

The Boolean and Profinite Hulls of Reduced Special Groups

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October 2011

Abstract

This work is a development of some logical-categorical aspects of the theory of Special Groups – a first order axiomatization of the algebraic theory of quadratic forms. We discuss the interaction between two functors defined on reduced special groups (**RSG**): the Boolean Hull Functor ([5], [6], [7], [8], [14]) and the Profinite Hull Functor ([14], [16], [17], [18]). Both functors are left adjoints to certain inclusion functors and encode local-global principles related to classes of positive-existential formulas. We describe the Boolean hull of a profinite RSG as a certain Boolean subalgebra of a profinite Boolean algebra (BA); prove that the Profinite hull functor preserves stability index; identify the restriction of the Profinite hull functor to the subcategory $\mathbf{BA} \leftrightarrow \mathbf{RSG}$; study the interaction between these hulls (in particular, when they “commute”) and apply it to prove that the Profinite hull functor preserves and reflects complete embeddings.

Key words: *Algebraic Theory of Quadratic Forms, Special Groups, Profinite Groups, Boolean Hull, Profinite Hull, Subform Reflection Property, Pure Embeddings, Local-Global Principle*

The following is a summary of the results presented at the Séminaire de Structures Algébriques Ordonnés of the Équipe de Logique of the University of Paris VII, in February, 2012. Proofs will appear elsewhere (the Logic Journal of the IGPL).

1 Preliminaries

1.1 Special Groups. We recall the definitions and results on special groups needed in the sequel, all set down and proven in [5], [12] or [14].

Special Groups (SGs) are a first-order axiomatization of the Algebraic Theory of Quadratic Forms. The suitable first-order language, L_{SG} , contains two symbols for constants (1 and -1), one symbol for binary operation (multiplication) and one symbol for quaternary relation (\equiv , the isometry between quadratic forms of dimension 2); the axioms for special groups are sentences of the form $\forall \vec{x}(\psi_0(\vec{x}) \rightarrow \psi_1(\vec{x}))$, where $\psi_0(\vec{x}), \psi_1(\vec{x})$ are positive-existential L_{SG} -formulas (see Definition 1.2 in [5]).

Many concepts in the theory of SGs can be described by positive-existential L_{SG} -formulas. Examples include the isometry between n -forms, the notion of isotropic form and the relation of a form being a subform of another.

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The authors would like to thank the referees for their valuable suggestions and references.

The SG-morphisms are the L_{SG} -morphisms of their underlying L_{SG} -structures. If $f : G \rightarrow H$ is a SG-morphism and $\varphi = \langle a_1, \dots, a_n \rangle$ is a n -form over G , write $f \star \varphi = \langle f(a_1), \dots, f(a_n) \rangle$ for the f -image form over H . Some special kinds of L_{SG} -monomorphisms appear naturally:

- (i) complete embeddings, i.e., the SG-morphisms that preserve and reflect isometry of n -forms, $n \geq 1$;
- (ii) SG-morphisms that preserve and reflect isotropy; (ii)' SG-morphisms that preserve Witt index;
- (ii)'' SG-morphisms that preserve and reflect subforms;
- (iii) Elementary (resp., pure) SG-morphisms, i.e., those L_{SG} -morphisms that preserve and reflect arbitrary (resp., positive-existential) L_{SG} -formulas;
- (iv) L_{SG} -sections, i.e., the SG-morphisms, $G \xrightarrow{s} H$, such that there is a SG-morphism, $H \xrightarrow{r} G$, satisfying $r \circ s = Id_G$.

The notions of monomorphisms (i), (ii), (iii), (iv) are pairwise different. By Proposition 3 in [17] and the Witt cancellation law, the notions (ii), (ii)', (ii)'' coincide.

The most useful notion of SG-epimorphism is that of the projection of a reduced special group (RSG) on a quotient by a *saturated* subgroup; the (proper) saturated subgroups classify the congruences on a RSG whose associated quotient is a RSG. If G is a RSG and Σ is a saturated subgroup of G , write $p_\Sigma : G \rightarrow G/\Sigma$ for the canonical SG-quotient morphism. If $\varphi = \langle a_1, \dots, a_n \rangle$ is a n -form over G , whenever convenient, we write $\varphi/\Sigma = \langle a_1/\Sigma, \dots, a_n/\Sigma \rangle = p_\Sigma \star \varphi$ for the image form over G/Σ . \square

From Corollary 2.29 and Theorem 2.11 in [5], we obtain:

Proposition 1.2 *Let G be a special group and let Δ be a subgroup of G .*

- a) *The proper saturated subgroups of G are, precisely, the kernels of SG-morphisms from G into some reduced special group.*
- b) *The proper maximal saturated subgroups of G are precisely the kernels of the SG-characters of G , i.e. the kernels of the elements of*

$$X_G = \text{Hom}_{SG}(G, \mathbb{Z}_2),$$

the space of orderings of G . Moreover, this association is bijective.

- c) (Separation Theorem) *Let Δ^{sat} be the least saturated subgroup of G containing Δ , then $\Delta^{\text{sat}} = \bigcap \{ \ker \sigma : \sigma \in X_G \text{ and } \Delta \subseteq \ker \sigma \}$.* \square

1.3 A (non first-order) abstract presentation of the algebraic theory of quadratic forms is given by Murray Marshall's notion of *space of orderings*. The central reference on this subject is [20]. In Chapter 3 of [5] it is shown that there is a *duality* between the category of reduced special groups and the category of spaces of orderings. \square

The following technical result is known by the quadratic form community for groups of exponent two (cf. Lemma 3.17, p. 55, in [5]); a proof in the context of special groups is given in Lemma 6.7 in [14].

Proposition 1.4 *Let G be a RSG and let Γ be a proper saturated subgroup of G . If Γ has codimension m in G – i.e., $\dim_{\mathbb{F}_2}(G/\Gamma) = m$ –, then there is a linearly independent subset $\{\tau_1, \dots, \tau_m\} \subseteq X_G = \text{Hom}_{SG}(G, \mathbb{Z}_2)$, such that $\Gamma = \bigcap_{k=1}^m \ker \tau_k$.* \square

In all that follows, write

$$\begin{cases} \mathbb{P}(U) & := \{V : V \subseteq U\} \\ \mathbb{P}_{fin}(U) & := \{V \in \mathbb{P}(U) : V \text{ is finite}\}. \end{cases}$$

1.5 If G is a SG, $Ssat(G)$ is the set of saturated subgroups of G , $X(G)$ is the set of *proper maximal* saturated subgroups of G and $\mathcal{F}(G)$ is the set of saturated subgroups of G of *finite index* in G . \square

Proposition 1.6 (Proposition 4.15 in [14]) *Let G be a RSG. Consider the mappings:*

$$(i) \quad \mathbb{P}(X(G)) \longrightarrow Ssat(G) \quad \text{given by} \quad S \mapsto \bigcap S;$$

$$(ii) \quad Ssat(G) \longrightarrow \mathbb{P}(X(G)) \quad \text{given by} \quad \Delta \mapsto X_\Delta(G) := \{\Sigma \in X(G) : \Delta \subseteq \Sigma\}.$$

a) *The functions above are decreasing and define a contravariant Galois connection, i.e., for each $\Delta \in Ssat(G)$ and $S \in \mathbb{P}(X(G))$ we have*

$$\Delta \subseteq \bigcap S \quad \Leftrightarrow \quad S \subseteq X_\Delta(G).$$

In particular, $S \subseteq X_{\bigcap S}(G)$.

b) *For each $\Delta \in Ssat(G)$, $\Delta = \bigcap X_\Delta(G)$.*

c) *$\mathcal{F}(G)$ is a filter in $Ssat(G)$, being the closure under finite intersections of $X(G)$ in $Ssat(G)$. In particular, $\mathcal{F}(G)$ is the filter generated by $X(G)$ in $Ssat(G)$ and $\langle \mathcal{F}(G), \subseteq \rangle$ is a downward directed poset.*

d) *The contravariant Galois connection in (a), $\langle \mathbb{P}(X(G)), \subseteq \rangle \rightleftarrows \langle Ssat(G), \subseteq \rangle$ restricts to a Galois connection: $\langle \mathbb{P}_{fin}(X(G)), \subseteq \rangle \rightleftarrows \langle \mathcal{F}(G), \subseteq \rangle$. \square*

Definition 1.7 (Definition 4.1 in [14]) **(The induced adjunction)** *Let $f : G \longrightarrow G'$ be a morphism in the category **RSG**. We define*

$$f_* : Ssat(G) \longrightarrow Ssat(G') \quad \text{and} \quad f^* : Ssat(G') \longrightarrow Ssat(G),$$

given, for $\Delta \in Ssat(G)$ and $\Delta' \in Ssat(G')$, by

$$\begin{cases} f_*(\Delta) & = (f[\Delta])^{sat}; \\ f^*(\Delta') & = f^{-1}[\Delta']. \end{cases}$$

f_ is the (saturated) direct image of f and f^* the (saturated) inverse image of f . \square*

Proposition 1.8 (Proposition 4.2 in [14]) *Let $G \xrightarrow{f} G' \xrightarrow{g} G''$ **RSG**-morphisms.*

a) *f_* and f^* are increasing functions, constituting an adjoint pair, i.e., for each $\Delta \in Ssat(G)$ and $\Delta' \in Ssat(G')$,*

$$f_*(\Delta) \subseteq \Delta' \quad \Leftrightarrow \quad \Delta \subseteq f^*(\Delta').$$

In particular, $\Delta \subseteq f^(f_*(\Delta))$ and $f_*(f^*(\Delta')) \subseteq \Delta'$.*

b) (1) $(g \circ f)_* = g_* \circ f_*$; (2) $(g \circ f)^* = f^* \circ g^*$;

$$(3) (Id_G)^* = Id_{Ssat(G)} = (Id_G)_*.$$

c) *The following conditions are equivalent:*

(1) *f is a complete embedding;*

(2) *For any $\Delta \in Ssat(G)$, $\Delta = f^*(f_*(\Delta))$.*

d) *If f is surjective and $\Delta' \in Ssat(G')$, then $f_*(f^*(\Delta')) = \Delta'$.*

e) If f is regular surjective¹, then for all $\Delta \in \text{Ssat}(G)$

$$\ker f \subseteq \Delta \Rightarrow f_*(\Delta) = f[\Delta] \text{ and } f^*(f_*(\Delta)) = \Delta. \quad \square$$

1.9 Boolean Algebras as Special Groups. In Chapters 4, 5 and 7 in [5] there is an extensive analysis of the interaction between Boolean algebras and special groups. In particular, the Boolean hull functor and its properties are an essential tool in the solutions of many questions in quadratic form theory (see [6], [7], [8]). We provide here just the definitions and the results needed below; all this material was developed in the references above and in [14].

Let $\langle B, \vee, \wedge, \perp, \top \rangle$ be a Boolean algebra (BA). Then $\langle B, \Delta, \perp \rangle$ is a group of exponent 2 and $\langle B, \Delta, \wedge, \perp, \top \rangle$ is a Boolean unitary ring.

For each $a, b, c, d \in B$, define

$$[\equiv_B] \quad \langle a, b \rangle \equiv_B \langle c, d \rangle \Leftrightarrow a \wedge b = c \wedge d \text{ and } a \vee b = c \vee d.$$

By Corollary 4.4.(b) in [5], $\langle \langle B, \Delta, \perp \rangle, \equiv_B, \top \rangle$ is RSG, where $1 := \perp, -1 := \top$.

The following table describes the correspondence between BA concepts and SG concepts:

Special groups	Boolean algebras
\cdot	Δ
1	\perp
-1	\top
$a \in D_G(1, b)$	$a \leq b$
Saturated subgroup	Ideal
SG-morphism	BA-morphism
L_{SG} -pure embedding	L_{BA} -pure embedding

Let **BA** be the category of Boolean algebras and **BA**-morphisms. We have a functor, $\gamma : \mathbf{BA} \rightarrow \mathbf{RSG}$, identifying **BA** with a full subcategory of **RSG**, justifying its frequent omission from the notation. \square

1.10 The Boolean hull of RSGs. Let G be a reduced special group, $X_G = \text{Hom}_{SG}(G, \mathbb{Z}_2)$ be its space of orders and let $B_G := \mathcal{B}(X_G)$ be BA of clopen subsets of X_G . The map $\varepsilon_G : G \rightarrow B_G$, given by

$$\varepsilon_G(a) = [a = -1] = \{\sigma \in X_G : \sigma(a) = -1\} = [-a = 1],$$

is a RSG-embedding and the diagram $G \xrightarrow{\varepsilon_G} B_G$ is the **Boolean hull of G** . Moreover, if $G \xrightarrow{f} G'$ is a RSG-morphism, let $X_G \xleftarrow{X(f)} X_{G'}$ be the induced continuous map. Now let $B_G \xrightarrow{B(f)} B_{G'}$ be the BA-morphism dual to $X(f)$, given by $B(f)(U) := X(f)^{-1}[U], U \in \mathcal{B}(X_G)$. \square

The main properties of the above construction are described in the following

Theorem 1.11 (Chapter 4, [5]) *Let G be a RSG.*

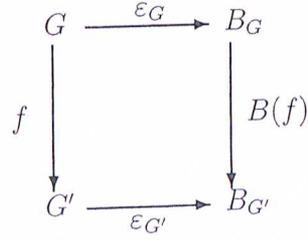
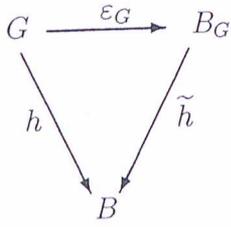
a) *The set $\varepsilon_G[G]$ generates the BA B_G , i.e., for each $u \in B_G$, there is a finite family $\{F_1, \dots, F_n\}$ of finite subsets of G such that*

$$u = \bigcup_{i=1}^n \bigcap_{a \in F_i} \varepsilon_G(a).$$

b) *If G is a BA, then $\varepsilon_G : G \rightarrow B_G$ is a RSG-isomorphism.*

¹I.e., f is the composition of an isomorphism with a quotient by a saturated subgroup.

c) If B is a BA and $h : G \rightarrow B$ is a SG-morphism, there is a unique BA-morphism, $\tilde{h} : B_G \rightarrow B$, such that the triangle below is commutative:



d) The map $(G \xrightarrow{f} G') \mapsto (B_G \xrightarrow{B(f)} B_{G'})$, is a covariant functor, $B : \mathbf{RSG} \rightarrow \mathbf{BA}$, the Boolean Hull Functor.

e) The family $\{(G \xrightarrow{\varepsilon_G} B_G) : G \in \text{Obj}(\mathbf{RSG})\}$ is a natural transformation, $\varepsilon : \text{Id}_{\mathbf{RSG}} \rightarrow \gamma \circ B$, i.e., the square above is commutative, for all RSG-morphism $f : G \rightarrow G'$.

f) The Boolean hull functor is left adjoint to the “inclusion” functor $\gamma : \mathbf{BA} \rightarrow \mathbf{RSG}$ and the natural transformation $\varepsilon : \text{Id}_{\mathbf{RSG}} \rightarrow \gamma \circ B$ is the unit of this adjunction. \square

Remark 1.12 Since the “inclusion”, $\gamma : \mathbf{BA} \rightarrow \mathbf{RSG}$ is a right adjoint it preserves all limits. Hence, when we consider limits of BAs, we need not specify the limit as a BA or a RSG. \square

It is shown in Theorem 5.16 (p. 86) of [5], that the operations of taking Boolean hull and quotients of RSGs “commute”; the following result, proved in Proposition 6.10 and Fact 6.21 of [14], is a refinement of the former:

Theorem 1.13 Let G be a RSG, let $\varepsilon_G : G \rightarrow B_G$ be its Boolean hull and let $\langle \varepsilon_{G*}, \varepsilon_G^* \rangle$ be the induced dual pair as in Definition 1.7.

a) If $\Delta \in \text{Ssat}(G)$, then

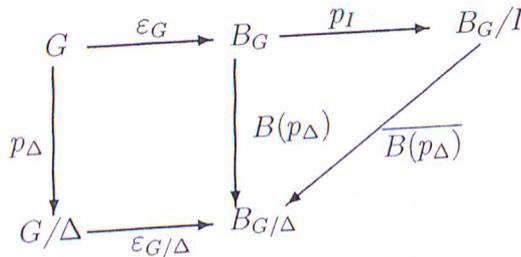
$$\varepsilon_{G*}(\Delta) = \{b \in B_G : \exists \{x_1, \dots, x_n\} \subseteq \Delta \text{ such that } b \leq \bigcup_{j=1}^n \varepsilon_G(x_j)\},$$

i.e., $\varepsilon_{G*}(\Delta)$ is the ideal of B_G generated by $\varepsilon_G[\Delta]$.

b) Given $\{\sigma_1, \dots, \sigma_n\} \subseteq X_G$ let $\{\tilde{\sigma}_j : 1 \leq j \leq n\} \subseteq X_{B_G}$ be the set of extensions, as in Theorem 1.11.(c). If $\Delta = \bigcap_{i=1}^n \ker \sigma_i$, then

$$\varepsilon_{G*}(\Delta) = \bigcap_{i=1}^n \ker \tilde{\sigma}_i.$$

c) If I is an ideal of finite index in B_G and $\Delta := \varepsilon_G^*(I)$, then by Proposition 1.8.(c) $\varepsilon_{G*}(\Delta) = I$ and there is a unique BA-morphism, $\overline{B(p_\Delta)} : B_G/I \rightarrow B_{G/\Delta}$, making the following diagram commutative.



Moreover $\overline{B(p_\Delta)}$ is a BA-isomorphism with inverse $(\varepsilon_G)_I : B_{G/\Delta} \rightarrow B_G/I$, the “extension” of the (induced on quotients) RSG-morphism $(\varepsilon_G)_I : G/\Delta \rightarrow B_G/I$, given by $g/\Delta \mapsto \varepsilon_G(g)/I$.

d) The dual pair $\langle \varepsilon_{G*}, \varepsilon_G^* \rangle$ establish increasing correspondences between $\mathcal{F}(G)$ and $\mathcal{F}(B_G)$ that are inverse bijections. \square

Remark 1.14 In Proposition 4.15.(b) in [5] it is shown that the dual pair $\langle \varepsilon_{G^*}, \varepsilon_G^* \rangle$ yields inverse bijections between $X(G)$ and $X(B_G)$; note that Theorem 1.13.(d) improves this result. On the other hand, by Proposition 7.28 in [5], ε_{G^*} and ε_G^* establish inverses bijective correspondences between $Ssat(G)$ and $Ssat(B_G)$ iff G is a BA. \square

1.15 Profinite Special Groups. Profinite RSGs appeared (in the dual setting of orderings spaces) in [12]; after they have been studied in [13]; [15] and [16] describe logical-categorical properties of profinite structures and the profinite hull functor in a general setting; [14], [17] and [18] describe properties and applications of the notions of being profinite and of the profinite hull in the context of special groups, while [1] establishes a representation of profinite RSGs by Pythagorean fields. As above, we register the definitions and the results needed in what follows; all this material is established in the aforementioned references.

Definition 1.16 Let L be a first-order language and let $L\text{-str}$ be the category of L -structures and L -morphisms.

a) An L -formula is geometrical if it is the negation of an atomic formula or logically equivalent to one of the form $\forall \bar{x}(\varphi(\bar{x}, \bar{y}) \rightarrow \psi(\bar{x}, \bar{y}))$, where φ, ψ are positive-existential L -formulas. An L -theory is geometrical if it can be axiomatized by geometric sentences.

a) A downward directed poset is a poset $\langle I, \leq \rangle$ that is non-empty and such that, for each $i, j \in I$, there is $k \in I$ with $k \leq i, j$.

b) A cofiltered system of L -structures is a functor (or diagram), \mathcal{D} , from a downward directed poset $\langle I, \leq \rangle$ into the category $L\text{-str}$.

c) An L -structure is profinite if it is L -isomorphic to the limit, in the category $L\text{-str}$, of a cofiltered system of finite L -structures. \square

1.17 Notation. We assume the reader is familiar with limits of cofiltered systems and set down notation to be used hereafter. Let $\langle I, \leq \rangle$ be a downward directed poset and let $\mathcal{M} = \langle M_i; \{f_{ij} : i \leq j\} \rangle$ be a cofiltered system of L -structures over I . Let $M = \prod_{i \in I} M_i$ be the product L -structure and, for each $i \in I$, let $q_i : M \rightarrow M_i$ be the corresponding coordinate projection.

The limit of \mathcal{M} is a pair $P = \langle P; \{\mathfrak{p}_i : i \in I\} \rangle$, where P is a L -structure and $\mathfrak{p}_i : P \rightarrow M_i$ are L -morphisms satisfying a well-known universal property. Moreover, there is a natural L -monomorphism,

$$\iota_P : P \rightarrow M, \text{ given by } \iota_P(x) = \langle \mathfrak{p}_i(x) \rangle_{i \in I},$$

such that for all i and all $i \leq j$ in I :

$$f_{ij} \circ \mathfrak{p}_i = \mathfrak{p}_j \quad \text{and} \quad \pi_i \circ \iota_P = \mathfrak{p}_i.$$

Whenever P is clear from context, we write ι in place of ι_P . The limit P may (and often is) identified with the closed² L -substructure $\iota[P]$ of M given by:

$$(*) \quad \iota[P] = \{ \langle a_i : i \in I \rangle \in M : \forall i \leq j \text{ in } I, f_{ij}(a_i) = a_j \}.$$

With this identification, the L -morphism \mathfrak{p}_i is the restriction to $\iota[P]$ of the projection $\pi_i, i \in I$. \square

A very useful characterization of profinite SGs comes from the following general result in [15]³:

²I.e. $\iota[P]$ is a closed subset of $\prod_{i \in I} M_i$ endowed with the product (Boolean) topology.

³Generalizing to the category $L\text{-str}$ Lemma 4.4 in [12], stated for spaces of orderings.

Theorem 1.18 ([15]) *Let L be a first-order language.*

a) *Profinite L -structures are retracts of ultraproducts of finite L -structures. More precisely, if P is the limit of a cofiltered system $\mathcal{M} = (M_i, \{f_{ij} : i \leq j \in I\})$ of finite L -structures over the downward directed poset $\langle I, \leq \rangle$, there is an ultrafilter U over I together with L -morphisms,*

$$P \xrightarrow{\iota} \prod_{i \in I} M_i \xrightarrow{q} \prod_{i \in I} M_i/U \xrightarrow{\gamma^U} P$$

such that $\gamma^U \circ q \circ \iota = \text{Id}_P$, where ι is the canonical embedding of P into the product of the M_i (cf. 1.17) and q is the natural quotient morphism.

b) *If T is a geometrical L -theory, then $\text{Mod}(T)$, the full subcategory of L -str consisting of models of T , is closed under profinite limits.* \square

1.19 It is natural to consider the category $\mathbf{RSG}^{\text{top}}$ of all topological RSGs (i.e., the product operation is continuous) and continuous SG-morphisms. $\mathbf{RSG}_{\text{dis}}$, the full subcategory of all discrete topological RSGs, is clearly isomorphic to \mathbf{RSG} . We will denote $\mathbf{RSG}_{\text{fin}}$ the full subcategory of $\mathbf{RSG}^{\text{top}}$ of all finite and discrete topological RSGs and \mathbf{RSG}_{pf} denotes the full subcategory of $\mathbf{RSG}^{\text{top}}$ of all profinite topological RSGs. $\mathbf{RSG}_{\text{fin}}$ is a full subcategory of \mathbf{RSG}_{pf} that is isomorphic to a full subcategory of \mathbf{RSG} .

If G is a topological RSG we will denote $X_G^c := \{\sigma \in \text{SG}(G, \mathbb{Z}_2) : \sigma \text{ is continuous}\}$, where $\mathbb{Z}_2 \in \text{Obj}(\mathbf{RSG}_{\text{fin}})$, i.e. it is a finite and discrete topological RSG. \square

1.20 The Profinite Hull of a Topological RSG. ⁴ Let G be a topological RSG. Since $(\mathcal{F}(G), \subseteq)$ is a filter in $(\text{Ssat}(G), \subseteq)$ (Proposition 1.6.(c)), it is a downward directed poset and hence

$$\mathcal{V}(G) := \{\Delta \in \mathcal{F}(G) : \text{the } \mathbf{RSG}^{\text{top}}\text{-quotient } G/\Delta \text{ is (finite and) discrete}\}$$

is a downward directed poset; furthermore, we have an obvious diagram:

$$(\Delta \subseteq \Gamma) \in \mathcal{V}(G) \quad \mapsto \quad (G/\Delta \xrightarrow{q_{\Delta\Gamma}} G/\Gamma) \in \mathbf{RSG}_{\text{fin}},$$

where $q_{\Delta\Gamma}$ is the unique SG-morphism such that:

$$(G \xrightarrow{p_\Delta} G/\Delta \xrightarrow{q_{\Delta\Gamma}} G/\Gamma) = (G \xrightarrow{p_\Gamma} G/\Gamma)$$

where p_Δ, p_Γ are the canonical SG-quotient morphisms.

We now define: $\mathcal{P}(G) := \varprojlim_{\Delta \in \mathcal{V}(G)} G/\Delta$ and $p_\Delta : \mathcal{P}(G) \rightarrow G/\Delta$ as the continuous coordinate

projection. For $\Sigma \in \text{Ssat}(\mathcal{P}(G))$, we denote $P_\Sigma : \mathcal{P}(G) \rightarrow \mathcal{P}(G)/\Sigma$, the canonical SG-quotient.

Clearly, $\mathcal{P}(G)$ is a profinite RSG. The map

$$\eta_G : G \rightarrow \mathcal{P}(G), \quad \text{given by } \eta_G(g) = (g/\Delta)_{\Delta \in \mathcal{V}(G)},$$

is a continuous RSG-morphism. The diagram $\eta_G : G \rightarrow \mathcal{P}(G)$ is the **profinite hull of G** .

Note that if G is a discrete RSG (all points are open), then $\mathcal{V}(G) = \mathcal{F}(G)$ and $X_G^c = X_G$. \square

The main properties of the preceding construction are described below:

Theorem 1.21 ([15], [16]) *Let G, G' be topological RSGs.*

a) *The set $\eta_G[G]$ is a dense subset of $\mathcal{P}(G)$.*

⁴Generalizing the construction in Remark 1.4 in [12].

b) If G is a profinite RSG, then $\eta_G : G \rightarrow \mathcal{P}(G)$ is a \mathbf{RSG}_{pf} -isomorphism.

c) If P is a profinite reduced special group and $h : G \rightarrow P$ is a continuous RSG-morphism, then there is a unique \mathbf{RSG}_{pf} -morphism, $\widehat{h} : \mathcal{P}(G) \rightarrow P$ making the triangle below commutative. In particular, if $f : G \rightarrow G'$ is continuous RSG-morphism, there is a unique continuous RSG-morphism, $\mathcal{P}(f) : \mathcal{P}(G) \rightarrow \mathcal{P}(G')$, making the square below commutative.

$$\begin{array}{ccc} G & \xrightarrow{\eta_G} & \mathcal{P}(G) \\ & \searrow h & \nearrow \widehat{h} \\ & & P \end{array}$$

$$\begin{array}{ccc} G & \xrightarrow{\eta_G} & \mathcal{P}(G) \\ f \downarrow & & \downarrow \mathcal{P}(f) \\ G' & \xrightarrow{\eta_{G'}} & \mathcal{P}(G') \end{array}$$

d) The association $(G \xrightarrow{f} G') \mapsto (\mathcal{P}(G) \xrightarrow{\mathcal{P}(f)} \mathcal{P}(G'))$, is a functor, $\mathbf{RSG}^{top} \xrightarrow{\mathcal{P}} \mathbf{RSG}_{pf}$, the **Profinite Hull Functor**, left adjoint to the inclusion, $\iota : \mathbf{RSG}_{pf} \rightarrow \mathbf{RSG}^{top}$. Moreover, the natural transformation $\{(G \xrightarrow{\eta_G} \mathcal{P}(G)) : G \in \text{Obj}(\mathbf{RSG})\}$ is the unit of this adjunction. \square

From the universal property in the Theorem 1.21.(c) above, we obtain:

Corollary 1.22 Let G be a topological RSG. Then:

a) $\eta_G^*| : X_{\mathcal{P}(G)}^c \rightarrow X_G^c$, given by $\sigma \mapsto \sigma \circ \eta_G^*$ is an well defined continuous bijection.

b) We have well defined increasing and inverse maps:

* $u_G : (\mathcal{V}(G), \subseteq) \rightarrow (\mathcal{V}(\mathcal{P}(G)), \subseteq) : \Delta \mapsto u_G(\Delta) = \ker(\mathfrak{p}_\Delta)$;

* $d_G : (\mathcal{V}(\mathcal{P}(G)), \subseteq) \rightarrow (\mathcal{V}(G), \subseteq) : \Sigma \mapsto d_G(\Sigma) = \ker(P_\Sigma \circ \eta_G)$.

Moreover, for each $\Delta \in \mathcal{V}(G)$ we have well defined inverse \mathbf{RSG}_{fin} -isomorphisms:

* $(\eta_G)_\Delta : G/\Delta \rightarrow \mathcal{P}(G)/u_G(\Delta) : g/\Delta \mapsto \eta_G(g)/u_G(\Delta)$;

* $\bar{\mathfrak{p}}_\Delta : \mathcal{P}(G)/u_G(\Delta) \rightarrow G/\Delta : \vec{s}/u_G(\Delta) \mapsto \mathfrak{p}_\Delta(\vec{s})$. \square

1.23 Recall that the *stability index* of a RSG, G , is $stab(G) = \sup\{k \in \mathbb{N} : \exists \Delta \in \mathcal{F}(G), G/\Delta \text{ is a fan and } \dim_{\mathbb{F}_2}(G/\Delta) = k + 1\} \in \mathbb{N} \cup \infty$; in particular, G is (the RSG associated to) a BA iff $stab(G) \leq 1^5$.

It follows directly from the paragraph above Theorem 2.5 in [12] that if \widehat{G} is a profinite RSG, then $stab(\widehat{G}) = \sup\{k \in \mathbb{N} : \exists \Sigma \in \mathcal{V}(\widehat{G}), \widehat{G}/\Sigma \text{ is a fan and } \dim_{\mathbb{F}_2}(\widehat{G}/\Sigma) = k + 1\}$. \square

The result below provides a (very useful) characterization of the *spaces of orderings* of a profinite RSGs with *finite stability index*.

Theorem 1.24 (Theorem 3.2 in [12]) Let $\mathcal{G} = \langle G_i, (f_{ij})_{i \leq j \in I} \rangle$ be a cofiltered system of finite RSGs over the downward directed poset $\langle I, \leq \rangle$ and let the profinite RSG $\langle \widehat{G}, (\pi_i)_{i \in I} \rangle = \varprojlim \mathcal{G}$. Then, the following are equivalent:

(1) $stab(\widehat{G}) < \infty$;

(2) $X_{\widehat{G}}^c := \{\sigma \in SG(\widehat{G}, \mathbb{Z}_2) : \sigma \text{ is continuous}\} = {}^6 \bigcup_{i \in I} \pi_i^*[X_{G_i}]$ is a discrete subspace of the (Boolean) space $X_{\widehat{G}}$ and the inclusion $i_{\widehat{G}} : X_{\widehat{G}}^c \hookrightarrow X_{\widehat{G}}$ is canonically identified with the Stone-Čech compactification embedding $SC_{\widehat{G}} : X_{\widehat{G}}^c \hookrightarrow \beta(X_{\widehat{G}}^c)$. \square

⁵See section 3.4 in [20].

⁶By Theorem 1.5 in [12].

⁷I.e., if $\check{i}_{\widehat{G}} : \beta(X_{\widehat{G}}^c) \rightarrow X_{\widehat{G}}$ is the unique continuous function such that $\check{i}_{\widehat{G}} \circ SC_{\widehat{G}} = i_{\widehat{G}}$, then $\check{i}_{\widehat{G}}$ is a homeomorphism.

1.25 The Boolean and Profinite hulls and local-global principles. Local-global principles in the algebraic theory of quadratic forms were developed, initially, in the context of fields: Pfister's local-global principle, a generalization of Sylvester's inertia law for the real field, appeared in the sixties. It was generalized to special groups in [5].

Equivalent renderings of Pfister's local-global principle in the theory of RSGs are, where G is a any reduced special group:

(1) Usual description (Theorem 2.30 in [5]): for all forms φ, ψ of the the same dimension over G ,

$$\varphi \equiv_G \psi \Leftrightarrow \text{for all } \sigma \in X_G, \sigma \star \varphi \equiv_{\mathbb{Z}_2} \sigma \star \psi.$$

(2) (Proposition 3.7 in [5]): for all forms φ, ψ of the same dimension over G

$$\varphi \equiv_G \psi \Leftrightarrow \text{for each } \Sigma \in X(G), \varphi/\Sigma \equiv_{G/\Sigma} \psi/\Sigma.$$

(3) The diagonal arrow

$$\delta_G^m : G \longrightarrow \prod_{\Sigma \in X(G)} G/\Sigma, \text{ given by } g \mapsto \langle g/\Sigma : \Sigma \in X(G) \rangle$$

is a complete embedding.

(4) (Corollary 5.4 in [5]) : The canonical SG-morphism, $G \xrightarrow{\varepsilon_G} B_G$, is a *complete embedding*.

There is a formulation of (3) for quotients that will be useful below. If $\Delta_0 \subseteq \Delta_1$ are saturated subgroups of G , we have the usual "double quotient Theorem" for RSGs: $(G/\Delta_0)/(\Delta_1/\Delta_0) \cong G/\Delta_1$. Hence, if $\Delta \in \mathcal{F}(G)$, we get a natural bijection of finite Boolean spaces

$$X(G/\Delta) \approx X_\Delta(G) = \{\Sigma \in X(G) : \Delta \subseteq \Sigma\}.$$

(5) By (3) and the "double quotient Theorem" the SG-morphism

$$\alpha_\Delta^G : G/\Delta \longrightarrow \prod_{\Sigma \in X_\Delta(G)} G/\Sigma, \text{ given by } g/\Delta \mapsto \langle g/\Sigma : \Sigma \in X_\Delta(G) \rangle$$

is a complete embedding.

In [19], M. Marshall establishes a local-global principle, stronger than Pfister's, and also related to a certain class of positive-existential L_{SG} -formulas, namely:

Marshall's Isotropy Theorem: If G is a RSG and φ is a form over G , then

$$\varphi \text{ is isotropic over } G \Leftrightarrow \text{For all } \Delta \in \mathcal{F}(G), \varphi/\Delta \text{ is isotropic over } G/\Delta.$$

In [17], we prove that the profinite hull functor, considered as a functor $\mathbf{RSG} \cong \mathbf{RSG}_{dis} \hookrightarrow \mathbf{RSG}^{top} \rightarrow \mathbf{RSG}_{pf}$, also encodes a general local-global principle⁸: **the reflection of subforms**. Recall that if φ and ψ are forms over a SG G , we say that φ is a subform of ψ , written $\varphi \preceq_G \psi$, if there is a form θ over G such that $\psi \equiv_G \varphi \oplus \theta$. □

Theorem 1.26 ([17]) *Let G be a RSG.*

a) *For each G -forms φ, ψ , the following are equivalent:*

$$(1) \varphi \preceq_G \psi; \quad (2) \text{For all } \Delta \in \mathcal{F}(G), \varphi/\Delta \preceq_{G/\Delta} \psi/\Delta;$$

(3) *Let $\Pi(G) := \prod_{\Delta \in \mathcal{F}(G)} G/\Delta$ and consider the "diagonal" SG-morphism, $\delta_G^f : G \longrightarrow \Pi(G)$, given by $g \mapsto (g/\Delta)_{\Delta \in \mathcal{F}(G)}$, then*

$$\delta_G^f \star \varphi \preceq_{\Pi(G)} \delta_G^f \star \psi.$$

b) *The canonical SG-morphism, $\eta_G : G \longrightarrow \mathcal{P}(G)$, preserves and reflects subforms. In particular, it preserves and reflects isotropy and is a complete embedding of RSGs.* □

⁸In fact, equivalent to Marshall's isotropy theorem; see also the Remark 1.4 in [12].

2 Profinite Boolean algebras and the Boolean Hull of a Profinite Reduced Special Group

In this section we show that the Boolean hull of a profinite RSG is a Boolean subalgebra of a profinite BA. We begin by recalling basic properties of profinite Boolean algebras.

2.1 Let BA_{compl} be the category of complete Boolean algebras (cBA) and *complete* BA-morphisms⁹. It is clear that if $\{B_i : i \in I\}$ is a family of cBAs and $B := \prod_{i \in I} B_i$ is the product BA, then B is a cBA and the coordinate projections, π_i , are complete BA-morphisms. It is straightforward to check that this remains true for *general* projective limits instead of products¹⁰. Since the category BA_{fin} of *finite* BAs and BA-morphisms is a *full* subcategory of BA_{compl} , profinite Boolean algebras are *complete*. \square

Some of the basic properties of the well-known class of profinite Boolean algebras are collected in the following

Fact 2.2 a) If B is a Boolean algebra, then the following are equivalent:

- (1) B is the underlying BA of an object of BA_{Bool} , the category of all topological BA endowed with a Boolean topology and continuous BA-homomorphisms.
- (2) B is profinite.
- (3) $Stone(B)$ is an extremally disconnected Boolean space and $Stone(B)^{iso} := \{U \in Stone(B) : U \text{ is an isolated point of } Stone(B)\}$ is a dense subset of $Stone(B)$.
- (4) The inclusion $Stone(B)^{iso} \hookrightarrow Stone(B)$ is canonically identified with the Stone-Ćech compactification $Stone(B)^{iso} \xrightarrow{\beta} \beta(Stone(B)^{iso})$.
- (5) B is a complete atomic BA.
- (6) B is isomorphic to the Boolean algebra $\mathbb{P}(Y)$, for some set Y .

b) The contravariant powerset functor $\mathbb{P} : Set \rightarrow BA_{Bool}$ is a duality of categories with quasi-inverse the functor $(A \xrightarrow{f} B) \in BA_{Bool} \mapsto (Stone(A)^{co} \xleftarrow{f^{-1}} Stone(B)^{co}) \in Set$, where $Stone(B)^{co} := \{U \in Stone(B) : U \text{ is a clopen subset of } B\}$. \square

2.3 Construction. Let $\langle I, \leq \rangle$ be a downward directed poset and $\mathcal{G} = \langle G_i; f_{ij} \rangle$ be a cofiltered system of RSGs and SG-morphisms over I . Applying the Boolean hull functor to this diagram, we obtain a cofiltered system of BAs over I ,

$$\mathcal{B} = \langle B_{G_i}; B(f_{ij}) \rangle.$$

Moreover, if \mathcal{G} is a diagram in RSG_{fin} , then \mathcal{B} is a diagram in BA_{fin} : if G_i is a finite RSG, the space of orders X_{G_i} is finite ($X_{G_i} \subseteq \mathbb{Z}_2^{G_i}$) and B_{G_i} , the BA of clopens of X_{G_i} , is a finite BA, $i \in I$.

Since the collection $\{G \xrightarrow{\varepsilon_G} B_G : G \text{ is a RSG}\}$ is a natural transformation from Id_{RSG} into B , the collection $\varepsilon = \{\varepsilon_{G_i} : i \in I\}$ is a morphism of cofiltered systems

⁹I.e., that preserves all joins and all meets (by the de Morgan law, meet and join preservation in BAs are equivalent).

¹⁰In fact, this works even in the category of complete lattices and complete morphisms.

$$\varepsilon : \mathcal{G} \longrightarrow \mathcal{B}$$

such that ε_{G_i} is a complete embedding ($i \in I$). Let

$$\langle \widehat{\mathcal{G}}, g_i \rangle = \varprojlim \mathcal{G} \quad \text{and} \quad \langle \widehat{\mathcal{B}}, \beta_i \rangle = \varprojlim \mathcal{B}.$$

By items (a) and (b) in Theorem 1.18, $\widehat{\mathcal{G}}$ is a RSG and the inclusion SG-morphism

$$\iota : \widehat{\mathcal{G}} \longrightarrow G = \prod_{i \in I} G_i$$

is a SG-section, whence a complete embedding. By 2.1 and 2.2, $\widehat{\mathcal{B}}$ is a complete atomic BA and the BA-morphisms β_i preserve arbitrary meets and joins. We thus obtain a unique *continuous* SG-morphism of *profinite* RSGs

$$(e) \quad e = \varprojlim \varepsilon : \varprojlim \mathcal{G} \longrightarrow \varprojlim \mathcal{B}$$

$$\begin{array}{ccc}
 \widehat{\mathcal{G}} & \xrightarrow{g_i} & G_i \\
 \downarrow e & & \downarrow \varepsilon_{G_i} \\
 \widehat{\mathcal{B}} & \xrightarrow{\beta_i} & B_{G_i}
 \end{array}$$

such that the square above commutes, for each $i \in I$. □

Theorem 2.4 *With notation as in 2.3,*

a) *The (continuous) SG-morphism $e : \widehat{\mathcal{G}} \longrightarrow \widehat{\mathcal{B}}$ is a complete embedding.*

b) *The induced BA-morphism $\tilde{e} : B_{\widehat{\mathcal{G}}} \longrightarrow \widehat{\mathcal{B}}$ (1.11.(c)) is injective and establishes a BA-isomorphism between the Boolean hull of $\widehat{\mathcal{G}}$, $B_{\widehat{\mathcal{G}}}$, and the Boolean subalgebra of $\widehat{\mathcal{B}}$ generated by $e[\widehat{\mathcal{G}}]$.*

c) *Suppose that the set $I^s := \{i \in I : g_i : \widehat{\mathcal{G}} \longrightarrow G_i \text{ is surjective}\}$ is coinital in $\langle I, \leq \rangle$ ¹¹. Then $\tilde{e} : B_{\widehat{\mathcal{G}}} \longrightarrow \widehat{\mathcal{B}}$ is a BA-isomorphism iff the stability index of $\widehat{\mathcal{G}}$ is finite.* □

3 The Profinite Hull Functor on the category of Boolean algebras

This section is essentially devoted to a characterization of the action of Profinite Hull functor on the category of Boolean algebras¹². In particular, for each Boolean algebra B : (i) the profinite BA, $\mathcal{P}(B)$, is shown to be (naturally) isomorphic, as profinite BAs, to the BA of parts of the Stone space of B ; (ii) under this isomorphism, the embedding $\eta_B : B \longrightarrow \mathcal{P}(B)$ is (naturally) identified with the well-known BA-embedding taking B to the standard base of its Stone space. The reader should keep in mind the correspondence table in 1.9.

Remark 3.1 In [3] Theorem 2.11.(2) and first paragraph of section 3 (or Corollary 3.3 in [4]) it is shown that $\mathcal{P}(B)$ is isomorphic to $\mathbb{P}(\text{Stone}(B))$; however, there is lack of an explicit characterization of the profinite hull functor's action on BA-morphisms and of the canonical embedding of a BA into its profinite hull¹³. Although this functorial behavior can be described “dually”, by an appropriate combination of Corollary 1.22.(a), Theorem 1.24 and the universal property of the

¹¹E.g., by 1.21.(b), any profinite RSG can be described as the projective limit of a system of finite RSGs such that all the projections are surjective and *regular* (see 1.8.(e)).

¹²This seems to be carried out for the first time in Theorem 6.16 in [14] (in Portuguese).

¹³The emphasis in [2], [3] and [4] is on lattice theoretical aspects and deep results are obtained on the profinite and profinite completions of Heyting algebras and distributive lattices in general.

4 Some Applications

In this section we explore the interaction between the Boolean hull and the profinite hull of RSGs, providing a natural transformation $B \circ \mathcal{P} \rightarrow \mathcal{P} \circ B$ and show that the profinite hull functor preserves complete embeddings.

4.1 Let $f : G \rightarrow G'$ be a SG-morphism. Since η_G is a SG-morphism that *reflects subforms* and $\eta_{G'} \circ f = \mathcal{P}(f) \circ \eta_G$, it is straightforward that if $\mathcal{P}(f)$ is a complete embedding, a L_{SG} -monomorphism, reflects isotropy, or reflects subforms then the same is true of f .

On the other hand, the proof that $\mathcal{P}(f)$ is a complete embedding iff f is a complete embedding requires considerably more work. \square

If $f : G \rightarrow G'$ is a complete embedding and $\Delta \in \mathcal{F}(G)$, then, in general, we cannot ensure that $f_*(\Delta) \in \mathcal{F}(G')$. However, by Theorem 1.13.(b), this property is satisfied when $G' = B_G$ and $f = \varepsilon_G$. This fact is essential in the:

Proposition 4.2 For any reduced special group, G , the continuous SG-morphism $\mathcal{P}(\varepsilon_G) : \mathcal{P}(G) \rightarrow \mathcal{P}(B_G)$ is canonically identified with the limit morphism

$$e_G = \lim_{\leftarrow \Delta \in \mathcal{F}(G)} \varepsilon_{G/\Delta} : \lim_{\leftarrow \Delta \in \mathcal{F}(G)} G/\Delta \rightarrow \lim_{\leftarrow \Delta \in \mathcal{F}(G)} B_{G/\Delta} \quad (2.3).$$

In particular, $\mathcal{P}(\varepsilon_G)$ a complete embedding. \square

Theorem 4.3 a) Let G be a RSG and let, $\widetilde{\mathcal{P}(\varepsilon_G)} : B_{\mathcal{P}(G)} \rightarrow \mathcal{P}(B_G)$ be the unique BA-morphism such that $\widetilde{\mathcal{P}(\varepsilon_G)} \circ \varepsilon_{\mathcal{P}(G)} = \mathcal{P}(\varepsilon_G)$ (1.11.(c)). Then, $\widetilde{\mathcal{P}(\varepsilon_G)}$ is injective and the Boolean hull of $\mathcal{P}(G)$, $B_{\mathcal{P}(G)}$, is isomorphic to the Boolean subalgebra of $\mathcal{P}(B_G)$ generated by $\mathcal{P}(\varepsilon_G)[\mathcal{P}(G)]$.

b) The family of (injective) BA-morphisms $\{\widetilde{\mathcal{P}(\varepsilon_G)} : G \in \text{Obj}(\mathbf{RSG})\}$ is a natural transformation $B \circ \mathcal{P} \rightarrow \mathcal{P} \circ B$, i.e., for each $f : G \rightarrow G'$, the diagram below commutes:

$$\begin{array}{ccccc}
 & & \mathcal{P}(\varepsilon_G) & & \\
 & & \curvearrowright & & \\
 \mathcal{P}(G) & \xrightarrow{\varepsilon_{\mathcal{P}(G)}} & B\mathcal{P}(G) & \xrightarrow{\widetilde{\mathcal{P}(\varepsilon_G)}} & \mathcal{P}B(G) \\
 \downarrow \mathcal{P}(f) & & \downarrow B\mathcal{P}(f) & & \downarrow \mathcal{P}B(f) \\
 \mathcal{P}(G') & \xrightarrow{\varepsilon_{\mathcal{P}(G')}} & B\mathcal{P}(G') & \xrightarrow{\widetilde{\mathcal{P}(\varepsilon_{G'})}} & \mathcal{P}B(G') \\
 & & \curvearrowleft & & \\
 & & \mathcal{P}(\varepsilon_{G'}) & &
 \end{array}$$

c) $RSG_{stfin} := \{G \in \text{Obj}(\mathbf{RSG}) : \text{stab}(G) < \infty\}$ defines the largest full subcategory of \mathbf{RSG} where B and \mathcal{P} “commute”, i.e., for each $G \in \text{Obj}(\mathbf{RSG})$, the following are equivalent:

- (1) $\widetilde{\mathcal{P}(\varepsilon_G)} : B_{\mathcal{P}(G)} \rightarrow \mathcal{P}(B_G)$ is a SG-isomorphism;
- (2) $\text{stab}(G) < \infty$.

\square

Theorem 4.4 The functor \mathcal{P} preserves complete embeddings. \square

5 Final remarks

We provide here some counterexamples showing that, in general, the functor \mathcal{P} does not preserve infinitary products and that the natural map η_G is not an elementary embedding. We also establish a sharpening of Proposition 1.6 in the Boolean algebra setting.

Remark 5.1 By a Proposition in [18], the functor $\mathcal{P} : \mathbf{RSG} \rightarrow \mathbf{RSG}_{pf}$ preserves finite products. However, in general, it *does not* preserve infinite products. There is a simple counter-example using the characterization of the profinite hull of BAs in section 3. If B is a BA, the profinite hull $\eta_B : B \rightarrow \mathcal{P}(B)$ is isomorphic to $\delta_B^m : B \rightarrow 2^{X(B)}$. If $B = \prod_{i \in \mathbb{N}} 2 \cong 2^{\mathbb{N}}$, a result due to A. Tarski guarantees that $\text{card}(X(B)) = 2^{2^\omega}$ and $\text{card}(\mathcal{P}(\prod_{i \in \mathbb{N}} 2)) = \text{card}(2^{X(B)}) = 2^{2^{2^\omega}}$; but note that $\text{card}(\prod_{i \in \mathbb{N}} (\mathcal{P}(2))) = \text{card}(\prod_{i \in \mathbb{N}} 2) = 2^\omega$. \square

Remark 5.2 In Theorem 1.26 we saw that if G is a RSG, the map $\eta_G : G \rightarrow \mathcal{P}(G)$ reflects subforms; and, in Theorem 3.4.(f) we saw that if G is a BA, then η_G is a pure embedding. However, in general, η_G is *not* an elementary embedding; moreover, G and $\mathcal{P}(G)$ might not even be elementarily equivalent. To see this, we once again use the characterization of the profinite hull of BAs in section 3. Let B be any infinite BA *without atoms*; then, $2^{X(B)}$ is an infinite, complete atomic BA and, since the existence of atoms is first-order expressible, it is clear that B and $\mathcal{P}(B)$ are not elementarily equivalent. In particular, η_B cannot be an elementary embedding. Since the order relation in a BA, \leq , can be described by a L_{SG} -formula ($x \leq y$ iff $\langle x, xy \rangle \equiv \langle 1, y \rangle$), there is a corresponding L_{SG} -sentence expressing the existence of atoms, showing that B and $\mathcal{P}(B)$ are not elementary equivalent as *reduced special groups*. \square

In Proposition 1.6.(d) we saw that, for each RSG G , we have a Galois connection:

$$\mathbb{P}_{fin}(X(G)) \rightleftarrows \mathcal{F}(G)$$

$$* S \in \mathbb{P}_{fin}(X(G)) \mapsto \bigcap S \in \mathcal{F}(G); \quad * \Delta \in \mathcal{F}(G) \mapsto X_\Delta(G) \in \mathbb{P}_{fin}(X(G)),$$

such that

$$* \forall \Delta \in \mathcal{F}(G) \ (\Delta = \bigcap X_\Delta(G)); \quad * \forall S \in \mathbb{P}_{fin}(X(G)) \ (S \subseteq X_{\bigcap S}(G)).$$

When we are restricted to the Boolean algebra setting we have a stronger result:

Proposition 5.3 *For each RSG G the following are equivalent:*

- (1) G is a BA;
- (2) The Galois connection $\mathbb{P}_{fin}(X(G)) \rightleftarrows \mathcal{F}(G)$ is a duality, i.e., it is given by inverse decreasing bijections.

\square

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