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Web modules and applications

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Dedicated to the memory of
Prof. Hans Zassenhaus

Abstract

If k is a field and $\mathbf{Q}'$ is a finite connected quiver without oriented cycles, its k - *proper web* module is the representation with a k at each vertex and an identity map at each arrow. Given a k-algebra Λ of the form $k\mathbf{Q}/I$, the Λ-web modules are the ones induced by proper web modules and certain functors $F : k\mathbf{Q}' \to \Lambda$ (which we call *qfaithful*). One of the main results is that web modules are indecomposable. An application is given to show that all indecomposable modules of certain biserial algebras are web modules.

1 Introduction and notations.

This paper is contributed as an homage to Prof. Hans Zassenhaus. We have tried to choose our contents in a way that avoids as much as possible

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technical developments, staying close to combinatorics, and to write it following the elementary algorithmic lines that our dear professor would have liked.

1.1

Let Λ be a k-category of the form $\frac{k\mathbf{Q}}{\mathbf{I}}$ where $\mathbf{Q}$ is a quiver and $\mathbf{I}$ an admissible ideal of $k\mathbf{Q}$.

In general, if $\mathbf{Q}$ is a quiver, let us denote by $\overline{\mathbf{Q}}$ de graph subjacent to $\mathbf{Q}$. Given $\alpha \in \mathbf{Q}_1$ (α is an arrow of $\mathbf{Q}$) and $i \in \mathbf{Q}_0$ (i is a vertex of $\mathbf{Q}$), we say they are *incident* if i is one of α endpoints ($i \in \{o(\alpha), e(\alpha)\}$).

A *representation* M of Λ is a functor of Λ into the category mod k of finite-dimensional k-vector spaces, or, equivalently, a *(right)* Λ*-module*. It is defined by associating a finite-dimensional k-vector space $M(i)$ to each vertex i of $\mathbf{Q}$ and a k-linear map $M(\alpha) : M(i) \to M(j)$ to each arrow $i \xrightarrow{\alpha} j$ of $\mathbf{Q}$, in such a way that all maps corresponding as a result to elements of the relations ideal $\mathbf{I}$ be equal to 0.

The *quiver support*, $qsup(M)$, of such a representation M is the subquiver of $\mathbf{Q}$ whose vertices are those i such that $M(i) \neq 0$ and whose arrows are those α for which $M(\alpha) \neq 0$. The *graph support*, $gsup(M)$ of M is the graph that is subjacent to $qsup(M)$, in other words, $gsup(M) = \overline{qsup(M)}$.

Let us now fix a subquiver $\mathbf{Q}'$ of $\mathbf{Q}$. Let us denote, for the moment, by $\mathbf{I}''$ the ideal of $k\mathbf{Q}$ that is generated by the arrows which do not belong to $\mathbf{Q}'_1$. Next, let us consider the set $\mathbf{I}'$ of all elements σ' (that is, linear combinations of paths) in $k\mathbf{Q}'$, such that there is some $\sigma'' \in \mathbf{I}''$ with the property that $\sigma' + \sigma'' \in \mathbf{I}$. It is easy to see that in this way we have constructed the least possible ideal $\mathbf{I}'$ f $k\mathbf{Q}'$ whose corresponding quotient category $\Lambda' = \frac{k\mathbf{Q}'}{\mathbf{I}'}$ has the following property.

Every Λ'-module M', if "extended" to $\mathbf{Q}$ by associating the 0 space to each vertex not in $\mathbf{Q}'_0$ and the 0 map to each arrow not in $\mathbf{Q}'_1$, defines a Λ module; and, conversely, any Λ module with quiver support contained in $\mathbf{Q}'$ defines a Λ'-module. In fact, this correspondence comes from a fully faithful functor from Λ'-mod to Λ-mod which defines an equivalence between Λ'-mod and the full subcategory of Λ-mod defined by the representations with support contained in $\mathbf{Q}'$.

We will use the following notations. The ideal $\mathbf{I}'$ just defined above will be denoted by $\mathbf{I}_{\mathbf{Q}'}$ and will be called the *restriction* of $\mathbf{I}$ to $\mathbf{Q}'$; and the corresponding k-category Λ' will be denoted by $\Lambda_{\mathbf{Q}'}$ and will be called the *restriction* of Λ to $\mathbf{Q}'$.

Also, given a Λ-module M with $qsup(M) = \mathbf{Q}'$, the notations $\mathbf{I}_M$, Λ_M may be used in place of $\mathbf{I}_{\mathbf{Q}'}$, $\Lambda_{\mathbf{Q}'}$, respectively.

The following definition will facilitate our exposition in later sections.

Definition 1 *Keeping the above notations, let $F : \Lambda' \longrightarrow \Lambda$ be a (k-linear) functor of k-categories. F is said to be* qfaithful *(for* quiver-faithful*) if it is induced by a morphism of quivers (which will be denoted by the same symbol F) $\mathbf{Q}' \longrightarrow \mathbf{Q}$ that is surjective on the arrows and that is locally injective in the following sense:*

> *If $\alpha, \beta \in \mathbf{Q}'_1$ have $o(\alpha) = o(\beta)$ or $e(\alpha) = e(\beta)$ (are* incident*)*
> *then $F(\alpha) = F(\beta)$ implies $\alpha = \beta$.*

1.2

Let us recall now the concept of an *induced representation* (or induced module). Let $F : \Lambda' \longrightarrow \Lambda$ be a (k-linear) functor between two k-categories.

By definition, if M is a Λ-module, the corresponding Λ'-module *coinduced* by F, denoted by M_F or by $M_{\Lambda'}$, is $M \circ F$, and, if M' is a Λ'-module, the corresponding Λ-module *induced* by F, denoted by M'^F or by M'^Λ, is $M' \otimes_{\Lambda'} \Lambda$, where here, to be short, we denote again by Λ the left Λ'-module corresponding to the left Λ-module Λ coinduced by F: $\Lambda \circ F$. (Needless to say that, by definition, a *left* module over a category is a (*right*) module over the opposite category.)

Remark 1 *On induced modules.*

As it is well known, an induced module M^F may be characterized also as the solution of an universal problem.

Let $\iota : M' \longrightarrow M'^F$ be the Λ'-morphism (*i. e.* it is implicitly assumed there that M'^F denotes actually the coinduced module $(M^F)_F$) defined naturally by $\iota : m' \mapsto m' \otimes 1$. Then ι is the (unique, up to isomorfismo) solution of the following.

Given a Λ-module N, for each Λ'-morphism $\theta' : M' \to N_F$, there exists a unique Λ-morphism $\theta : M^F \to N$ satisfying $\theta' = \theta \circ \iota$.

The proof is direct and is left to the reader.

Example 1 *Modules induced by qfaithful functors.*

We show a particular situation where the induced module M'^F is easily constructed. Let us keep our notations for Λ', Λ as defined by quivers with relations and let us be given a qfaithful functor $F : \Lambda' \to \Lambda$ and a Λ'-module M'.

First we construct a $k\mathbf{Q}$-module $\tilde{M}$ in the following way. On the one hand, to each vertex $i \in \mathbf{Q}_0$, we associate the vector space $\tilde{M}(i)$ which is the direct sum of the spaces $(M(i'))_{F(i')=i}$ (it should be noted that the fact that F is qfaithful implies that it is surjective on vertices). On the other hand, to each arrow $i \xrightarrow{\alpha} j \in \mathbf{Q}_1$ we define the corresponding linear map

$$\tilde{M}(\alpha) : \oplus_{F(i')=i} M(i')) \to \oplus_{F(j')=j} M(j'))$$

by saying that, if there is an arrow $i' \xrightarrow{\alpha'} j'$, then the (i', j')-component of $\tilde{M}(\alpha)$ is precisely $M'(\alpha')$, and, if there is no such arrow α', then that component is equal to 0.

At this moment, it is convenient to introduce also $\tilde{\iota}$, the family of linear applications $(\tilde{\iota}_{i'})_{i'} \in \mathbf{Q}'_0$ where $\tilde{\iota}_{i'} : M'(i') \to M(F(i'))$ is the natural inclusion. As it is easily seen, $\tilde{\iota}$ is a Λ'-morphism from M' to $\tilde{M}$

Next, we define naturally M as the Λ-module induced by $\tilde{M}$, *i. e.* $M = \frac{\tilde{M}}{I\tilde{M}}$, and ι as the passage of $\tilde{\iota}$ to the quotient, *i. e.* as the composition of $\tilde{\iota}$ with the canonical map $\tilde{M} \to M$.

Lemma 1 *Keeping the preceeding notations,* $M \cong M'^F = M' \otimes_{\Lambda'} \Lambda$.

Proof. It is enough to show that (M, ι) is a solution of the induced module universal problem for M'. Let $\theta' : M' \to N_{\Lambda'}$ be a Λ'-morphism, where N is an arbitrary Λ-module. In case $\theta : M \to N$ is a Λ-morphism with $\theta' = \theta \circ \iota$, then, for each $i' \in \mathbf{Q}'_0$, we would have $\theta'_{i'} = \theta_{F(i')} \circ \iota_{i'}$, so that θ is uniquely determined. On the other hand, these equalities, as it is easily verifieed, may be used to define a morphism θ satiafying the desired condition. $\square$

2 Webs.

Definition 2 1. *We say that a representation M' of a connected quiver* $\mathbf{Q}'$ *with no oriented cycles is a* proper web *(or that M' is a* proper web module*) if the following are satisfied.*

$$\forall i' \in \mathbf{Q}'_0,\ M'(i') = k;$$
$$\forall \alpha' \in \mathbf{Q}'_1,\ M'(\alpha') = 1_k.$$

(Notice that a proper web module is determined by its support, and conversely.)
2. *Given the k-category $\Lambda = \frac{k\mathbf{Q}}{\mathbf{I}}$, we say that a faithful Λ-module M is a* web *if there is a proper web module M' (as above) and a qfaithful functor $F : k\mathbf{Q}' \to \Lambda_{qsup(M)}$ such that*

$$M = M'^F,$$

and where we require also that, for all paths τ' of $\mathbf{Q}'$, *the map of M corresponding to τ' does not kill its source web point* (in other words: $M(F(\tau'))(1_{o(\tau')}) \neq 0$, *see* **Remark 2** below).
3. *Given the k-category $\Lambda = \frac{k\mathbf{Q}}{\mathbf{I}}$, we say that a Λ-module M is a* web *if it is a web as a $\Lambda/\mathbf{N}$-module, where* $\mathbf{N}$ *denotes the anihilator of M in Λ.*

(Notice that, obviously, the concept of a web module is defined only up to isomorphism.) Sometimes in what follows, to simplify our exposition, we will call *web quivers* the quivers which are the support of a proper web, *i. e.* those which are connected and have no oriented cycles.

Example 2 For any algebra all the indecomposable projective and all the indecomposable injective modules are web modules.

Example 3 Let us consider a quiver $\mathbf{Q}'$ such that $\hat{\mathbf{Q}}'$ is a Dynkin diagram of type $\mathbf{A}_n$, with such an orientation that there are no oriented paths with length greater than 1, and let Λ be the algebra

$$\begin{pmatrix} k & k \oplus k \\ 0 & k \end{pmatrix},$$

the path algebra of the Kronecker quiver

$$o \quad \overset{\alpha}{\underset{\vec{\beta}}{\rightarrow}} \quad \omega .$$

Let us assume that n is odd: $n = 2t + 1$ and that the extremal points of $\mathbf{Q}'$ are terminal points of the corresponding arrows. Next, let us fix the notations in such a way that $\mathbf{Q}'_0 = \{1, ..., 2t + 1\}$, the arrow from $2s - 1$ to $2s$ is α'_s $(s = 1, ..., t)$ and the arrow from $2s + 1$ to $2s$ is β'_s $(s = 1, ..., t)$. Then, clearly, the functor F sending the α' arrows to α and the β' arrows to β is qfaithful. The web Λ-module M defined by these data has at o the vector space k^{t+1} and at ω the vector space k^t. The matrices $M(\alpha)$, $M(\beta)$ are, respectively, the following.

$$\begin{pmatrix} 1 & 0 & . & . & . & 0 & 0 \\ 0 & 1 & . & . & . & 0 & 0 \\ . & . & . & . & . & . & . \\ 0 & 0 & . & . & . & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 & . & . & . & 0 \\ 0 & 0 & 1 & . & . & . & 0 \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & 1 \end{pmatrix}.$$

These are exactly the indecomposable *preinjective* Λ-modules (*see* [Ri, 3.2, p. 122]). It is easy to see that all the indecomposable *preprojective* and some of the indecomposable *regular* Kronecker modules are also webs.

Remark 2 Let us introduce some vocabulary which will simplify our writing. Keeping our previous general notations the vectors of the form $\iota_{i'}(1) \in M(i)$ (where $i = F(i')$) (*see* **Example 1**) will be called *(web) points* of the web M. On the other hand, the sequences which correspond to maximal paths of M' (in a sense that we will proceed right away to characterize properly) will be called *threads* of the web M.

Let $(i'_0|\alpha'_1\alpha'_2...\alpha'_t|i'_t)$ be a maximal path τ' of $\mathbf{Q}'$, $(i'_\lambda = o(\alpha'_\lambda)$, $(\lambda = 0, 1, ..., t)$; $i'_t = e(\alpha'_t))$. According to the definitions, each $M'(i'_\lambda)$ is equal to k, so that it is convenient to distinguish all these "equal" spaces by calling, say, $x'_\lambda \in M'(i'_\lambda)$ the 1 of k laying in that vector space. Next, let us denote $x_\lambda \in M(i_\lambda)$ the image $\iota_{i'_\lambda}(x'_\lambda)$, where we assume that $i_\lambda = F(i'_\lambda)$. The thus obtained sequences of web points $(x_0x_1...x_t)$ are, by definition, the *threads* of M (so that, also, the $(x'_0x'_1...x'_t)$'s are the threads of M'). If we denote

$\alpha_\lambda = F(\alpha'_\lambda)$, then $(i_0|\alpha_1\alpha_2...\alpha_t|i_t)$ is the path support of our thread of M, and it may be called a *threadpath* of M.

It is clear that if $i \xrightarrow{\alpha} j \in \mathbf{Q}_1$ is not a loop, then there are bases of $M(i)$, $M(j)$, formed by web points with respect to which $M(\alpha)$ has a matrix in the canonical form

$$M(\alpha) = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$$

(where $\mathbf{1}$, $\mathbf{0}$ denote, respectively, identity and zero blocks of convenient size).

The following result must be well known but we give a proof of it for lack of easy reference. In the statement, the notation $\mathbf{Q} \setminus \{i\}$ (where $\mathbf{Q}$ is a quiver and $i \in \mathbf{Q}_0$) denotes the full subquiver determined by the vertices $\mathbf{Q}_0 \setminus \{i\}$.

Lemma 2 *Let* $\mathbf{Q}$ *be a web quiver (*i. e. $\mathbf{Q}$ *is connected and has no oriented cycles) with at least two vertices. Then, there exist* $i_0, j_0 \in \mathbf{Q}_0$, $i_0 \neq j_0$, *that are either sink or sources of* $\mathbf{Q}$, *such that* $\mathbf{Q} \setminus \{i_0\}$ *and* $\mathbf{Q} \setminus \{j_0\}$ *are also web quivers.*

PROOF. The result is clear if $\mathbf{Q}$ has exactly two vertices because they have to be sinks or sources. Hence, we proceed by induction on $n = |\mathbf{Q}_0|$ assuming that $n \geq 3$ and that the result is true for web quivers with less than n vertices. It is well known that $\mathbf{Q}$ must have a sink and a source. If they are not the desired vertices, we can assume that there exists a sink w such that

$$\mathbf{Q} \setminus \{w\} = \mathbf{Q}_1 \cup \mathbf{Q}_2 \cup ... \cup \mathbf{Q}_s \quad (s > 1)$$

(decomposition into connected components), and it is clear then that all the subquivers $\mathbf{Q}^*_\lambda$ of $\mathbf{Q}$ defined by sets of vertices of the form $(\mathbf{Q}_\lambda)_0 \cup \{w\}$ ($\lambda = 1, 2, ..., s$), are connected. Hence, if $\{i_1, j_1\}$ and $\{i_2, j_2\}$ solve our question for $\mathbf{Q}^*_1$, $\mathbf{Q}^*_2$ respectively, then, picking in each pair one vertex different from w, we get the vertices we were looking for. □

The following is one of the main results in this paper.

Proposition 1 *If M is a web Λ-module, then M is indecomposable.*

PROOF. We keep our above notations. By the lemma and by an obvious duality argument, we can assume that $\mathbf{Q}'$ has a sink w' such that $\mathbf{Q}' \setminus \{w'\}$ is also a web quiver. Let us denote w the image of w' under the qfaithful functor F that defines our web module. We proceed by induction on $|\mathbf{Q}'_0|$ and divide the argument in the following two cases.

$x_{w'}$ is a linear combination of the other web points $x_{i'}$ in $M(w)$. In this case, the same family of vector spaces $M(i)_i$ is obtained as the web module defined through F for the web quiver $\mathbf{Q}' \setminus \{w'\}$ which is indecomposable by the induction hypothesis.

$x_{w'}$ is linearly independent of the remaining web points in $M(w)$. Let us suppose, by contradiction, that $M = M_1 \oplus M_2$ is a proper decomposition of M and let us assume, for example, that the projection $x_{w'1}$ of $x_{w'}$ into M_1 is linearly independent of the remaining web points. Then we can see that M can be redefined as a web module M^* by changing only the web point $x_{w'}$ for $x_{w'1}$. In fact, let us consider the linear k-isomorphism that takes $x_{w'}$ into $x_{w'1}$ and leaves all other web points fixed. Then, remembering the condition that F satisfies for being qfaithful (see **Def. 1**), we easily deduce that our two web modules are isomorphic. Now, this leads to a contradiction to the induction hypothesis and, so, the proof is complete. □

3 Applications.

One interesting question is to study, for some types of algebras, different ways of obtaining some or all the indecomposable modules starting from web modules. In particular, it would be interesting to know for which algebras it happens that all the indecomposable modules are webs. These and other relevant subjects require much more technical aproaches and are left to be developed in a forthcoming paper.

It is clear that the above mentioned property is very especial. For instance, let Λ be the hereditary algebra of type $\mathbf{E}_6$ corresponding to the following orientation.

$$\begin{array}{ccccccccc} & & & & 4 & & & & \\ & & & & \downarrow & & & & \\ 1 & \to & 2 & \to & 3 & \to & 5 & \to & 6 \end{array}$$

Then Λ has indecomposable modules (as the one whose dimension vector is given below) that are not webs. On the other hand, if one introduces, for example, a relation saying that the path from 4 to 5 is zero, then all the indecomposables are webs.

$$\begin{array}{ccccc} & & 2 & & \\ 1 & 2 & 3 & 2 & 1 \end{array}$$

We will consider here only two particular situations. First, we will study algebras of biserial kind and of finite representation type and, secondly, a very interesting class of algebras of infinite representation type with just one indecomposable projective.

3.1

Let us assume here that, keeping our general notations, $\Lambda = \frac{kQ}{I}$ has the following properties.

- Λ is of finite representation type;
- for each $i \in Q_0$ there are at most two arrows, α, β with $e(\alpha) = e(\beta) = i$, and at most two arrows, α_1, β_1 with $o(\alpha_1) = o(\beta_1) = i$;
- for each $\alpha \in Q_1$ there is at most one arrow, α' with $e(\alpha') = o(\alpha)$ and at most one arrow α'' with $e(\alpha) = o(\alpha'')$ satisfying $\alpha\alpha' \notin \mathbf{I}$ and $\alpha''\alpha \notin \mathbf{I}$.

Algebras with these properties are *biserial* in the sense that every indecomposable projective P is either uniserial, or rad$P$ is the direct sum of two uniserial submodules or soc$P$ is simple and rad$P$/soc$P$ is the direct sum of two uniserial modules, with the dual versions of these properties holding for every indecomposable injective module.

These algebras have been studied for several researchers. of whom we mention, for example, [P-S], [S-W] and [E].

Definition 3 *We say that a proper web module M' is a* proper zig-zag *if $gsup(M)$ is a Dynkin diagram of type $\mathbf{A}_n$. A web module M is a* zig-zag *if the proper web defining it, M', is a proper zig-zag (cf.* **Definition 2***).*

A proper zig-zag has possible three kinds of web points. There are *terminal* points, characterized as corresponding to vertices that are incident with exactly one arrow; *hat* points, which correspond to vertices that are the origin of two arrows and the *wedge* points, associated to vertices that are the end points of two arrows. These denominations are also used for the corresponding points of a zig-zag module defined by a given proper zig-zag. It is easy to see that hats are at the top and wedges at the socle of the zig-zag module. Also, a terminal point from which an arrow goes out, is at the top, and one where an arrow ends is at the socle.

Remark 3 It is easy to see that, for algebras with the properties listed above, all web modules are necessarily zig-zag modules. Let M be such a module and let $x, y \in M(i)$ be two web points which correspond respectively to, say, web points $x' \in M'(i_1'), y' \in M'(i_2')$. Since $gsup(M')$ is connected, there is a (non necessarily oriented) walk from i_1' to i_2', so that there is also a walk in $\mathbf{Q}$ beginning and ending at i. Since this walk supports threads of our zig-zag, and since Λ is of finite representation type, we deduce that either x, y are linearly dependent vectors, and the walk consists of two parallel threadpaths which are linearly dependent modulo $\mathbf{I}$, or the walk is just an oriented cycle. In fact, otherwise we would have as Λ-modules all representations of some algebra of type $\tilde{\mathbf{A}}_n$, which has an infinite number of indecomposables.

Lemma 3 *Let*

$$0 \to S \hookrightarrow \hat{M} \to M \to 0$$

be a non split exact sequence of Λ-modules with S simple and M a zig-zag module. Then either $\hat{M}$ is a zig-zag module again or it is the direct sum of two zig-zag modules.

PROOF. Let us show that, when $\hat{M}$ is indecomposable, the only possible "link" between S and M is through a terminal point of the latter. We fix notations assuming that $S = kx \subset \hat{M}(i)$, that the two possible arrows ending up at i are α, β, with origins, resp., at j, h and that the two possible arrows starting at i are α_1, β_1, with end points, resp., at j_1, h_1. Let us assume also that the paths $\beta_1\alpha, \alpha_1\beta$ are in the relations ideal **I**. The hypothesis allows us to suppose that $\hat{M}(u) = M(u) \oplus k.x$, so that we can fix a k-basis $x_1, x_2, ..., x_t$ of $M(u)$ composed by web points of M. As for the arrows, we have, for instance, that, if π is the projection of $\hat{M}(u)$ onto $M(u)$, $M(\alpha) = \pi \circ \hat{M}(\alpha)$.

Some of the points x_λ may be terminal, others may be hats and others, wedges. But it is easy to see from **Remark 3** that there is at most one socle point (terminal or wedge) and at most one top point (terminal or hat). We can assume that, for example, the first t of them are images of web points in $\hat{M}(j) = M(j)$ which we will denote by y_λ keeping the same index for corresponding points. Hence we have, for example, for $\lambda \leq t$ and for some $a_\lambda \in k$, $\hat{M}(\alpha)(y_\lambda) = x_\lambda + a_\lambda x$, $M(\alpha)(y_\lambda) = x_\lambda$.

Also, if the arrow β does exist, we can enumerate the x_λ's in such a way that from, say, $\lambda = t+1$ up to some value. they correspond to non-wedge points linked to $M(h)$ by means of $M(\beta)$. If there is a wedge, it would be the point x_t. The remaining points are a hat or terminal points. In doing this, we denote in general by z_λ a web point of $M(h)$ which $M(\beta)$ applies onto some x_λ.

If there is no wedge, or there is one but we have $\hat{M}(\alpha)(y_t) = \hat{M}(\beta)(z_t)$, then M is isomorphic to the representation that changes each $M(\alpha)(y_\lambda)$, $M(\beta)(z_\lambda)$ for the corresponding image under the $\hat{M}$ map. Hence, if this is a submodule of $\hat{M}$, $\hat{M}$ decomposes into the direct sum of it plus S. Otherwise, there is a terminal point of M which goes into an element of S, and in this case $\hat{M}$ is again a web module.

On the other hand, if there is a wedge point but $\hat{M}(y_t) \neq \hat{M}(z_t)$, since these vectors are linearly independent, we see that $\hat{M}$ decomposes as the direct sum of two zig-zags, one with a terminal point at $\hat{M}(y_t)$ and the other with a terminal point at $\hat{M}(z_t)$. □

Lemma 4 *Let*

$$0 \to S \hookrightarrow \sum_\rho M_\rho$$

be an exact sequence where S is a simple Λ-module, where each M_ρ is a zig-zag module and where, for each ρ, $M_\rho \cap \sum_{\rho' \neq \rho} M_{\rho'} \subset S$. Let S bi concentrated at the vertex i and let us call α and, if it exists, β the arrows ending at i. Let us assume further that each M_ρ has a socle terminal web *point $x_\rho \in S$, so that we will say that it is of "type" α (resp. β) if the corresponding threadpath ends with α (resp. β) (it follows from* **Remark 3** *that each M_ρ is of type α or of type β but not both). Then, if there are more than one zig-zag modules M_ρ of one type, say, α, $\sum_\rho M_\rho$ is a decomposable module.*

In other words, if there are more than two summands, the sum decomposes.

PROOF. Let us call M_1 a M_ρ of type α whose threadpath ending at S has minimum length. Let us denote then by M' the sum of all other zig-zag modules. Let $\alpha_t, ..., \alpha_1, \alpha$ be the sequence of arrows that defines the threadpath of M_1 ending up at a point of S, and let us call $y_t, ..., y_1, x$ the corresponding poits of this thread. Hence, let us choose a corresponding sequence of vectors in M', $y'_t, ..., y'_1, x'$ in such a way that $x' = M'(\alpha \circ \alpha_1 \circ ... \circ \alpha_t)(y'_t) \in S$. It should be observed that all vectors $y_\lambda, y'_\lambda, x, x'$ with $\lambda \neq t$ are anihilated by any arrow different from α and from the α_λ's.

We now define an isomorphism from the direct sum of M_1/kx and M' into $M_1 + M'$, whose existence gives the desired proof. This isomorphism will have the identity map at any vertex differente from the $e(\alpha_\lambda)$, $\lambda = 1, ..., t$, and an automorphism σ_λ at $e(\alpha_\lambda)$. If, say, $x = ax'$, for some $a \in k, a \neq 0$, we define $\sigma_\lambda(y_\lambda) = y_\lambda - a.y'_\lambda$, stipulating that it is the identity in a basis complement of y_λ containing y'_λ. The representation from which this isomorphism starts is the one that has the same arrow maps as $M_1 + M'$, except for α and the α_λ's, where the map is obtained by composing with the apropriate σ_λ. The remaining details are straightforward and are left to the reader. □

As an easy consequence of these lemmas we have the following proposition which is the main result of this subsection.

Proposition 2 *If Λ is a k-algebra with the properties listed at the beginning of this subsection, then all indecomposable Λ-modules are zig-zags.*

PROOF. We proceed by induction on the dimension of the indecomposable module $\hat{M}$. Let S by a simple submodule of $\hat{M}$ and let $M = \oplus_\rho M_\rho$ be the corresponding quotient, already decomposed as a direct sum of web modules. Then, by **Lemma 3**, each preimage $\hat{M}_\rho$ of M_ρ is a direct sum of zig-zag modules, so that $\hat{M}$ is as in **Lemma 4**. It follows then that it is also a zig-zag module. □

3.2

Let us consider now a biserial type algebra $\Lambda = k\mathbf{Q}/\mathbf{I}$ where $\mathbf{Q}$ has only one vertex with two loops, α, β and where the admissible ideal $\mathbf{I}$ is defined by the relations

$$\alpha^n = \beta^m = \alpha\beta = \beta\alpha = 0.$$

Then Λ is an algebra of infinite representation type.

These algebras are being or have been studied by Raymundo Bautista and his student María Alicia Aviñó, who have proven that all indecomposable modules are zig-zags or certain close generalizations of them which are called *closed* zig-zags (*see* [Ba]).

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