

RT - MAT 92-01

The automorphism Group of a Structural
Matrix Algebra

Sônia P. Coelho

Janeiro 1992.

The Automorphism Group of a Structural Matrix Algebra

Sônia P. Coelho*
Instituto de Matemática e Estatística
Universidade de São Paulo
Caixa Postal 20570 - Ag. Iguatemi
01498 - São Paulo - Brazil

1. Introduction

Automorphisms of certain subalgebras of matrix algebras have been the object of several recent papers. Jøndrup [6] has shown that if A is a simple algebra, finite dimensional over its center K , then all K -automorphisms of the algebra of upper triangular matrices over A are inner. Similarly, Barker and Kezlan [3] have proved that every R -automorphism of the algebra of upper triangular matrices with entries from an integral domain R is inner.

Related results appeared in [1], [2], [5] and [7].

In this paper we characterize the group of K -automorphisms of certain subalgebras of a matrix algebra with entries from a field K . Also, we give necessary and sufficient conditions for every K -automorphism of such a subalgebra to be inner. In this sense, our results include those from [3] and [6], when the ring of coefficients is a field (see Corollary to Theorem D).

*The author was partially supported by the Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq).

We would like to remark that the arguments of section 3 actually hold when entries are taken on a simple algebra, finite dimensional over its center (with very slight changes). Therefore, theorems A and B remain valid when the ring of coefficients is such an algebra.

2. Notations and main results

Let $M_n(K)$ be the ring of $n \times n$ -matrices over a field K . The unity element of $M_n(K)$ is denoted by I_n . Given $A \in M_n(K)$, A_{ij} denotes the (i, j) -entry of A . The matrix unit having 1 in the (i, j) th position and zeros elsewhere is denoted by E^{ij} . The matrix E^{ii} is written simply E^i .

If ρ is a relation on $I = \{1, 2, \dots, n\}$ which is reflexive and transitive, then the set

$$S = S(\rho, K) = \{A \in M_n(K) \mid A_{ij} = 0 \text{ if } (i, j) \notin \rho\}$$

is a subalgebra of $M_n(K)$ [see 10].

Following [10], we call $S(\rho, K)$ the *structural matrix algebra over K defined by the relation ρ* .

The group of K -automorphisms of S is denoted by $\text{Aut}(S)$. In what follows, the word automorphism means K -automorphism.

For each invertible matrix $A \in S$, we denote by C_A the inner automorphism induced by A .

Then, the set

$$\mathcal{C} = \{C_A \mid A \in S \text{ is invertible}\}$$

is a normal subgroup of $\text{Aut}(S)$.

A permutation σ of I is said to be an *automorphism of ρ* if $(\sigma(i), \sigma(j)) \in \rho$, for all $(i, j) \in \rho$.

Such a permutation gives rise to an automorphism $\hat{\sigma}$ of S , defining:

$$\hat{\sigma}(E^{ij}) = E^{\sigma(i)\sigma(j)}, \quad (i, j) \in \rho,$$

and extending linearly.

The set of automorphisms $\hat{\sigma}$ of S such that $\sigma(i) < \sigma(j)$ whenever $(i, j), (j, i) \in \rho$ and $i < j$ is a subgroup of $\text{Aut}(S)$, which we shall denote by \mathcal{P} .

In the case when S is semi-simple, these groups allow us to describe $\text{Aut}(S)$.

Theorem A. Let S be a semisimple structural matrix algebra. Then,

$$\text{Aut}(S) = \mathcal{C} \rtimes \mathcal{P},$$

the semi-direct product of \mathcal{C} by \mathcal{P} .

When S is a semisimple algebra, it follows that ρ is an equivalence relation and we have:

Theorem B. Let S be a semisimple structural matrix algebra. Then, every K -automorphism of S is inner if and only if all the equivalence classes of ρ are of different sizes.

Following [8], we say that a function $g : \rho \rightarrow K^*$ is *transitive* if

$$g(i, j)g(j, k) = g(i, k)$$

for all $(i, j), (j, k) \in \rho$. Such a function is said to be *trivial* if there exists a function $s : I \rightarrow K^*$ such that $g(i, j) = s(i)s(j)^{-1}$, for all $(i, j) \in \rho$.

Every transitive function $g : \rho \rightarrow K^*$ gives rise to an automorphism g^* of S , defining:

$$g^*(E^{ij}) = g(i, j)E^{ij}, \quad (i, j) \in \rho,$$

and extending linearly.

Now, let $\bar{\rho}$ be the following relation on I :

$$(i, j) \in \bar{\rho} \text{ if and only if } (i, j), (j, i) \in \rho,$$

and let Δ be the graph associated to the relation $\rho \setminus \bar{\rho}$ (that is, the vertices of Δ are the elements i of I such that either (i, j) or $(j, i) \in \rho \setminus \bar{\rho}$, for some $j \in I$, and the edges of Δ are the unordered pairs $\{i, j\}$ such that either (i, j) or $(j, i) \in \rho \setminus \bar{\rho}$).

For each connected component Δ_ℓ of Δ , we consider a tree $T_\ell \subset \Delta_\ell$ containing every vertex of Δ_ℓ [see 4, §2, Corollary 5]. We fix one such tree T_ℓ for each index ℓ .

Let $\bar{\bar{\rho}}$ be the subset of ρ such that

$$(i, j) \in \bar{\bar{\rho}} \text{ if and only if } (i, j) \in \rho \text{ and } \{i, j\} \text{ is an edge of } \cup_\ell T_\ell.$$

Furthermore, let V be the set of vertices of $\cup T_\ell$. Let also $J = I \setminus V$ and set $\bar{\bar{\bar{\rho}}} = \bar{\bar{\rho}} \cap J \times J$.

Then, the set

$$\mathcal{G} = \{g^* \in \text{Aut}(S) | g : \rho \rightarrow K^* \text{ and } g(i, j) = 1 \text{ for all } (i, j) \in \bar{\bar{\rho}} \cup \bar{\bar{\bar{\rho}}}\}$$

is a subgroup of $\text{Aut}(S)$.

Theorem C. Let S be a structural matrix algebra. Then,

$$\text{Aut}(S) = (\mathcal{C} \rtimes \mathcal{G}) \rtimes \mathcal{P}.$$

Theorem D. Let S be a structural matrix algebra. Then, every K -automorphism of S is inner if and only if the following conditions hold:

- (i) every transitive mapping $g : \rho \rightarrow K^*$ is trivial;
- (ii) every automorphism of ρ fixes the equivalence classes of $\bar{\rho}$.

3. The semisimple case

We begin with some lemmas.

(3.1) Lemma. Let $e \neq 0$ be a central idempotent of S . Then, there exists a subset J of I such that

$$e = \sum_{j \in J} E^j .$$

Proof. For all $i \in I$, there exists $x_i \in K$ such that $E^i e E^i = x_i E^i$.

Since $E^i e E^i$ is an idempotent, we must have that either $x_i = 0$ or $x_i = 1$.

Writing

$$e = I_n e I_n = \sum_{i \in I} E^i e E^i ,$$

the conclusion follows. □

(3.2) Lemma. The set

$$\mathcal{I} = \{A \in S \mid A_{ij} = 0 \text{ if } (i, j) \in \bar{\rho}\}$$

is an ideal of S .

Proof. Take $B \in S$ and $A \in \mathcal{I}$, and set $C = BA$.

Suppose now that $(i, j) \in \bar{\rho}$. Then,

$$C_{ij} = \sum_{k \in I} B_{ik} A_{kj}.$$

If, for some k , we had that both $B_{ik} \neq 0$ and $A_{kj} \neq 0$, then we would have that $(j, i), (i, k) \in \rho$. Hence, we can conclude that $(j, k) \in \rho$.

But, since $A_{kj} \neq 0$, we must have that $(k, j) \notin \rho$, a contradiction.

Then, $C_{ij} = 0$ and $C \in \mathcal{I}$.

A similar argument shows that $AB \in \mathcal{I}$. □

(3.3) Lemma. If S is a semisimple algebra, then $\mathcal{I} = 0$.

Proof. In fact, by Lemma (3.2), we have that \mathcal{I} is an ideal, hence a direct summand of S , i.e.:

$$S = \mathcal{I} \oplus \mathcal{I}',$$

where \mathcal{I}' is an ideal of S .

Let e be the unity element of \mathcal{I} . We have that either

$$e = \sum_{j \in J} E^j \text{ or } e = 0,$$

due to Lemma (3.1).

But \mathcal{I} contains no diagonal matrices; hence $e = 0$. □

We are now ready to give a characterization of a semisimple structural matrix algebra. This result was proved in [9] following a different approach.

(3.4) Theorem. Let $S = S(\rho, K)$ be a structural matrix algebra. Then S is semisimple if and only if ρ is symmetric.

Proof. Suppose that S is semisimple. By Lemma (3.3), we must have that $\mathcal{I} = 0$; thus, ρ is symmetric.

For the converse, let $L \neq 0$ be an ideal of S . We shall show that L is a direct summand of S .

Given $A \in S$, with $A_{kl} \neq 0$, we have that

$$E^k A E^l = A_{kl} E^{kl} \in L.$$

Hence, $E^{kl} \in L$. Since ρ is symmetric, we conclude that $E^{lk} \in S$. Therefore, $E^l = E^{lk} E^{kl} \in L$.

Now, let $T = \{t \in I \mid E^t \in L\}$ and set $e = \sum_{t \in T} E^t$. We claim that $L = Se$.

Clearly, we have that $Se \subset L$. Now, if $A \in L$ with $A_{kl} \neq 0$, we have that $A_{kl} E^{kl} \in L$ as we have shown above.

Then, in order to prove that $L \subset Se$, it is enough to show that $E^{kl} \in Se$. But this follows immediately from the equation: $E^{kl} = E^{kl} e$.

Now, if $E^{kl} \in L$, we have that $l \in T$, as before. A similar argument shows that also $k \in T$.

Hence, it is easy to conclude that e is a central idempotent of S , and the claim follows. \square

Before proving Theorem A, we need to characterize the simple components of S .

(3.5) Lemma. Let S be semisimple and let $e \in S$ be an idempotent such that Se is a simple component of S . Then, there exists one and only one equivalence class C of ρ such that

$$e = \sum_{x \in C} E^x.$$

Proof. By Lemma (3.1), we have that $e = \sum_{j \in J} E^j$, for a suitable subset J of I .

Pick $j \in J$ and $(k, j) \in \rho$. We have that

$$E^{kj}e = E^{kj}.$$

Then, we must have $eE^{kj} = E^{kj}$, and we conclude that $k \in J$.

We have shown that, if $j \in J$, the equivalence class of j is contained in J .

Then, J is a union of equivalence classes. But, if there were two or more such classes, it is easy to see that e would be decomposable. \square

(3.6) Lemma. With the notations of Lemma (3.5), we have that

$$Se = \sum_{x_k, x_l \in C} KE^{x_k x_l},$$

and, consequently, $\dim_K Se = |C|^2$ (where $|C|$ denotes the order of C).

Proof. If $x_k, x_l \in C$, we have that $(x_k, x_l) \in \rho$. then, $E^{x_k x_l} = E^{x_k x_l}e \in Se$, due to the expression of e given by Lemma (3.5). we have shown that $\sum_{x_k, x_l} KE^{x_k x_l} \subset Se$.

For the other inclusion, we observe that, if $A \in Se$, $A = Ae = eA$. From this, it is easy to conclude that A is of the required form. \square

Proof of Theorem A. Let $S = S_1 \oplus S_2 \oplus \cdots \oplus S_r$, where the S_i are the simple components of S , and let ϕ be an automorphism of S .

We have that

$$\phi(S_i) = S_{f(i)}, \quad 1 \leq i \leq r,$$

where f is a bijection of $\{1, 2, \dots, r\}$.

Furthermore, if C_i denotes the equivalence class of ρ corresponding to the unity element of S_i , given by Lemma (3.5), we have that $|C_i| = |C_{f(i)}|$, in view of Lemma (3.6).

We now enumerate explicitly $C_i = \{x_1, x_2, \dots\}$ and $C_{f(i)} = \{y_1, y_2, \dots\}$, where $x_k < x_l$ and $y_k < y_l$ if $k < l$.

We can define a bijection from C_i to $C_{f(i)}$, mapping x_k to the corresponding y_k . As this can be done in each equivalence class of ρ , this process defines a permutation of I , which we shall denote by σ .

It is easy to see that σ is an automorphism of ρ ; hence, it induces an automorphism $\hat{\sigma}$ of S .

We now fix a simple component S_i of S . As we saw above, we have that $\phi(S_i) = S_{f(i)}$.

Given an element $X \in S_{f(i)}$, we have that

$$(\hat{\sigma})^{-1}(X) \in S_i,$$

since

$$(\hat{\sigma})^{-1}(E^{y_k y_l}) = E^{x_k x_l} \in S_i,$$

for all $E^{y_k y_l} \in S_{f(i)}$ (because of Lemma (3.6)).

Then, the map

$$X \in S_{f(i)} \xrightarrow{(\hat{\sigma})^{-1}} (\hat{\sigma})^{-1}(X) \in S_i \xrightarrow{\phi} (\phi \circ (\hat{\sigma})^{-1})(X) \in S_{f(i)}$$

is an automorphism of $S_{f(i)}$.

But $S_{f(i)}$ is a simple algebra finite dimensional over its center. By the Skolem-Noether Theorem, there exists an invertible matrix $A_{f(i)} \in S_{f(i)}$ such that this automorphism is conjugation by $A_{f(i)}$.

Set now $A = A_{f(1)} + \cdots + A_{f(n)}$. We claim that $\phi \circ (\hat{\sigma})^{-1} = C_A$.

In fact, it is enough to prove the equality in each component $S_{f(i)}$.

Given a matrix unit $E^{v_k w_l} \in S_{f(i)}$, we have that:

$$C_A(E^{v_k w_l}) = A E^{v_k w_l} A^{-1} = A_{f(i)} E^{v_k w_l} A_{f(i)}^{-1} = (\phi \circ (\hat{\sigma})^{-1})(E^{v_k w_l}),$$

and the claim is proved.

Therefore, $\phi = C_A \circ \hat{\sigma}$.

To conclude the proof, we notice first that $\hat{\sigma} \in \mathcal{P}$.

Also, $\mathcal{C} \cap \mathcal{P} = 1$. In fact, given $\hat{\sigma} \in \mathcal{C} \cap \mathcal{P}$, we have that $\hat{\sigma}$ is a conjugation by matrix $A \in S$. Therefore, $\hat{\sigma}(S_i) = S_i$, for all i . Then, $\sigma(C_i) = C_i$, for all i , and by the definition of σ it is easy to conclude that $\sigma = 1$.

Since $\mathcal{C} \triangleleft \text{Aut}(S)$, the proof is finished. \square

A little extra effort gives Theorem B.

Proof of Theorem B. First, we observe that if σ is an automorphism of ρ and C is an equivalence class of ρ , then $\sigma(C)$ is also an equivalence class of ρ .

Suppose first that the equivalence classes of ρ are of different sizes.

In view of Theorem B, it is enough to prove that $\mathcal{P} = 1$.

Set $\hat{\sigma} \in \mathcal{P}$. By the hypothesis and the above, we have that $\sigma(C) = C$, for every equivalence class C of ρ .

Now, let $C = \{x_1, x_2, \dots\}$, where $x_k < x_l$ if $k < l$. As $\hat{\sigma} \in \mathcal{P}$, we have that $\sigma(x_k) < \sigma(x_l)$. Thus, it is easy to see that $\sigma(x_k) = x_k$, for all x_k , and hence $\hat{\sigma} = 1$.

For the converse, we observe that $\mathcal{P} = 1$ by hypothesis.

Suppose there are two equivalence classes C_1 and C_2 of ρ such that $|C_1| = |C_2|$ and write:

$$C_1 = \{x_1, x_2, \dots\}, \quad C_2 = \{y_1, y_2, \dots\},$$

where $x_k < x_l$ and $y_k < y_l$ if $k < l$.

Then, we can define the following permutation σ of I :

$$\begin{aligned} \sigma(x_k) &= y_k, & \forall x_k \in C_1, \\ \sigma(y_l) &= x_l, & \forall y_l \in C_2, \\ \sigma(z) &= z, & \forall z \notin C_1 \cup C_2. \end{aligned}$$

Then, $\hat{\sigma} \in \mathcal{P}$, a contradiction. □

4. The general case

First, we need to describe $\mathcal{J}(S)$, the Jacobson radical of S . In what follows, we shall denote $\mathcal{J}(S)$ simply by \mathcal{J} .

The following proposition can be obtained as a consequence of the results in [9]. We offer here a direct proof.

(4.1) Proposition. With the notation above, we have that

$$\mathcal{J} = \{A \in S \mid A_{ij} = 0 \text{ if } (i, j) \in \bar{\rho}\}.$$

Proof. Let

$$\mathcal{I} = \{A \in S \mid A_{ij} = 0 \text{ if } (i, j) \in \bar{\rho}\}.$$

By Lemma (3.2), \mathcal{I} is an ideal of S .

We claim that

$$\frac{S}{\mathcal{I}} = \{A + \mathcal{I} \mid A_{ij} = 0 \text{ if } (i, j) \notin \bar{\rho}\}.$$

In fact, if $B + \mathcal{I} \in \frac{S}{\mathcal{I}}$, choose $B' \in S$ such that $B'_{ij} = B_{ij}$ if $(i, j) \in \bar{\rho}$, and $B'_{ij} = 0$ if $(i, j) \notin \bar{\rho}$. Then, $B - B' \in \mathcal{I}$, and $B + \mathcal{I} = B' + \mathcal{I} \in \{A + \mathcal{I} \mid A_{ij} = 0 \text{ if } (i, j) \notin \bar{\rho}\}$.

The opposite inclusion is trivial.

Now, let

$$S' = \{A \in M_n(K) \mid A_{ij} = 0 \text{ if } (i, j) \notin \bar{\rho}\}.$$

Then, the mapping $f : S' \rightarrow \frac{S}{\mathcal{I}}$ given by $f(A) = A + \mathcal{I}$, $\forall A \in S'$, is an isomorphism of algebras.

By Theorem (3.4), S' is semisimple, since it is a structural matrix algebra defined by the symmetric relation $\bar{\rho}$.

Then, $\frac{S}{\mathcal{I}}$ is semisimple, hence $\mathcal{J} \subset \mathcal{I}$.

Now, $\frac{\mathcal{I}}{\mathcal{J}}$ is a semisimple component of $\frac{S}{\mathcal{J}}$. Let $e + \mathcal{J}$ be its unity element, where $e \in \mathcal{I}$. We compute:

$$\begin{aligned} e + \mathcal{J} &= (I_n + \mathcal{J})(e + \mathcal{J})(I_n + \mathcal{J}) = \left(\sum_{i \in I} E^i + \mathcal{J}\right)(e + \mathcal{J})\left(\sum_{i \in I} E^i + \mathcal{J}\right) = \\ &= \sum_{i \in I} E^i e E^i + \mathcal{J}. \end{aligned}$$

That is,

$$e - \sum_{i \in I} E^i e E^i = \eta \in \mathcal{J}.$$

Since $\mathcal{J} \subset \mathcal{I}$, we have that $\sum E^i e E^i = e - \eta \in \mathcal{I}$. But \mathcal{I} does not contain diagonal matrices. Hence, $e - \eta = 0$ and $e = \eta \in \mathcal{J}$. That is, $\mathcal{I} \subset \mathcal{J}$. \square

In order to deal with the automorphisms of S , we observe that given $\phi \in \text{Aut}(S)$, we have that $\phi(\mathcal{J}) = \mathcal{J}$. So, ϕ induces an automorphism $\bar{\phi}$ of $\frac{S}{\mathcal{J}}$, namely:

$$\bar{\phi}(A + \mathcal{J}) = \phi(A) + \mathcal{J}, \text{ for all } A \in S.$$

Suppose now that e, e' are idempotents of S such that $\phi(e + \mathcal{J}) = e' + \mathcal{J}$.

Since $\bar{\phi}(e + \mathcal{J}) = \phi(e) + \mathcal{J}$, we have that $\phi(e)$ must be of the form: $\phi(e) = e' + \eta$, where $\eta \in \mathcal{J}$.

We need more information about these idempotents.

(4.2) Lemma. Let $E = E^j$, $\eta \in \mathcal{J}$ and let $\mathcal{E} = E + \eta$ be an idempotent of S . Then, for all $i \geq 3$ we have that

$$\begin{aligned} \mathcal{E} &= E + (\eta + \eta^2 + \cdots + \eta^{i-1})E + E(\eta + \eta^2 + \cdots + \eta^{i-1}) + \\ &\quad \sum_{k=1}^{i-2} (\eta + \eta^2 + \cdots + \eta^{i-1-k})E\eta^k + \eta^i. \end{aligned}$$

Proof. We shall proceed by induction on i . We have that

$$\mathcal{E} = \mathcal{E}^2 = E^2 + E\eta + \eta E + \eta^2 = E + E\eta + \eta E + \eta^2.$$

Multiplying the first and last members of the equation by $\mathcal{E}(= E + \eta)$,

we obtain:

$$\mathcal{E} = E + E\eta E + \eta E + \eta^2 E + E\eta + E\eta^2 + \eta E\eta + \eta^3 .$$

But $E\eta E = \eta_{jj}E^j$, $\eta_{jj} \in K$. As $E\eta E \in \mathcal{J}$ and \mathcal{J} does not contain diagonal matrices, we conclude that $E\eta E = 0$.

So, $\mathcal{E} = E + (\eta + \eta^2)E + E(\eta + \eta^2) + \eta E\eta + \eta^3$, and the claim is proved for $i = 3$.

Suppose now that equality holds for a given index i , and multiply both sides of the equality by $\mathcal{E}(= E + \eta)$.

Observing that the terms $E(\eta + \eta^2 + \dots + \eta^{i-1})E$ and $\sum_k(\eta + \eta^2 + \dots + \eta^{i-k-1})E\eta^k E$ are both zero (by the argument used above), we get:

$$\begin{aligned} \mathcal{E} &= E + (\eta + \eta^2 + \dots + \eta^i)E + E(\eta + \eta^2 + \dots + \eta^i) + (\eta + \eta^2 + \dots \\ &\quad \eta^{i-1})E\eta + \sum_{k=1}^{i-2} (\eta + \eta^2 + \dots + \eta^{i-1-k})E\eta^{k+1} + \eta^{i+1} , \end{aligned}$$

as required. □

Notice that, since S is artinian, the elements of \mathcal{J} are nilpotent.

(4.3) Corollary. With the notation of Lemma (4.2), we have that:

- (i) if the index of nilpotency of η is 2, then $\mathcal{E} = E + \eta E + E\eta$;
- (ii) if the index of nilpotency of η is $s > 2$, then

$$\begin{aligned} \mathcal{E} &= E + (\eta + \eta^2 + \dots + \eta^{s-1})E + E(\eta + \eta^2 + \dots + \eta^{s-1}) \\ &\quad + \sum_{k=1}^{s-2} (\eta + \eta^2 + \dots + \eta^{s-1-k})E\eta^k . \end{aligned}$$

Proof. Since $\mathcal{E} = E + E\eta + \eta E + \eta^2$, (i) follows trivially.

In a similar way, (ii) follows from the lemma above. \square

(4.4) **Corollary.** Let \mathcal{E} be an idempotent of S in the conditions of Lemma (4.2). Then, there exists $\theta \in \mathcal{J}$ such that

$$\mathcal{E} = E + E\theta + \theta E + \theta E\theta .$$

Conversely, if $\theta \in \mathcal{J}$ and $E = E^j$ ($j \in I$), then

$$\mathcal{E} = E + E\theta + \theta E + \theta E\theta$$

is an idempotent of S .

Proof. Suppose \mathcal{E} is an idempotent as in Lemma (4.2).

If the index of nilpotency of η is 2, then $\mathcal{E} = E + E\eta + \eta E$, by Corollary (4.3).

Computing \mathcal{E}^2 , we get that:

$$E + E\eta + \eta E = \mathcal{E} = \mathcal{E}^2 = E + E\eta + \eta E + \eta E\eta .$$

Hence, $\eta E\eta = 0$ and it is enough to choose $\theta = \eta$.

If the index of nilpotency is $s > 2$, computing \mathcal{E}^2 , we obtain:

$$\begin{aligned} \mathcal{E} = \mathcal{E}^2 &= E + (\eta + \eta^2 + \cdots + \eta^{s-1})E + E(\eta + \eta^2 + \cdots + \eta^{s-1}) + \\ &(\eta + \eta^2 + \cdots + \eta^{s-1})E(\eta + \eta^2 + \cdots + \eta^{s-1}) . \end{aligned}$$

Then, it is enough to choose $\theta = \eta + \eta^2 + \cdots + \eta^{s-1}$.

The converse is trivial. \square

(4.5) **Lemma.** Let $\theta \in \mathcal{J}$, $E = E^j$ ($j \in I$) and let \mathcal{E} be the idempotent:

$$\mathcal{E} = E + E\theta + \theta E + \theta E\theta .$$

Then,

$$U = (I_n + E\theta)(I_n - \theta E) \in S$$

is invertible and

$$U\mathcal{E}U^{-1} = E.$$

Proof. We have that

$$(I_n - \theta E)(I_n + \theta E) = I_n,$$

$$(I_n + E\theta)(I_n - E\theta) = I_n.$$

Therefore, $U^{-1} = (I_n + \theta E)(I_n - E\theta)$.

An easy calculation shows that

$$(I_n - \theta E)\mathcal{E}(I_n + \theta E) = E + E\theta,$$

$$\begin{aligned}(I_n + E\theta)(E + E\theta)(I_n - E\theta) &= (I_n + E\theta)E(I_n + E\theta)(I_n - E\theta) = \\ &= (I_n + E\theta)E = E\end{aligned}$$

□

Now, for each $j \in I$, pick $\theta_j \in \mathcal{J}$ and consider:

$$\mathcal{E}_j = E^j + E^j\theta_j + \theta_j E^j + \theta_j E^j\theta_j,$$

$$U_j = (I_n + E^j\theta_j)(I_n - \theta_j E_j).$$

(4.6) Lemma. With the notations above, we have that:

(i) $\overline{C_{U_j}} = 1$, for all $j \in I$.

(ii) If $E^i \mathcal{E}_j = \mathcal{E}_j E^i = 0$, where $i, j \in I$, $i \neq j$, then $E^i \mathcal{U}_j = \mathcal{U}_j E^i = E_i$.

Proof. Since the elements \mathcal{U}_j are such that $\mathcal{U}_j + \mathcal{J} = I_n + \mathcal{J}$, for all $j \in I$, (i) follows.

In order to prove (ii), we compute:

$$\begin{aligned} E^i \mathcal{E}_j &= E^i \theta_j E^j + E^i \theta_j E^j \theta_j = E^i \theta_j E^j (I_n + \theta_j) = 0, \\ \mathcal{E}_j E^i &= E^j \theta_j E^i + \theta_j E^j \theta_j E^i = (I_n + \theta_j)(E^j \theta_j E^i) = 0. \end{aligned}$$

Since $\theta_j \in \mathcal{J}$, we have that $I_n + \theta_j$ is invertible. Hence, $E^i \theta_j E^j = E^j \theta_j E^i = 0$.

Now, we compute $E^i \mathcal{U}_j$ and $\mathcal{U}_j E^i$:

$$\begin{aligned} E^i \mathcal{U}_j &= E^i (I_n + E^j \theta_j)(I_n - \theta_j E^j) = E^i (I_n - \theta_j E^j) = E^i - E^i \theta_j E^j = E^i, \\ \mathcal{U}_j E^i &= (I_n + E^j \theta_j)(I_n - \theta_j E^j) E^i = (I_n + E^j \theta_j) E^i = E^i + E^j \theta_j E^i = E^i \end{aligned}$$

□

(4.7) Lemma. Let φ be an automorphism of S such that there exists a permutation σ of I satisfying:

$$\overline{\varphi}(E^j + \mathcal{J}) = E^{\sigma(j)} + \mathcal{J}, \text{ for all } j \in J.$$

Then, there exists an invertible element \mathcal{U} of S such that $\overline{C_{\mathcal{U}}} = 1$ and

$$(C_{\mathcal{U}} \circ \varphi)(E^j) = E^{\sigma(j)}, \text{ for all } j \in J.$$

Proof. We have that

$$\varphi(E^1) = E^{\sigma(1)} + \eta_1, \text{ where } \eta_1 \in \mathcal{J}.$$

Then $E^{\sigma(1)} + \eta_1$ is an idempotent of the type described in Lemma (4.2).

By Corollary (4.4), there exists $\theta_{\sigma(1)} \in \mathcal{J}$ such that

$$\mathcal{E}_{\sigma(1)} = E^{\sigma(1)} + \eta_1 = E^{\sigma(1)} + E^{\sigma(1)}\theta_{\sigma(1)} + \theta_{\sigma(1)}E^{\sigma(1)} + \theta_{\sigma(1)}E^{\sigma(1)}\theta_{\sigma(1)}.$$

Consider now the element $\mathcal{U}_{\sigma(1)}$, given by Lemma (4.5).

We have:

$$(C_{\mathcal{U}_{\sigma(1)}} \circ \varphi)(E^1) = C_{\mathcal{U}_{\sigma(1)}}(\mathcal{E}_{\sigma(1)}) = E^{\sigma(1)},$$

$$\overline{C_{\mathcal{U}_{\sigma(1)}} \circ \varphi} = \bar{\varphi},$$

due to Lemmas (4.5) and (4.6)(i).

Then,

$$\left(\overline{C_{\mathcal{U}_{\sigma(1)}} \circ \varphi}\right)(E^j + \mathcal{J}) = E^{\sigma(j)} + \mathcal{J}, \quad \forall j \in I.$$

Similarly,

$$(C_{\mathcal{U}_{\sigma(1)}} \circ \varphi)(E^2) = E^{\sigma(2)} + \eta_2, \text{ where } \eta_2 \in \mathcal{J}.$$

We now consider the element $\theta_{\sigma(2)} \in \mathcal{J}$, the idempotent $\mathcal{E}_{\sigma(2)} = E^{\sigma(2)} + \eta_2 = E^{\sigma(2)} + E^{\sigma(2)}\theta_{\sigma(2)} + \theta_{\sigma(2)}E^{\sigma(2)} + \theta_{\sigma(2)}E^{\sigma(2)}\theta_{\sigma(2)}$ and the element $\mathcal{U}_{\sigma(2)}$, given by Corollary (4.4) and Lemma (4.5) respectively.

We have that

$$E^{\sigma(2)}\mathcal{E}_{\sigma(2)} = (C_{\mathcal{U}_{\sigma(1)}} \circ \varphi)(E^1E^2) = 0,$$

$$\mathcal{E}_{\sigma(2)}E^{\sigma(2)} = (C_{\mathcal{U}_{\sigma(1)}} \circ \varphi)(E^2E^1) = 0.$$

By Lemma (4.6)(ii), we have that $C_{U_{\sigma(2)}}(E^{\sigma(1)}) = E^{\sigma(1)}$. Therefore, $(C_{U_{\sigma(2)}} \circ C_{U_{\sigma(1)}} \circ \varphi)(E^1) = E^{\sigma(1)}$. Furthermore,

$$\begin{aligned} (C_{U_{\sigma(2)}} \circ C_{U_{\sigma(1)}} \circ \varphi)(E^2) &= E^{\sigma(2)}, \\ \overline{C_{U_{\sigma(2)}} \circ C_{U_{\sigma(1)}} \circ \varphi} &= \bar{\varphi}. \end{aligned}$$

That is,

$$\overline{(C_{U_{\sigma(2)}} \circ C_{U_{\sigma(1)}} \circ \varphi)}(E^j + \mathcal{J}) = E^{\sigma(j)} + \mathcal{J}, \quad \forall j \in I.$$

Proceeding in this way, we get that the element $\mathcal{U} = \mathcal{U}_{\sigma(n)}\mathcal{U}_{\sigma(n-1)} \cdots \mathcal{U}_{\sigma(1)}$ verifies the thesis. \square

Now we shall deal with the subgroup \mathcal{G} .

Before stating the next lemma, we must fix some notation.

We point out that if C, C' are equivalence classes of $\bar{\rho}$ and $(x, y) \in \rho$, with $x \in C$ and $y \in C'$, then $C \times C' \subset \rho$.

Among the equivalence classes of $\bar{\rho}$, we may have some classes C with the following property: if C' is an equivalence class such that either $C' \times C \subset \rho$ or $C \times C' \subset \rho$, then $C' = C$.

Let C_1, C_2, \dots, C_q be such classes (if they do exist) and fix an element x_i in each C_i .

Choose also an element $y_\ell \in I$ such that y_ℓ is a vertex of T_ℓ , for each tree T_ℓ .

In regard to the classes C_i and the trees T_ℓ , we remark that:

- The sets of vertices of the trees T_ℓ are mutually disjoint.

- None of the classes C_i , $1 \leq i \leq q$, intercept these sets of vertices.
- The union of the set of vertices of the graph $\cup_{\ell} T_{\ell}$ with $\cup_{i=1}^q C_i$ is the set I .

We are ready now to state our next lemma.

(4.8) Lemma. Let $g : \rho \rightarrow K^*$ be a trivial function such that $g(j, k) = 1$, for all $(j, k) \in \bar{\rho}$ and let x_i , $1 \leq i \leq q$, and y_{ℓ} be the elements above. Then, there exists a map $s : I \rightarrow K^*$ such that

- (i) $s(x_i) = 1$, $1 \leq i \leq q$, and $s(y_{\ell}) = 1$, for all ℓ ;
- (ii) $g(j, k) = s(j)s(k)^{-1}$, for all $(j, k) \in \rho$.

Proof. There exists a map $s_1 : I \rightarrow K^*$ such that

$$g(j, k) = s_1(j)s_1(k)^{-1}, \text{ for all } (j, k) \in \rho,$$

since g is trivial.

Pick $j \in I$. Due to the remarks above, we have that either $j \in C_i$, for some i , which is uniquely determined, or there exists ℓ , also uniquely determined, such that j is a vertex of T_{ℓ} .

In the first case, we define $s(j) = s_1(j)s_1(x_i)^{-1}$, and in the second one, we set $s(j) = s_1(j)s_1(y_{\ell})^{-1}$. Hence, s satisfies condition (i).

To prove (ii), pick first $(j, k) \in \bar{\rho}$. Then, if $j \in C_i$, for some i , we must have that $k \in C_i$, and

$$s(j)s(k)^{-1} = s_1(j)s_1(x_i)^{-1}s_1(k)^{-1}s_1(x_i) = s_1(j)s_1(k)^{-1} = g(j, k).$$

Suppose now that $j \notin C_i$, for any i .

Then, on the one hand, we have that $j, k \in C$, for some equivalence class C of $\bar{\rho}$. On the other hand, there exists an equivalence class $C' \neq C$ such that either $C \times C' \subset \rho \setminus \bar{\rho}$ or $C' \times C \subset \rho \setminus \bar{\rho}$.

Then, it is easy to conclude that j and k are vertices of the same tree T_ℓ . Therefore,

$$s(j)s(k)^{-1} = s_1(j)s_1(y_\ell)^{-1}s_1(k)^{-1}s_1(y_\ell) = s_1(j)s_1(k)^{-1} = g(j, k).$$

Similarly, if $(j, k) \in \rho \setminus \bar{\rho}$, we have that j and k are vertices of the same tree T_ℓ , and the equation above shows that $g(j, k) = s(j)s(k)^{-1}$. \square

We remark that it can be proved that the function s in the conditions of Lemma (4.8) is uniquely determined.

We now observe that the set

$$G = \{g : \rho \rightarrow K^* \mid g \text{ is a transitive function}\}$$

is an abelian group, with respect to the pointwise multiplication of functions.

Furthermore, the subsets

$$D = \{g \in G \mid g \text{ is trivial}\},$$

$$F = \{g \in G \mid g(j, k) = 1, \text{ for all } (j, k) \in \bar{\rho} \cup \prod_{i=1}^q C_i \times C_i\}$$

are easily seen to be subgroups of G .

(4.9) Lemma. With the notations above, we have that

$$G = D \times F \text{ (direct product).}$$

Proof. First, consider $g \in D \cap F$. We claim that $g(j, k) = 1$, for all $(j, k) \in \bar{\rho}$.

In fact, given $(j, k) \in \bar{\rho}$ such that $(j, k) \notin \prod C_i \times C_i$, there exist equivalence classes C and C' of $\bar{\rho}$, $C \neq C'$, satisfying: $j, k \in C$ and either $C \times C' \subset \rho$ or $C' \times C \subset \rho$. Hence, j, k are vertices of the same tree T_ℓ .

Furthermore, there exists a function $s_1 : I \rightarrow K^*$ such that $g(x, y) = s_1(x)s_1(y)^{-1}$, for all $(x, y) \in \rho$. Since $g \in F$, an easy argument shows that $s_1(j) = s_1(k) = s_1(y_\ell)$, where y_ℓ is the fixed vertex of T_ℓ .

That is, $g(j, k) = s_1(j)s_1(k)^{-1} = 1$. The claim is proved.

Given the elements x_i and y_ℓ of Lemma (4.8), we have that there exists a map $s : I \rightarrow K^*$ such that s satisfies condition (i) and $g(j, k) = s(j)s(k)^{-1}$, for all $(j, k) \in \rho$.

Take $j \in I$ such that j is vertex of a (unique) tree T_ℓ . An easy argument shows that $s(j) = 1$, since $s(y_\ell) = 1$ and $g(j, k) = s(j)s(k)^{-1}$, for all $(j, k) \in \bar{\rho}$.

If $j \in C_i$, for some i , then $(x_i, j) \in \bar{\rho}$. Since $g(j, k) = 1$, for all $(j, k) \in \bar{\rho}$, we have that $s(j) = s(x_i) = 1$.

Hence, $g(j, k) = s(j)s(k)^{-1} = 1$, for all $(j, k) \in \rho$. Thus, $D \cap F = 1$.

Now, choose an element $g \in G$, and consider the restriction of g to $\bar{\rho}$. We claim that there exists $s : I \rightarrow K^*$ such that

$$g(j, k) = s(j)s(k)^{-1},$$

for all $(j, k) \in \bar{\rho}$.

In fact, if $j \in C_i$, for some i , we have that $(x_i, j) \in \rho$ and we define:

$s(j) = g(x_i, j)^{-1}$. On the other hand, if j is a vertex of a (unique) tree T_ℓ , consider the unique path $z_0 z_1 \cdots z_m$ connecting y_ℓ to j (that is, $z_0 = y_\ell$ and $z_m = j$).

In order to define $s(j)$, we proceed by induction on m .

If $m = 0$, then $j = y_\ell$ and we define $s(j) = 1$. Suppose now that $s(z_0), s(z_1), \dots, s(z_{m-1})$ are defined. As usually, either $(z_{m-1}, z_m) \in \rho$ or $(z_m, z_{m-1}) \in \rho$.

In the first case, we define

$$s(j) = s(z_m) = g(z_{m-1}, z_m)^{-1} s(z_{m-1}),$$

while in the second case, we set

$$s(j) = s(z_m) = g(z_m, z_{m-1}) s(z_{m-1}).$$

Then, s is as required.

In order to see this, we observe that if $(j, k) \in \bar{\rho}$, then j and k are vertices of the same tree T_ℓ and there exists a path joining y_ℓ to k .

Suppose that $w_0 w_1 \cdots w_t$ is such a path, with $w_0 = y_\ell$ and $w_t = k$. Due to the uniqueness of the paths joining vertices of T_ℓ , we conclude that either $w_{t-1} = j$ or $w_0 w_1 \cdots w_t w_{t+1}$, with $w_{t+1} = j$, is the unique path connecting y_ℓ to j .

In both cases, the definition of s shows that $g(j, k) = s(j)s(k)^{-1}$ for all $(j, k) \in \bar{\rho}$.

Now, let $\tilde{s}(j, k) = s(j)s(k)^{-1}$, for all $(j, k) \in \rho$. Then, $\tilde{s} \in D$.

Furthermore, if $(j, k) \in C_i \times C_i$, for some i , we have that:

$$(\tilde{s}^{-1}g)(j, k) = g(x_i, j)g(x_i, k)^{-1}g(j, k) = g(x_i, j)g(k, x_i)g(j, k) =$$

$$= g(k, j)g(j, k) = 1 .$$

Therefore, we conclude that:

$$(\tilde{s}^{-1}g)(j, k) = 1, \text{ for all } (j, k) \in \bar{\rho} \cup \prod C_i \times C_i .$$

Hence, $\tilde{s}^{-1}g = f \in F$, or $g = \tilde{s}f$, and the proof is complete. \square

(4.10) Lemma. Let $g : \rho \rightarrow K^*$ be a transitive mapping. Then, $g^* = C_A$ for some $A \in S$ if and only if g is trivial.

Proof. Suppose that $g^* = C_A$, for some $A \in S$. Then,

$$AE^{ij}A^{-1} = g(i, j)E^{ij}, \text{ i.e.,}$$

$$AE^{ij} = g(i, j)E^{ij}A,$$

for all $(i, j) \in \rho$.

It is now easy to see that A is a diagonal matrix and hence $A_{kk} \neq 0$, for all $k \in I$.

The (i, j) -entries of both sides of the equation above give us: $A_{ii} = g(i, j)A_{jj}$.

Hence, $g(i, j) = A_{ii}A_{jj}^{-1}$, and it is enough to define $s(i) = A_{ii}$, for all $i \in I$.

Suppose now that $g(i, j) = s(i)s(j)^{-1}$, for all $(i, j) \in \rho$.

Then, if $A = \sum_i s(i)E^i$, it is easy to see that

$$AE^{ij}A^{-1} = g^*(E^{ij}),$$

as we wished to prove. \square

Proof of Theorem C. Let ϕ be an automorphism of S and $\bar{\phi}$ be the automorphism induced on $\frac{S}{\mathcal{J}}$.

As we saw in the proof of Proposition (4.1), we have that $\frac{S}{\mathcal{J}} \cong S'$, where

$$S' = \{A \in M_n(K) \mid A_{ij} = 0 \text{ if } (i, j) \notin \bar{\rho}\}.$$

Furthermore, an isomorphism $f : S' \rightarrow \frac{S}{\mathcal{J}}$ is given by:

$$f(A) = A + \mathcal{J}, \text{ for all } A \in S'.$$

Due to Theorem A applied to the semisimple structural matrix algebra S' and the definition of f , we have that

$$\bar{\phi} = C_{A+\mathcal{J}} \circ \hat{\sigma},$$

where $A \in S' \subset S$ is an invertible matrix and σ is an automorphism of $\bar{\rho}$ such that

$$\hat{\sigma}(E^{ij} + \mathcal{J}) = E^{\sigma(i)\sigma(j)} + \mathcal{J}, \text{ for all } (i, j) \in \bar{\rho}.$$

Furthermore, σ verifies the condition: $\sigma(x) < \sigma(y)$ whenever $(x, y) \in \bar{\rho}$ and $x < y$.

Then,

$$\overline{C_{A^{-1}} \circ \bar{\phi}} = C_{A^{-1}+\mathcal{J}} \circ \bar{\phi} = C_{A^{-1}+\mathcal{J}} \circ C_{A+\mathcal{J}} \circ \hat{\sigma} = \hat{\sigma}.$$

Now, due to the definition of $\hat{\sigma}$, we have that:

$$\overline{C_{A^{-1}} \circ \bar{\phi}}(E^j + \mathcal{J}) = \hat{\sigma}(E^j + \mathcal{J}) = E^{\sigma(j)} + \mathcal{J}, \text{ for all } j \in I.$$

Hence, the automorphism $\varphi = C_{A^{-1}} \circ \phi$ satisfies the hypothesis of Lemma (4.7). So, there exists an invertible element $U \in S$ such that

$$(C_U \circ \varphi)(E^j) = E^{\sigma(j)}, \text{ for all } j \in I,$$

$$\overline{C_U} = 1 .$$

We set $\Psi = C_U \circ \varphi = C_{U_{A^{-1}}} \circ \phi$. Then,

$$\begin{aligned} \Psi(E^{ij}) &= \Psi(E^i E^{ij} E^j) = \Psi(E^i) \Psi(E^{ij}) \Psi(E^j) = \\ &= E^{\sigma(i)} \Psi(E^{ij}) E^{\sigma(j)} = a_{\sigma(i)} a_{\sigma(j)} E^{\sigma(i)\sigma(j)} , \end{aligned}$$

where $a_{\sigma(i)\sigma(j)} \in K^*$.

Then, $(\sigma(i), \sigma(j)) \in \rho$, for all $(i, j) \in \rho$; hence σ is actually an automorphism of ρ , and $\hat{\sigma} \in \mathcal{P}$.

Now, let $g : \rho \rightarrow K^*$ be the transitive function:

$$g(\sigma(i), \sigma(j)) = a_{\sigma(i)\sigma(j)}, \text{ for all } (i, j) \in \rho .$$

We have that:

$$\begin{aligned} \Psi(E^{ij}) &= g^*(E^{\sigma(i)\sigma(j)}) , \text{ i.e.,} \\ \Psi(E^{ij}) &= (g^* \circ \hat{\sigma})(E^{ij}), \text{ for all } (i, j) \in \rho . \end{aligned}$$

Hence $\Psi = g^* \circ \hat{\sigma}$, where $\hat{\sigma} \in \mathcal{P}$. Or:

$$\begin{aligned} C_{U_{A^{-1}}} \circ \phi &= g^* \circ \hat{\sigma}, \text{ i.e.,} \\ \phi &= C_{U_{A^{-1}}} \circ g^* \circ \hat{\sigma} . \end{aligned}$$

But, by Lemma (4.9), we have that $g = d f$, where $d \in D$ and $f \in F$.

Hence, $g^* = d^* \circ f^*$, where $f^* \in \mathcal{G}$ and $d^* = C_{A'}$ for some $A' \in S$, due to Lemma (4.10).

Then,

$$\phi = C_{U_{A^{-1}A'}} \circ f^* \circ \hat{\sigma} ,$$

where $C_{\mathcal{M}^{-1}\mathcal{A}} \in \mathcal{C}$, $f^* \in \mathcal{G}$ and $\hat{\sigma} \in \mathcal{P}$. Thus, ϕ has the required form.

In order to finish the proof, we must show that:

(a) $\mathcal{C} \cap \mathcal{G} = 1$, $g^* \circ C_{\mathcal{A}} \circ (g^*)^{-1} \in \mathcal{C}$, for all $g^* \in \mathcal{G}$ and $C_{\mathcal{A}} \in \mathcal{C}$.

(b) $(\mathcal{C} \rtimes \mathcal{G}) \cap \mathcal{P} = 1$, $\hat{\sigma} \circ \phi \circ (\hat{\sigma})^{-1} \in \mathcal{C} \rtimes \mathcal{G}$, for all $\hat{\sigma} \in \mathcal{P}$ and $\phi \in \mathcal{C} \rtimes \mathcal{G}$.

We start with claim (a). Clearly, \mathcal{C} is normal in $\text{Aut}(S)$. So, pick $g^* \in \mathcal{G}$ such that g^* is conjugation by a matrix of S . By Lemma (4.10), g is trivial. Hence, $g \in D \cap F$. By Lemma (4.9), we obtain $g = 1$, i.e., $g^* = 1$.

To prove (b), we observe that it is enough to show that $\hat{\sigma} \circ g^* \circ (\hat{\sigma})^{-1} \in \mathcal{C} \rtimes \mathcal{G}$, for all $g^* \in \mathcal{G}$.

We have that

$$(\hat{\sigma} \circ g^*)(E^{jk}) = \hat{\sigma}(g(j, k)E^{jk}) = g(j, k)E^{\sigma(j)\sigma(k)}, \text{ for all } (j, k) \in \rho.$$

Let $h : \rho \rightarrow K^*$ be such that $h(\sigma(j), \sigma(k)) = g(j, k)$ for all $(j, k) \in \rho$.

Trivially, h is a transitive function. Therefore, $h \in G$.

By Lemma (4.9), $h = df$, where $d \in D$ and $f \in F$.

Back on the equation above, we obtain:

$$(\hat{\sigma} \circ g^*)(E^{jk}) = (h^* \circ \hat{\sigma})(E^{jk}) = (d^* \circ f^* \circ \hat{\sigma})(E^{jk}) \text{ for all } (j, k) \in \rho.$$

That is,

$$\hat{\sigma} \circ g^* \circ (\hat{\sigma})^{-1} = d^* \circ f^*.$$

But, by Lemma (4.10), we have that $d^* \in \mathcal{C}$. Therefore, $d^* \circ f^* \in \mathcal{C} \rtimes \mathcal{G}$, as we wanted to prove.

Finally, pick $\hat{\sigma} \in (\mathcal{C} \rtimes \mathcal{G}) \cap \mathcal{P}$. We have that $\hat{\sigma} = C_{\mathcal{A}} \circ g^*$, where $C_{\mathcal{A}} \in \mathcal{C}$ and $g^* \in \mathcal{G}$. Then:

$$E^{\sigma(j)} = \hat{\sigma}(E^j) = (C_{\mathcal{A}} \circ g^*)(E^j) = C_{\mathcal{A}}(E^j), \text{ for all } j \in I.$$

Considering this equation in $\frac{S}{\mathcal{J}}$, we get that:

$$E^{\sigma(j)} + \mathcal{J} = C_{A+\mathcal{J}}(E^j + \mathcal{J}).$$

Now, we recall that $\frac{S}{\mathcal{J}} \cong S'$, and from the definition of the isomorphism, we get that

$$E^{\sigma(j)} = C_{A'}(E^j),$$

where $A' \in S'$ is such that $A' + \mathcal{J} = A + \mathcal{J}$.

But conjugation by A' fixes the simple components of S' . By Lemma (3.6), we conclude that $\sigma(j)$ belongs to the equivalence class of j defined by the relation $\bar{\rho}$. From the definition of \mathcal{P} , we get that $\sigma = 1$.

The proof of Theorem C is complete. \square

Proof of Theorem D. Suppose first that conditions (i) and (ii) hold. Then Lemmas (4.9) and (4.10) and the definition of \mathcal{P} imply respectively that $\mathcal{G} = 1$ and $\mathcal{P} = 1$. Hence, by Theorem C, every automorphism of S is inner.

Now, assume that every automorphism of S is inner, and consider a transitive function $g : \rho \rightarrow K^*$. By hypothesis g^* is inner. Hence, by Lemma (4.10), g is trivial.

For (ii), pick an automorphism σ of ρ . Since $\hat{\sigma}$ is inner, computing in $\frac{S}{\mathcal{J}}$, we have:

$$E^{\sigma(j)} + \mathcal{J} = \bar{\sigma}(E^j + \mathcal{J}) = C_{A+\mathcal{J}}(E^j + \mathcal{J}).$$

Now, $\frac{S}{\mathcal{J}} \cong S'$, where

$$S' = \{A \in M_n(K) \mid A_{ij} = 0 \text{ if } (i,j) \notin \bar{\rho}\},$$

as we saw in Proposition (4.1). So, $E^{\sigma(j)} = C_{A'}(E^j)$, where $A' \in S$ is such that $A' + \mathcal{J} = A + \mathcal{J}$.

Since S' is semisimple and $C_{A'}$ fixes the simple components of S' , we get from Lemma (3.6) that $\sigma(j)$ belongs to the equivalence class of j defined by the relation $\bar{\rho}$. This completes the proof. \square

We can now obtain the results from [3] and [6] in the case where the ring of coefficients is a field.

(4.11) Corollary. Every K -automorphism of the algebra of upper triangular matrices with entries on K is inner. \bullet

Proof. We observe first that this is a structural matrix algebra defined by the relation

$$\rho = \{(i, j) \mid i, j \in I \text{ and } i \leq j\}.$$

Hence, we must show that conditions (i) and (ii) of Theorem D hold.

To prove (i), pick a transitive function $g : \rho \rightarrow K^*$.

We define:

$$s(1) = 1, s(2) = g(1, 2)^{-1}, \dots, s(n) = g(1, n)^{-1}.$$

We have that $g(1, i)g(i, j) = g(1, j)$, that is, $g(i, j) = s(i)s(j)^{-1}$. Hence, g is trivial.

For (ii), pick an automorphism σ of ρ . Since the equivalence classes of $\bar{\rho}$ are singleton, we must show that $\sigma = 1$.

We observe first that the relation ρ can be written as: $\rho = \bigcup_{i=1}^n \rho_i$, where $\rho_i = \{(i, i), (i, i+1), \dots, (i, n)\}$, $1 \leq i \leq n$.

Then, the image of ρ_1 by σ is: $\{(\sigma(1), \sigma(1)), (\sigma(1), \sigma(2)), \dots, (\sigma(1), \sigma(n))\}$.

Now, this set has n elements and is contained in ρ_i , for some i . But $|\rho_i| = n - i + 1$, so $i = 1$. Therefore, $\sigma(1) = 1$.

Proceeding in the same way with ρ_2 , we get that $\sigma(2) = 2$ and, similarly, $\sigma(i) = i$, for all i . Hence, $\sigma = 1$. \square

5. Examples

We now show that conditions (i) and (ii) of Theorem D are mutually independent.

Example 1. Let K be an arbitrary field. We exhibit a relation ρ where every transitive mapping $g : \rho \rightarrow K^*$ is trivial, but not all automorphisms of ρ fix the equivalence classes of $\bar{\rho}$.

Set $\rho = \{(1, 1), (1, 2), (2, 2), (3, 2), (3, 3)\}$; so $\bar{\rho} = \{(1, 1), (2, 2), (3, 3)\}$.

Now, the permutation $\sigma = (1\ 3)$ is easily seen to be an automorphism of ρ , but $\sigma(1) \neq 1$.

Let now $g : \rho \rightarrow K^*$ be a transitive function.

We observe that the graph Δ associated to $\rho \setminus \bar{\rho}$ is:



Hence, it is connected and coincides with a tree containing all its vertices.

Therefore, with the notations of Lemma (4.9), we have that $G = D$, $F = 1$. That is, every transitive function is trivial.

We point out that, for the corresponding structural matrix algebra, we have that $\mathcal{G} = 1$ and $\mathcal{P} = \{\hat{1}, \hat{\sigma}\}$. \square

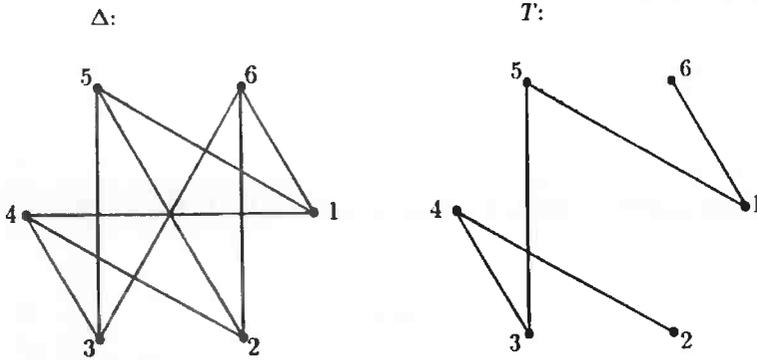
Example 2. We shall now show that we can find a field K and a relation ρ where the automorphisms fix the equivalence classes of $\bar{\rho}$, but not all transitive functions $g : \rho \rightarrow K^*$ are trivial.

Set

$$\rho = \{(1, 1), (1, 4), (1, 5), (1, 6), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6), (4, 4), (5, 5), (5, 6), (6, 5), (6, 6)\},$$

and let K be a field such that $|K| > 2$.

For such a ρ , we have that the graph Δ and a tree T which contains all its vertices are given by:



Furthermore, we have that $\bar{\rho} = \{(1, 5), (1, 6), (2, 4), (3, 4), (3, 5)\}$ and $\bar{\bar{\rho}} = \emptyset$.

Then, the function $g : \rho \rightarrow K^*$ such that $g(1,4) = \alpha \in K^*$, $\alpha \neq 1$, $g(i,j) = 1$, for all $(i,j) \neq (1,4)$, is easily seen to be a non-trivial transitive function.

We now observe that the equivalence classes of $\bar{\rho}$ are: $\{1\}$, $\{2,3\}$, $\{4\}$, $\{5,6\}$.

Let σ be an automorphism of ρ . Since equivalence classes go to equivalence classes under σ , we have that either $\sigma(1) = 1$ or $\sigma(1) = 4$.

But, if $\sigma(1) = 4$, we must have that $\sigma(4) = 1$, and $(\sigma(1), \sigma(4)) = (4,1)$, which does not belong to ρ . Then, $\sigma(1) = 1$ and therefore $\sigma(4) = 4$.

Similarly, either $\sigma(\{2,3\}) = \{2,3\}$ or $\sigma(\{2,3\}) = \{5,6\}$. If $\sigma(2) = 5$, we have that $(\sigma(2), \sigma(4)) = (5,4) \notin \rho$, a contradiction. Hence, $\sigma(\{2,3\}) = \{2,3\}$ and, consequently, $\sigma(\{5,6\}) = \{5,6\}$.

An easy calculation shows that for the corresponding structural matrix algebra, we have that $\mathcal{G} \cong K^*$ and $\mathcal{P} = 1$. □

Acknowledgment

The author is grateful to her colleagues Nair A. Fernandes, Maria Lúcia S. Singer and César Polcino Milies for introducing her to the subject.

References

1. G.P. BARKER and T.P. KEZLAN, Automorphism of algebras of upper triangular matrices, *Arch. Math.* **55** (1990), 38-43.
2. G.P. BARKER, Automorphism groups of algebras of triangular matrices, *Linear Algebra Appl.* **121** (1989), 207-215 .

3. G.P. BARKER and T.P. KEZLAN, The automorphism group of a matrix algebra in "Current Trends in Matrix Theory", pp. 33-39, Proceedings Auburn, 1986.
4. B. BOLLOBÁS, "Graph theory - An introductory course", Springer-Verlag, New York - Berlin - Heidelberg - Tokyo, 1979.
5. I.M. ISAACS, Automorphisms of matrix algebras over commutative rings, *Linear Algebra Appl.* **31** (1980), 215-231.
6. S. JØNDRUP, Automorphisms of upper triangular matrix rings, *Arch. Math.* **49** (1987), 497-502.
7. S. JØNDRUP, The group of automorphisms of certain subalgebras of matrix algebras, *J. Algebra* **141** (1991), 106-114.
8. A. NOWICKI, Derivations of special subrings of matrix rings and regular graphs, *Tsukuba J. Math.* **7** n^o 2 (1983), 289-297.
9. A.D. SANDS, Radicals of structural matrix rings, *Quaestiones Mathematicae* **13** (1990), 77-81.
10. L. VAN WYK, Maximal left ideals in structural matrix rings, *Comm. Alg.* **16** (1988), 399-419.

TRABALHOS DO DEPARTAMENTO DE MATEMÁTICA

TÍTULOS PUBLICADOS

- 91-01 COELHO S.P. & POLCINO MILIES, C. Derivations of Upper Triangular Matrix Rings. 7p.
- 91-02 JESPERS E., LEAL G. & POLCINO MILIES, C. Idempotents in Rational Abelian Groups Algebras. 10p.
- 91-03 FALBEL E. Non-embeddable CR-structures and dilations. 7p.
- 91-04 JESPERS E., LEAL G. & POLCINO MILIES, C. Indecomposable R. A. Loops. 36p.
- 91-05 COELHO F., HAPPEL D. & UNGER L. Complements to partial tilting modules. 21p.
- 91-06 FALBEL, E. The topology of envelopes of holomorphy and Hartogs Theorem. 9p.
- 92-01 COELHO, S.P. The automorphism group of a structural matrix algebra. 33p.

NOTA: Os títulos publicados dos Relatórios Técnicos dos anos anteriores (1980 a 1990) estão à disposição no Departamento de Matemática do Instituto de Matemática e Estatística - USP.

Cidade Universitária "Armando de Salles Oliveira"
Rua do Matão, 1010 - Butantã
Caixa Postal - 20.570 (Ag. Iguatemi)
CEP - 01498 - São Paulo - Brasil