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Some Remarks on the Spectrum of Sub-Riemannian Symmetric Spaces

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0 Introduction

There has been considerable interest in computing the spectrum of the Laplace operator acting both on functions and on differential forms for a compact Riemannian manifold, see e. g. [12, 2, 9, 22, 18, 24]. The natural method for such computations in Lie groups and homogeneous spaces essentially uses the Peter-Weyl theorem and the Frobenius reciprocity law to reduce the problem to a representation-theoretic one, namely, one wants to know: *how does an irreducible representation of a compact Lie group decompose into irreducible summands when restricted to a smaller subgroup?* ("branching law"). Once that is solved, one can compute the eigenvalues from Freudenthal's formula and the multiplicities from Weyl's dimension formula. But generally branchings laws are not easy to find.

The purpose of this note is to examine the corresponding problem in the sub-Riemannian setting. Let $(M, \mathcal{D}, g)$ be a contact sub-Riemannian manifold, that is, M is a smooth $2n+1$-dimensional manifold, $\mathcal{D}$ is a contact distribution of hyperplanes in M (a subbundle of TM) and g is a positive definite metric on $\mathcal{D}$. Associated to $(M, \mathcal{D}, g)$, there is a sub-elliptic operator of the simplest kind, its sub-Laplacian. This operator has been defined and studied by a number of authors (see e. g. [20, 14, 19, 10, 1]). We determine its eigenvalues acting on functions on the odd-dimensional spheres and on the real rank two Stiefel manifolds.

1 The sub-Riemannian Laplace operator on a sub-Riemannian manifold

Let $(M, \mathcal{D}, g)$ be a contact sub-Riemannian (sR) manifold, that is, M is a smooth $2n+1$-dimensional manifold, $\mathcal{D}$ is a contact distribution of hyperplanes in M (a subbundle of TM) and g is a positive definite metric on $\mathcal{D}$. Set $\dim M = 2n+1$ and let dV be the volume form of $\mathcal{D}$. The normalized contact form is the unique differential 1-form θ such that $\ker\theta = \mathcal{D}$ and $(d\theta|_{\mathcal{D}})^n = n!\,2^n\,dV$. Since $d\theta$ has maximal rank, there is a unique vector field ξ on M such that $\theta(\xi) = 1$ and $\iota_\xi d\theta = 0$. It is called the *characteristic* vector field. Note that the sR metric g has a natural extension to a Riemannian metric on M by setting ξ to be orthonormal to $\mathcal{D}$. In particular, there is a canonical volume form defined on M.

Let the operator d_D be defined on $C^\infty(M)$ as the usual differential of functions followed by the projection $\Omega^1(M) \to \Omega^1(M)/<\theta>$, and consider the sR connection ∇ (see [8, 5]). The (real) *sR Laplace operator* (or *sub-Laplacian*, for short) on functions is the second order linear sub-elliptic[1] differential operator

$$\Delta_D f = -\mathrm{trace}_D(\nabla d_D f), \quad f \in C^\infty(M)$$

(see also [14, 19, 20]). If $\{E_i\}_{i=1}^{2n}$ is a local frame on $\mathcal{D}$ and $g_{ij} = g(E_i, E_j)$, $(g^{ij}) = (g_{ij})^{-1}$, then we have that $\Delta_D f = -\sum_{i,j=1}^{2n} g^{ij}(E_iE_jf - df(\nabla_{E_i}E_j))$. In particular, if $\{E_i\}_{i=1}^{2n}$ is orthonormal at $x_0 \in M$ and we choose ∇-geodesics γ_i in M with $\gamma_i(0) = x_0$ and $\gamma_i'(0) = E_i$ for $i : 1, \ldots, 2n$, then $\Delta_D f = \sum_{i=1}^{2n} \frac{d^2}{dt^2}|_{t=0}(f\circ\gamma_i)(t)$.

EXAMPLE 1 Let H^3 be the 3-dimensional Heisenberg group with its unique left invariant sR structure, that is, X, Y, Z is a basis for its Lie algebra where $[X, Y] = Z$, Z is central and X, Y is an orthonormal basis for $\mathcal{D}$. Then we compute this frame to be parallel so that $\Delta_D = X^2 + Y^2$.

The celebrated theorem of Hörmander ([16]) implies that the sub-Laplacian is hypoelliptic. One can also show that Δ_D is self-adjoint and positive with

[1] This means that its symbol it positive semidefinite.

respect to the natural L^2-inner product in $C^\infty(M)$. It follows that if M is compact, the eigenvalues of Δ_D form a discrete sequence $0 = \lambda_0 < \lambda_1 < \ldots \to +\infty$ with finite multiplicities and the corresponding eigenfunctions form a complete orthonormal set in $L^2(M)$ ([21]).

Next we show that the sub-Laplacian is a sR invariant that characterizes the isometries among the diffeomorphisms of a sR manifold.

Theorem 1 *Let $\phi : M \to M$ be a diffeomorphism. Then ϕ leaves the sub-Laplacian Δ_D invariant if and only if it is an isometry (in the sR sense).*

Proof. Let $p \in M$. We must show that $\phi_*\mathcal{D} = \mathcal{D}$ an $\phi^* g = g$ at p. By the Darboux theorem, there are local coordinates $(x_1, y_1, \ldots, x_n, y_n, z)$ centered at p such that $\mathcal{D} = \ker\theta$ where $\theta = dz - \frac{1}{2}\sum_{i=1}^n (x_i dy_i - y_i dx_i)$. Now the vector fields $X_i = \frac{\partial}{\partial x_i} - \frac{y_i}{2}\frac{\partial}{\partial z}$, $Y_i = \frac{\partial}{\partial y_i} + \frac{x_i}{2}\frac{\partial}{\partial z}$, $\xi = \frac{\partial}{\partial z}$ are such that $[X_i, Y_i] = \xi$, $i : 1, \ldots, n$, and the other brackets equal zero. Define $E_{2i-1} = X_i$, $E_{2i} = Y_i$ for $i : 1, \ldots, n$. Then $\{E_i\}_{i=1}^{2n}$ is a local frame on $\mathcal{D}$ and we have the following convenient expression for the sub-Laplacian: $\Delta_D f = -\sum_{i,j} g^{ij}[(E_i E_j f) - \frac{1}{2}\sum_{k,l} g^{kl}(E_i g_{jk} + E_j g_{ki} - E_k g_{ij})(E_l f)]$. We can always assume that $g_{ij}(\phi(p)) = \delta_{ij}$. Therefore, at $\phi(p)$ the expression becomes

$$\Delta_D f(\phi(p)) = \sum_i [(E_i^2 f)(\phi(p)) - \sum_k (E_i g_{ik} - \frac{1}{2} E_k g_{ii})(\phi(p))(E_k f)(\phi(p))].$$

Now, by applying this formula, the hypothesis $\Delta_D(f \circ \phi) = \Delta_D f \circ \phi$ and the product formula $\Delta_D(ff') = (\Delta_D f)f' + f(\Delta_D f') + 2g(\mathrm{grad}_{\mathcal{D}} f, \mathrm{grad}_{\mathcal{D}} f')$ (for f, $f' \in C^\infty(M)$, where $\mathrm{grad}_{\mathcal{D}}$ is the $\mathcal{D}$-component of the gradient) to the coordinate functions on M, we easily see that $d\phi_p$ is an isometry $\mathcal{D}_p \to \mathcal{D}_{\phi(p)}$. □

Let Diff(M) be the group of diffeomorphisms of M and consider its action on $L^2(M)$ by composition with the inverse, namely, $\phi \cdot f(x) = f(\phi^{-1}(x))$ for $\phi \in \mathrm{Diff}(M)$, $f \in L^2(M)$, $x \in M$.

Corollary 1 *An element $\phi \in$ Diff(M) is a sR isometry if and only the Δ_D-eigenspace decomposition of $L^2(M)$ is invariant under the action of ϕ.*

2 The case of a Hermitean sub-Riemannian symmetric space

The sub-Laplacian on a homogeneous sR manifold is certainly an invariant differential operator. In this section we want to describe the sub-Laplacian in the case of sR symmetric spaces of Hermitean type. These are homogeneous sR manifolds classified in ([5, 7, 3]). We recall the construction.

Let G/H be a compact semisimple Hermitean Riemannian symmetric space and let $\mathfrak{g} = \mathfrak{h} + \mathfrak{p}$ be the Cartan decomposition. Choose an element $\xi \in \mathfrak{h}$ such that the centralizer of ξ in $\mathfrak{g}$ is $\mathfrak{h}$, define $\mathfrak{k}$ to be the Cartan-Killing orthogonal complement of ξ in $\mathfrak{h}$ and let K be the connected subgroup of G defined by $\mathfrak{k}$. Then $M = G/K$ is the total space of a circle bundle over G/H, and the Ad_K-invariant complement $\mathfrak{p}$ to the fibers defines a G-invariant distribution $\mathcal{D}$ on M. It is easy to check that $\mathcal{D}$ is a contact distribution. Now the restriction of the Cartan-Killing form sign changed to $\mathfrak{p}$ defines a G-invariant sR metric on $\mathcal{D}$. In this way, $(M, \mathcal{D}, g)$ becomes a sR manifold. Since the Riemannian central symmetries of G/H can be lifted to sR isometries of M which are central symmetries when restricted to $\mathcal{D}$, M becomes — as it is called in [3] — a sR symmetric space of Hermitean type. The reader may keep in mind the example of an odd-dimensional sphere fibering over a complex projective space.

View $\mathfrak{g}$ as the Lie algebra of left-invariant first-order differential operators on G, $Xf(g) = \frac{d}{dt}\big|_{t=0} f(g \exp tX)$ for $X \in \mathfrak{g}$, $f \in C^\infty(G)$ an $g \in G$, and identify the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ of $\mathfrak{g}$ with the associative algebra $D(G)$ of all left-invariant differential operators on G. Now, view $C^\infty(G/K)$ as the subspace of K-right-invariant functions in $C^\infty(G)$, that is, $C^\infty(G/K) = \{f \in C^\infty(G) : f(gk) = f(g) \quad \text{for } g \in G, k \in K\}$. Then it is easy to see that the elements of $\mathfrak{U}(\mathfrak{g})^K = \{a \in \mathfrak{U}(\mathfrak{g}) : \mathrm{Ad}_k a = a \quad \text{for } k \in K\}$ induce differential operators on G/K. More precisely, we have

Theorem 2 ([15]) *The associative algebra $D(G/K)$ of G-invariant differential operators on G/K is naturally isomorphic to $\mathfrak{U}(\mathfrak{g})^K / (\mathfrak{U}(\mathfrak{g})^K \cap \mathfrak{U}(\mathfrak{g})\mathfrak{k})$.*

The *sub-Casimir* element of $\mathfrak{g}$ is the element $\omega_D = \omega + \xi^2$ in the enveloping algebra $\mathfrak{U}(\mathfrak{g})$ of $\mathfrak{g}$, where ω is the Casimir element of $\mathfrak{g}$ and ξ is a unit vector in the orthogonal complement to $\mathfrak{k}$ in $\mathfrak{h}$ (with respect to the Cartan-Killing form). Note that ω_D is Ad_K-invariant, so it defines a G-invariant second order differential operator on M.

Proposition 1 *Let M be equipped with the G-invariant sR metric induced from the negative of Cartan-Killing form of G. Then the sub-Laplacian of M is just the action of the Casimir element on functions, that is, $\Delta_D = \omega_D$ on $C^\infty(M)$.*

Proof. Since both operators are invariant, it suffices to check that they coincide at the base-point x_0. Write $\pi : G \to M$ for the projection and let $X_1, \ldots, X_r; \xi; X_{r+1}, \ldots, X_{r+2n}$ be an orthonormal basis of $\mathfrak{g}$ compatible with its decomposition $\mathfrak{g} = \mathfrak{k} + <\xi> + \mathfrak{p}$. Since the Cartan-Killing form of $\mathfrak{g}$ is negative definite, we have that $\omega_D = -\sum_{i=1}^{r+2n} X_i^2$. Now $\gamma_i : t \mapsto \exp_{x_0} tX_i$ is the ∇-geodesic[2] through x_0 with initial velocity X_i for $i : r+1, \ldots, r+2n$. Therefore:

$$
\begin{aligned}
\Delta_D f(x_0) &= -\sum_{i=r+1}^{r+2n} \frac{d^2}{dt^2}\Big|_{t=0} f(\exp tX_i\, x_0) = -\sum_{i=1}^{r+2n} \frac{d^2}{dt^2}\Big|_{t=0} f(\exp tX_i\, x_0) \\
&= -\sum_{i=1}^{r+2n} \frac{d^2}{dt^2}\Big|_{t=0} (f \circ \pi)(\exp tX_i\, 1) = -\sum_{i=1}^{r+2n} \frac{d^2}{dt^2}\Big|_{t=0} (f \circ \pi)(\exp 1\, tX_i) \\
&= \omega_D (f \circ \pi)(1)
\end{aligned}
$$

□

We know that H is a subgroup of maximal rank in G, so there is a CSA $\mathfrak{t}$ of $\mathfrak{g}$ such that $\xi \in \mathfrak{t} \subset \mathfrak{h}$. Let Δ^+ be a positive root system and write $\pi = \{\alpha_1, \ldots, \alpha_r\}$ for the corresponding system of simple roots. The Borel-de Siebenthal theory says that the highest root $\beta = \alpha_{i_0} + \sum_{i \neq i_0} n_i \alpha_i$ and $\alpha_{i_0}(\xi) \neq 0$, $\alpha_i(\xi) = 0$ if $i \neq i_0$ for some index i_0. Let $\lambda_1, \ldots, \lambda_r$ be the fundamental highest weights and let $\Lambda^+ = \sum_i \mathbb{Z}^+ \lambda_i$, $\Lambda^+(G)$, respectively, denote the dominant integral weight lattices of $\mathfrak{g}$ and G (recall that $\Lambda^+/\Lambda^+(G)$ is an abelian group isomorphic to the fundamental group of G). Finally, write $\rho = (1/2) \sum_{\alpha \in \Delta^+} \alpha$. Now we have

[2] For sections X, Y of $\mathcal{D}$ we have the following relation between the Levi-Cività connection $\bar{\nabla}$ and the sR connection ∇: $\bar{\nabla}_X Y - \nabla_X Y = -\frac{1}{2} d\theta(X, Y)$, because the subtorsion τ is zero, see [8, 5]; it follows that γ_i is a geodesic, *qua* Riemannian, *qua* sub-Riemannian.

Theorem 3 *If π_λ denotes the irreducible complex representation of G of highest weight $\lambda \in \Lambda^+(G)$, then the sub-Laplacian Δ_D has eigenvalues*

$$||\lambda+\rho||^2 - ||\rho||^2 - ||\lambda_{i_0}||^{-2}(\lambda_{i_0},\mu)^2,$$

where μ is a weight of π_λ such that its weightspace has K-fixed vectors. The multiplicity of the eigenvalue is equal to the dimension of representation space of π_λ multiplied by the dimension of the space of K-fixed vectors in the weight space of μ.

Proof. For a compact manifold, the C^∞ functions are dense in the L^2 functions. By the Peter-Weyl theorem,

$$L^2(G) = \sum_{\lambda\in\Lambda^+(G)} V_{\pi_\lambda} \otimes V_{\pi_\lambda^*}$$

as $G\times G$-modules, where the left hand side is acted on by the (left-and-right-) regular representation and where V_{π_λ} is the representation space of π_λ, so that $V_{\pi_\lambda} \otimes V_{\pi_\lambda^*}$ can be considered to be the space of coefficients of π_λ: the embedding $V_{\pi_\lambda} \otimes V_{\pi_\lambda^*} \to L^2(G)$ is given by $u\otimes v \mapsto f_{u,v}$ where $f_{u,v}(g) = \,<u, \pi_\lambda(g)>$, $g \in G$. Viewing $L^2(M)$ as the subspace of K-right-invariant functions in $L^2(G)$, we get

$$L^2(M) = \sum_{\lambda\in\Lambda_K^+(G)} V_{\pi_\lambda} \otimes V_{\pi_\lambda^*}^K,$$

where the superscript indicates the subspace of K-fixed vectors and $\Lambda_K^+(G)$ is the sub-lattice of $\Lambda^+(G)$ corresponding to representations that have K-fixed vectors, the so-called *class 1 representations* relative to K. By Proposition 1, $-\Delta_D$ coincide with ω_D on $L^2(M)$. Since ω_D acts on functions on the right, we are left to compute the eigenvalues of $\pi_\lambda(\omega_D)$ on $V_{\pi_\lambda}^K$ for $\lambda \in \Lambda_K^+(G)$.

Now it is well known that $\pi_\lambda(\omega)$ is the scalar $||\lambda+\rho||^2 - ||\rho||^2$ on V_{π_λ} (see e. g. [17, 11, 23]), so we have only to compute the action of $\pi_\lambda(\xi)$. Since $2(\lambda_i,\alpha_j)/(\alpha_j,\alpha_j) = \delta_{ij}$, we get that ξ corresponds to $\sqrt{-1}||\lambda_{i_0}||^{-1}\lambda_{i_0}$ under the Cartan-Killing form. Therefore, if v_μ is a vector in the eigenspace corresponding to μ, then

$$\pi_\lambda(\xi)v_\mu = \mu(\xi)v_\mu = \sqrt{-1}||\lambda_{i_0}||^{-1}(\lambda_{i_0},\mu).$$

This completes the proof of the theorem. □

Remark 1 We note here, for future reference, that the weights of π_λ whose weightspace has K-fixed weight vectors are all multiples of λ_{i_0}. That is because the α_i's with $i \neq i_0$ are simple roots for $\mathfrak{k}$ relative to the CSA $\mathfrak{t}\cap\mathfrak{k}$.

3 Calculations

3.1 S^3

We shall use the notation and results in [6]. We identify

$$S^3 = SU(2) = \{\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} : \alpha, \beta \in \mathbb{C}, |\alpha|^2 + |\beta|^2 = 1\}$$

so that the unit sphere metric corresponds to $-1/8$ times the Cartan-Killing form. Let

$$X_1 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad X_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad X_3 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

be a basis for $\mathfrak{su}(2)$ such that $[X_1, X_2] = X_3$ and other cyclic permutations of this identity hold. The most general sR symmetric space structure on S^3 is then given by the horizontal space $\mathcal{D} = \langle X_2, X_3 \rangle$ of the Höpf fibration $U(1) \to S^3 \to \mathbb{C}P^2$ and the sR metric on $\mathcal{D}$ is given by the matrix

$$\begin{pmatrix} 1/a & 0 \\ 0 & 1/b \end{pmatrix}, \quad a \geq b > 0.$$

Note that this example does not conform to the construction described in Section 2 since the sR metric is not induced by a multiple of the Cartan-Killing form of $\mathfrak{su}(2)$ if $a \neq b$ (in fact, the eigenvalue of the sub-torsion is $\tau_0 = (a-b)/4$, see [8, 5]). Nonetheless, we can make use of the group structure of the space to perform the calculations. The Peter-Weyl theorem says that $L^2(SU(2)) = \sum_{n=0}^{\infty}(n+1)V_n$ as left $SU(2)$-modules, where the left hand side is acted on by the left regular representation and in the right hand side V_n is the nth symmetric power of the vector representation which can be identified with the complex homogeneous polynomials of degree n in two variables z_1, z_2. The horizontal one-parameter subgroups are geodesics (see [6]), so we get no linear terms and $\Delta_D = -aX_2^2 - bX_3^2 : V_n \to V_n$. Define $v_j = \begin{pmatrix} n \\ j \end{pmatrix}^{1/2} z_1^{n-j} z_2^j$, for $j : 0, \ldots, n$. Then $\{v_0, \ldots, v_n\}$ is an orthonormal basis of V_n and

$$\Delta_D v_j = \frac{1}{2}\{[(n-j)(n-j-1)(j+2)(j+1)]^{1/2}(a-b)v_{j+2} \\ -[n+2j(n-j)](a+b)v_j + [(n-j+2)(n-j+1)j(j-1)]^{1/2}(a-b)v_{j-2}\}.$$

In particular, if $a = b = 4$, then we recover the result in [23, 10]:

$$\Delta_D v_j = 4[n + 2j(n-j)]v_j.$$

Note that the matrix $(\Delta_D|v_n)$ is "bi-symmetric" (symmetric with respect to both diagonals), so the eigenvalues have even multiplicities, with the possible exception of the ones that are also eigenvalues of the Riemannian Laplacian of $\mathbb{C}P^2$. It is not an easy problem to diagonalize that matrix. We can list the first few eigenvalues and their multiplicities:

- $n = 1 : (a+b)/2$, multiplicity 4;
- $n = 2 : 2a$, $2b$, $2(a+b)$, multiplicity 3;
- $n = 3 : (5(a+b) \pm (7(a^2+b^2)+2ab)^{1/2})/2$, multiplicity 8;
- $n = 4 : 2(a+b)$, $2(a+4b)$, $2(4a+b)$, $4(a+b\pm(a^2+b^2-ab)^{1/2})$, multiplicity 5, etc.

REMARK 2 Notice the phenomenon of "changing multiplicities" of the eigenvalues, as we vary the parameters a, b defining the metric. Corollary 1 points out to the fact that this should be related to the existence of metrics which are more "symmetric" than others, but the precise relation is not clear as it is easy to see that all such metrics have the same full group of isometries, with the obvious exception of the one with $a = b$ ([6]).

3.2 The odd-dimensional spheres S^{2n+1}, $n \geq 2$

Consider the $(2n+1)$-dimensional sphere as a sub-Riemannian symmetric space, $S^{2n+1} = SU(n+1)/SU(n)$ with the sub-Riemannian structure pulled back from $\mathbb{C}P^n$, $n \geq 2$. Here $G = SU(n+1)$, $H = S(U(1) \times U(n))$ and $K = S(\{1\} \times U(n))$. The Cartan-Killing form of $\mathfrak{g}$ is $<X, Y> = 2(n+1)\mathrm{trace}(XY)$. Choose the CSA to be the diagonal matrices in $\mathfrak{g}$,

$$\mathfrak{t} = \left\{ H = \begin{pmatrix} \sqrt{-1}\,a_1 & & \\ & \ddots & \\ & & \sqrt{-1}\,a_{n+1} \end{pmatrix} : \quad a_i \in \mathbb{R},\, a_1 + \ldots + a_{n+1} = 0 \right\}.$$

Define $\theta_i(H) = \sqrt{-1}\,a_i$, $i : 1, \ldots, n+1$. Then $\Delta^+ = \{\theta_i - \theta_j : 1 \leq i < j \leq n+1\}$ is a system of positive roots and $\{\alpha_i = \theta_i - \theta_{i+1} : 1 \leq i \leq n+1\}$ is the corresponding system of simple roots, so the rank of $\mathfrak{g}$ is n. The fundamental highest weights are $\lambda_i = \theta_1 + \ldots + \theta_i$, $i : 1, \ldots, n$ and G is simply-connected, so $\Lambda^+(G) = \Lambda^+$. Note that $\alpha_1(\xi) \neq 0$ and $\alpha_i(\xi) = 0$ for $i : 2, \ldots, n$, so ξ corresponds to $\lambda_1 = \theta_1$. The complex irreducible representations of $U(n)$ can be constructed as follows. Consider $U(n) = U(1) \times_{\mathbb{Z}_n} SU(n)$, where $\mathbb{Z}_n$ is the center of $SU(n)$. The complex irreducible representations of $U(1) \times SU(n)$ are the tensor products $\rho_m \otimes \tau_{k_2\theta_2+\ldots+k_n\theta_n}$ where $\tau_{k_2\theta_2+\ldots+k_n\theta_n}$ denotes the representation of $SU(n)$ of highest weight $k_2\theta_2+\ldots+k_n\theta_n$, $k_2 \geq \ldots \geq k_n \geq 0$, and ρ_m denotes the mth power of the circle action of $U(1)$, m an integer. In order for it to induce a representation of $U(n)$ we require $\mathbb{Z}_n$ to act trivially, so we must have $m = k_2 + \ldots + k_n + nk_1$ for an integer k_1. This gives the set of complex irreducible representations of $U(n)$ as

$$\{\rho_{k_2+\ldots+k_n+nk_1} \otimes \tau_{k_2\theta_2+\ldots+k_n\theta_n} : \quad k_2 \geq \ldots \geq k_n \geq 0,\ k_1 \text{ integers}\}. \quad (1)$$

Now $U(n)$ embeds in G as H, so θ_1 maps to $-\frac{1}{n}\theta_1$ and θ_i maps to $\theta_i + \frac{1}{n}\theta_1$, $i : 2, \ldots, n+1$. Therefore, as a representation of $H \subset G$, (1) has highest weight $-k_1\theta_1 + k_2\theta_2 + \ldots + k_n\theta_n$. Fix a representation of G on V of highest weight $\lambda = m_1\lambda_1 + \ldots + m_n\lambda_n \in \Lambda^+$, $m_1, \ldots, m_n \geq 0$. We need the following "branching theorem":

Theorem 4 ([18]) *We have that V decomposes, as a left H-module, into irreducible H-modules as follows:*

$$V = \sum V_{k_1\theta_1+k_2\theta_2+\ldots+k_n\theta_n}$$

where the summation runs over all integers k_1, $k_2 \geq k_3 \geq \ldots \geq k_n \geq 0$ for which there exists an integer k satisfying

$$m_1 + \ldots + m_n \geq k_2 + k \geq$$
$$m_2 + \ldots + m_n \geq k_3 + k \geq m_3 + \ldots + m_n \geq \ldots \geq k_n + k \geq m_n \geq k \geq 0$$

and

$$(m_1 + 2m_2 + \ldots + nm_n) - (k_1 + k_2 + \ldots + k_n) = (n+1)k.$$

Now we can find the representations of G that have K-fixed vectors. In the decomposition of Theorem 4, we must have the n-tuple $(k_1, 0, \ldots, 0)$ allowed. Unravelling the inequalities in the theorem easily gives that $m_2 =$

$\ldots = m_{n-1} = 0$, and then $k_1 = m_1 - m_n$ turns out to be the unique possibility. We conclude that a representation of G which is class 1 relative to K has a highest weight $a\lambda_1 + b\lambda_n$, $a, b \geq 0$ integers, and its subspace of K-fixed vectors is one-dimensional with corresponding weight given by $(a-b)\lambda_1 = (a-b)\theta_1$. Therefore, Theorem 3 says that the eigenvalues of the sub-Laplacian for the sphere S^{2n+1} are

$$\frac{1}{2(n+1)}(2ab + n(a+b)),$$

or, after dividing the metric by $4(n+1)$ in order to get the unit sphere metric (compare [21]),

$$2(2ab + n(a+b)),$$

and the multiplicity is the dimension of the representation $\pi_{a\theta_1+b\theta_n}$ which can be easily computed using Weyl's dimension formula to be

$$\frac{(a+n-1)!(b+n-1)!(a+b+n)}{a!b!n!(n-1)!}.$$

It is interesting to observe that the representation space of $\pi_{a\theta_1+b\theta_n}$ can be realized to be the space of harmonic polynomials of bi-degree (a, b) in $\mathbb{C}^{n+1}$. So we refine the usual $SO(2n+2)$-irreducible decomposition of $L^2(S^{2n+1})$ into the spaces of harmonic polynomials of degree m in $\mathbb{R}^{2n+2}$, $m \geq 0$ (compare [10]).

3.3 The real rank two Stiefel manifolds

The next example is the sub-Riemannian symmetric space $SO(N+2)/SO(2) \times SO(N)$ of orthonormal 2-frames in $N+2$-dimensional Euclidean space. We divide the analysis into two cases.

3.3.1 The odd dimensional case

Here, $G = SO(2n+1)$, $H = SO(2) \times SO(2n-1)$ and $K = \{1\} \times SO(2n-1)$ where 1 is an identity 2×2 block. The Cartan-Killing form of $\mathfrak{g}$ is $<X, Y> =$

$(2n-1)\mathrm{trace}(XY)$. Choose the CSA to be

$$\mathfrak{t} = \left\{ H = \begin{pmatrix} 0 & a_1 & & & & \\ -a_1 & 0 & & & & \\ & & \ddots & & & \\ & & & 0 & a_n & \\ & & & -a_n & 0 & \\ & & & & & 0 \end{pmatrix} : \quad a_i \in \mathbb{R} \right\}.$$

Define $\theta_i(H) = \sqrt{-1}\, a_i$, $i : 1, \dots, n$. Then $\Delta^+ = \{\theta_i \pm \theta_j : 1 \leq i < j \leq n\} \cup \{\theta_i : 1 \leq i \leq n\}$ is a system of positive roots and $\{\alpha_i = \theta_i - \theta_{i+1} : 1 \leq i \leq n-1; \alpha_n = \theta_n\}$ is the corresponding sysem of simple roots, so the rank of $\mathfrak{g}$ is n. We have that G is not simply-connected and its fundamental group has order two; so $\Lambda^+(G)$ has index two in Λ^+ and we have

$$\Lambda^+(G) = \{\lambda = \sum_{i=1}^{n} a_i\theta_i : \quad m_1 \geq m_2 \geq \dots m_n \geq 0 \text{ integers}\}.$$

Note that $\alpha_i(\xi)$ is not zero precisely for $i = 1$, so ξ corresponds to θ_1.

Let π_λ be the complex irreducible representation of G of highest weight $\lambda \in \Lambda^+(G)$.

Lemma 1 **([13])** *We have that π_λ is of class 1 relative to K if and only if $\lambda = a\theta_1 + b\theta_2$ for $a \geq b \geq 0$ integers.*

Proof. This is an easy consequence of the "branching theorems" for the orthogonal groups (cf. [4, 11]). □

Let $\pi^n_{a,b}$ denote the representation $\pi_{a\theta_1+b\theta_2}$ of $G = SO(2n+1)$, $a \geq b \geq 0$ integers. We need to identify the K-fixed weights of $\pi^n_{a,b}$.

Lemma 2 *The K-fixed weights of $\pi^n_{a,b}$ are $(a-b-2j)\theta_1$ for $j : 0, 1, \dots, a-b$.*

Proof. We proceed by induction on the rank n. If $n = 2$ then, just by looking at the weight diagrams, it is easy to see that the K-fixed weights for $\pi^2_{a,b}$ are the same as those for $\pi^2_{a-b,0}$, which have the required form. Let $n \geq 3$ and assume the result true if the rank is less than n. Consider the subgroups G_1, G_2 of $G = SO(2n+1)$, isomorphic to $SO(2n-1)$ and spanned by $\theta_1, \dots, \theta_n$ and by $\theta_2, \dots, \theta_{n+1}$, respectively. Then $G_1 \cap G_2$ is isomorphic to

$SO(2n-3)$. Let v be a weight-vector in the representation space of $\pi^n_{a,b}$ of weight μ which is G_2-fixed. We know that μ is a multiple of θ_1, so v is in the representation space of $\pi^{n-1}_{a,b}$ of G_1, if we consider it as a sub-representation of $\pi^n_{a,b}$. But we have that v is $G_1 \cap G_2$-fixed, so we can apply the inductive hypothesis for $(G_1, G_1 \cap G_2)$. □

Theorem 3 now says that the eigenvalues of the sub-Laplacian are

$$\frac{1}{2(2n-1)}(2ab + (2n+2j-1)a + (2n-2j-3)b - 4j^2),$$

where $a \geq b \geq 0$ and $j : 0, \ldots, [\frac{a-b}{2}]$ (here $[\cdot]$ denotes the "greatest integer less than"). The multiplicity of each eigenvalue is $2 \dim V_{a,b}$, except in the case where $a-b$ is even and $j = \frac{a-b}{2}$, when it is $\dim V_{a,b}$. Weyl's dimension formula gives

$$\dim V_{a,b} = \frac{\Pi_{i<j}(m_i - m_j - i + j)(m_i + m_j + 2n - i - j + 1)\Pi_i(2m_i + 2n - 2i + 1)}{(2n-1)!(2n-3)!\ldots 1!},$$

where $m_1 = a$, $m_2 = b$ and $m_i = 0$ if $3 \leq i \leq n$. In particular, when $n = 2$ we get

$$\dim V_{a,b} = \frac{1}{6}(2a+3)(2b+1)(a-b+1)(a+b+2).$$

3.3.2 The even dimensional case

Here $G = SO(2n)$, $H = SO(2) \times SO(2n-2)$ and $K = \{1\} \times SO(2n-2)$ where 1 is an identity 2×2 block. The Cartan-Killing from of $\mathfrak{g}$ is $<X, Y> = 2(n-1)\mathrm{trace}(XY)$. Choose the CSA to be

$$\mathfrak{t} = \left\{ H = \begin{pmatrix} 0 & a_1 & & & \\ -a_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & a_n \\ & & & -a_n & 0 \end{pmatrix} : a_i \in \mathbb{R} \right\}.$$

Define $\theta_i(H) = \sqrt{-1}\, a_i$, $i : 1, \ldots, n$. Then $\Delta^+ = \{\theta_i \pm \theta_j : 1 \leq i < j \leq n\}$ is a system of positive roots and $\{\alpha_i = \theta_i - \theta_{i+1} : 1 \leq i \leq n-1; \alpha_n = \theta_{n-1} + \theta_{n+1}\}$

is the corresponding sysem of simple roots, so the rank of $\mathfrak{g}$ is n. We have that G is not simply-connected and its fundamental group has order two; so $\Lambda^+(G)$ has index two in Λ^+ and we have

$$\Lambda^+(G) = \{\lambda = \sum_{i=1}^{n} a_i\theta_i : \quad m_1 \geq m_2 \geq \ldots m_{n-1} \geq |m_n| \text{ integers}\}.$$

Note that $\alpha_i(\xi)$ is not zero precisely for $i = 1$, so ξ corresponds to θ_1.

Let π_λ be the complex irreducible representation of G of highest weight $\lambda \in \Lambda^+(G)$. As in the odd-dimensional case, we can show

Lemma 3 ([13]) *We have that π_λ is of class 1 relative to K if and only if $\lambda = a\theta_1 + b\theta_2$ for $a \geq b \geq 0$ integers.*

and

Lemma 4 *The K-fixed weights of $\pi_{a\theta_1+b\theta_2} = \pi_{a,b}$ are $(a-b-2j)\theta_1$ for $j : 0, 1, \ldots, a-b$.*

Theorem 3 now says that the eigenvalues of the sub-Laplacian are

$$\frac{1}{4(n-1)}(2ab + 2(n+j-1)a + 2(n-j-2)b - 4j^2),$$

where $a \geq b \geq 0$ and $j : 0, \ldots, [\frac{a-b}{2}]$. The multiplicity of each eigenvalue is $2\dim V_{a,b}$, except in the following cases: if $n > 2$, $a-b$ is even and $j = \frac{a-b}{2}$ or if $n = 2$, a is even, $b = 0$ and $j = \frac{a}{2}$ then it is half that value; if $n > 2$, $a - b$ is odd and $b > 0$ then it is twice that value. Weyl's dimension formula gives

$$\dim V_{a,b} = 2^{n-1}\frac{\Pi_{i<j}(m_i - m_j - i + j)(m_i + m_j + 2n - i - j)}{(2n-2)!(2n-4)!\ldots 2!},$$

where $m_1 = a$, $m_2 = b$ and $m_i = 0$ if $3 \leq i \leq n$. In particular, if $n = 2$ we get

$$\dim V_{a,b} = (a+1)^2 - b^2.$$

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