5.16
	7.5	6.02	4.74	4.74	4.72	4.98	4.86
		Nominal level 10%					
α	β	S_{LR}	S^*_{LR1}	S^*_{LR2}	S^*_{LR3}	$S_{LR_{Boot}}$	$S^*_{LR_{Boot}}$
0.5	0.5	12.10	10.52	10.46	10.38	10.62	10.48
	5.0	11.66	10.24	10.20	10.20	10.10	10.22
	7.5	10.66	9.20	9.12	9.10	9.24	9.14

Table 4. Rejection rates for $n = 25$ and $H_0 : \alpha = \alpha_0, \beta = \beta_0$.

		Nominal level 1%					
α	β	S_{LR}	S^*_{LR1}	S^*_{LR2}	S^*_{LR3}	$S_{LR_{Boot}}$	$S^*_{LR_{Boot}}$
0.5	0.5	1.22	0.98	0.98	0.98	1.26	1.12
	5.0	1.22	1.06	1.06	1.06	1.22	1.08
	7.5	1.14	0.92	0.92	0.92	1.08	0.94
		Nominal level 5%					
α	β	S_{LR}	S^*_{LR1}	S^*_{LR2}	S^*_{LR3}	$S_{LR_{Boot}}$	$S^*_{LR_{Boot}}$
0.5	0.5	6.28	5.72	5.72	5.68	5.72	5.58
	5.0	5.08	4.50	4.50	4.50	4.64	4.46
	7.5	5.50	5.00	4.98	4.98	5.20	5.02
		Nominal level 10%					
α	β	S_{LR}	S^*_{LR1}	S^*_{LR2}	S^*_{LR3}	$S_{LR_{Boot}}$	$S^*_{LR_{Boot}}$
0.5	0.5	11.12	10.42	10.38	10.36	10.44	10.36
	5.0	9.64	8.70	8.70	8.70	8.92	8.76
	7.5	10.68	9.76	9.74	9.72	10.06	9.78

Table 5. Rejection rates for $n = 40$ and $H_0 : \alpha = \alpha_0, \beta = \beta_0$.

		Nominal level 1%					
α	β	S_{LR}	S^*_{LR1}	S^*_{LR2}	S^*_{LR3}	$S_{LR_{Boot}}$	$S^*_{LR_{Boot}}$
0.5	0.5	1.36	1.24	1.24	1.24	1.46	1.32
	5.0	0.96	0.78	0.78	0.76	0.98	0.84
	7.5	1.00	0.96	0.96	0.96	1.20	0.86
		Nominal level 5%					
α	β	S_{LR}	S^*_{LR1}	S^*_{LR2}	S^*_{LR3}	$S_{LR_{Boot}}$	$S^*_{LR_{Boot}}$
0.5	0.5	6.24	5.78	5.78	5.78	5.70	5.80
	5.0	5.66	5.06	5.06	5.04	5.46	5.16
	7.5	5.56	5.24	5.24	5.22	5.18	5.24
		Nominal level 10%					
α	β	S_{LR}	S^*_{LR1}	S^*_{LR2}	S^*_{LR3}	$S_{LR_{Boot}}$	$S^*_{LR_{Boot}}$
0.5	0.5	11.18	10.70	10.70	10.70	10.64	10.70
	5.0	10.66	10.22	10.22	10.22	10.36	10.26
	7.5	10.60	10.04	10.02	10.02	10.14	10.10

Table 6. Rejection rates for $n = 60$ and $H_0 : \alpha = \alpha_0, \beta = \beta_0$.

		Nominal level 1%					
α	β	S_{LR}	S^*_{LR1}	S^*_{LR2}	S^*_{LR3}	$S_{LR_{Boot}}$	$S^*_{LR_{Boot}}$
0.5	0.5	1.32	1.28	1.28	1.28	1.52	1.16
	5.0	1.18	1.08	1.08	1.08	1.22	1.10
	7.5	1.16	1.10	1.10	1.10	1.32	1.08
		Nominal level 5%					
α	β	S_{LR}	S^*_{LR1}	S^*_{LR2}	S^*_{LR3}	$S_{LR_{Boot}}$	$S^*_{LR_{Boot}}$
0.5	0.5	5.48	5.16	5.16	5.16	5.34	5.22
	5.0	5.42	5.04	5.04	5.04	5.32	5.16
	7.5	4.98	4.82	4.82	4.80	4.92	4.84
		Nominal level 10%					
α	β	S_{LR}	S^*_{LR1}	S^*_{LR2}	S^*_{LR3}	$S_{LR_{Boot}}$	$S^*_{LR_{Boot}}$
0.5	0.5	10.80	10.40	10.40	10.40	10.70	10.60
	5.0	10.62	10.32	10.32	10.32	10.44	10.32
	7.5	10.08	9.52	9.52	9.50	9.72	9.58

In general, the LR test is somewhat liberal, with rejection rates higher than the considered nominal level. On the other hand, the tests based on the statistics S^*_{LR1}, S^*_{LR2}, S^*_{LR3}, and $S^*_{LR_{Boot}}$, presented rejection rates closer to the fixed nominal levels, even for $n = 15$. This result is due to the fact that the correction factor leads to a reduction in the rejection rate. Although the four corrected tests and the *bootstrap* test are competitive, it is possible to observe, for all the simulated cases, that in general, the analytical correction performed better than the *bootstrap* correction. Furthermore, for the nominal level equal to 1% and $n \geq 25$, the test based on $S_{LR_{Boot}}$ had similar or worse behavior than the test based on S_{LR}.

We also point out that, in general, as the sample size grows, the respective rejection rates move closer to the considered nominal level, as expected. In some cases, especially when the sample size is large, the rejection rate based on S_{LR} is very close to the nominal level, showing that the statistic does not need correction.

5. Concluding remarks

In recent years, the Kumaraswamy distribution has gained considerable attention in the literature. It is as flexible as the beta distribution, but is more computationally attractive because it has a simple and closed expression for the distribution function, which allows generating data more easily. Furthermore, a quantile regression can be constructed based on its quantile function.

In this work, we derived the skewness coefficient of order $\mathcal{O}(n^{-1/2})$ for the MLE distribution of the Kumaraswamy distribution parameters. We obtained a simple and easily implementable analytical expression. We performed a simulation study which showed that for small sample sizes and for values of $\beta > 1$ the distribution of $\hat{\beta}$ is far from the normal distribution. As expected, the approximation to the normal distribution was reached as n increased. On the other hand, the distribution of $\hat{\alpha}$ was close to the normal distribution for any sample size.

We proposed a Bartlett correction factor and a *bootstrap* Bartlett correction factor for the LR test, as well as a *bootstrap* LR test for the parameters of the $Kum(\alpha, \beta)$ distribution. We carried out a Monte Carlo simulation study to evaluate the performance of the tests studied, considering the null hypothesis $H_0 : \alpha = \alpha_0,\ \beta = \beta_0$. The results showed that the corrected tests performed better than the usual LR test and none of the corrections substantially stood out from the others. However, it is possible to note that, in general, the analytic correction performed better than the bootstrap correction.

The normality assumption of the distribution of the maximum likelihood estimator is important for constructing confidence intervals based on the Wald statistic (Wald 1943), commonly used by practitioners due its simplicity. As seen in this work, this assumption is not always guaranteed. In this way, we have solid evidence to recommend the likelihood ratio statistic, more precisely, its corrected version, over the Wald statistic. Consequently, our findings make a relevant contribution to applied and methodological statistical research.

As future work, it is possible to obtain Bartlett-type corrections for score and gradient statistics by calculating others cumulants in addition to those presented in Appendix. Another topic of interest may be the study of the power of the tests in question.

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Appendix

A. Quantities in the Fisher information matrix

The quantities $\overline{A}$ and $\overline{B}$ in (3) are given by:

$$\overline{A} = \overline{A}(\beta) = 1 + \frac{\beta}{\beta - 2}\left\{[\psi(\beta) - \psi(2)]^2 - \left[\psi'(\beta) - \psi'(2)\right]\right\},$$
$$\overline{B} = \overline{B}(\beta) = -\frac{1}{\beta - 1}\left[\psi(\beta + 1) - \psi(2)\right],$$

and where $\psi(\cdot)$ is the digamma function, that is, $\psi(z) = d\log\Gamma(z)/dz$, its derivative is $\psi'(z) = d\psi(z)/dz$, the trigamma function.

The inverse of Fisher's information matrix for $\boldsymbol{\theta}$, Equation (3), is given by:

$$\boldsymbol{K}_{\boldsymbol{\theta}}^{-1} = \begin{bmatrix} k^{\alpha,\alpha} & k^{\alpha,\beta} \\ k^{\beta,\alpha} & k^{\beta,\beta} \end{bmatrix}, \tag{A.1}$$

where $k^{\alpha,\alpha} = \alpha^2/n(\overline{A} - \beta^2\overline{B})$, $k^{\alpha,\beta} = k^{\beta,\alpha} = -\alpha\beta\overline{B}/n(\overline{A} - \beta^2\overline{B})$ and $k^{\beta,\beta} = \beta^2\overline{A}/n(\overline{A} - \beta^2\overline{B})$.

B. Skewness quantities

The quantities a_0 to a_2 and b_0 to b_3, in (5), are given by:

$$a_0 = a_0(\beta) = 4 + 6(\overline{A} - 1) - 2\overline{B}\beta^2\left(3 + 2\overline{B}\beta\right),$$
$$a_1 = a_1(\beta) = 3\overline{B}\beta^2[2\overline{BC} - (\overline{A} - 1)],$$
$$a_2 = a_2(\beta) = -6\beta\overline{B}(\overline{A} - 1) + 3\overline{D},$$
$$a_3 = a_3(\beta) = 2(\overline{G} - \overline{E} - \overline{F}),$$
$$b_0 = b_0(\beta) = 4\overline{A}^3 + 2\beta^3\overline{B}^3,$$
$$b_1 = b_1(\beta) = 3\beta\overline{AB}\left[\beta^2\overline{B}(\overline{A} - 1) - 2\overline{AC}\right],$$
$$b_2 = b_2(\beta) = 3\beta\overline{AB}[2\beta(\overline{A} - 1)\overline{B} - \overline{D}],$$
$$b_3 = b_3(\beta) = 2\beta^3\overline{B}^3(\overline{E} - \overline{G} + \overline{F}).$$

where

$$\overline{C} = \overline{C}(\beta) = \beta^2\left[\psi'(\beta + 1) + \overline{B}\right],$$
$$\overline{D} = \overline{D}(\beta) = \beta^3\overline{B}\left\{2\psi'(\beta)[\psi(\beta) - \psi(2)] - \psi''(\beta)\right\},$$
$$\overline{E} = \overline{E}(\beta) = \frac{\beta(\beta + 1)}{2(\beta - 2)} \times\left\{[\psi(\beta) - \psi(1)]^3 + \left[\psi''(\beta) - \psi''(1)\right] - 3\left[\psi'(\beta) - \psi'(1)\right][\psi(\beta) - \psi(1)]\right\},$$
$$\overline{F} = \overline{F}(\beta) = \frac{6\beta}{\beta - 2}[\psi(\beta) - \psi(1)],$$
$$\overline{G} = \overline{G}(\beta) = \frac{3\beta(\beta + 3)}{2(\beta - 2)}\left\{[\psi(\beta) - \psi(1)]^2 - \left[\psi'(\beta) - \psi'(1)\right]\right\}.$$

C. Test correction

Cumulants up to the fourth order for the Kumaraswamy distribution are:

$$
\begin{aligned}
\kappa_{\alpha\alpha} &= -\frac{n}{\alpha^2}\overline{A},\ \kappa_{\alpha\beta} = -\frac{n}{\alpha}\overline{B},\ \kappa_{\beta\beta} = -\frac{n}{\beta^2},\ \kappa_{\beta\beta\beta} = \frac{2n}{\beta^3},\ \kappa_{\alpha\beta\beta} = 0,\\
\kappa_{\alpha\alpha\beta} &= -\frac{n}{\alpha^2(\beta-1)}(\overline{A}-1), \kappa_{\alpha\alpha\alpha} = \frac{2n}{\alpha^3}\left[1+\frac{1}{\beta-3}(\overline{E}-\overline{G}+\overline{F})\right], \kappa_{\beta\beta}^{(\alpha)} = 0,\\
\kappa_{\beta\beta}^{(\beta)} &= \frac{2n}{\beta^3}, \kappa_{\alpha\alpha}^{(\alpha)} = \frac{2n}{\alpha^3}\overline{A},\ \kappa_{\alpha\alpha}^{(\beta)} = \frac{n}{\alpha^2\beta(\beta-2)}\left[2(\overline{A}-1)-\frac{\overline{D}}{\beta\overline{B}}\right], \kappa_{\alpha\beta}^{(\alpha)} = \frac{n}{\alpha^2}\overline{B},\\
\kappa_{\alpha\beta}^{(\beta)} &= \frac{n\overline{C}}{\alpha\beta^2(\beta-1)},\ \kappa_{\alpha\alpha\alpha}^{(\alpha)} = \frac{-6n}{\alpha^4}\left[1+\frac{1}{\beta-3}(\overline{E}-\overline{G}+\overline{F})\right],\\
\kappa_{\alpha\alpha\alpha}^{(\beta)} &= -\frac{2n(\beta^2-6)(\overline{E}-\overline{G}+\overline{F})}{\alpha^3\beta(\beta-2)(\beta-3)^2} + \frac{2n\left(\overline{E}+\overline{I}+\overline{J}+\overline{L}\right)}{\alpha^3(\beta-3)(\beta+1)}\ \kappa_{\alpha\alpha\beta}^{(\alpha)} = \frac{2n}{\alpha^3(\beta-1)}\left(\overline{A}-1\right),\\
\kappa_{\alpha\alpha\beta}^{(\beta)} &= \frac{n}{\alpha^2(\beta-1)^2(\beta-2)}\left[\frac{(\beta^2-2)(\overline{A}-1)}{\beta} - \frac{(\beta-1)\overline{D}}{\beta^2\overline{B}}\right],\ \kappa_{\beta\beta\alpha}^{(\alpha)} = 0,\ \kappa_{\beta\beta\beta}^{(\alpha)} = 0,\\
\kappa_{\beta\beta\alpha}^{(\beta)} &= 0,\ \kappa_{\beta\beta\beta}^{(\beta)} = -\frac{6n}{\beta^4},\ \kappa_{\alpha\alpha}^{(\alpha\alpha)} = -\frac{6n}{\alpha^4}\overline{A},\ \kappa_{\beta\alpha}^{(\alpha\alpha)} = -\frac{2n}{\alpha^3}\overline{B}\ \kappa_{\beta\beta}^{(\alpha\alpha)} = 0,\\
\kappa_{\alpha\alpha}^{(\alpha\beta)} &= -\frac{2n}{\beta(\beta-2)\alpha^3}\left[2(\bar{A}-1)-\frac{\bar{D}}{\beta\bar{B}}\right],\ \kappa_{\beta\alpha}^{(\beta\alpha)} = -\frac{n\overline{C}}{\alpha^2\beta^2(\beta-1)},\\
\kappa_{\alpha\alpha}^{(\beta\beta)} &= -\frac{4n\left[(\bar{A}-1)\beta^2\bar{B}-\bar{D}\right]}{\beta^3(\beta-2)^2\alpha^2\bar{B}} - \frac{n\beta\bar{H}}{(\beta-2)\alpha^2},\ \kappa_{\beta\beta}^{(\beta\alpha)} = 0,\ \kappa_{\beta\beta}^{(\beta\beta)} = -\frac{6n}{\beta^4},\\
\kappa_{\beta\alpha}^{(\beta\beta)} &= \frac{-2n\bar{C}}{\alpha\beta^2(\beta-1)^2} + \frac{n\psi''(\beta+1)}{\alpha(\beta-1)},\\
\kappa_{\alpha\alpha\alpha\alpha} &= -\frac{6n}{\alpha^4} - \frac{n\beta(\beta-1)}{\alpha^4}\\
&\quad\times\left[\frac{\bar{M}_1}{(\beta-3)(\beta-4)} + \frac{8\bar{M}_2}{(\beta-2)(\beta-3)(\beta-4)} + \frac{6\bar{M}_3}{(\beta-1)(\beta-2)(\beta-3)(\beta-4)}\right],\\
\kappa_{\beta\alpha\alpha\alpha} &= \frac{n\beta}{\alpha^3}\left[\frac{\bar{N}_1}{(\beta-2)(\beta-3)} + \frac{2\bar{N}_2}{(\beta-1)(\beta-2)(\beta-3)}\right],\ \kappa_{\beta\beta\alpha\alpha} = 0,\ \kappa_{\beta\beta\beta\alpha} = 0,\\
\kappa_{\beta\beta\beta\beta} &= -\frac{6n}{\beta^4},\ \kappa_{\alpha,\alpha,\alpha} = \frac{4n}{\alpha^3}\left(1+\frac{\bar{E}-\bar{G}+\bar{F}}{\beta-3}\right) - \frac{6n\bar{A}}{\alpha^3},\\
\kappa_{\beta,\alpha,\alpha} &= -\frac{2n(A-1)}{\alpha^2(\beta-1)} - \frac{n}{\alpha^2(\beta-2)}\left[\frac{2(\bar{A}-1)}{\beta} - \frac{\bar{D}}{\beta^2\bar{B}}\right] - \frac{2n\bar{B}}{\alpha^2},\\
\kappa_{\beta,\beta,\alpha} &= -\frac{2n\bar{C}}{\alpha\beta^2(\beta-1)},\ \kappa_{\beta,\beta,\beta} = -\frac{2n}{\beta^3},\ \kappa_{\alpha\alpha,\alpha} = -\frac{2n}{\alpha^3}\left(1+\frac{\bar{E}-\bar{G}+\bar{F}}{\beta-3}\right) + \frac{2n\bar{A}}{\alpha^3},\\
\kappa_{\alpha\alpha,\beta} &= \frac{n(\bar{A}-1)}{\alpha^2(\beta-1)} + \frac{n}{\alpha^2(\beta-2)}\left[\frac{2(\bar{A}-1)}{\beta} - \frac{\bar{D}}{\beta^2\bar{B}}\right],\ \kappa_{\alpha\beta,\alpha} = \frac{n(\bar{A}-1)}{\alpha^2(\beta-1)} + \frac{n\bar{B}}{\alpha^2},\\
\kappa_{\alpha\beta,\beta} &= \frac{n\bar{C}}{\alpha\beta^2(\beta-1)},\ \kappa_{\beta\beta,\alpha} = 0,\ \kappa_{\beta\beta,\beta} = 0,
\end{aligned}
$$

with

$$\overline{H} = \overline{H}(\beta) = 2\left[(\psi'(\beta))^2 + (\psi(\beta)-\psi(2))\psi''(\beta)\right] - \psi'''(\beta),$$

$$\bar{I} = \bar{I}(\beta) = \frac{\beta(\beta+1)^2}{2(\beta-2)} \times \left\{\psi'''(\beta) + 3\psi'(\beta)\left[(\psi(\beta)-\psi(1))^2 - (\psi'(\beta)-\psi'(1))\right] - 3\psi''(\beta)(\psi(\beta)-\psi(1))\right\},$$

$$\bar{J} = \bar{J}(\beta) = \frac{\beta(\beta+1)}{2(\beta-2)}\left\{3(\beta+3)\left[\psi''(\beta) - 2\psi'(\beta)(\psi(\beta)-\psi(1))\right]\right\},$$

$$\bar{L} = \bar{L}(\beta) = \frac{\beta(\beta+1)}{2(\beta-2)}\left\{-3(\psi(\beta)-\psi(1))^2 + 15\psi'(\beta) - 3\psi'(1)\right\},$$

$$\begin{aligned}\overline{M}_1 = \overline{M}_1(\beta) &= [\psi(\beta-2)-\psi(2)]^4 + 4[\psi(\beta-2)-\psi(2)]\left[\psi''(\beta-2)-\psi''(2)\right]\\ &\quad - 3\left[\psi'(\beta-2)-\psi'(2)\right]\left\{-\left[\psi'(\beta-2)-\psi'(2)\right] + 2[\psi(\beta-2)-\psi(2)]^2\right\}\\ &\quad - \left[\psi'''(\beta-2)-\psi'''(2)\right],\end{aligned}$$

$$\begin{aligned}\overline{M}_2 = \overline{M}_2(\beta) &= [\psi(\beta-1)-\psi(3)]^4 + 4[\psi(\beta-1)-\psi(3)]\left[\psi''(\beta-1)-\psi''(3)\right]\\ &\quad - 3\left[\psi'(\beta-1)-\psi'(3)\right]\left\{-\left[\psi'(\beta-1)-\psi'(3)\right] + 2[\psi(\beta-1)-\psi(3)]^2\right\}\\ &\quad - \left[\psi'''(\beta-1)-\psi'''(3)\right],\end{aligned}$$

$$\begin{aligned}\overline{M}_3 = \overline{M}_3(\beta) &= [\psi(\beta)-\psi(4)]^4 + 4[\psi(\beta)-\psi(4)]\left[\psi''(\beta)-\psi''(4)\right]\\ &\quad - 3\left[\psi'(\beta)-\psi'(4)\right]\left\{-\left[\psi'(\beta)-\psi'(4)\right] + 2[\psi(\beta)-\psi(4)]^2\right\}\\ &\quad - \left[\psi'''(\beta)-\psi'''(4)\right],\end{aligned}$$

$$\begin{aligned}\overline{N}_1 = \overline{N}_1(\beta) &= [\psi(\beta-1)-\psi(2)]^3 - 3[\psi(\beta-1)-\psi(2)]\left[\psi'(\beta-1)-\psi'(2)\right]\\ &\quad + \left[\psi''(\beta-1)-\psi''(2)\right],\end{aligned}$$

$$\overline{N}_2 = \overline{N}_2(\beta) = [\psi(\beta)-\psi(3)]^3 - 3[\psi(\beta)-\psi(3)]\left[\psi'(\beta)-\psi'(3)\right] + \left[\psi''(\beta)-\psi''(3)\right],$$

where $\psi(z) = \frac{d\log(\Gamma(z))}{dz}$, where $\Gamma(z)$ is the Gamma Function and $\psi^{'}(z) = \frac{d\psi(z)}{dz}$, $\psi^{''}(z) = \frac{d\psi^{'}(z)}{dz}$ and so on.