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FINITE CONJUGACY AND NILPOTENCY IN LOOPS OF UNITS

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ABSTRACT. Let $\mathcal{U}(RL)$ denote the Moufang loop of units in an alternative loop ring RL. In this paper, we give necessary and sufficient conditions for $\mathcal{U}(RL)$ to be nilpotent or to have the finite conjugacy property when R is the ring of rational integers or a field.

1. **Introduction.** An *alternative* ring is a ring which satisfies the left and right alternative laws, $x(xy) = x^2y$ and $(yx)x = yx^2$. Any associative ring is alternative, but in this paper we are concerned primarily with alternative rings which are not associative. The Cayley numbers is undoubtedly the best known example of such a ring.

A *Moufang* loop is a loop in which $x(y \cdot xz) = (xy \cdot x)z$ is an identity. Any loop of units (invertible elements) contained in an alternative ring is a Moufang loop. For example, the standard basis elements of the Cayley numbers, together with their negatives, form a Moufang loop of order 16, and one which is *Hamiltonian* (all its subloops are normal). We refer the reader to [17] and [13] for information about alternative rings and Moufang loops, respectively.

Generalizing the terminology of group theory, we say that a Moufang loop L is FC, or has the *finite conjugacy property*, if, for all $\ell \in L$, the set $\{x^{-1}\ell x \mid x \in L\}$ is finite. The concept of nilpotency in loop theory, like that for groups, is a measure of the deviation of a loop from an abelian group, so it involves associators as well as commutators. If a, b and c are elements of a loop L, the *commutator* of a and b and the *associator* of a, b and c are the elements (a,b) and (a,b,c) of L, respectively, defined by

$$ab = ba(a,b) \quad \text{and} \quad (ab)c = (a \cdot bc)(a,b,c).$$

If X, Y, Z are subsets of L, we write (X,Y) for the set of all commutators (x,y), $x \in X$, $y \in Y$, (X,Y,Z) for the set of all associators (x,y,z), $x \in X$, $y \in Y$, $z \in Z$, and $\langle X \rangle$

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for the subloop of L generated by X. Let $\gamma_0(L) = L$, $\gamma_1(L) = \langle (L,L),(L,L,L)\rangle$ and, for $i \geq 1$,

$$\gamma_{i+1}(L) = \langle (L,\gamma_i(L)),(\gamma_i(L),L),(L,L,\gamma_i(L)),(L,\gamma_i(L),L),(\gamma_i(L),L,L)\rangle.$$

The subloop $\gamma_1(L)$ is also denoted L' and called the *commutator/associator subloop* of L. The loop L is *nilpotent* (Bruck uses the term "centrally nilpotent" in Chapter VI of his well-known treatise [1]) if $\gamma_n(L) = \{1\}$ for some positive integer n, which is then called the *nilpotency class* of L.

Let L be a loop and suppose that the loop ring RL is alternative, but not associative, for any commutative and associative ring R with unity. Then the loop L (which, as we have observed, is necessarily Moufang) has many special properties, including nilpotence and finite conjugacy [2]. In fact, L is nilpotent of class 2 and, for any $\ell \in L$, the set $\{x^{-1}\ell x \mid x \in L\}$ has cardinality at most 2.

The complete set $\mathcal{U}(RL)$ of units in RL is a Moufang loop containing L and it is natural to wonder if $\mathcal{U}(RL)$ inherits any of the properties of L. In this connection, and for various rings R, we have recently explored the possibility that $\mathcal{U}(RL)$ is nilpotent or has the finite conjugacy property and it is our purpose here to report our findings.

2. Integral Alternative Loop Rings.

Over the ring Z of rational integers, nilpotency and finite conjugacy in $\mathcal{U}(ZL)$ are equivalent. In fact, we have established the following theorem [9].

Theorem 2.1. *Suppose ZL is an alternative, but not associative, ring. Then the following are equivalent:*

1. $\mathcal{U}(ZL)$ *is FC;*
2. $\mathcal{U}(ZL)$ *is nilpotent;*
3. $\mathcal{U}(ZL)$ *is nilpotent of class* 2;
4. *The set T of torsion elements of L form an abelian group or a Moufang Hamiltonian 2-loop such that for any $t \in T$ and any $x \in L$, we have $x^{-1}tx = t^{\pm 1}$. Moreover, if T is an abelian group and $x \in L$ is any element which does not centralize T, then $x^{-1}tx = t^{-1}$ for all $t \in T$.*

A *torsion* element in a loop is an element of finite order. As a consequence of Theorem 2.1, if $\mathcal{U}(\mathbb{Z}L)$ is nilpotent or FC, then the only torsion elements of $\mathbb{Z}L$ are *trivial*; that is, of the form $\pm\ell$, $\ell \in L$ [10]. In particular, the torsion units of $\mathbb{Z}L$ form a subloop.

3. Alternative Loop Algebras over Fields. More recently, we have examined nilpotency and finite conjugacy in alternative loop algebras over fields and found the situation to be quite different from the case of loop rings over $\mathbb{Z}$. It is interesting to contrast our results for the cases that L is or is not a torsion loop.

Theorem 3.1. *Let L be a torsion loop and F a field such that FL is alternative. Then*

1. *$\mathcal{U}(FL)$ is an FC loop if and only if both F and L are finite* [6].
2. *$\mathcal{U}(FL)$ is nilpotent if and only if F has characteristic* 2 [7].

Thus we see, for example, that if L is a finite loop, it is the field which alone determines whether or not $\mathcal{U}(FL)$ is nilpotent or FC. We do not know if nilpotency or finite conjugacy of a Moufang loop implies that the torsion units form a subloop, but, as with loop rings over the integers, such is the case for unit loops in the alternative loop algebras of torsion loops.

Theorem 3.2. [8] *Let L be a torsion loop and F a field such that FL is alternative. Then the torsion units of FL form a subloop if and only if F has positive characteristic p and either $p = 2$ or F is algebraic over its prime field.*

Turning to the case that L is not a torsion loop, we use T to denote the set of torsion units in L and note that, for any loop considered in this paper, T is always a subloop [9, Lemma 2.1]. We consider finite conjugacy and nilpotency of the unit loop $\mathcal{U}(FL)$ separately.

Theorem 3.3. [6] *Let L be a loop with torsion subloop $T \neq L$. Let F be a field such that FL is an alternative algebra.*

1. *If the characteristic of F is 0, $\mathcal{U}(FL)$ is FC if and only if T is central in L and, if it is also infinite, then $T = \mathbb{Z}(2^\infty) \times B$ where B is a finite group, and there exists an integer k such that F does not contain roots of unity of order 2^k.*

2. *If the characteristic of F is $p > 0$ and L contains an element of order p, then $\mathcal{U}(FL)$ is FC if and only if $p = 2$ and $T = L' \times A$, where A is a finite abelian group of odd order.*
3. *If the characteristic of F is $p > 0$ and L does not contain an element of order p, then $\mathcal{U}(FL)$ is FC if and only if T is an abelian group and one of the following occurs:*
   (i) *FT is finite and, for all $t \in T$ and all $x \in L$, we have $xtx^{-1} = t^{p^r}$ for some integer $r \geq 0$, a multiple of $[F\colon \mathcal{P}]$, where $\mathcal{P}$ denotes the prime field of F.*
   (ii) *T is finite and central.*
   (iii) *T is central and of the form $\mathbb{Z}(2^\infty) \times B$ with B finite, and there exists an integer k such that F does not contain roots of unity of order 2^k.*

Theorem 3.4. [7] *Let L be a loop with torsion subloop $T \neq L$. Let F be a field such that FL is an alternative algebra.*

1. *If the characteristic of F is 0, or if* char $F = p > 0$ *and L contains no element of order p, then $\mathcal{U}(FL)$ is nilpotent if and only if either T is central or $|F| = p = 2^\beta - 1$ for some positive integer β, T is an abelian group of exponent $2(p-1)$ and, for all $x \in L$ and all $t \in T$, we have $x^{-1}tx = t$ or t^p.*
2. *If the characteristic of F is $p > 0$ and L contains an element of order p, then $\mathcal{U}(FL)$ is nilpotent if and only if $p = 2$.*

Once again, nilpotency or finite conjugacy of $\mathcal{U}(FL)$ implies that the torsion units of $\mathcal{U}(FL)$ form a subloop, for we have

Theorem 3.5. [8] *Let L be a loop with torsion subloop $T \neq L$ and F a field such that FL is alternative. Then*

1. *If the characteristic of F is 0, then the product of torsion units in FL is a torsion unit if and only if T is an abelian group, for each $t \in T$ and $x \in L$, we have $xtx^{-1} = t^i$ for some i and, for each noncentral element $t \in T$, F contains no root of unity whose order is the order of t.*

2. *If the characteristic of F is $p > 0$, then the product of torsion units in FL is a torsion unit if and only if $p = 2$ or T is an abelian group and, if it is not central, then $\overline{\mathcal{P}}$, the algebraic closure in F of the prime field of F, is finite and, for all $x \in L$ and all $t \in T$ of order relatively prime to p, we have $xtx^{-1} = t^{p^r}$ for some positive integer r, a multiple of $[\overline{\mathcal{P}} : \mathcal{P}]$.*

4. **Conclusion.** It is appropriate to observe that the questions we have considered in this paper have all previously been settled in the case of group rings. In fact, the literature is rather extensive. For finite conjugacy of unit groups over fields or the integers, we refer the reader to [4] and [15] respectively. Nilpotence in group rings is the subject of [5] and [16]. The interested reader should also consult [14, Chapter VI]. Both finite conjugacy and nilpotency of the unit group are related to the property that the torsion units in a group ring form a subgroup; see [12], [11] and [3].

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