Elementary Properties of the Boolean Hull Functor

M. A. Dickmann F. Miraglia

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1 Introduction

In [DM3] we proved the following isotropy-reflection principle:

Theorem. *Let F be a formally real field and let F^p denote its Pythagorean closure. The natural embedding of reduced special groups from $G_{red}(F)$ into $G_{red}(F^p) = G(F^p)$ induced by the inclusion of fields, reflects isotropy.* □

Here $G_{red}(F)$ denotes the reduced special group (with underlying group $\dot{F}/\Sigma\dot{F}^2$) associated to the field F, henceforth assumed formally real; cf. [DM1], Ch. 1, §3, for details.

The result proved in [DM3] is, in fact, more general. For example, the Pythagorean closure F^p can be replaced in the statement above by the so-called order-closure of F, i.e., the smallest algebraic extension (inside a fixed algebraic closure) of F to which every order extends uniquely. Similar statements hold for even more general relative Pythagorean closures (in the sense of Becker [B], Ch. 3).

Since the notion of isotropy of a quadratic form can be expressed by a first-order formula in the natural language L_{SG} for special groups (with the coefficients as parameters), this result raises the question whether the embedding ι_{FF^p}: $G_{red}(F) \hookrightarrow G(F^p)$ is elementary. Further, since the L_{SG}-formula expressing isotropy is positive-existential, one may also ask whether ι_{FF^p} reflects all (closed) formulas of that kind with parameters in $G_{red}(F)$.

In this paper we give a negative answer to the first of these questions, for a vast class of formally real (non-Pythagorean) fields F (Prop. 5.1). Concerning the second question the situation is more nuanced; we shall deal with it in another paper.

The result just stated follows from two, rather general preservation results concerning the "Boolean hull" and the "reduced quotient" operations on special groups.

In Chapter 4, section 2 of [DM1] we introduced the Boolean hull operation which to every reduced special group (rsg) G associates a Boolean algebra (BA), its Boolean hull B_G. The correspondence $G \hookrightarrow B_G$ is functorial. In §2 below we show that this functor preserves elementary equivalence and elementary SG-morphisms (Proposition 3.2). The technique to prove this consists in, first, showing that the Boolean dual $B(\delta_G)$ of the diagonal embedding $\delta_G : G \longrightarrow G^I/\mathfrak{U}$ of the rsg G into any ultrapower is an elementary morphism of Boolean algebras (Proposition 3.1); then, the results announced above follow by use of the Keisler-Shelah ultrapower theorem.

Similar results hold with "elementary equivalence" replaced by "positive-existential equivalence" and "elementary morphism" by "pure morphism" (i.e., a morphism reflecting positive-existential sentences with parameters). However, these results are, in a sense, trivial. Indeed, in §1 we prove the seemingly unknown (at least, previously unknown to us), though relatively straightforward fact that any injective (SG–, or Boolean) morphism between BA's is pure.

In §3 we shall consider the operation that to each formally real special group (frsg) G associates its reduced quotient $G_{red} = G/Sat(G)$, where $Sat(G) = \bigcup_{n \geq 1} D_G(n\langle 1 \rangle)$ is the smallest saturated subgroup of G; for details on this notion see Chapter 2 of [DM1], especially Definition 2.7. This operation is functorial, and we shall prove that it preserves purity of SG-morphisms (Proposition 4.1(1)) and positive-existential equivalence (Corollary 4.3(b)); under the additional assumption that the underlying groups have finite Pythagoras number [1], this functor preserves elementary morphisms (Proposition 4.2(b)) and elementary equivalence (Corollary 4.3.(c)). The technique of proof for this case is similar to that of the Boolean hull functor.

Another by-product of these results is that the first-order theory of formally real fields of fixed, finite Pythagoras number having at least n orders ($n \in \mathbb{N} \cup \{\infty\}$, $n \geq 1$) is axiomatizable in the first-order language of fields; we do not give an explicit set of axioms for this class, when $n \geq 3$.

1.1 Notation and Preliminaries. (I) Concerning special groups, spaces of orderings and quadratic forms, we shall adhere to the notation used in the monography [DM1]. The notions and results therein will also be used systematically.

Let G be a frsg, I a set and $\mathfrak{U}$ an ultrafilter on I. Elements of the ultrapower $G^I/\mathfrak{U}$ will be written $\langle g_i : i \in I \rangle/\mathfrak{U}$.

(II) Observe that any SG-morphism $f : G \longrightarrow H$ extends to ultrapowers, as follows : For $\langle g_i : i \in I \rangle/\mathfrak{U}$,

(**) $$\widehat{f}(\langle g_i : i \in I \rangle/\mathfrak{U}) = \langle f(g_i) : i \in I \rangle/\mathfrak{U}.$$

Routine verification shows that $\widehat{f}$ is well-defined and a SG-morphism. Fur-

[1] The *Pythagoras number* of a special group G, $p(G)$, is the least $n \in \mathbb{N}$ such that $Sat(G) = D_G(n\langle 1 \rangle)$, if it exists; otherwise, $p(G) = \infty$.

ther, a routine application of Łoś' Theorem yields

Lemma 1.2 *The map $f \mapsto \widehat{f}$ preserves injectivity, completeness, purity and elementarity.* □

(III) a) Since I, $\mathfrak{U}$ remain fixed throughout the proofs, we may safely omit them to ease notation; thus, we set $\widehat{G} = G^I/\mathfrak{U}$. We write $\delta_G : G \longrightarrow \widehat{G}$ for the diagonal embedding which sends $g \in G$ to the class $\langle g \rangle/\mathfrak{U}$ of the constant I-sequence with value g. δ_G is an elementary embedding (Łoś' Theorem.)

b) In particular, for the Boolean hull B_G of a rsg G, we have $\widehat{B_G} = {B_G}^I/\mathfrak{U}$, to be distinguished from the (much smaller) BA $B_{\widehat{G}}$.

(IV) a) Let us right away note that the "reduced quotient" operation on frsg's is well-defined. Since any SG-morphism $f : G \longrightarrow H$ preserves isometry of and representation by forms of arbitrary dimension, it follows that

$$f(Sat(G)) \subseteq Sat(H).$$

Thus, if G, H are frsg's, the rule

$$f_{red}(a/Sat(G)) = f(a)/Sat(H) \qquad (a \in G)$$

defines a SG-morphism $f_{red} : G_{red} \longrightarrow H_{red}$ (preservation of binary isometry follows from Definition 2.13 and Proposition 2.28 in [DM1]). It is clear that the correspondence

$$G \longmapsto G_{red}; \quad f \longmapsto f_{red}$$

is a functor from the category of **FRSG** of formally real special groups to the category of **RSG** of reduced special groups, which we call the "reduced quotient" functor. Clearly, this functor is the identity on the sub-category **RSG** of **FRSG**.

b) Write $\pi_G : G \longrightarrow G_{red}$ for the canonical quotient map. For a SG-morphism $f : G \longrightarrow H$ the functional equation defining f_{red} is :

(+) $$f_{red} \circ \pi_G = \pi_H \circ f.$$

c) In the case of the reduced quotient G_{red} we shall use the traditional notation $G^I_{red}/\mathfrak{U}$ —instead of $\widehat{G_{red}}$— in order to avoid typographical confusion with $\widehat{G}_{red}$ (i.e., $(\widehat{G})_{red}$), which will also be in use. Likewise, we employ δ^*_G instead of $(\delta_G)_{red}$, to avoid confusion with the diagonal embedding $\delta_{G_{red}}$: $G_{red} \longrightarrow (G_{red})^I/\mathfrak{U}$, which will also be used.

(V) For structures $\mathfrak{A}$, $\mathfrak{B}$, with language L, an L-morphism $f : \mathfrak{A} \longrightarrow \mathfrak{B}$ is **pure** if for every positive-existential formula $\varphi(v_1, \ldots , v_n)$ and all $a_1, \ldots , a_n \in \mathfrak{A}$,

(*) $$\mathfrak{B} \models \varphi[f(a_1), \ldots , f(a_n)] \quad \Rightarrow \quad \mathfrak{A} \models \varphi[a_1, \ldots , a_n].$$

The converse is always true, since f is an L-morphism. As noted in [DM1], Ch. 5, §3, to establish purity it suffices to check condition (*) on positive-primitive (pp) L-formulas, i.e. for φ's of the form

$$\varphi(v_1, \ldots, v_n) := \exists\, w_1, \ldots, w_m \bigwedge_{i=1}^{k} \psi_i(v_1, \ldots, v_n, w_1, \ldots, w_m),$$

where the ψ_i's are atomic. This stems from the fact that any positive-existential formula is logically equivalent to a disjunction of conjunctions of pp-formulas.

(VI) The following model-theoretic fact will be used below :

Fact 1.3 Let $f : \mathfrak{A} \longrightarrow \mathfrak{B}$, $g : \mathfrak{B} \longrightarrow \mathfrak{C}$ be L-morphisms. Then,

(1) $g \circ f$ pure $\Rightarrow$ f pure.

(2) Assume :

(a) g is a L-monomorphism, i.e., for every atomic L-formula $\varphi(v_1, \ldots, v_n)$ and $b_1, \ldots, b_n \in \mathfrak{B}$

$$\mathfrak{B} \models \varphi[b_1, \ldots, b_n] \;\Leftrightarrow\; \mathfrak{C} \models \varphi[g(b_1), \ldots, g(b_n)]$$

(b) $g \circ f$ is elementary.

Then, f is elementary. □

Remark. If f is surjective, then in 1.3.(1) we may conclude, in addition, that g is pure and, in 1.3.(2), that g is elementary and f is an isomorphism. □

(VII) We write $\mathfrak{A}\ (\exists^+)\ \mathfrak{B}$ to indicate that every positive-existential L-sentence (no parameters) holding in $\mathfrak{A}$ also holds in $\mathfrak{B}$. We write $\mathfrak{A} \equiv_{\exists^+} \mathfrak{B}$ for $\mathfrak{A}\ (\exists^+)\ \mathfrak{B}$ and $\mathfrak{B}\ (\exists^+)\ \mathfrak{A}$.

The proof of the next result is similar to that of Proposition 5.2.2 (p. 307) in [CK] :

Proposition 1.4 *a) The following conditions are equivalent :*

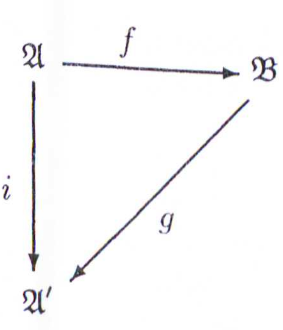
(1) $\mathfrak{A}\ (\exists^+)\ \mathfrak{B}$.

(2) *There is an elementary extension $\mathfrak{B} \prec \mathfrak{B}'$ and a L-morphism $g : \mathfrak{A} \longrightarrow \mathfrak{B}'$.*

b) The following conditions are equivalent :

(3) *$f : \mathfrak{A} \longrightarrow \mathfrak{B}$ is pure.*

(4) *There is an elementary extension $i : \mathfrak{A} \longrightarrow \mathfrak{A}'$ of $\mathfrak{A}$ and a L-morphism $g : \mathfrak{B} \longrightarrow \mathfrak{A}'$ such that $i = g \circ f$.*

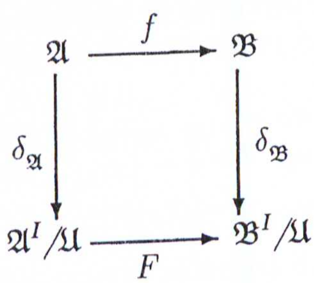


In (4), the elementary extension may be taken as an ultrapower of $\mathfrak{A}$, in which case the L-morphism g will satisfy $g \circ f =$ the diagonal embedding of $\mathfrak{A}$ into its ultrapower. □

(VIII) We shall use the following major model-theoretic result (Theorem 6.1.15, p. 398-401, in [CK]) :

Theorem 1.5 (*The Keisler-Shelah ultrapower theorem*)

Let $f : \mathfrak{A} \longrightarrow \mathfrak{B}$ be a L-morphism of L-structures. Then, f is elementary iff there is a set I, an ultrafilter $\mathfrak{U}$ on I and a L-isomorphism $F : \mathfrak{A}^I/\mathfrak{U} \longrightarrow \mathfrak{B}^I/\mathfrak{U}$ so that the following diagram commutes :



where $\delta_{\mathfrak{A}}$, $\delta_{\mathfrak{B}}$ are the respective diagonal embeddings. □

2 SG-Embeddings of Boolean Algebras and SAP Fields

In this section it will be shown that a SG-embedding of a Boolean algebra into any reduced special group is pure (2.3). This in turn will imply that if F is a formally real SAP field and G is a reduced special group, any complete embedding of $G_{red}(F)$ into G is pure.

Our argument proves, in fact, a fairly general result on first-order structures of continuous functions. Let $\mathcal{L}$ be a first-order language with equality, M a $\mathcal{L}$-structure and Z a Boolean space (compact, Hausdorff, with a basis of clopens). We can associate to M and Z two $\mathcal{L}$-structures, $\mathcal{C}(Z, M)$ and M^Z, where

$$\mathcal{C}(Z, M) = \{Z \xrightarrow{f} M : f \text{ is continuous}\},$$

with M endowed with the discrete topology. M^Z and $\mathcal{C}(Z, M)$ can be naturally considered as $\mathcal{L}$-structures, by defining operations and relations pointwise. Moreover, if c is a constant symbol in $\mathcal{L}$, then its interpretation in both of these structures is the constant function with value c^M, the interpretation of c in M. Hence, if R is a n-ary relation symbol in $\mathcal{L}$ and $\overline{f} = \langle f_1, \ldots, f_n \rangle$ is sequence of elements in $\mathcal{C}(Z, M)$ or M^Z, then

$$\mathcal{C}(X, M) \models R[\overline{f}] \quad \text{iff} \quad \text{For all } z \in Z, \ \ M \models R[f_1(z), \ldots, f_n(z)],$$

with an analogous definition holding for M^Z. We then see that the natural injection

$$\gamma : \mathcal{C}(Z, M) \longrightarrow M^Z,$$

that sends a continuous map to its underlying set map, is an embedding of $\mathcal{L}$-structures, that is, for all terms $\tau_1, \ldots, \tau_m$ in (at most) the n free variables $v_1, \ldots, v_n$, all m-ary relations R in $\mathcal{L}$ and $\overline{f} = \langle f_1, \ldots, f_n \rangle$ in $\mathcal{C}(Z, M)$

$$[\text{atom}] \begin{cases} \mathcal{C}(Z,M) \models R(\tau_1, \ldots, \tau_m)[\langle f_1, \ldots, f_n \rangle] & \text{iff} \\ M^Z \models R(\tau_1, \ldots, \tau_m)[\langle f_1, \ldots, f_n \rangle] & \text{iff} \\ \text{For all } z \in Z, \ \ M \models R(\tau_1, \ldots, \tau_m)[f_1(z), \ldots, f_n(z)]. \end{cases}$$

We have

Proposition 2.1 *The embedding $\gamma : \mathcal{C}(Z, M) \longrightarrow M^Z$ is pure, that is, it reflects positive-existential formulas with parameters in $\mathcal{C}(Z, M)$.*

Proof : As usual, we write $\varphi(v_1, \ldots, v_n)$ to mean that the free variables in φ are among the $v_1, \ldots, v_n$. For $\overline{g} = \langle g_1, \ldots, g_n \rangle$ in M^Z and $z \in Z$, set

$$\overline{g}(z) = \langle g_1(z), \ldots, g_n(z) \rangle.$$

Let Γ be the set of all formulas $\varphi(v_1, \ldots, v_n)$ in $\mathcal{L}$ ($n \geq 0$ is arbitrary) verifying the following conditions:

(1) For $\overline{g} = \langle g_1, \ldots, g_n \rangle$ in M^Z, $\ M^Z \models \varphi[\overline{g}]$ iff $\forall\, z \in Z, \ M \models \varphi[\overline{g}(z)]$;

(2) For $\overline{f} = \langle f_1, \ldots, f_n \rangle$ in $\mathcal{C}(Z, M)$, $\ M^Z \models \varphi[\overline{f}] \ \Rightarrow \ \mathcal{C}(Z, M) \models \varphi[\overline{f}]$.

The comments preceding the statement of 2.1 ([atom]) guarantee that all atomic formulas belong to Γ. It is straightforward that Γ is closed under conjunction and disjunction. To finish the proof, it suffices to check that if $\varphi(v_1, \ldots, v_n) \in \Gamma$, then $\exists v_j\, \varphi \in \Gamma$. Without loss of generality, we may assume that $j = 1$.

<u>$\exists v_1\, \varphi$ verifies (1)</u> : If $\overline{h} = \langle h_2, \ldots, h_n \rangle$ is in M^Z, then since $\varphi \in \Gamma$,

$$\begin{aligned} M^Z \models \exists v_1\, \varphi[\overline{h}] \ \ &\text{iff} \ \ \exists\, g \in M^Z, M^Z \models \varphi[g; \overline{h}] \\ &\text{iff} \ \ \forall\, z \in Z, \ \ M \models \varphi[g(z); \overline{h}(z)] \\ &\text{iff} \ \ \forall\, z \in Z, \ \ M \models \exists v_1\, \varphi[\overline{h}(z)], \end{aligned}$$

where the last equivalence is true because any map that associates to $z \in Z$ a witness in M of $\exists v_1\, \varphi(v_1; \overline{h}(z))$ belongs to M^Z.

<u>$\exists v_1\, \varphi$ verifies (2)</u> : Fix a $(n-1)$ tuple $\overline{f} = \langle f_2, \ldots, f_n \rangle$ in $\mathcal{C}(Z, M)$; since each f_j is a locally constant function, there is a finite partition P of Z into non-empty clopens such that each f_j is constant in every $u \in P$. Now, assuming that

$$M^Z \models \exists v_1\, \varphi[\overline{f}],$$

there is a map $g : Z \longrightarrow M$ (not necessarily continuous) such that $M^Z \models \varphi[g; \overline{f}]$. Since φ verifies (1), it follows that for all $z \in Z$

$$M \models \varphi[g(z); \overline{f}(z)]. \tag{*}$$

For $u \in P$, select $z_u \in u$; then, because $\overline{f}$ is constant on u, (*) entails

$$\text{For all } t \in u, \quad M \models \varphi[g(z_u); \overline{f}(t)]. \tag{**}$$

Define $\widetilde{g} : Z \longrightarrow M$ by setting the value of $\widetilde{g}$ equal to $g(z_u)$ on $u \in P$. Then, $\widetilde{g}$ is locally constant in Z and thus an element of $\mathcal{C}(Z, M)$. Moreover, since (**) is valid on each $u \in P$, we get

$$\text{For all } z \in Z, \quad M \models \varphi[\widetilde{g}(z); \overline{f}(z)].$$

Since $\varphi \in \Gamma$, we obtain $M^Z \models \varphi[\widetilde{g}; \overline{f}]$ (from (1)) and so $\mathcal{C}(Z, M) \models \varphi[\widetilde{g}; \overline{f}]$ (from (2)). Hence, $\mathcal{C}(Z, M) \models \exists v_1 \varphi[\overline{f}]$, as desired.

The preceding argument shows that every positive-existential formula in $\mathcal{L}$ belongs to Γ, concluding the proof. □

Let B be a Boolean algebra and let $S(B)$ be its Stone space, consisting of the (proper) ultrafilters on B, or equivalently, the SG-characters of B. Moreover, B is isomorphic to $\mathcal{C}(S(B), 2)$ where $2 = \{\pm 1\}$ is the 2-element BA. Hence, 2.1 yields the first assertion in

Corollary 2.2 *a) If B is a Boolean algebra and $S(B)$ is its Stone space, then the canonical monic*

$$b \in B \longmapsto \check{b} \in 2^{S(B)},$$

where, for $\sigma \in S(B)$, $\check{b}(\sigma) = \sigma(b)$, is a pure embedding.

b) Any injective BA-morphism is a pure embedding.

Proof : To verify (b), let $h : B \longrightarrow C$ be an injective BA-morphism. By Stone duality, h induces a continuous surjection

$$h^* : S(C) \longrightarrow S(B), \quad \sigma \in S(C) \longmapsto \sigma \circ h,$$

which in turn gives rise to an injective BA-morphism, $\widehat{h} : 2^{S(B)} \longrightarrow 2^{S(C)}$ (again by composition). It is readily verified that the following diagram is commutative :

$$\begin{array}{ccc} B & \xrightarrow{\;h\;} & C \\ \big\downarrow \check{(\cdot)} & & \big\downarrow \check{(\cdot)} \\ 2^{S(B)} & \xrightarrow[\;\widehat{h}\;]{} & 2^{S(C)} \end{array} \tag{I}$$

By (a), both vertical arrows in (I) are pure. Since for any set J, 2^J is a complete BA and complete BA's are the injective objects in the category of Boolean algebras (Thm. 5.13, p. 71 in [HBA]), the injection $\widehat{h}$ has a retract[2], and so is also pure. It is then straightforward that h must be a pure embedding. □

Proposition 2.3 *Any injective SG-morphism from a Boolean algebra to a reduced special group is a pure embedding.*

[2] If A is the image of $\widehat{h}$ in $2^{S(C)}$, the inverse of $\widehat{h}$ from A to $2^{S(B)}$ can be extended to $2^{S(C)}$ to yield a left inverse to $\widehat{h}$.

Proof : Let $B \xrightarrow{f} G$ be a SG-injection, where B is a BA and G a rsg. Let $\varepsilon_G : G \longrightarrow B_G$ be the Boolean hull of G (Thm. 4.17 in [DM1]). The composition $\varepsilon_G \circ f$ yields an injective BA-morphism of B into B_G, which by 2.2.(b) is a pure embedding. Hence, f must also be a pure embedding. □

Remark. The converse to Proposition 2.3 also holds: Boolean algebras are the only rsg's such that every SG-embedding into another such group is pure. Indeed, if the canonical embedding $\varepsilon_G : G \longrightarrow B_G$ is pure, Proposition 7.17 of [DM1] together with the fact that the statement "the form $\langle 1, a, b, -ab \rangle$ is isotropic" $(a, b \in G)$ is expressed by a positive-existential L_{SG}-sentence, entail that G is a BA. □

Corollary 2.4 *Any two Boolean algebras have the same positive-existential true sentences (in both the languages for special groups and for Boolean algebras).*

Proof : Given BA's A, B, let C be a joint extension of A and B, and let $f : A \hookrightarrow C$, $g : B \hookrightarrow C$ be (SG– or Boolean) embeddings. Let φ be a positive-existential sentence such that $A \models \varphi$; since f is a homomorphism, $C \models \varphi$, and since g is pure, $B \models \varphi$. This shows that $A\ (\exists^+)\ B$. Symmetrically, we have $B\ (\exists^+)\ A$, whence $A \equiv_{\exists^+} B$. □

As a particular case of 2.3 we have:

Proposition 2.5 *Let F be a formally real SAP field.*

a) If L is a formally real extension of L such that $\Sigma \dot{L}^2 \cap \dot{F} = \Sigma \dot{F}^2$, then the SG-morphism $\iota_{FL} : G_{red}(F) \longrightarrow G_{red}(L)$ is a pure embedding.

b) In particular, the reduced special group of F is purely embedded in the (necessarily reduced) special groups of its Pythagorean closure and of its order closure.

Proof : Item (b) is an immediate consequence of (a) by taking L to be either the Pythagorean closure or the order closure of F. For (a), recall that if F is a SAP field, then its reduced special group is a Boolean algebra (see item (1) in the proof of Prop. 5.5 in [DM1]) and observe that the condition $\dot{F} \cap \Sigma \dot{L}^2 = \Sigma \dot{F}^2$ implies that ι_{FL} is a SG-embedding. The conclusion then follows from 2.3. □

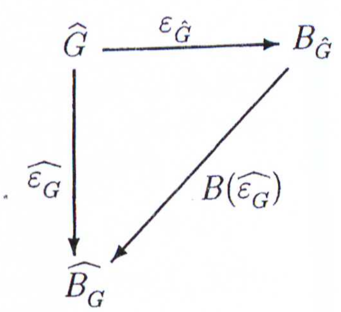
Remark. Statement (b) holds, more generally, when L is the relative Pythagorean closure of F inside a prime-closed extension; cf. [B], Ch. 3. □

3 Elementary Behavior of the Boolean Hull Functor

In this section we show that the functor of the title preserves both elementary SG-morphisms and elementary equivalence in the language L_{SG}.

Proposition 3.1 *Let G be a rsg. With notation as in 1.1.(III.a), the Boolean dual $B(\delta_G) : B_G \longrightarrow B_{\widehat{G}}$ of the diagonal embedding $\delta_G : G \longrightarrow \widehat{G}$ is an elementary embedding (both in language L_{SG} of special groups and in the language L_{BA} of Boolean algebras).*

Proof : Let $\widehat{\varepsilon_G} : \widehat{G} \longrightarrow \widehat{B_G}$ be the extension of the canonical embedding $\varepsilon_G : G \longrightarrow B_G$, as defined in item (II) of 1.1. By Lemma 1.2 and Corollary 5.4.(a) in [DM1], $\widehat{\varepsilon_G}$ is complete. Theorem 4.17.(4) in [DM1] shows that $\widehat{\varepsilon_G}$ factors through $B_{\widehat{G}}$, i.e. the following diagram is commutative :



where $B(\widehat{\varepsilon_G})$ is the BA homomorphism functorially associated to $\widehat{\varepsilon_G}$ by Theorem 4.14.(1) in [DM1]. We prove :

(A) $B(\widehat{\varepsilon_G}) \circ B(\delta_G) : B_G \longrightarrow \widehat{B_G}$ is the diagonal embedding δ_{B_G}.

Let $b \in B_G$; write $\langle b \rangle$ for the constant I-sequence with value b. We must verify that

$$B(\widehat{\varepsilon_G}) \circ (\delta_G)(b) = \langle b \rangle/\mathfrak{U}. \tag{I}$$

Since $Im(\varepsilon_G)$ generates B_G as a BA (Proposition 4.10 in [DM1]) and $B(\widehat{\varepsilon_G}) \circ B(\delta_G)$ is a BA-morphism, it suffices to check the identity in (I) for elements $b \in Im(\varepsilon_G)$, i.e., $b = \varepsilon_G(g)$, $g \in G$. Now, since $\delta_G(g) = \langle g \rangle/\mathfrak{U}$ and $\widehat{\varepsilon_G}(\langle g \rangle/\mathfrak{U}) = \langle \varepsilon_G(g) \rangle/\mathfrak{U}$, it is clear that (I) is satisfied for $g \in G$, as needed.

Remark. More formally, note that (**) in 1.1.(II) gives, for a constant function $\langle g \rangle \in G^I$, $\widehat{f}(\langle g \rangle/\mathfrak{U}) = \langle f(g) \rangle/\mathfrak{U}$, i.e.

$$\widehat{f} \circ \delta_G = \delta_H \circ f.$$

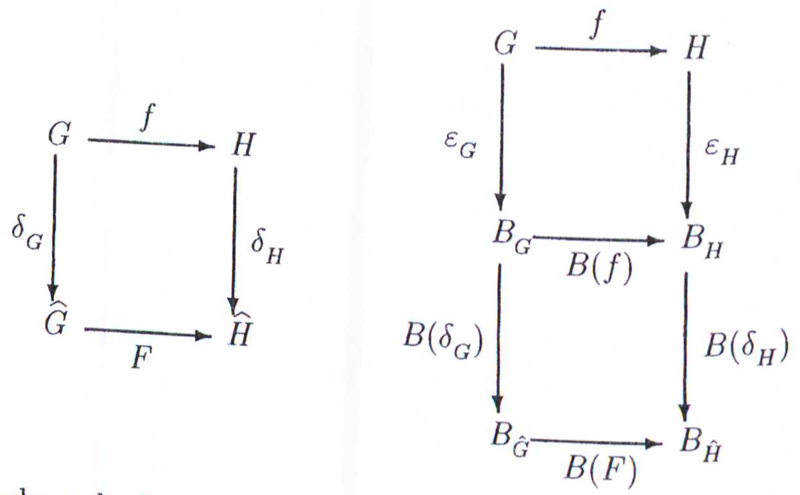
For $f = \varepsilon_G : G \longrightarrow B_G$, we get $\widehat{\varepsilon_G} \circ \delta_G = \delta_{B_G} \circ \varepsilon_G$. Applying the Boolean hull functor to this identity and observing that $B(\delta_{B_G}) = \delta_{B_G}$ and $B(\varepsilon_G) = Id_{B_G}$, proves (A). □

Item (A) shows that $B(\widehat{\varepsilon_G}) \circ B(\delta_G)$ is elementary. By Fact 1.3.(2), the proof of 3.1 reduces to showing that $B(\widehat{\varepsilon_G})$ is a L_{SG}-monomorphism, but this follows immediately from the fact that $\widehat{\varepsilon_G}$ is complete (cf. Lemma 1.2 and Theorem 5.1 in [DM1]).

Remark. We have just shown that $B(\delta_G)$ is L_{SG}-elementary. For L_{BA}, recall that the Boolean operations are first-order definable in terms of representation, and hence in L_{SG}. □

Proposition 3.2 *If $f : G \longrightarrow H$ is an elementary morphism of reduced special groups, the same is true of $B(f) : B_G \longrightarrow B_H$.*

Proof: We use Theorem 1.5 to get an isomorphism $F : \widehat{G} \longrightarrow \widehat{H}$, so that the diagram on the left commutes :



Taking Boolean duals and using Theorem 4.17 in [DM1], the diagram on the right also commutes. Item (2) of Theorem 4.17 in [DM1] (applied twice, to F and F^{-1}) shows that $B(F)$ is an isomorphism. Since the maps $B(\delta_G)$ and $B(\delta_H)$ are elementary (Proposition 3.1), $B(F) \circ B(\delta_G)$ is elementary. Then, the commutative diagram above right and 1.3.(2) entail that $B(f)$ is elementary. □

Corollary 3.3 *If G, H are rsg's, then $G \equiv H$ implies $B_G \equiv B_H$.*

Proof: By 1.5, G and H have isomorphic ultrapowers, $\widehat{G} \approx \widehat{H}$. The Boolean dual of this isomorphism gives $B_{\widehat{G}} \approx B_{\widehat{H}}$. Since the maps $B(\delta_G) : B_G \longrightarrow B_{\widehat{G}}$ and $B(\delta_H) : B_H \longrightarrow B_{\widehat{H}}$ are elementary, we must have $B_G \equiv B_H$. □

4 Elementary Properties of the Reduced Quotient Functor

This section is devoted to examine the behavior of the reduced quotient functor with respect to first-order notions, proving the preservation results announced in the Introduction.

Proposition 4.1 *Let G be a frsg. With notation as in 1.1.(IV.c), the SG-morphism $(\delta_G)_{red} = \delta^*_G : G_{red} \longrightarrow \widehat{G}_{red}$ induced by the diagonal embedding $\delta_G : G \longrightarrow \widehat{G}$ has the following properties :*

(1) *δ^*_G is pure.*

(2) *If G has finite Pythagoras number, then δ^*_G is elementary.*

Proof : As in the proof of 3.1, the idea is to define a SG-morphism $\theta : \widehat{G}_{red} \longrightarrow (G_{red})^I/\mathfrak{U}$ such that $\theta \circ \delta^*_G : G_{red} \longrightarrow (G_{red})^I/\mathfrak{U}$ is the diagonal embedding $\delta_{G_{red}}$. Since this is an elementary map, Fact 1.3.(1) yields that δ^*_G is pure, proving (1). For (2), we show that if G has finite Pythagoras number, then θ is a L_{SG}-monomorphism; Fact 1.3.(2) then entails that δ^*_G is elementary.

The map θ will be induced by the ultrapower extension $\widehat{\pi_G} : \widehat{G} \longrightarrow (G_{red})^I/\mathfrak{U}$ of the canonical quotient map $\pi_G : G \longrightarrow G_{red}$, as defined in (**) of 1.1.(II). To ease notation, write $\pi = \pi_G$ and $\widehat{\pi} = \widehat{\pi_G}$. We first check :

(A) $Sat(\widehat{G}) \subseteq ker(\widehat{\pi})$.

Proof of (A). Let $\langle g_i : i \in I \rangle/\mathfrak{U} \in Sat(\widehat{G})$, i.e., there is $k \geq 0$ such that $\langle g_i : i \in I \rangle/\mathfrak{U} \in D_{\widehat{G}}(1,\ k\langle 1 \rangle)$. Since representation by forms of arbitrary dimension is first-order definable in L_{SG}, by Łoś' Theorem

$$A = \{i \in I : g_i \in D_G(1,\ k\langle 1 \rangle)\} \in \mathfrak{U}.$$

For $i \in A$, we clearly have $\pi(g_i) = 1$. Thus,

$$A \subseteq \{i \in I : \pi(g_i) = 1\},$$

and so this last set must be in $\mathfrak{U}$. But then, $\langle \pi(g_i) : i \in I \rangle/\mathfrak{U} = \widehat{\pi}(\langle g_i : i \in I \rangle/\mathfrak{U})$ $= 1$, as desired. □

By the inclusion in (A), $\widehat{\pi}$ induces a SG-morphism $\theta : \widehat{G}_{red} \longrightarrow (G_{red})^I/\mathfrak{U}$; namely, for $\alpha \in \widehat{G}$,

$$\theta(\alpha/Sat(\widehat{G})) = \widehat{\pi}(\alpha).$$

In other words, θ is defined by the functional equation

(++) $$\theta \circ \pi_{\widehat{G}} = \widehat{\pi}.$$

Next, we prove :

(B) $\theta \circ \delta^*_G$ is the diagonal embedding $\delta_{G_{red}}$ of G_{red} into $(G_{red})^I/\mathfrak{U}$.

Proof of (B). δ^*_G is defined by the functional equation (+) of 1.1.(IV.b), namely

$$\delta^*_G \circ \pi = \pi_{\widehat{G}} \circ \delta_G.$$

Composing on the left by θ and using (++) yields

(+++) $$\theta \circ \delta^*_G \circ \pi = \theta \circ \pi_{\widehat{G}} \circ \delta_G = \widehat{\pi} \circ \delta_G.$$

The diagonal embedding $\delta_{G_{red}}$ is defined, for $g \in G$, by :

$$\delta_{G_{red}}(\pi(g)) = \langle \pi(g) \rangle/\mathfrak{U}.$$

Since $\widehat{\pi}(g) = \widehat{\pi}(\langle g \rangle/\mathfrak{U}) = \langle \pi(g) \rangle/\mathfrak{U}$, we get $\widehat{\pi} \circ \delta_G = \delta_{G_{red}} \circ \pi$. From (+++) we get $\theta \circ \delta^*_G \circ \pi = \delta_{G_{red}} \circ \pi$, verifying $\theta \circ \delta^*_G = \delta_{G_{red}}$, as required. □

As observed at the beginning, this establishes (1) in 4.1.

Returning to item (A), note that *equality may not hold* in (A) (equivalently, θ might not be injective). In fact, $\widehat{\pi}(\langle g_i : i \in I \rangle/\mathfrak{U}) = 1$ means that

$$B = \{i \in I : \pi(g_i) = 1\} = \{i \in I : g_i \in Sat(G)\} \in \mathfrak{U},$$

that is, for each $i \in B$, there is an integer $k_i \geq 0$ such that $g_i \in D_G(1, k_i\langle 1 \rangle)$; the k_i may well be unbounded in $\mathbb{N}$. However :

(C) If G has finite Pythagoras number, then $Sat(\widehat{G}) = ker(\widehat{\pi})$. Moreover, $p(G) = p(\widehat{G})$.

Proof of (C). The second assertion follows by observing that $p = p(G)$ is expressed by first-order L_{SG}-sentences (without parameters) and hence goes over from G to $\widehat{G}$. Namely, p is smallest $m \in N$ such that $Sat(G) = D_G(m\langle 1 \rangle)$ iff the following L_{SG}-sentences hold in G :

* For $k > p$, $\forall\, x\, [x \in D(k\langle 1 \rangle) \rightarrow x \in D(p\langle 1 \rangle)]$;[3]
* $\exists\, x\, [x \in D(p\langle 1 \rangle) \wedge x \notin D((p-1)\langle 1 \rangle)]$.

To prove the first assertion in (C), let $\langle g_i : i \in I \rangle/\mathfrak{U} \in ker(\widehat{\pi})$, i.e.,

$$C = \{i \in I : g_i \in Sat(G)\} \in \mathfrak{U}.$$

Since $Sat(G) = D_G(p\langle 1 \rangle)$,

$$C = \{i \in I : g_i \in D_G(p\langle 1 \rangle)\},$$

and since the condition $x \in D(p\langle 1 \rangle)$ is first-order, Łoś' Theorem yields $\langle g_i : i \in I \rangle/\mathfrak{U} \in D_{\widehat{G}}(p\langle 1 \rangle) \subseteq Sat(\widehat{G})$, as needed. □

As observed above, (C) yields :

(D) If G has finite Pythagoras number, then θ is injective.

Proof of (D). The functional equation (++) implies

$$ker(\widehat{\pi}) = \pi_{\widehat{G}}^{-1}[ker\, \theta];$$

this and (C) entail, for $\alpha \in \widehat{G}$:

$$\alpha/Sat(\widehat{G}) = \pi_{\widehat{G}}(\alpha) \in ker(\theta) \text{ iff } \alpha \in ker(\widehat{\pi}) \text{ iff } \alpha \in Sat(\widehat{G}) \text{ iff } \alpha/Sat(\widehat{G}) = 1.$$ □

To complete the proof of (2) in 4.1 we show :

(E) If G has finite Pythagoras number, then θ is a SG-monomorphism.

For the proof of (E) we need the following immediate consequence of Definition 2.13 and Proposition 2.28 in [DM1] :

Fact. *If Δ is a saturated subgroup of a special group G and $a, b \in G$, then*

[3] It suffices to require it for $k = p + 1$.

$$a/\Delta \in D_{G/\Delta}(1, b/\Delta) \quad \textit{iff} \quad \begin{cases} \exists\, a_0, b_0 \in G, \text{ so that } aa_0, bb_0 \in \Delta \\ \qquad \text{and } a_0 \in D_G(1, b_0). \end{cases}$$

Proof of (E). To ease notation, set $H = (G_{red})^I/\mathfrak{U}$. Since θ is a SG-morphism we need only verify that θ reflects atomic L_{SG}-formulas with parameters in $\widehat{G}_{red}$; that is, we must show that for $\xi = \langle g_i : i \in I\rangle$, $\eta = \langle g'_i : i \in I\rangle \in G^I$,

$$\theta(\pi_{\widehat{G}}(\xi/\mathfrak{U})) \in D_H(1, \theta(\pi_{\widehat{G}}(\eta))) \quad \Rightarrow \quad \pi_{\widehat{G}}(\xi/\mathfrak{U}) \in D_{\widehat{G}_{red}}(1, \pi_{\widehat{G}}(\eta/\mathfrak{U})).$$

By virtue of the functional equation (++) above, the assumption translates into

$$\widehat{\pi}(\xi/\mathfrak{U}) \in D_H(1, \widehat{\pi}(\eta/\mathfrak{U}));$$

since $\widehat{\pi}(\xi/\mathfrak{U}) = \widehat{\pi}(\langle g_i : i \in I\rangle/\mathfrak{U}) = \langle \pi(g_i) : i \in I\rangle/\mathfrak{U}$ (cf. definition of $\widehat{\pi}$ just before the statement of (A)), we get

$$E = \{i \in I : \pi(g_i) \in D_{G_{red}}(1, \pi(g'_i)\} \in \mathfrak{U}.$$

Using the preceding Fact, with $\Delta = Sat(G)$, for each $i \in E$ there are elements $h_i, h'_i \in G$ such that $g_i h_i, g'_i h'_i \in Sat(G)$ and $h_i \in D_G(1, h'_i)$. For $j \in I \setminus E$ set $h_j = g_j$ and $h'_j = g'_j$. By assumption, $Sat(G) = D_G(p\langle 1\rangle)$, for some integer $p \geq 1$, whence $g_i h_i, g'_i h'_i \in D_G(p\langle 1\rangle)$ for all $i \in I$. Since $v \in D(1, p\langle 1\rangle)$ is a L_{SG}-formula, Łoś' Theorem gives

$$\langle g_i h_i : i \in I\rangle/\mathfrak{U} \in D_{\widehat{G}}(p\langle 1\rangle) \subseteq Sat(\widehat{G}),$$

i.e., $\langle g_i : i \in I\rangle/\mathfrak{U} \cdot \langle h_i : i \in I\rangle/\mathfrak{U} \in Sat(\widehat{G})$, which implies

$$(*) \qquad \pi_{\widehat{G}}(\langle g_i : i \in I\rangle/\mathfrak{U}) = \pi_{\widehat{G}}(\langle h_i : i \in I\rangle/\mathfrak{U}),$$

and similarly for g'_i, h'_i. Further, since $h_i \in D_G(1, h'_i)$ for $i \in E \in \mathfrak{U}$, we get

$$\langle h_i : i \in I\rangle/\mathfrak{U} \in D_{\widehat{G}}(1, \langle h'_i : i \in I\rangle/\mathfrak{U}).$$

Since $\pi_{\widehat{G}}$ is a SG-morphism, (*) yields $\pi_{\widehat{G}}(\xi/\mathfrak{U}) \in D_{\widehat{G}_{red}}(1, \pi_{\widehat{G}}(\eta/\mathfrak{U}))$, as required to complete the proof of Proposition 4.1. □

By arguments similar to those proving 3.2, 3.3, we obtain analogous statements for the reduced quotient functor.

Proposition 4.2 *Let G, H be frsg's and let $f : G \longrightarrow H$ be a SG-morphism. Then,*

a) f pure $\Rightarrow$ f_{red} pure.

b) If G, H have finite Pythagoras number, f elementary $\Rightarrow$ f_{red} elementary.

Proof : a) By Proposition 1.4.(b) there is an ultrapower $\widehat{G}$ of G and a SG-morphism $g : H \longrightarrow \widehat{G}$ so that $g \circ f = \delta_G$, the diagonal embedding of G into $\widehat{G}$. Since the reduced quotient is a functor, $g_{red} \circ f_{red} = (\delta_G)_{red} = \delta^*_G$. Since δ^*_G is pure (4.1.(1)), Fact 1.3.(1) entails that f_{red} is also pure.

b) By Theorem 1.5 there are a set I, an ultrafilter $\mathfrak{U}$ on I and a SG-morphism F such that the diagram below left is commutative :

(x)
$$\begin{array}{ccc} G & \xrightarrow{f} & H \\ {\scriptstyle \delta_G}\downarrow & & \downarrow{\scriptstyle \delta_H} \\ \widehat{G} & \xrightarrow[F]{} & \widehat{H} \end{array} \qquad\qquad \begin{array}{ccc} G_{red} & \xrightarrow{f_{red}} & H_{red} \\ {\scriptstyle \delta^*_G}\downarrow & & \downarrow{\scriptstyle \delta^*_H} \\ \widehat{G}_{red} & \xrightarrow[F_{red}]{} & \widehat{H}_{red} \end{array}$$

where δ_G, δ_H are the diagonal embeddings. Below we prove :

(1) The diagram above right is commutative;

(2) F_{red} is a SG-isomorphism.

This suffices to prove (b). Indeed, δ^*_G, δ^*_H are elementary by 4.1.(2); then $F_{red} \circ \delta^*_G = \delta^*_H \circ f_{red}$ is elementary, which implies f_{red} elementary by Fact 1.3.(2).

Proof of (1). We need to show, for $g \in G$:

$$F_{red} \circ \delta^*_G(\pi_G(g)) = \delta^*_H \circ f_{red}(\pi_G(g)),$$

i.e.,

(α)
$$F_{red} \circ \delta^*_G \circ \pi_G = \delta^*_H \circ f_{red} \circ \pi_G.$$

The functional equation $f_{red} \circ \pi_G = \pi_H \circ f$ defines f_{red} (cf. (+) in 1.1.(IV.b)); replacing in (α) we are reduced to show :

(β)
$$F_{red} \circ \delta^*_G \circ \pi_G = \delta^*_H \circ \pi_H \circ f.$$

Likewise (+) also yields :

$$\delta^*_G \circ \pi_G = (\delta_G)_{red} \circ \pi_G = \pi_{\hat{G}} \circ \delta_G \quad \text{and} \quad \delta^*_H \circ \pi_H = \pi_{\hat{H}} \circ \delta_H.$$

Substituting these equalities in (β) reduces the problem to :

(γ)
$$F_{red} \circ \pi_{\hat{G}} \circ \delta_G = \pi_{\hat{H}} \circ \delta_H \circ f.$$

Yet another application of (+) to the definition of F_{red} gives :

$$F_{red} \circ \pi_{\hat{G}} = \pi_{\hat{H}} \circ F.$$

Substituting this in the left-hand side of (γ) we obtain :

(δ)
$$\pi_{\hat{H}} \circ F \circ \delta_G = \pi_{\hat{H}} \circ \delta_H \circ f.$$

Diagram (x) gives $F \circ \delta_G = \delta_H \circ F$, which composed on the left with $\pi_{\hat{H}}$ proves (δ) and (1).

Proof of (2). It suffices to prove that $(F^{-1})_{red} = (F_{red})^{-1}$, which is a consequence of the fact that the reduced quotient is a functor. □

The proof of the following result is analogous to that of 3.3.

Corollary 4.3 *Let G, H be formally real special groups.*

a) $G\ (\exists^+)\ H \ \Rightarrow\ G_{red}\ (\exists^+)\ H_{red}$.

b) $G \equiv_{\exists^+} H \ \Rightarrow\ G_{red} \equiv_{\exists^+} H_{red}$.

c) If G, H have finite Pythagoras number, $G \equiv H$ implies $G_{red} \equiv H_{red}$. □

5 Some Applications

We shall derive from the preceding analysis two interesting consequences in the case of fields.

I. The Pythagorean closure of a formally real field.

Proposition 5.1 *Let F be a non-Pythagorean formally real field whose space of orders has isolated points. Let F^p be the Pythagorean closure of F. Then $G_{red}(F) \not\equiv G(F^p)$. In particular, the map $i : G_{red}(F) \longrightarrow G(F^p)$ induced by the field inclusion of F into F^p is not elementary.*

Proof : By Corollary 3.3, the assumption $G_{red}(F) \equiv G(F^p)$ implies $B_{G_{red}(F)} \equiv B_{G(F^p)}$. Let us denote these BA's by B_1 and B_2, respectively. By Corollary 5.4.(b) in [DM1], $S(B_1)$, the Stone space of B_1, is homeomorphic to the space of orders $\chi(F)$, while $S(B_2)$ is homeomorphic to $\chi(F^p)$. By Theorem 3.2.(d) of [DM3], $\chi(F^p)$ is a perfect space, whenever F is formally real and non-Pythagorean. By the assumption on F, B_1 has atoms, while B_2 is atomless, contradicting $B_1 \equiv B_2$. □

There are many fields meeting the assumptions of 5.1; for example, any non-Pythagorean formally real field with finitely many orders. Examples are all formally real algebraic number fields (finite extensions of $\mathbb{Q}$). In fact, we have

Proposition 5.2 *Every Boolean space is homeomorphic to the space of orders of a* **non-Pythagorean** *field.*

We shall give a sketch of the proof of 5.2, indicating modifications on the proof presented in [P] of the result proved by Craven in [Cr], that every Boolean space is homeomorphic to the space of orders of a field, which turns out to be Pythagorean. The proof presented in [P] proceeds along the following steps :

<u>Step 1</u> ([P], Thm. 6.9) *Given a real closed field R, the field*

$$F = R(X)(\{\sqrt{X-a} \ : \ a \in R\});$$

is SAP ([P], Thm. 9.4), and $\chi(F) \approx 2^{\kappa}$, where $\kappa = card(R)$.

Since every Boolean space is homeomorphic to a closed subset of 2^{κ} for some infinite cardinal κ, in order to get Craven's Theorem it suffices to construct an extension F_1 of F such that $\chi(F_1)$ is homeomorphic to a given proper closed non-empty subset A of $\chi(F)$. This is achieved by :

<u>Step 2</u> ([P], Thm. 6.7) *Let A be a closed subset of $\chi(F)$, where F is a SAP field, and A is distinct from $\emptyset$ and $\chi(F)$. Suppose $C \subseteq \dot{F}$ is such that $\chi(F) \setminus A = \bigcup_{c \in C} H(c)$, with $H(c) \neq \emptyset$, $H(c) = \{P \in \chi(F) : c >_P 0\}$, $c \in C$. Consider the algebraic extension*

$$F_1 = F(\{(-c)^{1/2^n} \ : \ c \in C, \ n \in \mathbb{N}\})$$

of F, where the square roots are taken in a fixed real closure $\overline{(F, P_0)}$ with $P_0 \in A$. Then, $A \approx \chi(F_1)$.

We now indicate the changes needed in this proof to get Proposition 5.2.

The base field $R(X)$ of Step 1 is non-Pythagorean; for example, $1 + X^2$ is not a square in $R(X)$, by the Pfister-Cassel Theorem (Remark 2.4 and Corollary 2.6 in [KS]). We fix a sum of squares $s = \sum_{i=1}^{n} f_i^2$ in $R(X)$ which is not a square, and will check that s is not a square in the field F constructed in Step 1. Then, we modify Step 2 to get an extension F_2 of F, not necessarily algebraic, where the given element s remains a non-square. The following lemmas are designed to achieve the first objective (they are a bit more general than needed).

Lemma 5.3 *Let $K \subseteq L$ be fields, char$(K) \neq 2$, $a_1, \dots, a_n \in \dot{K}$, $b \in L$. If $b \notin \Sigma K^2$ and $\sqrt{b} \notin K(b)$, (i.e., $b \notin K(b)^2$), then*

$$\textstyle\sum_i a_i^2 \notin K^2 \quad \Rightarrow \quad \sum_i a_i^2 \notin K(\sqrt{b})^2.$$

Proof : Let $a = \sqrt{\sum_i a_i^2}$ (in some algebraic closure of K). Assume that $a \in K(\sqrt{b})$; then $a = x + y\sqrt{b}$, with $x, y \in K$. If $y = 0$, then $a = x \in K$, contrary to assumption; so $y \neq 0$.

1) $x = 0$. Then $a = y\sqrt{b}$, whence $b = \dfrac{a^2}{y^2} = \sum_i (\frac{a_i}{y})^2 \in \Sigma K^2$, contrary to assumption.

2) $x \neq 0$. Then, $a^2 = x^2 + by^2 + 2xy\sqrt{b}$. Since $x, y \neq 0$, $\sqrt{b} = \dfrac{a^2 - x^2 - by^2}{2xy} \in K(b)$, contrary to assumption. □

Iterating 5.3 yields :

Lemma 5.4 *Let $K \subseteq L$ be fields, char$(K) \neq 2$. Let $a_1, \dots, a_n \in \dot{K}$, $b_1, \dots, b_l \in \dot{L}$. For $0 \leq i \leq l-1$, set inductively*

$$K_0 = K \quad \text{and} \quad K_{i+1} = K_i(\sqrt{b_{i+1}}),$$

so that $K_i = K(\sqrt{b_1}, \dots, \sqrt{b_i})$, $i \geq 1$. Assume that for $i = 1, \dots, l$:

(1_i) $b_i \notin \Sigma K_{i-1}^2$; (2_i) $\sqrt{b_i} \notin K_{i-1}(b_i)$.

Then, $\sum_j a_j^2 \notin K^2 \Rightarrow \sum_j a_j^2 \notin K_l^2$.

Proof : The hypotheses (1_i) and (2_i) guarantee successive applicability of Lemma 5.3 to the fields K_i, $i = 1, \dots, l$. □

The following lemma gives more manageable conditions under which the assumptions of 5.4 hold.

Lemma 5.5 *Let $K \subseteq L$ be fields, K formally real; let $b_1, \dots, b_l \in \dot{L}$. Assume that there are orders $P_1, \dots, P_l \in \chi(K)$ such that*

(1) P_1 *extends to an order* $P'_1 \in \chi(K(b_1))$ *such that* $b_1 <_{P'_1} 0$,

and for every i, $2 \leq i \leq l$,

(2^*_i) P_i *extends to an order* $P'_i \in \chi(K(b_1, \ldots, b_i))$ *so that* $b_1, \ldots b_{i-1} >_{P'_i} 0$ *and* $b_i <_{P'_i} 0$.

Then, the assumptions (1_i) *and* (2_i) *of 5.4 are verified for* $i = 1, \ldots, l$. *In particular,* $\sum_j a_j^2 \notin K_l^2$, *if* $a_1, \ldots, a_n \in K$ *are such that* $\sum_j a_j^2 \notin K^2$.

Proof : (1_i) Since $b_1, \ldots, b_{i-1} >_{P'_i} 0$, the order P'_i extends to $K_{i-1} = K(\sqrt{b_1}, \ldots, \sqrt{b_{i-1}})$ (indeed, it has 2^{i-1} extensions). Let P''_i be an order of $K_{i-1}(b_i)$ extending P'_i. Since $b_i <_{P'_i} 0$, then $b_i <_{P''_i} 0$, and $b_i \notin \Sigma K^2_{i-1}$.
(2_i) Since $b_i <_{P''_i} 0$, b_i cannot be a square in $K_{i-1}(b_i)$. □

Proposition 5.6 *Let* $F = R(X)(\{\sqrt{X-a} \ : \ a \in R\})$, R *a real closed field, be as in Step 1. Let* $f_1, \ldots, f_n$ *be non-zero elements in* $R(X)$ *such that* $\sum_j f_j^2 \notin R(X)^2$. *Then* $\sum_j f_j^2 \notin F^2$.

Proof : It is enough to verify that for any finite set $\{a_1, \ldots, a_l\} \subseteq R$, $\sum_j f_j^2$ is not a square in $R(X)(\sqrt{X-a_1}, \ldots, \sqrt{X-a_l})$. We prove that the elements $b_i = X - a_i$ ($i = 1, \ldots, l$) verify the conditions (1) and (2^*_i) of Lemma 5.5.

Let $a_1 < \ldots < a_l$ in the order of R. Pick elements $r_1, \ldots, r_l \in R$ so that

$$r_1 < a_1 < r_2 < a_2 < \ldots < r_l < a_l.$$

The orders $P_i = P_{r_i^+}$ (or $P_{r_i^-}$) satisfy the conditions of 5.5. Indeed, for $a, r \in R$,

$$X - a <_{P_{r^+}} 0 \quad \text{iff} \quad a > r.$$

Then, $b_1 = X - a_1 <_{P_{r_1^+}} 0$; for $2 \leq i \leq l$, $X - a_j >_{P_{r_i^+}} 0$ whenever $j = 1, \ldots, i-1$, while $X - a_i <_{P_{r_i^+}} 0$. □

To modify the construction of Step 2 we use Proposition 8.7, p. 478, in [HJ], namely :

Proposition 5.7 *Let* K *be a field and* A *a closed subset of* $\chi(K)$. *Then, there is a pseudo real closed extension* L *of* K *such that*

(i) K *is relatively algebraically closed in* L.

(ii) *The restriction* $\rho_{L|K} : \chi(L) \longrightarrow \chi(K)$ *maps* $\chi(L)$ *homeomorphically onto* A. □

Let F_2 be the PRC extension of the field F constructed in Step 1 meeting the conditions of 5.7 (F_2 may not be an algebraic extension of F, while the field F_1 of Step 2 in Craven's theorem is). If $\sum_j f_j^2 \notin F^2$ (with, for example $f_1, \ldots, f_n$ as in 5.6), then $\sum_j f_j^2 \notin F_2^2$, by item ($i$) of 5.7. This completes the sketch of proof of Proposition 5.2. □

II. Formally real fields with prescribed Pythagoras number and a lower bound on the number of orders.

Proposition 5.8 *Let F_1, F_2 be formally real fields such that $G_{red}(F_1) \equiv G_{red}(F_2)$ (as special groups). Then either F_1 and F_2 have the same finite number of orders, or both have infinitely many orders. If, in addition, F_1, F_2 have finite Pythagoras number, the same conclusion holds under either of the assumptions $G(F_1) \equiv G(F_2)$ (as special groups) or $F_1 \equiv F_2$ (as fields).*

Proof: Observe first that the language L_{SG} of special groups is interpretable in the language of fields (i.e., unitary rings). This is just a matter of reproducing the standard definitions of the primitive notions of L_{SG} (representation, etc.; see section 3, chapter 1 in [DM1]) for the special group $G(F)$, F a field; we omit the details (some care has to be exercised in interpreting the quantifiers). At any rate, this shows, without any extra assumptions :

$$F_1 \equiv F_2 \text{ (as fields)} \quad \Rightarrow \quad G(F_1) \equiv G(F_2) \text{ (as special groups)}.$$

If F_1, F_2 have finite Pythagoras number, Corollary 4.3 gives :

$$G(F_1) \equiv G(F_2) \quad \Rightarrow \quad G_{red}(F_1) \equiv G_{red}(F_2).$$

Alternatively, if F has finite Pythagoras number, $G_{red}(F)$ is directly interpretable in F.

From $G_{red}(F_1) \equiv G_{red}(F_2)$ we have (Corollary 3.3) $B(F_1) \equiv B(F_2)$, where $B(F_i) = B_{G_{red}(F_i)}$, $i = 1, 2$. Now we consider two cases :

1) $\chi(F_1)$ is finite : Then, $B(F_1)$ is finite and $B(F_1) \approx B(F_2)$. Hence, $B(F_2)$ is finite and $|\chi(F_1)| = |\chi(F_2)|$ (= cardinal of $\chi(F_2)$).

2) $\chi(F_1)$ is infinite : Then $B(F_1)$ is infinite, and $B(F_1) \equiv B(F_2)$ implies $B(F_2)$ infinite, which, in turn, entails that $\chi(F_2)$ is infinite. □

For an integer $p \geq 1$ and $n \in \mathbb{N} \cup \{\aleph_0\}$, $n \geq 1$, write $FRF_{p,n}$ for the class of formally real fields with Pythagoras number exactly p, and having at least n (possibly infinitely many) orders.

The preceding result shows that $FRF_{p,n}$ is closed under elementary equivalence. It is also closed under ultraproducts; we sketch this proof :

Lemma 5.9 *Let $n \in \mathbb{N} \cup \{\aleph_0\}$, $n \geq 1$. Let $\{F_i : i \in I\}$ be a collection of formally real fields such that $|\chi(F_i)| \geq n$, for all $i \in I$. If $\mathfrak{U}$ is an ultrafilter on I, then $|\chi(\prod_{i\in I} F_i/\mathfrak{U})| \geq n$.*

Proof : It is enough to construct an injection

$$P : \textstyle\prod_{i\in I} \chi(F_i)/\mathfrak{U} \longrightarrow \chi(\prod_{i\in I} F_i/\mathfrak{U}).$$

For then, if $\kappa_i = |\chi(F_i)|$, we have $|\prod_{i\in I} \kappa_i/\mathfrak{U}| \geq |n^I/\mathfrak{U}| \geq n$, by Lemma 3.6 of Chapter 6 in [BS], as required. Observe, in passing, that if

$$\{i \in I : \kappa_i = m\} \in \mathfrak{U},$$

for some $m \in \mathbb{N}$, $m \geq n$, then $|\prod_{i \in I} \kappa_i/\mathfrak{U}| = m$ (Lemma 3.6, Chapter 6, [BS]). On the other hand, if this condition is not met (in particular if $n = \aleph_0$), then $|\prod_{I \in I} \kappa_i/\mathfrak{U}| \geq 2^{\aleph_0}$ (Lemma 3.11, Corollary 3.14, Chapter 6, [BS]).

We shall view orders of F as ± 1–characters whose kernels are closed under addition and which send -1 to -1. The map P is constructed in the obvious way : given $\sigma \in \prod_{i \in I} \chi(F_i)$, i.e., a map that selects an order in $\chi(F_i)$ for each $i \in I$, define, for $\langle f_i : i \in I \rangle \in \prod_{i \in I} F_i$;

$$P_\sigma(\langle f_i : i \in I \rangle) = \begin{cases} 1 & \text{if } \{i \in I : \sigma(i)(f_i) = 1\} \in \mathfrak{U}; \\ -1 & \text{if } \{i \in I : \sigma(i)(f_i) = 1\} \notin \mathfrak{U}. \end{cases}$$

Routine checking proves :

a) P_σ is a well defined ± 1-character on the multiplicative group of $\prod_{i \in I} F_i/\mathfrak{U}$, sending $-1/\mathfrak{U}$ to -1.

b) $ker(P_\sigma)$ is closed under addition, and so P_σ is an order on $\prod_{i \in I} F_i$.

c) $\sigma/\mathfrak{U} = \sigma'/\mathfrak{U} \;\Rightarrow\; P_\sigma = P_{\sigma'}$, that is, P_σ depends only on the class of σ modulo $\mathfrak{U}$, and we may unambiguously write $P_{\sigma/\mathfrak{U}} = P_\sigma$.

d) $\sigma/\mathfrak{U} \neq \sigma'/\mathfrak{U} \;\Rightarrow\; P_\sigma \neq P_{\sigma'}$, and hence the map $\sigma/\mathfrak{U} \mapsto P_\sigma$ is injective, as claimed. □

Corollary 5.10 *The class $FRF_{p,n}$ is axiomatizable in the language of fields.*

Proof : Immediate from 5.8 and 5.9, using the standard characterization of elementary classes (Theorem 4.1.12, p. 220, in [CK]). □

Obviously, this raises the question of finding an explicit axiom system for $FRF_{p,n}$ in the language of fields. This is easy for $n = 1, 2$; namely :

— $FRF_{p,1}$ is axiomatized by the sentences expressing that the Pythagoras number is exactly p :

(1) $$\forall x_1 \ldots x_{p+1} \, \exists y_1 \ldots y_p \, (\textstyle\sum_{i=1}^{p+1} x_i^2 = \sum_{i=1}^{p} y_j^2);$$

(2) $$\exists x_1 \ldots x_p \, \forall y_1 \ldots y_{p-1} \, (\textstyle\sum_{i=1}^{p} x_i^2 \neq \sum_{i=1}^{p-1} y_j^2);$$

plus :

(3) -1 is not a sum of p squares.

— $FRF_{p,2}$ is axiomatized by the sentences above plus the statement that the set of sums of squares is not (the positive cone of) an order (recall that a field F has a unique order iff ΣF^2 is an order). This is equivalent to $\Sigma F^2 \cup -\Sigma F^2 \neq F$ which, with the axioms above, boils down to :

$$\exists x \, \forall y_1 \ldots y_p \, (x \neq \textstyle\sum_{i=1}^{p} y_i^2 \;\wedge\; -x \neq \sum_{i=1}^{p} y_i^2).$$

It would be interesting to determine axioms for $FRF_{p,n}$ in general.

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M. Dickmann
Equipe de Logique Mathématique
Université de Paris VII
Paris — FRANCE
dickmann@logique.jussieu.fr

F. Miraglia
Departamento de Matemática
Universidade de São Paulo
São Paulo — BRAZIL
miraglia@ime.usp.br