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ORIGINAL ARTICLE

Multiple solutions for a Schrödinger–Bopp–Podolsky system with positive potentials

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Abstract
In this paper, we prove existence of solutions for a Schrödinger–Bopp–Podolsky system under positive potentials. We use the Ljusternick–Schnirelmann and Morse Theories to get multiple solutions with a priori given "interaction energy."

KEYWORDS
Ljusternick–Schnirelmann category, Morse Theory, multiplicity of solutions, nonlocal Schrödinger equation

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1 | INTRODUCTION

In this paper, we are concerned with existence and multiplicity results to the following system in $\mathbb{R}^3$:

$$\begin{cases} -\varepsilon^2 \Delta u + Vu + \lambda\phi u + f(u) = 0 \\ -\varepsilon^2\Delta\phi + \varepsilon^4\Delta^2\phi = u^2, \end{cases} \tag{P_ε}$$

where $\varepsilon > 0$ is a parameter, $V : \mathbb{R}^3 \to \mathbb{R}^3$ is a given external potential, and $f : \mathbb{R} \to \mathbb{R}$ is a nonlinearity satisfying suitable assumptions that will be given below. The unknowns are

$$u, \phi : \mathbb{R}^3 \to \mathbb{R} \quad \text{and} \quad \lambda \in \mathbb{R}.$$

Such a problem has been introduced in [13] and describes the physical interaction of a charged particle driven by the Schrödinger equation in the Bopp–Podolsky generalized electrodynamics. In particular, one arrives to a system like (P_ε) when looking at standing waves solutions in the purely electrostatic situation; indeed u represents the modulus of the wave function of the particle and ϕ is the electrostatic field. We refer the reader to [13] for more details and the physical origin of the system.

Actually there are few papers on Schrödinger–Bopp–Podolsky systems. We cite also [9, 17] where the authors study the critical case, [16] where the problem has been studied in the Proca setting on 3 closed manifolds, and [21] where the fibering method of Pohozaev has been used to deduce existence of solutions (depending on a parameter) and even nonexistence.

Coming back to our problem, we see that, for any fixed $\varepsilon > 0$, it is equivalent to the following one:

$$\begin{cases} -\Delta u + V(\varepsilon x)u + \lambda\phi u + f(u) = 0 \\ -\Delta\phi + \Delta^2\phi = u^2, \end{cases} \qquad (\widetilde{P}_\varepsilon)$$

in the sense that, once we find solutions (λ, u, ϕ) for $(\widetilde{P}_\varepsilon)$, the triple

$$\lambda, \quad u(\cdot/\varepsilon), \quad \phi(\cdot/\varepsilon)$$

will be a solution of (P_ε). We give now a first set of assumptions.

On V, we start by assuming that

(V0) $V : \mathbb{R}^3 \to \mathbb{R}$ is in $L^\infty_{loc}(\mathbb{R}^3)$ and satisfies

$$0 < \text{ess inf}_{x\in\mathbb{R}^3} V(x) =: V_0.$$

The function $f : \mathbb{R} \to \mathbb{R}$ is continuous and

(f1) $f(u) \geq 0$ for $u \geq 0$ and $f(u) = 0$ for $u \leq 0$

or alternatively

(f1)' $f(u) \geq 0$ if $u \geq 0$ and f is odd,

and moreover

(f2) $\exists\, q \in (2, 6)$ such that $\lim_{u\to\infty} f(u)/u^{q-1} = 0$,
(f3) $\lim_{u\to 0} f(u)/u = 0$.

As usual, we will denote with F the primitive of f such that $F(0) = 0$.

The natural functional spaces in which we find the solutions u, ϕ of $(\widetilde{P}_\varepsilon)$ are

$$u \in W_\varepsilon := \left\{ u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(\varepsilon x)u^2 < +\infty \right\},$$

$$\phi \in \mathcal{D} := \left\{ \phi \in D^{1,2}(\mathbb{R}^3) : \Delta\phi \in L^2(\mathbb{R}^3) \right\} = \overline{C_0^\infty(\mathbb{R}^3)}^{|\nabla\cdot|_2 + |\Delta\cdot|_2}.$$

The space W_ε is an Hilbert space with (squared) norm

$$\|u\|^2_{W_\varepsilon} := \int_{\mathbb{R}^3} |\nabla u|^2 + \int_{\mathbb{R}^3} V(\varepsilon x)u^2$$

and is continuously embedded into $H^1(\mathbb{R}^3)$.

The space $\mathcal{D}$ has been introduced and deeply studied in [13], where it is proved that $\mathcal{D} \hookrightarrow L^p(\mathbb{R}^3)$ for $p \in [6, +\infty]$.

Actually problem $(\widetilde{P}_\varepsilon)$ can be simplified more. Indeed, as it is standard in these kind of systems (see [13] for details), a usual *reduction argument* transforms $(\widetilde{P}_\varepsilon)$ into the following nonlocal equation:

$$-\Delta u + V(\varepsilon x)u + \lambda\phi_u u + f(u) = 0 \quad \text{in } \mathbb{R}^3, \tag{1.1}$$

where

$$\phi_u(x) = \int_{\mathbb{R}^3} \frac{1 - e^{-|x-y|}}{|x-y|} u^2(y)dy. \tag{1.2}$$

Moreover, $\phi_u \in \mathcal{D}$ if $u \in H^1(\mathbb{R}^3)$. Hence, from now on, we will refer always to (1.1) in the only unknowns u and λ, since ϕ_u is determined by u by the above formula.

Fixed $\varepsilon > 0$, by a solution of (1.1), we mean a pair $(u,\lambda) \in W_\varepsilon \times \mathbb{R}$ such that

$$\int_{\mathbb{R}^3} \nabla u \nabla v + \int_{\mathbb{R}^3} V(\varepsilon x) uv + \lambda \int_{\mathbb{R}^3} \phi_u uv + \int_{\mathbb{R}^3} f(u)v = 0, \quad \forall v \in W_\varepsilon. \tag{1.3}$$

Note that under our assumptions, all the integrals appearing in (1.3) are finite and the relation between λ and u is given, for $u \not\equiv 0$, by

$$\lambda = -\frac{\|u\|^2_{W_\varepsilon} + \int_{\mathbb{R}^3} f(u)u}{\int_{\mathbb{R}^3} \phi_u u^2}$$

and so in particular λ is negative.

It is also clear that $(0,\lambda)$, $\lambda \in \mathbb{R}$, is a solution of (1.1), that we call *trivial*. Of course we are interested in nontrivial solutions, namely, solutions with $u \not\equiv 0$. See also Remark 3.1 where a simple bifurcation result is stated.

Our next assumption is as follows:

(C) $W_\varepsilon \hookrightarrow\hookrightarrow L^p(\mathbb{R}^3)$ for $p \in (2,6)$.

The compact embedding can be achieved in various ways. For example,

(1) by imposing that V is coercive—in this case it is known that W_ε has compact embedding into $L^p(\mathbb{R}^3)$, $p \in [2,6)$;
(2) by imposing that for any $c, r > 0$

$$\text{meas}\{x \in B_r(y) : V(x) \le c\} \to 0 \quad \text{as } |y| \to +\infty.$$

Hereafter, $B_r(y)$ is the ball in $\mathbb{R}^3$ with radius $r > 0$ centered in y. Also in this case, the embedding is compact into $L^p(\mathbb{R}^3)$, $p \in [2,6)$, see [5, p. 553].
(3) By imposing that V is radial; in this case, the natural setting to work with is the radial framework, namely, the subspace of radial functions in W_ε (if u is radial, also ϕ_u is), which has compact embedding into $L^p(\mathbb{R}^3)$, $p \in (2,6)$. This setting is justified by the Palais' Principle of Symmetric Criticality and then the solutions found will satisfy (1.3) even when tested on nonradial functions of W_ε. Then, if V is radial, all the solutions u found in the theorems below are radial too.

In the following, we will simply speak of "negative, one sign or sign-changing solutions" to say that u is negative, one sign, or sign-changing.

The solutions (u,λ) of (1.1) will be found as critical points of a C^1 energy functional I_ε restricted to the surface energy (known in physics also as Fermi surface)

$$\left\{ u \in W_\varepsilon : \int_{\mathbb{R}^3} \phi_u u^2 = 1 \right\}$$

and then λ will be the associated Lagrange multiplier. In this context, using a standard terminology, we mean by a *ground state solution* a solution u whose energy $I_\varepsilon(u)$ is minimal (on the constraint) among all the solutions.

The results proved here are of two types, depending essentially if (f1) or (f1)' is assumed.

We start with the assumption (f1)'. In this case, infinitely many solutions with divergent energy are found and they are possible sign-changing.

Theorem 1.1. *Assume (f1)', (f2), (f3), (V0), and (C). Then, for any $\varepsilon > 0$, problem (1.1) possesses infinitely many solutions (u_n, λ_n) with*

$$\|u_n\|_{W_\varepsilon} \to +\infty, \qquad \frac{1}{2}\|u_n\|^2_{W_\varepsilon} + \int_{\mathbb{R}^3} F(u_n) \to +\infty,$$

$$\lambda_n = -\left(\|u_n\|^2_{W_\varepsilon} + \int_{\mathbb{R}^3} f(u_n)u_n \right) \to -\infty.$$

The ground state solutions can be assumed of one sign.

The next three theorems deal with the existence of solutions (u, λ) under assumption (f1).

We state explicitly a first result on the existence of ground state.

Theorem 1.2. *Assume (f1)–(f3), (V0), and (C). Then, for any $\varepsilon > 0$, problem (1.1) admits a ground state solution, which is negative.*

To get multiplicity results, the smallness of ε and the topological properties of the set of minima of the potential V, when achieved, will be important. Then, our next assumption is stronger than (V0):

(V1) $V : \mathbb{R}^3 \to \mathbb{R}$ is continuous and satisfies

$$0 < \min_{x\in\mathbb{R}^3} V(x) =: V_0, \quad \text{with } M := \left\{ x \in \mathbb{R}^3 : V(x) = V_0 \right\}$$

and ∂M is a smooth manifold.

Recall that $\mathrm{cat}_Y(X)$ denotes the Ljusternick–Schnirelmann category of the set X in Y; that is, it is the least number of closed and contractible sets in Y, which cover X. If $X = Y$, we just write $\mathrm{cat}(X)$.

Theorem 1.3. *Assume (f1)–(f3), (V1), and (C). Then, there exists $\varepsilon^* > 0$ such that for every $\varepsilon \in (0, \varepsilon^*]$, problem (1.1) has at least $\mathrm{cat}(M)$ negative solutions with low energy.*

Moreover, if M is bounded and $\mathrm{cat}(M) > 1$, there is another negative solution with high energy.

The meaning of "low energy" or "high energy" will be clear during the proof.

A second multiplicity result of negative solutions is obtained by making use of the Morse Theory. In this case, we introduce the next set of assumptions stronger than the previous one on f:

(f4) f is C^1, $f(u) \geq 0$ for $u \geq 0$, and $f(u) = 0$ for $u \leq 0$;
(f5) $\exists\, q \in (2, 6)$ such that $\lim_{u\to\infty} f'(u)/u^{q-2} = 0$;
(f6) $\lim_{u\to 0} f'(u) = 0$.

In the following, $\mathcal{P}_t(M)$ is the Poincaré polynomial of M.

Theorem 1.4. *Assume (f4)–(f6), (V1), and (C). Then, there exists $\varepsilon^* > 0$ such that for every $\varepsilon \in (0, \varepsilon^*]$, problem (1.1) has at least $2\mathcal{P}_1(M) - 1$ negative solutions, possibly counted with their multiplicity.*

It is clear that in general, we get a better result using the Morse Theory. For example,

(1) if M is obtained by a contractible domain cutting off k disjoint contractible sets, it is

$$\mathrm{cat}(M) = 2, \quad \text{and} \quad \mathcal{P}_1(M) = 1 + k;$$

(2) if M is obtained as a union of l spheres $\{S_i\}_{i=1,\dots l}$ and m anuli $\{A_j\}_{j=1,\dots,m}$ all pairwise disjoint, then, since $\mathrm{cat}(S_i) = \mathrm{cat}(A_j) = \mathcal{P}_1(S_i) = \mathcal{P}_1(A_j) = 2$, we get

$$\mathrm{cat}(M) = 2(l + m) \quad \text{and} \quad 2\mathcal{P}_1(M) - 1 = 2 \cdot 2(l + m) - 1.$$

As we said above, our approach is variational. In particular, to prove Theorem 1.3 and Theorem 1.4, a fundamental role is played by the autonomous problem

$$-\Delta u + V_0 u + \lambda \phi_u u + f(u) = 0 \quad \text{in } \mathbb{R}^3, \tag{1.4}$$

and especially by its ground state solution $\mathfrak{u}$, that is, the minimum of the associated energy functional (denoted with E_{V_0}) on the functions $u \in H^1(\mathbb{R}^3)$ satisfying

$$\int_{\mathbb{R}^3} \phi_u u^2 = 1.$$

En passant, we then prove existence and multiplicity results for (1.4), see Theorem 4.2 and Theorem 4.3 in Section 4.

Remark 1.5. Observe finally that all the solutions we find satisfy

$$\int_{\mathbb{R}^3} \phi_u u^2 = 1$$

but indeed the results are evenly true if we consider solutions with

$$\int_{\mathbb{R}^3} \phi_u u^2 = c, \quad c > 0.$$

Remark 1.6. Our theorems are true also if the potential f depends explicitly on $x \in \mathbb{R}^3$. In this case, the limits in (f2), (f3), (f5), (f6) have to be uniform in x. In this case, some degeneracy in x is also permitted, in the sense that f can be zero for x in some region $\mathcal{R}$ of $\mathbb{R}^3$. Physically speaking, it means that the potential f is acting only on $\mathbb{R}^3 \setminus \mathcal{R}$.

Let us briefly comment now on our assumptions and see the differences with the usual approach used in the literature to prove existence of multiple solutions

First of all observe that, under (f1) or (f1)', if $\lambda \geq 0$ is given a priori, we do not have any nontrivial solution. Indeed if u is a solution of (1.1), just multiplying the equation by the same u and integrating, we reach $u \equiv 0$. Moreover, the positivity of f in case (f1) will be important in proving that the ground state solution of the autonomous problem (1.4) is radial. Note that the constraint on which we will restrict E_{V_0} is not closed under the radial decreasing rearrangements.

Assumptions (f2) and (f3) are standard when using variational methods: They will allow to define a C^1 energy functional related to the problem. Analogously, the stronger assumptions (f4)–(f6) will be useful to deal with the second derivative of the functional and in implementing the Morse Theory.

In particular, our assumptions on f cover the case $f \equiv 0$.

As we have seen, assumption (V0) is useful to define the right functional spaces and (V1) will be useful to deal with the multiplicity result via the category of Ljusternick and Schnirelmann.

Finally, assumption (C) will be important in order to recover the compactness condition of Palais and Smale, recalled in Section 2.

To prove the result, we will be mainly inspired by the classical papers [6–8] where a general method to obtain multiplicity of solutions depending on the topology of the "domain" has been developed. Later on, many other problems (involving quasilinear or fractional equations, among many others) have been treated with the same ideas: We just recall here [1–4, 10–12, 14, 15, 19, 20, 23]. However, there are evident differences with our paper.

In these last cited papers, the functional is unbounded below on the space and the constraint is the well-known Nehari manifold. The advantage of working on the Nehari manifold is that the functional becomes bounded below. Moreover, this constraint is introduced as the set of zeroes of a function, which involves the same energy functional (actually its derivative) and is a natural constraint. In this way, suitable conditions on the nonlinearity f (e.g., Ambrosetti–Rabinowitz–type condition) permit to obtain the boundedness of the Palais–Smale sequence and then the compactness results. We recall that in this cases, an additional assumption on V is set at infinity:

$$V_0 < \liminf_{|x| \to +\infty} V(x) =: V_\infty \leq +\infty,$$

which is useful to obtain compactness. Moreover, when dealing with the constraint of the Nehari manifold, a great help is given by the fact that there is a minimax characterization of the projection of any nonzero element on the constraint.

In our case, the functional is positive on the whole space (hence interesting from a physical point of view since it represents an energy) and the constraint has nothing to do with the functional. However, it is always possible to project nonzero functions on the constraint, and this is done without using the assumption on $f(u)/u$. Moreover, although we have again the uniqueness of the projection, the minimax characterization is lost.

Observe finally that we do not need any Ambrosetti–Rabinowitz–type condition and even more, our nonlinearity can vanish somewhere. For these reasons, although we follow the general strategy of the cited papers, many classical proofs do not work and need to be readjusted. Another difference from the classical papers is that our solutions of Theorem 1.3 and Theorem 1.4 are negative.

Then, to the best of our knowledge, this is the first paper dealing with the "photography method" of Benci, Cerami, and Passaseo with assumptions on the nonlinearity different from the usual ones and applied to a functional restricted to a manifold, which is not the Nehari one.

We believe that an interesting problem will be the study of multiplicity of solutions in other cases in which f is negative, as well as, remove assumption (C) and address the problem with the approach of Lions by using concentration compactness arguments.

The organization of the paper is the following.

In Section 2, we introduce the related variational setting, the constraint, and its fundamental properties, and we show the compactness property of the functional.

In the brief Section 3, we prove Theorem 1.1 and Theorem 1.2.

In Section 4, we study the autonomous problem (1.4), with a general positive constant μ instead of V_0. We obtain multiplicity results of infinitely many solutions if (f1)' holds (see Theorem 4.2). In case (f1) holds, we found the important result concerning the ground state solution (see Theorem 4.3) that will be used later on to implement the barycenter machinery.

In the final Section 5, after defining the barycenter maps and its properties, we prove Theorem 1.3 and Theorem 1.4.

Notations. Here, we list few notations that will be used throughout the paper. Others will be introduced whenever we need.

(1) $|\cdot|_p$ is the L^p-norm;
(2) $H^1(\mathbb{R}^3)$ is the usual Sobolev space with norm $\|\cdot\|$;
(3) the conjugate exponent of r is denoted by r';
(4) $o_n(1)$ denotes a vanishing sequence;
(5) $C, C', \ldots$ stand to denote suitable positive constants whose values may also change from line to line.

2 PRELIMINARIES AND VARIATIONAL SETTING

Let us start with few preliminaries and recalling some well-known facts.

It is standard that from the growth conditions on f and f' given in (f2), (f3), (f5), and (f6), it follows that for any $\delta > 0$, there exists $C_\delta > 0$ such that for every $v, w \in H^1(\mathbb{R}^3)$,

$$\int_{\mathbb{R}^3} |f(u)v| \leq \delta \int_{\mathbb{R}^3} |uv| + C_\delta \int_{\mathbb{R}^3} |u|^{q-1}|v| \leq \delta |u|_2 |v|_2 + C_\delta |u|_q^{q-1} |v|_q \tag{2.1}$$

and

$$\int_{\mathbb{R}^3} |f'(u)vw| \leq \delta \int_{\mathbb{R}^3} |vw| + C_\delta \int_{\mathbb{R}^3} |u|^{q-2}|v||w| \leq \delta |v|_2 |w|_2 + C_\delta |u|_q^{q-2} |v|_q |w|_q. \tag{2.2}$$

For completeness, we recall the following properties of ϕ_u defined in (1.2). They are contained in [13, Lemma 3.4 and Lemma 5.1].

Lemma 2.1. *For every $u \in H^1(\mathbb{R}^3)$ we have the following:*

(i) for every $y \in \mathbb{R}^3$, $\phi_{u(\cdot+y)} = \phi_u(\cdot + y)$;
(ii) $\phi_u \geq 0$;
(iii) for every $s \in (3, +\infty]$, $\phi_u \in L^s(\mathbb{R}^3) \cap C_0(\mathbb{R}^3)$;
(iv) for every $s \in (3/2, +\infty]$, $\nabla\phi_u \in L^s(\mathbb{R}^3) \cap C_0(\mathbb{R}^3)$;
(v) $|\phi_u|_6 \leq C\|u\|^2$ for some constant $C > 0$;

(vi) ϕ_u *is the unique minimizer of the functional*

$$E(\phi) = \frac{1}{2}|\nabla\phi|_2^2 + \frac{1}{2}|\Delta\phi|_2^2 - \int_{\mathbb{R}^3} \phi u^2, \quad \phi \in \mathcal{D}.$$

Moreover if u *is radial also* ϕ_u *is, and if* $u_n \rightharpoonup u$ *in* $H^1_{rad}(\mathbb{R}^3)$*, then*

(vii) $\phi_{u_n} \to \phi_u$ *in* $\mathcal{D}$*;*
(viii) $\int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \to \int_{\mathbb{R}^3} \phi_u u^2$*;*
(ix) $\int_{\mathbb{R}^3} \phi_{u_n} u_n v \to \int_{\mathbb{R}^3} \phi_u uv$ *for any* $v \in H^1(\mathbb{R}^3)$*.*

Let us recall finally the following Hardy–Littlewood–Sobolev inequality, see, for example, [18, Theorem 4.3].

Theorem 2.2. *Assume that* $1 < a, b < \infty$ *satisfies*

$$\frac{1}{a} + \frac{1}{b} = \frac{5}{3}.$$

Then, there exists a constant $H > 0$ *such that*

$$\left|\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{f(x)g(y)}{|x-y|}\right| \le H|f|_a|g|_b, \quad \forall f \in L^a(\mathbb{R}^3), g \in L^b(\mathbb{R}^3).$$

As a consequence, we get the following:

Proposition 2.3. *Under assumption (V0), if* $\{u_n, u\} \subset W_\varepsilon$ *is such that* $u_n \to u$ *in* $L^{12/5}(\mathbb{R}^3)$*, then*

$$\int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \to \int_{\mathbb{R}^3} \phi_u u^2.$$

Proof. Indeed, by the Hardy–Littlewood–Sobolev inequality,

$$\begin{aligned}
\int_{\mathbb{R}^3} \left|\phi_{u_n} u_n^2 - \phi_u u^2\right| &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{1-e^{-|x-y|}}{|x-y|}\left|u_n^2(x)u_n^2(y) - u^2(x)u^2(y)\right| \\
&\le \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{1}{|x-y|}\left|u_n^2(x)u_n^2(y) - u^2(x)u^2(y)\right| \\
&\le \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\left|u_n^2(y) - u^2(y)\right|}{|x-y|}u_n^2(x) + \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\left|u_n^2(x) - u^2(x)\right|}{|x-y|}u^2(y) \\
&\le H\left|u_n^2 - u^2\right|_{6/5}\left(|u_n^2|_{6/5} + |u^2|_{6/5}\right) \\
&= o_n(1),
\end{aligned}$$

and the conclusion follows. □

The strategy to find solutions $(u_\varepsilon, \lambda_\varepsilon) \in W_\varepsilon \times \mathbb{R}$ for (1.1) will be to look at the critical points of the functional

$$I_\varepsilon(u) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 + \int_{\mathbb{R}^3} V(\varepsilon x)u^2 + \int_{\mathbb{R}^3} F(u) = \frac{1}{2}\|u\|^2_{W_\varepsilon} + \int_{\mathbb{R}^3} F(u) \tag{2.3}$$

restricted to the set

$$\mathcal{M}_\varepsilon = \{u \in W_\varepsilon : J(u) = 0\}, \quad \text{where } J(u) := \int_{\mathbb{R}^3} \phi_u u^2 - 1.$$

Observe that $\mathcal{M}_\varepsilon \neq \emptyset$. Indeed, fix $u \neq 0$ and define

$$h : t \in (0, +\infty) \to \mathbb{R} \quad \text{such that} \quad h(t) := t^4 \int_{\mathbb{R}^3} \phi_u u.$$

Then, there is a unique positive value $t_\varepsilon(u) > 0$ such that

$$1 = t_\varepsilon(u)^4 \int \phi_u u^2 = \int \phi_{t_\varepsilon(u)u}(t_\varepsilon(u)u)^2, \quad \text{that is, } t_\varepsilon(u)u \in \mathcal{M}_\varepsilon. \tag{2.4}$$

Of course, the value t_ε does not have any minimax characterization as it happens with the Nehari constraint. Note that $t_\varepsilon(u) = t_\varepsilon(-u)$ and it is clear that

$$\forall \varepsilon_1, \varepsilon_2 > 0 : \ \mathcal{M}_{\varepsilon_1} = \mathcal{M}_{\varepsilon_2},$$
$$u \in \mathcal{M}_\varepsilon \Longrightarrow \pm|u| \in \mathcal{M}_\varepsilon.$$

Moreover, we have immediately the following:

Lemma 2.4. *If $\{u_n\} \subset \mathcal{M}_\varepsilon$ is bounded in W_ε, then it cannot converge to zero in $L^{12/5}(\mathbb{R}^3)$.*

Proof. Otherwise by (v) of Lemma 2.1, we would have

$$1 = \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \leq |\phi_{u_n}|_6 |u_n|_{12/5}^2 \leq C|u_n|_{12/5}^2 = o_n(1),$$

which is a contradiction. □

The unknown λ will be deduced as the Lagrange multiplier associated to the critical point u of I_ε on $\mathcal{M}_\varepsilon$. Indeed this is justified by the next result.

Lemma 2.5. *Under assumptions (V0) and (C), the set $\mathcal{M}_\varepsilon$ is bounded away from zero in the weak topology and is weakly closed. Moreover, it is a C^1 manifold of codimension 1 homeomorphic to the unit sphere $\mathbb{S}_\varepsilon$ of W_ε.*

Proof. If there is $\{u_n\} \subset \mathcal{M}_\varepsilon$ weakly convergent to 0, then, due to condition (C), we get a contradiction with Lemma 2.4.

The fact that it is weakly closed follows again by condition (C) and Proposition 2.3.

Since (see [13])

$$J'(u)[v] = \frac{1}{4} \int_{\mathbb{R}^3} \phi_u uv, \quad \forall u, v \in W_\varepsilon,$$

we see that whenever $u \in \mathcal{M}_\varepsilon$, the operator $J'(u)$ is not the trivial one (since on the same u gives $1/4$). Hence, $\mathcal{M}_\varepsilon$ is a C^1 manifold of codimension 1.

To see that $\mathcal{M}_\varepsilon$ is homeomorphic to the unit sphere, consider the *projection map*

$$\xi_\varepsilon : \mathbb{S}_\varepsilon \mapsto \mathcal{M}_\varepsilon, \quad \text{such that } \xi_\varepsilon(u) = t_\varepsilon(u)u,$$

where $t_\varepsilon(u)$ is defined in (2.4). Note that ξ_ε is injective due to the unicity of $t_\varepsilon(u)$ and it is easy to see that its inverse is the continuous *retraction map*$\xi_\varepsilon^{-1}(u) = u/\|u\|_{W_\varepsilon}$.

Moreover, ξ_ε is continuous. Actually we show that it is weakly continuous. Let $\{u_n, u\} \subset \mathbb{S}_\varepsilon$ with $u_n \rightharpoonup u$ in W_ε. In particular, by condition (C) and Proposition 2.3, we infer

$$\int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \to \int_{\mathbb{R}^3} \phi_u u^2. \tag{2.5}$$

By using the Hölder inequality joint with (v) of Lemma 2.1, we have, for a suitable constant $C > 0$,

$$1 = t_\varepsilon(u_n)^4 \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \leq t_\varepsilon(u_n)^4 \|u_n\|_{W_\varepsilon}^4 \leq t_\varepsilon(u_n)^4 C, \tag{2.6}$$

and we infer that $\{t_\varepsilon(u_n)\}$ cannot tend to zero. On the other hand, if $t_\varepsilon(u_n) \to +\infty$, from the equality in (2.6), we deduce that

$$\int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \to 0$$

and from (2.5), $u = 0$, which is a contradiction.

As a consequence, $t_\varepsilon(u_n) \to t \neq 0$ (up to subsequence). Passing to the limit in (2.6), we deduce that

$$1 = t^4 \int_{\mathbb{R}^3} \phi_u u^2,$$

which means that $t = t_\varepsilon(u)$ and implies that $\xi_\varepsilon(u_n) \to \xi_\varepsilon(u)$. This shows that ξ_ε is a homeomorphism, concluding the proof. □

As by product of the proof of Lemma 2.5, we state explicitly the following result that will be useful later on.

Corollary 2.6. *Under the assumptions and notation of Lemma 2.5, if $\{u_n, u\} \subset W_\varepsilon$ are such that $u_n \rightharpoonup u \neq 0$ in $H^1(\mathbb{R}^3)$, then $t_\varepsilon(u_n) \to t_\varepsilon(u)$. In particular, if $u \in \mathcal{M}_\varepsilon$, then $t_\varepsilon(u_n) \to 1$.*

We know that the functional I_ε (under both assumptions (f1) or (f1)') is positive and indeed we have the following:

Lemma 2.7. *Assume (V0) and (C). Then,*

$$\mathfrak{m}_\varepsilon := \inf_{u \in \mathcal{M}_\varepsilon} I_\varepsilon(u) > 0.$$

Proof. If the infimum were zero, then there would exist $\{u_n\} \subset \mathcal{M}_\varepsilon$ such that

$$\int_{\mathbb{R}^3} \phi_{u_n} u_n^2 = 1, \quad I_\varepsilon(u_n) = \frac{1}{2}\|u_n\|^2_{W_\varepsilon} + \int_{\mathbb{R}^3} F(u_n) \to 0.$$

In particular, $|u_n|_{12/5} \to 0$ contradicting Lemma 2.4. □

Let us recall the notion of genus of Krasnoselsky. Given A a closed and symmetric subset of some Banach space, with $0 \notin A$, the *genus* of A, denoted as $\gamma(A)$, is defined as the least number $k \in \mathbb{N}$ such that there exists a continuous and even map $h : A \to \mathbb{R}^k \setminus \{0\}$. If such a map does not exist, the genus is set to $+\infty$ and finally $\gamma(\emptyset) = 0$. It is well known that the genus is a topological invariant (under odd homeomorphism) and that the genus of the sphere in $\mathbb{R}^N$ is N, while in infinite dimension, it is $+\infty$. Hence, by Lemma 2.5, we have the following:

Corollary 2.8. *Assume (V0) and (C). Then, the manifold $\mathcal{M}_\varepsilon$ (which is closed and symmetric with respect to the origin) has infinite genus.*

Proof. Just observe that $\mathcal{M}_\varepsilon$ is homeomorphic to the unit sphere via an odd homeomorphism. □

Let us pass now to study the functional I_ε defined in (2.3), namely,

$$I_\varepsilon(u) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 + \int_{\mathbb{R}^3} V(\varepsilon x)u^2 + \int_{\mathbb{R}^3} F(u).$$

Here, $\varepsilon > 0$ is fixed.

The compactness condition: As it is standard in variational methods, we will need a compactness condition, the so-called *Palais Smale condition*, that we recall here. In general given I, a C^1 functional on a Hilbert manifold $\mathcal{M}$, a sequence $\{u_n\} \subset \mathcal{M}$ is said to be a *Palais–Smale sequence* for I (briefly, a (PS) sequence) if $\{I(u_n)\}$ is bounded and $I'(u_n) \to 0$ in the tangent bundle. The functional I is said to satisfy the *Palais–Smale condition* if every (PS) sequence has a convergent subsequence to an element of $\mathcal{M}$.

The validity of this condition is strongly based on the compactness assumption (C).

Lemma 2.9. *Assume (f1) (or (f1)'), (f2), (f3), (V0), and (C). Then, the functional I_ε satisfies the (PS) condition on $\mathcal{M}_\varepsilon$.*

Proof. Let $\{u_n\} \subset \mathcal{M}_\varepsilon$ be a (PS) sequence for I_ε, then we can assume

$$I_\varepsilon(u_n) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u_n|^2 + \frac{1}{2}\int_{\mathbb{R}^3} V(\varepsilon x)u_n^2 + \int_{\mathbb{R}^3} F(u_n) \to c$$

and there exists $\{\lambda_n\} \subset \mathbb{R}$ such that

$$\forall v \in W_\varepsilon : \int_{\mathbb{R}^3} \nabla u_n \nabla v + \int_{\mathbb{R}^3} V(\varepsilon x)u_n v + \lambda_n \int_{\mathbb{R}^3} \phi_{u_n} u_n v + \int_{\mathbb{R}^3} f(u_n)v = o_n(1). \tag{2.7}$$

Since I_ε is coercive, the sequence $\{u_n\}$ is bounded in W_ε, then converges, after passing to a subsequence, weakly to u and being $\mathcal{M}_\varepsilon$ weakly closed, we have

$$\int_{\mathbb{R}^3} \phi_u u^2 = 1. \tag{2.8}$$

By choosing $v = u_n$ in (2.7), we have

$$\int_{\mathbb{R}^3} |\nabla u_n|^2 + \int_{\mathbb{R}^3} V(\varepsilon x)u_n^2 + \lambda_n + \int_{\mathbb{R}^3} f(u_n)u_n = o_n(1), \tag{2.9}$$

and since $\{u_n\}$ is bounded in W_ε, we infer that (using (2.1))

$$\left|\int_{\mathbb{R}^3} f(u_n)u_n\right| \le \delta|u_n|_2^2 + C_\delta|u_n|_q^q \le C.$$

Then, by (2.9), we deduce that $\{\lambda_n\}$ is bounded, hence converging, up to subsequences, to some λ.

By (2.7) again, we have

$$\forall v \in W_\varepsilon : \int_{\mathbb{R}^3} \nabla u \nabla v + \int_{\mathbb{R}^3} V(\varepsilon x)uv + \lambda \int_{\mathbb{R}^3} \phi_u uv + \int_{\mathbb{R}^3} f(u)v = 0.$$

In particular, by taking $v = u$, we see that $\int_{\mathbb{R}^3} V(\varepsilon x)u^2 < +\infty$, which joint to (2.8) gives that $u \in \mathcal{M}_\varepsilon$.

Finally, by taking $v = u_n - u$ in (2.7) and passing to the limit, since (as it is easy to see)

$$\int_{\mathbb{R}^3} \phi_{u_n} u_n(u_n - u) \to 0 \quad \text{and} \quad \int_{\mathbb{R}^3} f(u_n)(u_n - u) \to 0,$$

we infer that $\|u_n\|_{W_\varepsilon} \to \|u\|_{W_\varepsilon}$.

Then, $u_n \to u$ in W_ε, which concludes the proof. □

3 | PROOF OF THEOREM 1.1 AND THEOREM 1.2

As a consequence of the (PS) condition, we have existence of ground state, namely, a minimizer for I_ε on $\mathcal{M}_\varepsilon$, and actually infinitely many critical points under the oddness condition.

Proof of Theorem 1.1: The existence of the ground state is a consequence of the (PS) condition. Of course, $I_\varepsilon(\pm|u|) = I_\varepsilon(u)$ and we have actually a positive and a negative ground state.

Finally, by applying the Krasnoselski Genus Theory, we get the existence of infinitely many critical points $\{u_n\}$. That $\{u_n\}$ are at divergent critical levels follows from the abstract theory. Then, it is easy to see, since

$$\int_{\mathbb{R}^3} F(u_n) \le \int_{\mathbb{R}^3} (u_n^2 + C|u_n|^p),$$

that $\{u_n\}$ are divergent also in norm. By noticing that $f(t)t \ge 0$, the divergence of the Lagrange multipliers follows.

Proof of Theorem 1.2: It follows by the (PS) condition and the fact that $I_\varepsilon(-|u|) \le I_\varepsilon(u)$.

Remark 3.1. In case of negative ground states, since the functional essentially reduces to the (squared) norm, we can find an easy result concerning the bifurcation from the trivial solution $(0, \lambda)$ of the ground states.

To this aim, for any $\varepsilon, c > 0$, let us denote with $u_{\varepsilon,c}$ the negative ground state solution found in Theorem 1.1 or Theorem 1.2 (recall Remark 1.5) on the constraint

$$\int_{\mathbb{R}^3} \phi_u u^2 = c > 0,$$

and let $\lambda_{\varepsilon,c}$ be the associated Lagrange multiplier. Then, explicitly

$$I_\varepsilon(u_{\varepsilon,c}) = \frac{1}{2}\|u_{\varepsilon,c}\|^2_{W_\varepsilon}, \quad \int_{\mathbb{R}^3} \phi_{u_{\varepsilon,c}} u^2_{\varepsilon,c} = c, \quad \lambda_{\varepsilon,c} c = -\|u_{\varepsilon,c}\|^2_{W_\varepsilon} < 0. \tag{3.1}$$

We see that if $0 < c_1 < c_2$, then

$$\frac{1}{2}\|u_{\varepsilon,c_1}\|^2_{W_\varepsilon} = I_\varepsilon(u_{\varepsilon,c_1}) \le I_\varepsilon\big((c_1/c_2)^{1/4} u_{\varepsilon,c_2}\big) = \frac{1}{2}(c_1/c_2)^{1/2}\|u_{\varepsilon,c_2}\|^2_{W_\varepsilon},$$

which means that the map

$$c \in (0, +\infty) \mapsto \frac{\|u_{\varepsilon,c}\|^2_{W_\varepsilon}}{c^{1/2}} \in (0, +\infty) \quad \text{is increasing}$$

and then

$$\exists \lim_{c \to 0^+} \frac{\|u_{\varepsilon,c}\|^2_{W_\varepsilon}}{c^{1/2}} \in [0, +\infty).$$

In particular, $\lim_{c\to 0^+} \|u_{\varepsilon,c}\|^2_{W_\varepsilon} = 0$. Consequently by (3.1),

$$\lim_{c\to 0^+} \lambda_{\varepsilon,c} c = -\lim_{c\to 0^+} \|u_{\varepsilon,c}\|^2_{W_\varepsilon} = 0,$$

and we see that two cases hold:

(1) there exists a sequence $c_n \to 0^+$ such that $\lim_{n\to+\infty} \lambda_{\varepsilon,c_n} = \overline{\lambda} \in (-\infty, 0]$, or
(2) $\lim_{c\to 0^+} \lambda_{\varepsilon,c} = -\infty$.

In the first case, we have a bifurcation point $(0, \overline{\lambda})$; in the second case, we have a bifurcation "from $-\infty$."

4 | THE AUTONOMOUS PROBLEM

In order to prove the multiplicity results involving condition (f1), it will be important to consider the autonomous problem associated to (1.1).

For a given constant potential $\mu > 0$, consider the problem

$$-\Delta u + \mu u + \lambda \phi_u u + f(u) = 0 \quad \text{in} \quad \mathbb{R}^3. \tag{A_μ}$$

Let $H^1_\mu(\mathbb{R}^3)$ be the usual subspace of $H^1(\mathbb{R}^3)$ endowed with (squared) norm

$$\|u\|^2_\mu = \int_{\mathbb{R}^3} |\nabla u|^2 + \mu \int_{\mathbb{R}^3} u^2.$$

The solutions $(u,\lambda)\in H^1_\mu(\mathbb{R}^3)\times\mathbb{R}$ of (A_μ) are the critical points of the positive and C^1 functional

$$E_\mu(u)=\frac{1}{2}\int|\nabla u|^2+\frac{\mu}{2}\int u^2+\int_{\mathbb{R}^3}F(u)=\frac{1}{2}\|u\|^2_\mu+\int_{\mathbb{R}^3}F(u)$$

restricted to

$$\mathcal{M}_\mu=\left\{u\in H^1_\mu(\mathbb{R}^3):J(u)=0\right\},\quad J(u)=\int_{\mathbb{R}^3}\phi_u u^2-1.$$

It is clear that $\mathcal{M}_\mu$ is not empty and has the same properties of $\mathcal{M}_\varepsilon$ given in Lemma 2.4, Lemma 2.5, Corollary 2.6, and Corollary 2.8. Finally, as in Lemma 2.7, we have

$$\mathfrak{m}_\mu:=\inf_{u\in\mathcal{M}_\mu}E_\mu(u)>0.$$

Actually, in order to find solutions of (A_μ), we work in the subspace of radial functions since, by the Palais's Symmetric Criticality Principle, it is a natural constraint. Then, define

$$\mathcal{M}_{\mathrm{rad},\mu}:=\mathcal{M}_\mu\cap H^1_{\mathrm{rad},\mu}(\mathbb{R}^3)$$

(which evidently has the same properties of $\mathcal{M}_\mu$ and $\mathcal{M}_\varepsilon$) and

$$\mathfrak{m}_{\mathrm{rad},\mu}:=\inf_{u\in\mathcal{M}_{\mathrm{rad},\mu}}E_\mu(u)\geq\mathfrak{m}_\mu>0.$$

The advantage of the radial setting is that, due to the compact embedding of $H^1_{\mathrm{rad},\mu}(\mathbb{R}^3)$ into $L^p(\mathbb{R}^3)$, $p\in(2,6)$, the manifold $\mathcal{M}_{\mathrm{rad},\mu}$ is weakly closed. Then, we get the following compactness condition whose proof, being very similar to that of Lemma 2.9, is omitted.

Lemma 4.1. *Assume (f1) (or (f1)'), (f2), and (f3). Then, the functional E_μ satisfies the (PS) condition on $\mathcal{M}_{rad,\mu}$.*

Then, we deduce a result analogous to Theorem 1.1 for critical points of E_μ.

Theorem 4.2. *Assume (f1)', (f2), and (f3). Then, any minimizing sequence for E_μ on $\mathcal{M}_{rad,\mu}$ is convergent. So $\mathfrak{m}_{rad,\mu}$ is achieved and the ground state can be assumed of one sign.*

Indeed the functional E_μ possesses infinitely many critical points $\{u_n\}$ on $\mathcal{M}_{rad,\mu}$ with associated Lagrange multipliers $\{\lambda_n\}\subset(-\infty,0)$ satisfying

$$E_\mu(u_n)=\frac{1}{2}\|u_n\|^2_\mu+\int_{\mathbb{R}^3}F(u_n)\to+\infty,$$
$$\|u_n\|^2_\mu\to+\infty,$$
$$\lambda_n=-\left(\|u_n\|^2_\mu+\int_{\mathbb{R}^3}f(u_n)u_n\right)\to-\infty.$$

In particular (A_μ) has infinitely many solutions.

In case condition $(f1)$ holds, then we have the following:

Theorem 4.3. *Assume (f1) –(f3). Then, any minimizing sequence for E_μ on $\mathcal{M}_{rad,\mu}$ is convergent. So $\mathfrak{m}_{rad,\mu}$ is achieved on a radial function, hereafter denoted with $\mathfrak{u}$, and moreover*

$$\mathfrak{m}_{rad,\mu}=\mathfrak{m}_\mu=\min_{u\in\mathcal{M}_\mu}E_\mu(u)=E_\mu(\mathfrak{u})>0.$$

Finally, $\mathfrak{u}$ is negative, and then $E_\mu(\mathfrak{u})=\frac{1}{2}\|\mathfrak{u}\|^2_\mu$.

We stress the fact that $\mathfrak{u}$ has minimal energy on the whole $\mathcal{M}_\mu$, namely, even between nonradial functions.

Proof. We need just to prove that $\mathfrak{u}$ realizes the minimum among all functions in $\mathcal{M}_\mu$, and for this, it is sufficient to show that for any $u \in \mathcal{M}_\mu$, there is another function in $\mathcal{M}_{\text{rad},\mu}$ with less energy.

Then, let $u \in \mathcal{M}_\mu$, denote with u^* its Schwartz symmetrization and set $t_* > 0$ such that $t_* u^* \in \mathcal{M}_\mu$. By the rearrangement inequality (see [18, Theorem 3.7]), we get

$$\frac{1}{t_*^4} = \int_{\mathbb{R}^3} \phi_{u^*}(u^*)^2 \geq \int_{\mathbb{R}^3} \phi_u u^2 = 1$$

and deduce that $t_* \leq 1$. Consequently, using the properties of the spherical rearrangement and that f is positive, for a suitable $\xi \in (0,1)$:

$$\begin{aligned} E_\mu(t_* u^*) - E_\mu(u) &\leq \frac{1}{2}(t_*^2 - 1)\|u\|_\mu^2 + \int_{\mathbb{R}^3} (F(t_* u) - F(u)) \\ &= \frac{1}{2}(t_*^2 - 1)\|u\|_\mu^2 + (t_* - 1)\int_{\mathbb{R}^3} f(\xi u) \\ &\leq 0, \end{aligned}$$

which concludes the proof. The final part follows by $F(-|u|) \leq F(u)$. □

Remark 4.4. Analogously to Remark 3.1, we have bifurcation of the negative ground states found in Theorem 4.3 from the trivial solution also for the autonomous problem (A_μ).

The ground state $\mathfrak{u}$ found in Theorem 4.3 will have a special role from now on.

We observe that all we have seen up to now was valid for any fixed $\varepsilon > 0$ and it was never used that the infimum V_0 of V is achieved.

5 | THE BARYCENTER MAP AND PROOF OF THEOREM 1.3 AND THEOREM 1.4

Without the oddness assumption of f (namely, condition (f1)'), the multiplicity result is obtained, thanks to the smallness of ε and the fact that V_0 is achieved on a subset $M \subset \mathbb{R}^3$:

$$0 < \min_{x\in\mathbb{R}^3} V(x) =: V_0, \quad \text{with } M = \Big\{x \in \mathbb{R}^3 : V(x) = V_0\Big\}.$$

Without loss of generality, we assume $0 \in M$. Define the set of negative functions:

$$N := \big\{u : \mathbb{R}^3 \to (-\infty, 0]\big\}.$$

Consider the autonomous problem

$$-\Delta u + V_0 u + \lambda \phi_u u + f(u) = 0 \quad \text{in } \mathbb{R}^3$$

and let $\mathfrak{u}$ be the radial and negative function satisfying

$$\mathfrak{m}_{V_0} = \min_{u\in\mathcal{M}_{V_0}} E_{V_0}(u) = E_{V_0}(\mathfrak{u}) > 0,$$

see Theorem 4.3.

Finally, since $\mathcal{M}_\varepsilon \subset \mathcal{M}_{V_0}$ and $V(x) \geq V_0$,

$$E_{V_0}(\mathfrak{u}) \leq \mathfrak{m}_\varepsilon.$$

For $T > 0$, define η the smooth nonincreasing cut-off function defined in $[0,\infty)$ by

$$\eta(s) = \begin{cases} 1 & \text{if } 0 \leq s \leq T/2 \\ 0 & \text{if } s \geq T \end{cases}$$

and for any $y \in M$, set

$$\Psi_{\varepsilon,y}(x) := \eta(|\varepsilon x - y|)\mathfrak{u}\left(\frac{\varepsilon x - y}{\varepsilon}\right).$$

Let $t_{\varepsilon,y} := t_\varepsilon(\Psi_{\varepsilon,y}) > 0$ such that $t_{\varepsilon,y}\Psi_{\varepsilon,y} \in \mathcal{M}_\varepsilon$, and define the map

$$\Phi_\varepsilon : y \in M \mapsto t_{\varepsilon,y}\Psi_{\varepsilon,y} \in \mathcal{M}_\varepsilon,$$

which is easily seen to be continuous. By construction, for any $y \in M$, $\Phi_\varepsilon(y)$ has compact support and $\Phi_\varepsilon(y) \in \mathcal{M}_\varepsilon \cap N$. In particular,

$$I_\varepsilon(\Phi_\varepsilon(y)) = \frac{1}{2}\|\Phi_\varepsilon(y)\|^2_{W_\varepsilon}.$$

Lemma 5.1. *Assume (f1)–(f3), (V1), and (C). Then,*

$$\lim_{\varepsilon\to 0^+} I_\varepsilon(\Phi_\varepsilon(y)) = E_{V_0}(\mathfrak{u}), \quad \textit{uniformly in } y \in M.$$

Proof. Suppose by contradiction that there exist $\delta_0 > 0$, $\varepsilon_n \to 0^+$ and $\{y_n\} \subset M$ such that

$$|I_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) - E_{V_0}(\mathfrak{u})| \geq \delta_0. \tag{5.1}$$

From the Lebesgue's Theorem, we deduce

$$\lim_{n\to\infty}\int_{\mathbb{R}^3}|\nabla\Psi_{\varepsilon_n,y_n}|^2 = \int_{\mathbb{R}^3}|\nabla\mathfrak{u}|^2, \quad \lim_{n\to\infty}\int_{\mathbb{R}^3}V(\varepsilon_n x)\Psi^2_{\varepsilon_n,y_n} = V_0\int_{\mathbb{R}^3}\mathfrak{u}^2. \tag{5.2}$$

In particular, $\{\Psi_{\varepsilon_n,y_n}\}$ is bounded in W_{ε_n}, and so weakly convergent to some $v \in W_\varepsilon$ and a.e. in $\mathbb{R}^3$. By (5.2), it has to be $v = \mathfrak{u}$, and therefore we have, due to the compactness assumption (C),

$$\Psi_{\varepsilon_n,y_n} \rightharpoonup \mathfrak{u} \quad \text{in } W_\varepsilon, \quad \Psi_{\varepsilon_n,y_n} \to \mathfrak{u} \quad \text{in } L^p(\mathbb{R}^3),\ p \in (2,6).$$

Recalling that $\Phi_{\varepsilon_n}(y_n) = t_{\varepsilon_n,y_n}\Psi_{\varepsilon_n,y_n} \in \mathcal{M}_{\varepsilon_n}$ and Proposition 2.3, we get

$$\frac{1}{t^4_{\varepsilon_n,y_n}} = \int_{\mathbb{R}^3}\phi_{\Psi_{\varepsilon_n,y_n}}\Psi^2_{\varepsilon_n,y_n} = \int_{\mathbb{R}^3}\phi_{\mathfrak{u}}\mathfrak{u}^2 + o_n(1) = 1 + o_n(1),$$

which implies that $t_{\varepsilon_n,y_n} \to 1$. But then using (5.2), we conclude that

$$\begin{aligned} I_{\varepsilon_n}(t_{\varepsilon_n,y_n}\Psi_{\varepsilon_n,y_n}) &= \frac{t^2_{\varepsilon_n,y_n}}{2}\int_{\mathbb{R}^3}|\nabla\Psi_{\varepsilon_n,y_n}|^2 + \frac{t^2_{\varepsilon_n,y_n}}{2}\int_{\mathbb{R}^3}V(\varepsilon_n x)\Psi^2_{\varepsilon_n,y_n} \\ &\to E_{V_0}(\mathfrak{u}). \end{aligned}$$

contradicting (5.1). □

By Lemma 5.1, $h(\varepsilon) := |I_\varepsilon(\Phi_\varepsilon(y)) - E_{V_0}(\mathfrak{u})| = o_\varepsilon(1)$ for $\varepsilon \to 0^+$ uniformly in y, and then

$$I_\varepsilon(\Phi_\varepsilon(y)) - E_{V_0}(\mathfrak{u}) \leq \left|I_\varepsilon(\Phi_\varepsilon(y)) - E_{V_0}(\mathfrak{u})\right| \leq h(\varepsilon) = o_\varepsilon(1).$$

For simplicity, we do not write the explicit dependence of h by y. In particular, the sublevel set

$$\mathcal{M}_\varepsilon^{E_{V_0}(\mathfrak{u})+h(\varepsilon)} := \left\{u \in \mathcal{M}_\varepsilon : I_\varepsilon(u) \leq E_{V_0}(\mathfrak{u}) + h(\varepsilon)\right\} \tag{5.3}$$

is not empty, since for sufficiently small ε,

$$\Phi_\varepsilon(y) \in \mathcal{M}_\varepsilon^{E_{V_0}(\mathfrak{u})+h(\varepsilon)} \cap N. \tag{5.4}$$

5.1 | The barycenter map

We are in a position now to define the barycenter map that will send a convenient sublevel in $\mathcal{M}_\varepsilon$ in a suitable neighborhood of M. From now on, we fix a $T > 0$ in such a way that M and

$$M_{2T} := \left\{ x \in \mathbb{R}^3 : d(x, M) \le 2T \right\}$$

are homotopically equivalent (d denotes the euclidean distance). In particular, they are also homotopically equivalent to

$$M_T := \left\{ x \in \mathbb{R}^3 : d(x, M) \le T \right\}.$$

Let $\rho = \rho(T) > 0$ be such that $M_{2T} \subset B_\rho$ and $\chi : \mathbb{R}^3 \to \mathbb{R}^3$ be defined as

$$\chi(x) = \begin{cases} x & \text{if } |x| \le \rho \\ \rho \dfrac{x}{|x|} & \text{if } |x| \ge \rho. \end{cases}$$

Finally, let the *barycenter map*β_ε defined on functions with compact support $u \in W_\varepsilon$ by

$$\beta_\varepsilon(u) := \frac{\displaystyle\int_{\mathbb{R}^3} \chi(\varepsilon x) u^2}{\displaystyle\int_{\mathbb{R}^3} u^2} \in \mathbb{R}^3.$$

The next three lemmas give the behavior of β_ε and I_ε.

Lemma 5.2. *Under assumption (V1), the function β_ε satisfies*

$$\lim_{\varepsilon \to 0^+} \beta_\varepsilon(\Phi_\varepsilon(y)) = y, \text{ uniformly in } y \in M.$$

Proof. Suppose, by contradiction, that the lemma is false. Then, there exist $\delta_0 > 0$, $\varepsilon_n \to 0^+$ and $\{y_n\} \subset M$ such that

$$|\beta_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) - y_n| \ge \delta_0. \tag{5.5}$$

Using the definition of $\Phi_{\varepsilon_n}(y_n)$, β_{ε_n} and η given above, we have the equality

$$\beta_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) = y_n + \frac{\displaystyle\int_{\mathbb{R}^3} [\chi(\varepsilon_n z + y_n) - y_n] \left|\eta(|\varepsilon_n z|)\mathbf{u}(z)\right|^2}{\displaystyle\int_{\mathbb{R}^3} \left|\eta(|\varepsilon_n z|)\mathbf{u}(z)\right|^2}.$$

Using the fact that $\{y_n\} \subset M \subset B_\rho$ and the Lebesgue's Theorem, it follows

$$|\beta_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) - y_n| = o_n(1),$$

which contradicts (5.5) and the lemma is proved. □

Lemma 5.3. *Assume (f1)–(f3), (V1), and (C). If $\varepsilon_n \to 0$ and $\{u_n\} \subset \mathcal{M}_{\varepsilon_n}$ is such that $I_{\varepsilon_n}(u_n) \to E_{V_0}(\mathbf{u})$, then $\{u_n\}$ converges to $\mathbf{u}$ in $H^1_{V_0}(\mathbb{R}^3)$.*

Then, for n sufficiently large, $\{u_n\}$ can be assumed negative.

Proof. Since $\{u_n\} \subset \mathcal{M}_{\varepsilon_n} \subset \mathcal{M}_{V_0}$,

$$|I_{\varepsilon_n}(u_n) - E_{V_0}(u_n)| \le \int_{\mathbb{R}^3} (V(\varepsilon_n x) - V_0) u_n^2 \to 0,$$

we deduce that $E_{V_0}(u_n) \to E_{V_0}(\mathfrak{u})$, namely, $\{u_n\}$ is a minimizing sequence for E_{V_0} on $\mathcal{M}_{V_0}$. The result follows by Theorem 4.3. □

Lemma 5.4. *Assume (f1)–(f3), (V1), and (C). Then,*

$$\lim_{\varepsilon\to 0^+} \sup_{u\in\mathcal{M}_\varepsilon^{E_{V_0}(\mathfrak{u})+h(\varepsilon)}\cap N} \inf_{y\in M_T} \left|\beta_\varepsilon(u)-y\right| = 0.$$

Proof. Let $\{\varepsilon_n\}$ be such that $\varepsilon_n \to 0^+$. For each $n\in\mathbb{N}$, there exists $u_n \in \mathcal{M}_{\varepsilon_n}^{E_{V_0}(\mathfrak{u})+h(\varepsilon_n)} \cap N$ such that

$$\inf_{y\in M_T} \left|\beta_{\varepsilon_n}(u_n)-y\right| = \sup_{u\in\mathcal{M}_{\varepsilon_n}^{E_{V_0}(\mathfrak{u})+h(\varepsilon_n)}\cap N} \inf_{y\in M_T} \left|\beta_{\varepsilon_n}(u)-y\right| + o_n(1).$$

Thus, it suffices to find a sequence $\{y_n\}\subset M_T$ such that

$$\lim_{n\to\infty} \left|\beta_{\varepsilon_n}(u_n)-y_n\right| = 0. \tag{5.6}$$

Actually this holds for *any* sequence $\{y_n\}\subset M_T$. Indeed since $\{u_n\}\subset\mathcal{M}_{V_0}$ (and since under assumption (f1), $\mathfrak{u}$ is the ground state of E_{V_0} on the whole $\mathcal{M}_{V_0}$), we have

$$E_{V_0}(\mathfrak{u}) \le E_{V_0}(u_n) \le I_{\varepsilon_n}(u_n) \le E_{V_0}(\mathfrak{u}) + h(\varepsilon_n),$$

which implies that $I_{\varepsilon_n}(u_n) \to E_{V_0}(\mathfrak{u})$. Then, by Lemma 5.3,

$$\{u_n\} \text{ is convergent to } \mathfrak{u} \text{ in } H^1_{V_0}(\mathbb{R}^3). \tag{5.7}$$

Then, if $\{y_n\}$ is any sequence in M_T, since

$$\beta_{\varepsilon_n}(u_n) = y_n + \frac{\displaystyle\int_{\mathbb{R}^3} [\chi(\varepsilon_n z + y_n) - y_n] u_n^2(z)}{\displaystyle\int_{\mathbb{R}^3} u_n(z)^2},$$

by using (5.7), we see that $\{y_n\}$ verifies (5.6). □

In virtue of Lemma 5.4, there exists $\varepsilon^* > 0$ such that

$$\sup_{u\in\mathcal{M}_\varepsilon^{E_{V_0}(\mathfrak{u})+h(\varepsilon)}\cap N} d(\beta_\varepsilon(u), M_T) < T/2.$$

Define now

$$M^+ := M_{3T/2} = \left\{ x\in\mathbb{R}^3 \,:\, d(x,M) \le 3T/2 \right\}$$

so that M and M^+ are homotopically equivalent.

Now, reducing $\varepsilon^* > 0$ if necessary, we can assume that Lemma 5.2, Lemma 5.4, and (5.4) hold. Then, by standard arguments (see, e.g., [7, 8]) the composed map

$$M \xrightarrow{\Phi_\varepsilon} \mathcal{M}_\varepsilon^{E_{V_0}(\mathfrak{u})+h(\varepsilon)} \cap N \xrightarrow{\beta_\varepsilon} M^+ \quad \text{is homotopic to the inclusion map.} \tag{5.8}$$

At this point, we can finish the proof of the multiplicity result by implementing the Ljusternick–Schnirelmann Theory.

5.2 | The Ljusternick–Schnirelmann category: Proof of Theorem 1.3

By (5.8) and the very well-known properties of the category, we get, for any $\varepsilon \in (0, \varepsilon^*]$,

$$\mathrm{cat}(\mathcal{M}_\varepsilon^{E_{V_0}(\mathfrak{u})+h(\varepsilon)} \cap N) \geq \mathrm{cat}_{M^+}(M).$$

Then, since the (PS) condition holds (Lemma 2.9), the Ljusternik–Schnirelman Theory (see, e.g., [22]) applies and I_ε has at least $\mathrm{cat}_{M^+}(M) = \mathrm{cat}(M)$ critical points on $\mathcal{M}_\varepsilon$ with energy less than $E_{V_0}(\mathfrak{u}) + h(\varepsilon)$; so we have found $\mathrm{cat}(M)$ solutions for problem (1.1), which are negative.

To find the other solution, we argue as in [7]. Since M is not contractible, the compact set $\mathcal{A} := \overline{\Phi_\varepsilon(M)}$ cannot be contractible in $\mathcal{M}_\varepsilon^{E_{V_0}(\mathfrak{u})+h(\varepsilon)}$. Moreover, we can choose $v \leq 0, v \in \mathcal{M}_\varepsilon \setminus \mathcal{A}$ so v cannot be multiple of any element of $\mathcal{A}$. In particular, $I_\varepsilon(v) > E_{V_0}(\mathfrak{u}) + h(\varepsilon)$.

Let

$$\mathfrak{C} := \left\{ tv + (1-t)u \,:\, t \in [0,1], u \in \mathcal{A} \right\}$$

be the cone (hence compact and contractible) generated by v over $\mathcal{A}$. It follows that $0 \notin \mathfrak{C}$.

Consider also (see the map defined in the proof of Lemma 2.5)

$$\xi_\varepsilon(\mathfrak{C}) = \left\{ t_\varepsilon(w)w \,:\, w \in \mathfrak{C} \right\}$$

the projection of the cone on $\mathcal{M}_\varepsilon$, compact as well, and define

$$c := \max_{t_\varepsilon(\mathfrak{C})} I_\varepsilon > E_{V_0}(\mathfrak{u}) + h(\varepsilon).$$

Since $\mathcal{A} \subset \xi_\varepsilon(\mathfrak{C}) \subset \mathcal{M}_\varepsilon$ and $\xi_\varepsilon(\mathfrak{C})$ is contractible in $\mathcal{M}_\varepsilon^c := \{u \in \mathcal{M}_\varepsilon \,:\, I_\varepsilon(u) \leq c\}$, it follows that also $\mathcal{A}$ is contractible in $\mathcal{M}_\varepsilon^c$.

Summing up, we have a set $\mathcal{A}$, which is contractible in $\mathcal{M}_\varepsilon^c$ but not in $\mathcal{M}_\varepsilon^{E_{V_0}(\mathfrak{u})+h(\varepsilon)}$, and $c > E_{V_0}(\mathfrak{u}) + h(\varepsilon)$. The reason of that, since I_ε satisfies the (PS) condition, is due to the existence of another critical level between $E_{V_0}(\mathfrak{u}) + h(\varepsilon)$ and c. Then, we have another critical point in $\mathcal{M}_\varepsilon \cap N$ with higher energy.

The proof of Theorem 1.3 is thereby complete.

5.3 | The Morse Theory: Proof of Theorem 1.4

Here, we prove Theorem 1.4, hence assumptions (f4)–(f6) as well as (V1) and (C) are assumed here once for all.

Let us recall first few basic definitions and fix some notations.

Given a pair (X, Y) of topological spaces with $Y \subset X$, let $H_*(X, Y)$ be its singular homology with coefficients in some field $\mathbb{F}$ (from now on omitted) and

$$\mathcal{P}_t(X, Y) = \sum_k \dim H_k(X, Y) t^k$$

its Poincaré polynomial. Whenever $Y = \emptyset$, then it will be always omitted in all the objects, which involve the pair.

Recall also that if H is a Hilbert space, $I : H \to \mathbb{R}$ a C^2 functional, and u an isolated critical point with $I(u) = c$, the *polynomial Morse index* of u is defined as

$$\mathcal{I}_t(u) = \sum_k \dim C_k(I, u) t^k.$$

Here, given the sublevel $I^c = \{u \in H : I(u) \leq c\}$ and a neighborhood U of the critical point u, $C_k(I, u) = H_k(I^c \cap U, (I^c \setminus \{u\}) \cap U)$ denote the critical groups. The multiplicity of u is the number $\mathcal{I}_1(u)$.

When $I''(u)$ is associated to a self-adjoint isomorphism, then the critical point u is said to be nondegenerate and it holds $\mathcal{I}_t(u) = t^{m(u)}$, where $m(u)$ is the *(numerical) Morse index of* u: the maximal dimension of the subspaces where $I''(u)[\cdot, \cdot]$ is negative definite.

Lemma 5.5. *The functional I_ε is of class C^2 and for $u, v, w \in W_\varepsilon$,*

$$I''_\varepsilon(u)[v,w] = \int_{\mathbb{R}^3} \nabla v \nabla w + \int_{\mathbb{R}^3} V(\varepsilon x) vw + \int_{\mathbb{R}^3} f'(u)vw.$$

Moreover, $I''_\varepsilon(u)$ is represented by the operator

$$\mathrm{L}_\varepsilon(u) := \mathrm{R}(u) + \mathrm{K}(u) : W_\varepsilon \to W'_\varepsilon,$$

where $\mathrm{R}(u)$ is the Riesz isomorphism and $\mathrm{K}(u)$ is compact.

Proof. By (2.2) I''_ε is well defined and continuous. Then,

$$I_\varepsilon(u) \approx L_\varepsilon(u) := R(u) + \mathrm{K}(u) : W_\varepsilon \to W'_\varepsilon.$$

Let us show that, for $u \in W_\varepsilon$, $\mathrm{K}(u)$ is compact. Let then $v_n \rightharpoonup 0$ in W_ε and $w \in W_\varepsilon$. By (2.2), we get that given $\delta > 0$ for some constant $C_\delta > 0$:

$$\int_{\mathbb{R}^3} |f'(u)v_n w| \le \delta |v_n|_2 |w|_2 + C_\delta |u|_q^{q-2} |v_n|_q |w|_q$$

and the last term tends to zero due to assumption (C). By the arbitrarily of δ, we deduce

$$\|\mathrm{K}(u)[v_n]\| = \sup_{\|w\|_{W_\varepsilon}=1} \left| \int_{\mathbb{R}^3} f'(u) v_n w \right| \to 0,$$

namely, the compactness of $\mathrm{K}(u)$. □

Now for $a \in (0, +\infty]$, define the sublevels of the functional

$$I_\varepsilon^a := \left\{ u \in W_\varepsilon : I_\varepsilon(u) \le a \right\}, \qquad \mathcal{M}_\varepsilon^a := \mathcal{M}_\varepsilon \cap I_\varepsilon^a,$$

and the sets of critical points

$$\mathcal{K}_\varepsilon := \left\{ u \in W_\varepsilon : I'_\varepsilon(u) = 0 \right\}, \qquad \mathcal{K}_\varepsilon^a := \mathcal{K}_\varepsilon \cap I_\varepsilon^a, \qquad (\mathcal{K}_\varepsilon)_a := \left\{ u \in \mathcal{K}_\varepsilon : I_\varepsilon(u) > a \right\}.$$

In the remaining part of this section, we will follow [4, 8].

Let $\varepsilon^* > 0$ small as at the end of Section 5 and let $\varepsilon \in (0, \varepsilon^*]$ be fixed. In particular, I_ε satisfies the Palais–Smale condition. We are going to prove that I_ε restricted to $\mathcal{M}_\varepsilon$ has at least $2\mathcal{P}_1(M) - 1$ critical points.

We can assume, of course, that there exists a regular value $b_\varepsilon^* > E_{V_0}(\mathfrak{u})$ for the functional I_ε. Moreover, possibly reducing ε^*, we can assume that, see (5.3),

$$\Phi_\varepsilon : M \to \mathcal{M}_\varepsilon^{E_{V_0}(\mathfrak{u})+h(\varepsilon)} \cap N \subset \mathcal{M}_\varepsilon^{b_\varepsilon^*}.$$

Since Φ_ε is injective, it induces injective homomorphisms in the homology groups, then $\dim H_k(M) \le \dim H_k(\mathcal{M}_\varepsilon^{b_\varepsilon^*})$ and consequently

$$\mathcal{P}_t(\mathcal{M}_\varepsilon^{b_\varepsilon^*}) = \mathcal{P}_t(M) + \mathcal{Q}(t), \qquad \mathcal{Q} \in \mathbb{P}, \tag{5.9}$$

where $\mathbb{P}$ is the set of all polynomials with nonnegative integer coefficients.

As in [8, Lemma 5.2], we have the following:

Lemma 5.6. *Let $r \in (0, E_{V_0}(\mathfrak{u}))$ and $a \in (r, +\infty]$ a regular level for I_ε. Then,*

$$\mathcal{P}_t(I_\varepsilon^a, I_\varepsilon^r) = t\mathcal{P}_t(\mathcal{M}_\varepsilon^a). \tag{5.10}$$

Then, the following result holds.

Corollary 5.7. *Let $r \in (0, \mathfrak{m}_{V_0})$. Then,*

$$\mathcal{P}_t(I_\varepsilon^{b_\varepsilon^*}, I_\varepsilon^r) = t\Big(\mathcal{P}_t(M) + \mathcal{Q}(t)\Big), \qquad \mathcal{Q} \in \mathbb{P},$$

$$\mathcal{P}_t(W_\varepsilon, I_\varepsilon^r) = t.$$

Proof. The first equality follows by (5.9) and (5.10) simply by choosing $a = b_\varepsilon^*$. The second one follows by (5.10) with $a = +\infty$ and recalling that $\mathcal{M}_\varepsilon$ is contractible. □

To deal with critical points above the regular level b_ε^*, we recall also the following result whose proof is only based on notions of algebraic topology and is exactly as in [8, Lemma 5.6], see also [4, Lemma 2.4].

Lemma 5.8. *It holds*

$$\mathcal{P}_t(W_\varepsilon, I_\varepsilon^{b_\varepsilon^*}) = t^2\Big(\mathcal{P}_t(M) + \mathcal{Q}(t) - 1\Big), \qquad \mathcal{Q} \in \mathbb{P}.$$

Then, by using the Morse Theory, we arrive at the following fundamental result.

Corollary 5.9. *Suppose that the set $\mathcal{K}_\varepsilon$ is discrete. Then,*

$$\sum_{u \in \mathcal{K}_\varepsilon^{b_\varepsilon^*}} \mathcal{I}_t(u) = t\Big(\mathcal{P}_t(M) + \mathcal{Q}(t)\Big) + (1+t)\mathcal{Q}_1(t)$$

and

$$\sum_{u \in (\mathcal{K}_\varepsilon)_{b_\varepsilon^*}} \mathcal{I}_t(u) = t^2\Big(\mathcal{P}_t(M) + \mathcal{Q}(t) - 1\Big) + (1+t)\mathcal{Q}_2(t),$$

where $\mathcal{Q}, \mathcal{Q}_1, \mathcal{Q}_2 \in \mathbb{P}$.

Proof. Indeed the Morse Theory gives

$$\sum_{u \in \mathcal{K}_\varepsilon^{b_\varepsilon^*}} \mathcal{I}_t(u) = \mathcal{P}_t(I_\varepsilon^{b_\varepsilon^*}, I_\varepsilon^r) + (1+t)\mathcal{Q}_1(t)$$

and

$$\sum_{u \in (\mathcal{K}_\varepsilon)_{b_\varepsilon^*}} \mathcal{I}_t(u) = \mathcal{P}_t(W_\varepsilon, I_\varepsilon^{b_\varepsilon^*}) + (1+t)\mathcal{Q}_2(t)$$

so that, by using Corollary 5.7 and Lemma 5.8, we easily conclude. □

Then by Corollary 5.9, we get

$$\sum_{u \in \mathcal{K}_\varepsilon} \mathcal{I}_t(u) = t\mathcal{P}_t(M) + t^2\Big(\mathcal{P}_t(M) - 1\Big) + t(1+t)\mathcal{Q}(t)$$

for some $\mathcal{Q} \in \mathbb{P}$. We easily deduce that, if the critical points of I_ε are nondegenerate, then they are at least $2\mathcal{P}_1(M) - 1$, if counted with their multiplicity.

Then, the proof of Theorem 1.4 is complete.

CONFLICT OF INTEREST STATEMENT

The authors do not have any conflicts of interest to declare.

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