SYMBOLIC DYNAMICS OF PIECEWISE CONTRACTIONS

Benito Pires
Departamento de Computação e Matemática, Faculdade de Filosofia, Ciências e Letras
Universidade de São Paulo, 14040-901, Ribeirão Preto - SP, Brazil
benito@usp.br

Abstract. A map $f\colon [0,1) \to [0,1)$ is a *piecewise contraction of* n *intervals* (n-PC) if there exist $0 < \lambda < 1$ and a partition of $I = [0,1)$ into intervals $I_1, I_2, \ldots, I_n$ such that $|f(x) - f(y)| \le \lambda |x - y|$ for every $x, y \in I_i$ ($i = 1, 2, \ldots, n$). An infinite word $\theta = \theta_0\theta_1\ldots$ over the alphabet $\mathcal{A} = \{1, \ldots, n\}$ is a *natural coding of* f if there exists $x \in I$ such that $\theta_k = i$ whenever $f^k(x) \in I_i$. We prove that if θ is a natural coding of an injective n-PC, then some infinite subword of θ is either periodic or isomorphic to a natural coding of a topologically transitive m-interval exchange transformation (m-IET), where $m \le n$. Conversely, every natural coding of a topologically transitive n-IET is also a natural coding of some injective n-PC.

1. Introduction

Throughout this article, let $I = [0,1)$ denote the unit interval. A map $f\colon I \to I$ is a *piecewise contraction of* n *intervals* (n-PC) if there exist $0 < \lambda < 1$ and a partition of I into non-degenerate intervals $I_1, \ldots, I_n$ such that $f|_{I_i}$ is λ-Lipschitz for every $1 \le i \le n$. If, in particular, there exist $b_1, \ldots, b_n \in \mathbb{R}$ and $\sigma_1, \ldots, \sigma_n \in \{-1, 1\}$ such that $f(x) = \sigma_i \lambda x + b_i$ for every $x \in I_i$, then we say that f is a *piecewise* λ*-affine contraction.*

The *natural* f*-coding of a point* $x \in I$ is the infinite word $\theta_f(x) = \theta_0\theta_1\ldots$ defined by $\theta_k = i$ whenever $f^k(x) \in I_i$, where f^0 denotes the identity map. We say that an infinite word θ is a *natural coding of* f if $\theta = \theta_f(x)$ for some $x \in I$. We say that θ is *ultimately periodic* (respectively, *periodic*) if there exist finite subwords u, v of θ such that $\theta = uvv\ldots$ (respectively, $\theta = vv\ldots$). The *language* $\mathcal{L}(\theta)$ of a natural coding θ is the union of the sets $L_k(\theta) = \{\theta_m\theta_{m+1}\cdots\theta_{m+k-1} : m \ge 0\}$ of finite subwords of length k occuring in θ, where L_0 is the one-point-set formed by the empty word.

In this article, we give a complete and systematic description of the languages of injective n-PCs, $n \ge 2$, by providing a dictionary between these languages and the fairly well-understood languages of interval exchange transformations (IETs). We also provide converse results which enable us to construct n-PCS with any prescribed admissible coding.

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The first point addressed in this article consists in providing the list of all admissible natural codings of injective n-PCs. Natural codings of piecewise contractions defined on 2 intervals (or more generally, defined on 2 complete metric spaces) were provided by Gambaudo and Tresser [14] and are intrinsically related to natural codings of rotations of the circle. Concerning languages of injective n-PCs $f\colon I \to I$ for $n > 2$, some progress was made recently by Catsigeras, Guiraud and Meyroneinc [7]. They proved that for each natural coding θ of f, the complexity function of the language $\mathcal{L}(\theta)$, defined by $p_\theta(k) = \# L_k(\theta)$, where # denotes cardinality, is eventually affine.

The second point concerns the problem of how to construct n-PCs with any prescribed list of admissible natural codings. In this regard, it follows from the works [21, 22, 23] that a generic n-PC admits only ultimately periodic natural codings. Therefore, n-PCs with no ultimately periodic natural coding are exotic and their construction is a nontrivial issue. The existence of 2-PCs having no ultimately periodic natural coding is related to the existence of smooth flows on the 2-torus with pathological dynamics (see Cherry [8]). More generally, 2-PCs topologically semiconjugate to irrational rotations have being constructed and studied via a rotation number approach (see [4, 5, 6, 15, 16, 18]). Here we address the second point in full generality, by using another approach, based on the existence of an invariant measure (see [26]). In particular, we prove that every minimal n-IET, with $n \geq 2$, with or without flips, is a topological factor of an n-PC with no ultimately periodic natural coding. This combined with Keane's irrationality criteria [17, p. 27] provides a huge class of exotic n-PCs. Since every irrational rotation can be considered as a minimal 2-PC, the previous results fit into our framework.

As for the motivation to study n-PCs, it is worth remarking that they describe pretty well the dynamics of some Cherry flows on 2-manifolds, dissipative outer billiards, traffic systems, queueing systems and switched server systems.

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2. Statement of the results

A bijective map $T\colon I \to I$ is an *n-interval exchange transformation* (n-IET) if there exist a partition $I_1, \ldots, I_n$ of I into non-degenerate intervals, $b_1, \ldots, b_n \in \mathbb{R}$ and $\sigma_1, \ldots, \sigma_n \in \{-1, 1\}$ such that $T(x) = \sigma_i x + b_i$ for every $x \in I_i$. an n-IET T is *standard* if I_i a left-closed right-open interval and $T|_{I_i}$ is the translation $x \mapsto x + b_i$ for every $1 \leq i \leq n$. Following [20], we say that a non-standard n-IET T *has flips* if, for some $1 \leq i \leq n$, $T|_{I_i}$ is the map $x \mapsto -x + b_i$. We say that an n-IET $T\colon I \to I$ is *irreducible* if there is no $0 < \delta < 1$ such that $T\big([0, \delta)\big) \subset [0, \delta)$ or, equivalently, if there is no $1 \leq j \leq n-1$ such that $T\big(\cup_{i=1}^{j} I_i\big) \subset \cup_{i=1}^{j} I_i$.

An n-IET $T\colon I \to I$ is: *topologically transitive* if it has a dense T-orbit $\{x, T(x), \ldots\}$, *minimal* if every T-orbit is dense, and *aperiodic* if it has no periodic orbit. A periodic orbit γ is *attractive* if there exists an open set U_γ such that $\omega(x) = \gamma$ for every $x \in U_\gamma$. In a standard n-IET, every periodic orbit is attractive, thus in this case topological transitivity is equivalent to minimality. An n-IET T satisfies the *infinite distinct orbit condition* (i.d.o.c.) if the orbits $\{x_i, T(x_i), \ldots\}$ of its discontinuities x_i, $1 \le i \le n-1$, are infinite and pairwise disjoint. Keane [17] proved that every irreducible standard n-IET, $n \ge 2$, satisfying the i.d.o.c. is *minimal*. The *natural T-coding* of a point $x \in I$ is the infinite word $\theta_T(x) = \theta_0\theta_1\ldots$ defined by $\theta_k = i$ whenever $T^k(x) \in I_i$. If T is an irreducible standard n-IET satisfying the i.d.o.c., then the language $\mathcal{L}(\theta)$ of a natural T-coding θ is the same for any θ. In this case, we define the *language of T*, denoted by $\mathcal{L}$, to be the language $\mathcal{L}(\theta)$ of any of its natural T-codings.

The *alphabet* $\mathcal{A}(\theta)$ of an infinite word $\theta = \theta_0\theta_1\ldots$ is the set of letters that occur in θ. We say that two infinite words $\theta = \theta_0\theta_1\ldots$ and $\omega = \omega_0\omega_1\ldots$ are *isomorphic* if their alphabets $\mathcal{A}(\theta)$ and $\mathcal{A}(\omega)$ have the same cardinality and there is a bijection $\pi\colon \mathcal{A}(\theta) \to \mathcal{A}(\omega)$ such that $\omega_k = \pi(\theta_k)$ for every $k \ge 0$. For example, the infinite words

$$\theta = 010010001\ldots \quad \text{and} \quad \omega = 121121112\ldots$$

are isomorphic because $\mathcal{A}(\theta) = \{0,1\}$ and $\mathcal{A}(\omega) = \{1,2\}$ have the same cardinality and $\omega_k = \pi(\theta_k)$ for every $k \ge 0$, where the bijection $\pi : \mathcal{A}(\theta) \to \mathcal{A}(\omega)$ is given by $\pi(0) = 1$ and $\pi(1) = 2$.

Our main results are the following.

Theorem 2.1. *Let $f\colon I \to I$ be an injective n-PC, then there exist $2 \le m \le n$ and an m-IET $T\colon I \to I$ without attractive periodic orbits such that for each $x \in I$ there exists an integer $k \ge 0$ such that the natural f-coding of $f^k(x)$ is either periodic or isomorphic to a non ultimately periodic natural coding of T.*

Theorem 2.2. *Given any topologically transitive n-IET $T\colon I \to I$, there exist an injective piecewise $\frac{1}{2}$-affine contraction $f_T\colon I \to I$ of n intervals and a continuous, surjective, non-decreasing map $h\colon I \to I$ such that $\theta_{f_T}(x) = \theta_T\big(h(x)\big)$ for every $x \in I$. In particular, if T is an irreducible standard n-IET satisfying the i.d.o.c., then the language of each natural coding of f_T equals the language of T.*

In Theorem 2.1, the term "without attractive periodic orbits" may be replaced by "aperiodic" in the case in which $I_i = \big[x_{i-1}, x_i\big)$ and $f|_{I_i}$, $1 \le i \le n$, is (strictly) increasing, where $I_1, \ldots, I_n$ is the partition associated to f.

Theorem 2.1 turns out to be a dictionary between languages of PCs and languages of IETs. Languages of minimal IETs were studied by Belov and Chernyat'ev [2], Ferenczi [11], Ferenczi and Zamboni [12], and Dolce and Perrin [10]. In particular, it is known that if θ is a natural coding of an irreducible standard minimal n-IET, then the language $\mathcal{L}(\theta)$ does not depend on θ, is uniformly recurrent and has complexity function satisfying

$p_\theta(k) \le (n-1)k+1$, where the equality holds if the IET satisfies the i.d.o.c.. Languages generated by substitutions (e.g. languages of self-similar IETs) were studied by Lopez and Narbel [19]. Natural codings of aperiodic n-IETs are isomorphic to natural codings of topologically transitive n-IETs.

Theorem 2.2 provides examples of n-PCs without periodic or ghost orbits, or equivalently, without ultimately periodic natural codings. These examples are not easy to construct because generically n-PCs of the interval are asymptotically periodic (see [22, 23]). The following result is a corollary of Theorem 2.1.

Corollary 2.3. *Let θ be a natural coding of an injective n-PC, then some infinite subword of θ is either periodic or isomorphic to a non ultimately periodic natural coding of a topologically transitive m-IET, where $2 \le m \le n$.*

Theorems 2.1 and 2.2 imply, in particular, the result of Catsigeras, Guiraud and Meyroneinc [7] concerning the complexity function of languages of n-PCs, which is stated below in a more complete way, with f_T given by Theorem 2.2.

Corollary 2.4. *Let θ be a natural coding of an injective n-PC $f\colon I \to I$, then*

- (*i*) *There exist $\alpha \in \{0,1,\ldots,n-1\}$, $\beta \ge 1$ and $k_0 \ge 1$ such that the complexity function of θ satisfies $p_\theta(k) = \alpha k + \beta$ for every $k \ge k_0$ with $\beta = 1$ if $\alpha = n-1$;*
- (*ii*) *If $n \ge 2$, $T\colon I \to I$ is a standard n-IET satisfying the i.d.o.c. and $f = f_T$, then $p_\theta(k) = (n-1)k+1$ for every $k \ge 1$.*

The particular family of 2-PCs $f\colon I \to I$ defined by $f(x) = \lambda x + \delta \pmod 1$ was considered by Bugeaud [4, 5], Bugeaud and Conze [6] and, more recently, by Janson and Öberg [16], and also by Laurent and Nogueira [18], by means of a rotation number approach. Concerning such family, we provide the following corollary, which turns out to be a special case of [18, Corollary 7]. We recall that an n-PC $f{:}I \to I$ is *topologically semiconjugate* to an n-IET $T{:}I \to I$ if there exists a continuous, nondecreasing and surjective map $h{:}I \to I$ such that $h \circ f = T \circ h$.

Corollary 2.5. *For each irrational $0 < \alpha < 1$, there exists a transcendent $\delta \in \mathbb{R}$ such that the 2-PC $f_T\colon I \to I$ and the minimal 2-IET $T\colon I \to I$ defined by*

$$f_T(x) = \frac{1}{2}x + \delta \ (\mathrm{mod}\ 1) \quad \textit{and} \quad T(x) = x + \alpha \ (\mathrm{mod}\ 1)$$

are topologically semiconjugate and every natural coding of f_T is a Sturmian sequence. In particular, if $\alpha = 2 - \varphi$, where $\varphi = (1+\sqrt{5})/2$ is the golden ratio, then $\delta = 1 - \frac{R}{2}$, where R is the rabbit constant.

In Corollary 2.5, we have that $\delta = \frac{1}{4}\sum_{k\ge 0} \theta_k 2^{-k}$, where $\theta = \theta_0\theta_1\ldots$ is the natural coding of α under the action of the irrational rotation $x \mapsto x + \alpha \pmod 1$.

3. Preparatory lemmas

In this section, we present some results that will be used to prove Theorem 2.1, as the next table clarifies.

result	needs
Lemma 3.3	Lemma 3.2
Corollary 3.4	Lemma 3.3
Theorem 3.5	Lemma 3.6 and [26, Theorem 2.1]
Theorem 2.1	Lemma 3.1, Corollary 3.4 and Theorem 3.5

Throughout this section, let $I=[0,1)$ and $f\colon I\to I$ be an injective n-PC with associated partition $I_1,\ldots,I_n$ whose endpoints are $0=x_0<x_1<\cdots<x_n=1$.

The ω*-limit set* of $x\in I$ is defined by

$$\omega(x)=\bigcap_{\ell\geq 1}\overline{\bigcup_{k\geq\ell}\{f^k(x)\}},$$

where $\overline{S}$ denotes the topological closure in $\mathbb{R}$ of any set $S\subset I$.

Lemma 3.1. *Let $x\in I$ be such that $\omega(x)$ is finite, then there exists an integer $k\geq 0$ such that the natural f-coding of $f^k(x)$ is periodic.*

Proof. We may assume that $\{x,f(x),f^2(x),\ldots\}$ is an infinite set, otherwise x would be a periodic point, then we could take $k=0$. Since $\omega(x)$ is a finite set, we may write $\omega(x)=\{p_1,\ldots,p_r\}$. Without loss of generality, we suppose that $\omega(x)\subset(0,1)$, thus there exists $\epsilon>0$ so small that

$$\epsilon<\frac{1}{4}\min_{1\leq i<j\leq r}|p_i-p_j| \text{ and } \bigcup_{j=1}^{r}(p_j-\epsilon,p_j)\cup(p_j,p_j+\epsilon)\subset I\backslash\{x_0,x_1,\ldots,x_{n-1}\}.$$

In particular, if

$$\mathscr{I}=\{(p_1-\epsilon,p_1),(p_1,p_1+\epsilon),\ldots,(p_r-\epsilon,p_r),(p_r,p_r+\epsilon)\},$$

then $f(J)$ is an open interval for every $J\in\mathscr{I}$.

Let $\mathscr{I}'\subset\mathscr{I}$ denote the subcollection formed by the intervals that are visited infinitely many times by the f-orbit of x, that is,

$$\mathscr{I}'=\left\{J\in\mathscr{I}:\{x,f(x),f^2(x),\ldots\}\cap J \text{ is an infinite set}\right\}.$$

We claim that for each $J_1\in\mathscr{I}'$, there exists $J_2\in\mathscr{I}'$ such that $f(J_1)\subset J_2$. Without loss of generality, suppose that $J_1=(p-\epsilon,p)$, where $p\in\omega(x)$. As $J_1\subset I\backslash\{x_0,x_1,\ldots,x_{n-1}\}$, we have that $f|_{J_1}$ is a contraction, thus $f(J_1)$ is an open interval of length smaller than ϵ. On the other hand, since $J_1\in\mathscr{I}'$, there exists an increasing sequence of integers $0\leq k_1<k_2<\cdots$ such that $\left\{f^{k_j}(x)\right\}_{j\geq 1}\subset J$. Notice that $\lim_{j\to\infty}f^{k_j}(x)=p$, otherwise there would exist a point of $\omega(x)$ in J_1 different from p, which contradicts the first inequality in the definition of ϵ. Because $f|_{J_1}$ is injective and continuous, we have that $\left\{f^{k_j+1}(x)\right\}_{j\geq 1}\subset$

$f(J_1)$ converges to some point $q \in \omega(x) \cap \partial f(J_1)$, where $\partial f(J_1)$ denotes the endpoints of the open interval $f(J_1)$. Putting it all together, we conclude that $f(J_1)$ is an open interval that contains infinitely many points of the f-orbit of x, has length smaller than ϵ, and has an endpoint in $\omega(x)$. Therefore, there exists $J_2 \in \mathscr{I}'$ such that $f(J_1) \subset J_2$.

To finish the proof, let $J \in \mathscr{I}'$, then there exists $k' \geq 0$ such that $f^{k'}(x) \in J$. By the claim, there exist $1 \leq i_1 < i_2$ and intervals $J_1, \ldots, J_{i_1}, J_{i_1+1}, \ldots, J_{i_2} \in \mathscr{I}'$ such that J_1=J, $J_{i_1} = J_{i_2}$ and $f(J_i) \subset J_{i+1}$ for all $1 \leq i \leq i_2 - 1$, proving that $f^k(x)$ has a periodic natural f-coding for some $k \geq k'$. □

Lemma 3.2. *Let $J \subset I$ be an open interval, then there exists a finite set $B \subset I$ such that if $J_0 \subset J \backslash B$ is an open interval, then one of the following happens:*

(*i*) *$f(J_0)$, $f^2(J_0)$, ... are pairwise disjoint open intervals contained in $I \backslash J$;*

(*ii*) *$\exists m \geq 0$ such that $f^{m+1}(J_0)$ is open subinterval of J. Moreover, if $m \geq 1$, then $f(J_0), \ldots, f^m(J_0)$ are open subintervals of $I \backslash \big(J \cup \{x_0, x_1, \ldots, x_{n-1}\}\big)$.*

Proof. Let $J \subset I$ be an open interval. Given $x \in I$, set

$$\tau_x = \min \left\{k \geq 0 : f^{-k}\big(\{x\}\big) \subset J\right\}, \tag{1}$$

where by convention $\inf \emptyset = \infty$. Let

$$B = \bigcup \left\{f^{-\tau_x}(\{x\}) : x \in \{x_0, x_1, \ldots, x_{n-1}\} \cup \partial J \text{ and } \tau_x < \infty\right\}.$$

Let $J_0 \subset J \backslash B$ be an open interval, then one of the following alternatives happens: $\{f^k(J_0)\}_{k \geq 1} \subset I \backslash J$ or there exists an integer $\ell \geq 1$ such that $f^\ell(J_0) \cap J \neq \emptyset$. In the first case, by the injectivity of f and also because $J_0 \subset J \backslash B$, we have that J_0, $f(J_0)$, $f^2(J_0)$,... are pairwise disjoint sets contained in $I \backslash \{x_0, x_1, \ldots, x_{n-1}\}$. Since each $f|_{I_i}$ is Lipschitz continuous, we conclude that $f^k(J_0)$ is an open interval for every $k \geq 0$, which proves (*i*). As for the second alternative, let $m = \min \left\{\ell \geq 1 : f^\ell(J_0) \cap J \neq \emptyset\right\} - 1$. If $m = 0$, then $f(J_0) \cap J \neq \emptyset$, which together with the fact that $J_0 \subset J \backslash B$ implies that $f(J_0)$ is an open subset of J. Otherwise, if $m \geq 1$, then proceeding as in the first case yields that the sets $f(J_0), \ldots, f^m(J_0)$ are pairwise disjoint open subintervals of $I \backslash \big(J \cup \{x_0, x_1, \ldots, x_{n-1}\}\big)$. Moreover, because $J_0 \subset J \backslash B$, we have that $f^{m+1}(J_0) \cap J \neq \emptyset$ implies that $f^{m+1}(J_0)$ is an open subinterval of J. □

Remark. The item (*ii*) of Lemma 3.2 implies that $f^{m+1}|_{J_0} : J_0 \mapsto f^{m+1}(J_0)$ is a bijective contraction.

Lemma 3.3. *If for some $x \in I$ and $1 \leq i \leq n$, the set $\{x, f(x), f^2(x), \ldots\} \cap (x_{i-1}, x_i)$ is infinite and $\omega(x) \cap (x_{i-1}, x_i) = \emptyset$, then $\omega(x)$ is finite.*

Proof. By hypothesis, we have that

(H1) $\{x, f(x), f^2(x), \ldots\} \cap (x_{i-1}, x_i)$ is infinite;

(H2) $\{x, f(x), f^2(x), \ldots\} \cap K$ is finite for all compact set $K \subset (x_{i-1}, x_i)$.

By (H1), the orbit of x returns to $J = (x_{i-1}, x_i)$ infinitely many times. Let $1 \le k_1 < k_2 < \cdots$ denote the return times of x to J under the action of f. Because of (H2), we have only three cases to consider.

Case (a). $\lim_{j\to\infty} f^{k_j}(x) = x_{i-1}$.

Let B the finite set given by Lemma 3.2 considering $J = (x_{i-1}, x_i)$. Let $\epsilon > 0$ be so small that $J_0 = (x_{i-1}, x_{i-1} + \epsilon)$ is a subset of $J\backslash B$. Notice that the alternative (i) of Lemma 3.2 cannot occur. In fact, since $f^{k_j}(x) \downarrow x_{i-1}$, we have that $f^k(J_0) \cap J \ne \emptyset$ for many positive values of k. By exclusion, the item (ii) of Lemma 3.2 is true, then there exists $m \ge 0$ such that $f^{m+1}(J_0)$ is an open subinterval of J and, if $m \ge 1$, then $f(J_0), \ldots, f^m(J_0)$ are open subintervals of $I\backslash\big(J \cup \{x_0, x_1, \ldots, x_{n-1}\}\big)$. In particular, if $y \in J_0$, then $m+1$ is the first return time of y to J. This means that if $j_0 \ge 1$ is such that $\{f^{k_j}(x)\}_{j\ge j_0} \subset J_0$, then $\{f^{k_j}(x)\}_{j>j_0} \subset f^{m+1}(J_0)$, implying that x_{i-1} belongs to the boundary of the open interval $f^{m+1}(J_0)$. Moreover, since $f^{m+1}|_{J_0}\colon J_0 \to f^{m+1}(J_0)$ is a bijective contraction (see the Remark after Lemma 3.2), we have that $f^{m+1}(J_0) \subset J$ is an open interval with length smaller than ϵ and with an endpoint in x_{i-1}, thus $f^{m+1}(J_0) \subset (x_{i-1}, x_{i-1} + \epsilon) = J_0$. This implies that $\omega(x)$ is finite.

Case (b). $\lim_{j\to\infty} f^{k_j}(x) = x_i$.

Just proceed as in Case (a) considering now $J_0 = (x_i - \epsilon, x_i)$.

Case (c). $\bigcap_{\ell\ge1} \overline{\bigcup_{j\ge\ell} \{f^{k_j}(x)\}} = \{x_{i-1}, x_i\}$.

The proof presented here is a variation of that used in Case (a). Let $\epsilon > 0$ be so small that $J_0' = (x_{i-1}, x_{i-1} + \epsilon)$ and $J_0'' = (x_i - \epsilon, x_i)$ are contained in $J\backslash B$. Then by the same arguments used in Case (a), there exist $m', m'' \ge 0$ such that $f^{m'+1}(J_0')$ and $f^{m''+1}(J_0'')$ are disjoint open subintervals of J and, if $m' \ge 1$ (respectively, $m'' \ge 1$), then $f(J_0'), \ldots, f^{m'}(J_0')$ $\big($respectively, $f(J_0''), \ldots, f^{m''}(J_0'')\big)$ are open subintervals of $I\backslash\big(J \cup \{x_0, x_1, \ldots, x_{n-1}\}\big)$. In particular, if $y \in J_0'$ (respectively, if $y \in J_0''$), then $m'+1$ (respectively, $m''+1$) is the first return time of y to J. This means that if $j_0 \ge 1$ is such that $\{f^{k_j}(x)\}_{j\ge j_0} \subset J_0' \cup J_0''$, then $\{f^{k_j}(x)\}_{j>j_0} \subset f^{m'+1}(J_0') \cup f^{m''+1}(J_0'')$, implying that $x_{i-1} \in \partial f^{m''+1}(J_0'')$ and $x_i \in \partial f^{m'+1}(J_0')$. Moreover, since $f^{m'+1}|_{J_0'}\colon J_0' \to f^{m'+1}(J_0')$ and $f^{m''+1}|_{J_0''}\colon J_0'' \to f^{m''+1}(J_0'')$ are bijective contractions, we can argue in the same way as in Case (a) to conclude that $f^{m'+1}(J_0') \subset J_0''$ and $f^{m''+1}(J_0'') \subset J_0'$, proving that $\omega(x)$ is finite. □

Lemma 3.3 leads to the following result.

Corollary 3.4. *Let $x \in I$ be such that $\omega(x)$ is infinite. If for some $1 \le i \le n$, the set $\{x, f(x), f^2(x), \ldots\} \cap (x_{i-1}, x_i)$ is infinite, then $\omega(x) \cap (x_{i-1}, x_i) \neq \emptyset$.*

We will also need the following result, which is a variation of [26, Theorem 2.1].

Theorem 3.5. *Let $x \in I$ be such that $\Lambda = \omega(x)$ is infinite, then there exists a non-atomic f-invariant Borel probability measure whose support is Λ.*

The proof of Theorem 3.5 depends on Lemma 3.6 stated below. In what follows, let $x \in I$ be such that $\Lambda = \omega(x)$ is infinite. As x is not periodic, there exists $\ell \ge 0$ such that $\{f^k(x) : k \ge \ell\} \cap \{x_0, x_1, \ldots, x_{n-1}\} = \emptyset$. Hence, by replacing x by $f^\ell(x)$ if necessary, we assume that

$$\big\{x, f(x), f^2(x), \ldots\big\} \cap \{x_0, x_1, \ldots, x_{n-1}\} = \emptyset. \tag{2}$$

Denote by $\{\nu_m\}_{m\ge1}$ the sequence of Borel probability measures on I defined by

$$\nu_m = \frac{1}{m}\sum_{k=0}^{m-1} \delta_{f^k(x)},$$

where $\delta_{f^k(x)}$ is the Dirac probability measure on I concentrated at $f^k(x)$. By the Banach-Alaoglu Theorem, there exist a Borel probability measure on I, denoted henceforth by ν, and a subsequence of $\{\nu_m\}_{m\ge1}$, denoted henceforth by $\{\nu_{m_j}\}_{j\ge1}$, that converges to ν in the weak*- topology. We will keep these notations until the end of this section.

Lemma 3.6. *Let $y \in I$, then there exist an open subinterval J_y of I containing y and an integer $j_0 \ge 1$ such that $\nu_{m_j}(J_y) < \epsilon$ for every $j \ge j_0$. Moreover, the support of ν is $\Lambda = \omega(x)$.*

Proof. Let $y \in I$ and $\epsilon > 0$. We will prove that there exist $\delta > 0$ and $j_0 \ge 1$ such that the interval

$$J_y = \begin{cases} [0, \delta) & \text{if } y = 0 \\ (y - \delta, y + \delta) & \text{if } y > 0 \end{cases}$$

satisfies $J_y \subset I$ and $\nu_{m_j}(J_y) < \epsilon$ for all $j \ge j_0$. Without loss of generality, we may assume that $y > 0$ and $J_y = (y - \delta, y + \delta)$. Since ν is a probability measure, ν has at most countably many atoms, which means that the set

$$\Delta = \Big\{0 < \delta < \min\{y, 1 - y\} : \nu\big(\{y - \delta, y + \delta\}\big) = 0\Big\}$$

contains arbitrarily small values of δ. It follows from [25, Theorem 6.1, p. 40] that if $\delta \in \Delta$, then

$$J_y \subset I \quad \text{and} \quad \nu(J_y) = \lim_{j\to\infty} \nu_{m_j}(J_y). \tag{3}$$

Now have two cases to consider.

Case I: $y \notin \Lambda$, that is, $y \notin \omega(x)$.

In this case, there exist $\delta \in \Delta$ and $j_0 \geq 1$ such that $f^k(x) \notin J_y$ for every $k \geq m_{j_0}$. Let $j_1 \geq j_0$ be such that $m_j > m_{j_0}/\epsilon$ for every $j \geq j_1$, then

$$\nu_{m_j}(J_y) = \frac{1}{m_j}\#\left\{0 \leq k \leq m_{j_0} - 1 : f^k(x) \in J_y\right\} \leq \frac{m_{j_0}}{m_j} < \epsilon, \quad \forall j \geq j_1.$$

Moreover, making $j \to \infty$ and using (3) yield $\nu(J_y) = 0$, implying that y does not belong to the support of ν.

Case II: $y \in \Lambda$.

First assume that there exists an increasing sequence of integers $1 \leq k_1 < k_2 < \cdots$ such that $f^{k_j}(x) \uparrow y$. Since f is an injective piecewise contraction, the following limits are well-defined:

$$y_0 = y, \quad y_1 = \lim_{j\to\infty} f\big(f^{k_j}(x)\big), \quad y_2 = \lim_{j\to\infty} f^2\big(f^{k_j}(x)\big), \quad \ldots$$

We claim that $\#\{k \geq 1 : y_k = y\} \leq 1$. By way of contradiction, suppose that there exist $1 \leq p < q$ such that $y_q = y_p = y$. It is elementary to see that for every $\delta > 0$ small enough and $A_0 = (y - \delta, y)$, the sets $A_1 = f(A_0), A_2 = f^2(A_0), \ldots, A_q = f^q(A_0)$ are open intervals of length less than δ. Yet, $y_k \in \partial A_k$ for every $0 \leq k \leq q$. Hence, either $A_p \subset A_0$ or $A_q \subset A_0$, which contradicts the fact that $\omega(x)$ is infinite. In this way, the claim is true. Then, there exists $r_0 \geq 1$ such that $y_k \neq y$ for all $k \geq r_0$. In particular, given $r \geq 1$, there exists $\delta_1 = \delta_1(r)$ such that for every $0 < \delta < \delta_1$,

$$\#\left\{0 \leq k \leq r - 1 : f^k\big((y - \delta, y)\big) \cap J_y \neq \emptyset\right\} \leq 2.$$

Let $r > 0$ be such that $\frac{3}{r} < \frac{\epsilon}{3}$. Set $\delta_1 = \delta_1(r)$. Then, for all $0 < \delta < \delta_1$ with $\delta \in \Delta$ and for any j large enough,

$$\nu_{m_j}\big((y - \delta, y)\big) = \frac{1}{m_j}\#\left\{0 \leq k \leq m_j - 1 : f^k(x) \in J_y\right\} \leq \frac{3}{r} < \frac{\epsilon}{3}.$$

Now assume that the sequence $1 \leq k_1 < k_2 < \ldots$ does not exist, then for every δ small enough,

$$\nu_{m_j}\big((y - \delta, y)\big) = 0 < \frac{\epsilon}{3}.$$

Likewise, there exists $\delta_2 > 0$ such that for all $0 < \delta < \delta_2$ with $\delta \in \Delta$, we have that $\nu_{m_j}\big((y, y + \delta)\big) < \frac{\epsilon}{3}$ for any j large enough. Moreover, $\nu_{m_j}(\{y\}) < \frac{\epsilon}{3}$ for any j large enough. Putting all together, there exist $\delta > 0$ with $\delta \in \Delta$ and $j_0 \geq 1$ such that $\nu_{m_j}(J_y) < \epsilon$ for all $j \geq j_0$.

It remains to prove that in this case y belongs to the support of ν. By the above, we know that the orbit of x enters in J_y infinitely many times. If we prove that the return times of x to J_y are bounded, then we will conclude that $\inf_{j\geq j_0} \nu_{m_j}(J_y) > 0$, which together with (3) will imply that $\nu(J_y) > 0$. Let $S = \{x_0, x_1, \ldots, x_{n-1}\} \cup \partial J_y$ and $S' = \big\{z \in S : \cup_{k\geq 0} f^{-k}(\{z\}) \cap J_y \neq \emptyset\big\}$. Given $z \in S'$, let

$$\tau_z = \min\{k \geq 0 : f^{-k}(\{z\}) \subset J_y\}$$

and $B = \{f^{-\tau_z}(z) : z \in S'\}$. If U is a connected component of $J_y \backslash B$, then all points of U either never return to J_y or return to J_y at the same time. The second case always happens when $U \subset J_y$ is a small interval with an endpoint at y. In particular, the return times of the points of the orbit of x to J_y are bounded. □

Proof of Theorem 3.5. Theorem 3.5 is a variation of [26, Theorem 2.1] where the hypotheses of no connection and no periodic orbit were weakened. Here we just point out which change is necessary in the proof of [26, Theorem 2.1]. In this regard, [26, Lemma 3.2] ought to be replaced by Lemma 3.6. The hypothesis that f has no periodic orbit in the statement of [26, Theorem 2.1] is not necessary: all we need is that $\omega(x)$ is infinite. In this way, the claims of [26, Theorem 2.1] hold in our context, which proves Theorem 3.5. □

4. Proof of Theorem 2.1

Throughout this section, let $f\colon I \to I$ be an n-PC with associated partition $I_1, \ldots, I_n$ having endpoints $0 = x_0 < x_1 < \cdots < x_n = 1$. We will need the following elementary result.

Lemma 4.1 ([21, Lemma 3.6]). *There exist $r \le 2n$ pairwise disjoint open intervals $F_1, \ldots, F_r$ such that $f^k(F_j)$, $1 \le j \le r$, $k < 0$, are empty sets, and $f^k(F_j)$, $1 \le j \le r$, $k \ge 0$, are pairwise disjoint open intervals and $\Omega = \cup_{j=1}^r \cup_{k\ge 0} f^k(F_j)$ is a dense subset of $I\backslash\{x_0, x_1, \ldots, x_{n-1}\}$ having Lebesgue measure 1.*

A non-empty compact subset $\Lambda \subset [0,1]$ is an *attractor* of f if there exists $p \in I$ such that $\Lambda = \omega(p)$. Let $F_1, \ldots, F_r$ be as in the statement of Lemma 4.1, then for each $1 \le j \le r$, $\cup_{k\ge0} f^k(F_j) \cap \{x_0, x_1, \ldots, x_{n-1}\} = \emptyset$, implying that $\omega(p_j)$ is the same for any $p_j \in \cup_{k\ge0} f^k(F_j)$. In this way, the attractors

$$\Lambda_1 = \omega(p_1), \ldots, \Lambda_r = \omega(p_r) \tag{4}$$

do not depend on the choice of $(p_1, \ldots, p_r) \in F_1 \times \cdots \times F_r$.

Lemma 4.2. *Let $p \in I$. If $\omega(p)$ is infinite, then $\omega(p) \in \Lambda_1 \cup \cdots \cup \Lambda_r$.*

Proof. Since $\omega(p)$ is infinite, the f-orbit of p is not periodic. In particular, there exists $k_0 \ge 0$ such that the f-orbit of $f^{k_0}(p)$ does not pass through discontinuities. By the density of Ω, there exists $1 \le j \le r$ such that $f^{k_0}(p) \in \omega(p_j)$. Then, $\omega(p) = \omega\big(f^{k_0}(p)\big) \subset \omega(p_j) = \Lambda_j$. □

Without loss of generality, by replacing r by a smaller number, we may assume that the sets $\Lambda_1, \ldots, \Lambda_r$ are pairwise distinct. It follows from Lemma 4.1 that $S = I\backslash\Omega$ is a Lebesgue null set. Let $1 \le j \le r$. As $\overline{S} = S \cup \{1\}$ and $\Lambda_j \subset \overline{S}$, we have that $\overline{S}$ has empty interior, hence Λ_j is totally disconnected. By the Cantor-Bendixson Theorem, we conclude that Λ_j is either a finite set or the union of a Cantor set with a discrete set. If all the attractors $\Lambda_1, \ldots, \Lambda_r$ are finite, then, by Lemmas 3.1 and 4.2, all natural codings of f are ultimately periodic and we are done. Otherwise, there are $1 \le s \le r$ infinite attractors. Without loss of generality, assume that $\Lambda_1, \ldots, \Lambda_s$ are the infinite attractors. It follows from Theorem 3.5 that for each $1 \le j \le s$, there exists a non-atomic f-invariant Borel probability measure μ_j whose support is Λ_j. Hence, if

$$\mu = \frac{1}{s}\mu_1 + \cdots + \frac{1}{s}\mu_s, \quad A = \Lambda_1 \cup \cdots \cup \Lambda_s,$$

then μ is a non-atomic f-invariant Borel probability measure with support equal to A.

Let $h\colon [0,1] \to [0,1]$ be the nondecreasing, continuous, surjective map defined by $h(t) = \mu\big([0,t]\big)$, $t \in I$. Notice that h is strictly increasing on A and constant on each connected component of $I \backslash A$. Given $x, x' \in I_i$ with $h(x) = h(x')$, we claim that $h\big(f(x)\big) = h\big(f(x')\big)$. Since f is injective, $f|_{I_i}$ is either increasing or decreasing. Without loss of generality, in what follows, assume that $f|_{I_i}$ is increasing (and continuous) for every $1 \le i \le n$. Assume $x \le x'$, then $f(x) \le f(x')$. Moreover, since $f|_{I_i}$ is increasing and continuous,

$$[x, x') = f^{-1}\left(\big[f(x), f(x')\big)\right).$$

Hence, since μ is non-atomic and f-invariant, we have that for any $x \le x'$ in I_i,

$$h\big(f(x')\big) - h\big(f(x)\big) = \mu\Big(\big[f(x), f(x')\big)\Big) = \mu\Big(f^{-1}\big(\big[f(x), f(x')\big)\big)\Big) = h(x') - h(x), \tag{5}$$

which proves the claim.

We will use (5) to define an IET $T\colon I \to I$. Let

$$\mathcal{I} = \big\{1 \le i \le n : (x_{i-1}, x_i) \cap A \neq \emptyset\big\},$$

where x_{i-1} and x_i are the endpoints of I_i. Let $m \le n$ be the cardinality of $\mathcal{I}$, then we may write $\mathcal{I} = \{i_1, \ldots, i_m\}$. Let $0 = y_0 < y_1 < \cdots < y_m = 1$ be the points defined by $y_\ell = h(x_{i_\ell})$, $1 \le \ell \le m$. Let $T\colon I \to I$ be the map that at $h(x) \in I \backslash \{y_0, y_1, \ldots, y_{m-1}\}$ takes the value

$$T\big(h(x)\big) = h\big(f(x)\big). \tag{6}$$

The map T is well-defined on $I \backslash \{y_0, y_1, \ldots, y_{m-1}\}$. To see that, let $x, x' \in I$, $x < x'$, be such that $h(x) = h(x')$ is a point in $I \backslash \{y_0, y_1, \ldots, y_{m-1}\}$. Then, $\{x, x'\} \subset \bigcup_{i \in \mathcal{I}} (x_{i-1}, x_i)$, otherwise x or x' would belong to $\{y_0, y_1, \ldots, y_{m-1}\}$. In this way, there exist $i, j \in \mathcal{I}$ such that $x \in (x_{i-1}, x_i)$ and $x' \in (x_{j-1}, x_j)$. If $i \neq j$, then the hypothesis $h(x) = h(x')$ yields $h(x) = h(x_i) = h(x_{j-1}) = h(x')$, showing that $h(x) \in \{y_0, y_1, \ldots, y_{m-1}\}$, which is a contradiction. Hence, the only alternative left is $i = j$ and $x, x' \in (x_{i-1}, x_i)$. In this way, x, x' belong to the same interval I_i and (5) implies that $T\big(h(x)\big) = T\big(h(x')\big)$, thus T is well-defined on $I \backslash \{y_0, y_1, \ldots, y_{m-1}\}$.

Let us prove that $T|_{(y_{\ell-1}, y_\ell)}$, $1 \le \ell \le m$, is a translation. If y, y' are two points in $(y_{\ell-1}, y_\ell)$, there exist $x, x' \in (x_{i_\ell - 1}, x_{i_\ell})$ such that $y = h(x)$ and $y' = h(x')$, then (5) and (6) yield

$$T(y') - T(y) = T\big(h(x')\big) - T\big(h(x)\big) = h\big(f(x')\big) - h\big(f(x)\big) = h(x') - h(x) = y' - y,$$

proving that $T|_{(y_{\ell-1}, y_\ell)}$ is a translation. In particular, $T|_{(y_{\ell-1}, y_\ell)}$ is injective and $T\big((y_{\ell-1}, y_\ell)\big)$ is an open interval for each $1 \le \ell \le m$. Moreover, since h is order-preserving, if $\ell \neq k$, then

$h\big(x_{i_\ell-1}, x_{i_\ell}\big)$ and $h\big(x_{i_k-1}, x_{i_k}\big)$ are non-overlapping open intervals, implying that T is (globally) injective on $I\backslash\{y_0, y_1, \ldots, y_m\}$. As for the definition of T on the set $\{y_1, \ldots, y_{n-1}\}$, we can choose one of the lateral limits of f as we approach each of these points in such a way that T is, indeed, globally injective. In this way, T is a m-IET.

We claim that T has no attractive periodic orbit. In fact, if for each $1 \leq j \leq s$, γ_i is an infinite f-orbit dense in Λ_i, then the union of the infinite T-orbits $T(\gamma_1)$, ..., $T(\gamma_s)$ is a dense subset of I, ruling out attractive periodic T-orbits.

Let $x \in I$ be a point whose natural f-coding is $\theta = \theta_0\theta_1\ldots$, then we may assume that $\omega(x)$ is infinite, otherwise Lemma 3.1 says that θ would be ultimately periodic (i.e. $\exists k \geq 0$ such that the natural f-coding of $f^k(x)$ is periodic). By Corollary 3.4, there exists $k_0 \geq 0$ such that $f^k(x) \in (x_{i_1-1}, x_{i_1}) \cup \cdots \cup (x_{i_m-1}, x_{i_m})$ for all $k \geq k_0$. This means that the natural f-coding $\zeta = \zeta_0\zeta_1\ldots$ of $f^{k_0}(x)$ is an infinite word over the alphabet $\mathcal{A}' = \{i_1, \ldots, i_m\}$. Let $\eta = \eta_0\eta_1\ldots$ be the natural T-coding of $y = h(x)$, then $\zeta_j = i_\ell \in \{i_1, \ldots, i_m\}$ if and only if $\eta_j = \ell \in \{1, \ldots, m\}$, proving that ζ and η are isomorphic infinite words.

5. Proofs of Theorem 2.2 and Corollary 2.5

Proof of Theorem 2.2. Let $T : I \to I$ be a topologically transitive n-IET and $J_1, \ldots, J_n$ be the associated partition. Without loss of generality we may assume that the endpoints of J_i are y_{i-1} and y_i, where $0 = y_0 < y_1 < \cdots < y_n = 1$. Let $\{p_k\}_{k=1}^{\infty} \subset I\backslash\{y_0, y_1, \ldots, y_{n-1}\}$ be a dense T-orbit. Given $k \geq 1$, let

$$\mathcal{L}_k = \{\ell \geq 1 : p_\ell < p_k\} \quad \text{and} \quad G_k = \left[\sum_{\ell\in\mathcal{L}_k} 2^{-\ell}, 2^{-k} + \sum_{\ell\in\mathcal{L}_k} 2^{-\ell}\right]. \tag{7}$$

Notice that $p_k > 0$ and $\mathcal{L}_k \neq \emptyset$. Hence, $G_k \subset (0,1)$ is a well-defined interval of length $|G_k| = 2^{-k}$. We claim that $\{p_k\}_{k\geq1}$ and $\{G_k\}_{k\geq1}$ share the same ordering meaning that

$$p_k < p_j \iff \sup G_k < \inf G_j. \tag{8}$$

In fact, $p_k < p_j$ if and only if $\{k\} \cup \mathcal{L}_k \subset \mathcal{L}_j$, which is equivalent to

$$\sup G_k = 2^{-k} + \sum_{\ell\in\mathcal{L}_k} 2^{-\ell} < \sum_{\ell\in\mathcal{L}_j} 2^{-\ell} = \inf G_j.$$

In particular, we have that the intervals $G_1, G_2, \ldots$ are pairwise disjoint and their union is dense because $\sum_{k=1}^{\infty} |G_k| = 1$. Applying (8) we conclude that if $J \subset I$ is an interval and

$$\{m_k\}_{k\geq1} = \{\ell \geq 1 : p_\ell \in J\}, \quad \text{then } \overline{\cup_{k\geq1} G_{m_k}} \quad \text{is an interval.}$$

Let $\widehat{h}\colon \cup_{k\geq1} G_k \to I$ be the function that on G_k takes the constant value p_k. By (8), we have that $\widehat{h}$ is nondecreasing and has dense domain and dense range. Thus, $\widehat{h}$ admits a unique nondecreasing continuous surjective extension $h\colon [0,1] \to [0,1]$ to the whole interval $[0,1]$. It is elementary to see that $h^{-1}\big(\{p_k\}\big) = G_k$. Denote by $I_1, \ldots, I_n$ the partition of I defined by $I_i = h^{-1}(J_i)$. Notice that $x_i = h^{-1}(y_i)$, $0 \leq i \leq n$, are the endpoints of the partition $I_1, \ldots, I_n$.

Let $\widehat{f} : \cup_{k\geq 1} G_k \to \cup_{k\geq 2} G_k$ be such that $\widehat{f}|_{G_k}: G_k \to G_{k+1}$ is an affine bijection with slope $\frac{1}{2}T'(p_k)$ for every $k \geq 1$, where $T'(p_k) \in \{-1, 1\}$ is the derivative of T at p_k. We claim that for each $1 \leq i \leq n$, there exist a dense subset $\widehat{I_i}$ of I_i, $\lambda_i \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$ and $b_i \in \mathbb{R}$ such that

$$\widehat{f}(x) = \lambda_i x + b_i \quad \text{for all} \quad x \in \widehat{I_i}. \tag{9}$$

In order to show that (9) is true, fix $1 \leq i \leq n$ and let $\{m_k\}_{k\geq 1} = \{\ell \geq 1 : p_\ell \in J_i\}$, then $\widehat{J_i} = \cup_{k\geq 1}\{p_{m_k}\}$ is a dense subset of J_i and $\widehat{I_i} = \cup_{k\geq 1} G_{m_k}$ is a dense subset of I_i. Moreover, there exists $\lambda_i \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$ such that $T'(y) = 2\lambda_i$ for all $y \in J_i$. In particular, $T'(p_{m_k}) = 2\lambda_i$ for all $k \geq 1$. By definition, $\widehat{f}|_{G_{m_k}}: G_{m_k} \to G_{m_k+1}$ is an affine bijection with slope $\frac{1}{2}T'(p_{m_k}) = \lambda_i$ for all $k \geq 1$, which proves (9). We have proved that there exist $\lambda_i \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$ and $c_{m_k} \in \mathbb{R}$ such that

$$\widehat{f}(x) = \lambda_i x + c_{m_k} \quad \text{for all} \quad x \in G_{m_k}. \tag{10}$$

Let us prove that if $\lambda_i = \frac{1}{2}$ $\left(\text{respectively, } \lambda_i = -\frac{1}{2}\right)$ then $\widehat{f}$ is strictly increasing (respectively, strictly decreasing) on $\cup_{k\geq 1} G_{m_k}$. Without loss of generality, assume that $\lambda_i = -\frac{1}{2}$, then $\widehat{f}$ is strictly decreasing on each interval G_{m_k}. Let $y_k < z_j$ be such that $y_k \in G_{m_k}$ and $z_j \in G_{m_j}$, where $k \neq j$ and $\sup G_{m_k} < \inf G_{m_j}$. By (8), we have that $p_{m_k} < p_{m_j}$ and $\{p_{m_k}, p_{m_j}\} \subset J_i$. Then, since $T'(y) = 2\lambda_i = -1$ for all $y \in J_i$, we have that $T|_{J_i}$ is decreasing, thus $T(p_{m_k}) > T(p_{m_j})$, that is, $p_{m_k+1} > p_{m_j+1}$. By (8) once more, we get $\sup G_{m_j+1} < \inf G_{m_k+1}$. By definition, $f(y_k) \in G_{m_k+1}$ and $f(z_j) \in G_{m_j+1}$, thus $f(y_k) > f(z_j)$. This proves that $\widehat{f}$ is decreasing on $\cup_{k\geq 1} G_{m_k}$. It remains to prove that c_{m_k} in (10) is the same for all $k \geq 1$. Let $j \neq k$. We may assume that $a = \sup G_{m_j} < \inf G_{m_k} = b$. Notice that

$$\begin{aligned}
\frac{1}{2}(b-a) + \frac{|\lambda_i|}{\lambda_i}(c_{m_k} - c_{m_j}) &= \frac{|\lambda_i|}{\lambda_i}\big(\widehat{f}(b) - \widehat{f}(a)\big) = \sum_{G_{m_\ell} \subset [a,b]} \left|\widehat{f}(G_{m_\ell})\right| \\
&= \frac{1}{2} \sum_{G_{m_\ell} \subset [a,b]} |G_{m_\ell}| = \frac{1}{2}(b-a)
\end{aligned}$$

yielding $c_{m_k} = c_{m_j}$. Thus, (9) is true.

It follows from (9) that $\widehat{f}|_{\cup_{k\geq 1} G_{m_k}}$ admits a unique monotone continuous extension to the interval $I_i = h^{-1}(J_i)$. This extension is also an affine map with slope equal to $\frac{1}{2}$ in absolute value. Since i is arbitrary, we obtain an injective piecewise $\frac{1}{2}$-affine extension f of $\widehat{f}$ to the whole interval $I = \cup_{i=1}^n I_i$.

It remains to show that $h \circ f = T \circ h$. In fact, for every $y \in G_k$, we have that

$$h\big(f(y)\big) = \widehat{h}\big(\widehat{f}(y)\big) = p_{k+1} = T(p_k) = T\big(\widehat{h}(y)\big) = T\big(h(y)\big). \tag{11}$$

Hence, (11) holds for a dense set of $y \in I$. By continuity, (11) holds for every $y \in I$. $\square$

Proof of Corollary 2.5. Let $0 < \alpha < 1$ be irrational. Let $T{:}I \to I$ be the 2-IET defined by $T(y) = y + \alpha \pmod 1$, or equivalently, let $J_1 = [0, 1-\alpha)$, $J_2 = \big[1-\alpha, 1\big)$, and

$$T(y) = \begin{cases} y + \alpha & \text{if} \quad y \in J_1 \\ y + \alpha - 1 & \text{if} \quad y \in J_2 \end{cases}.$$

It is widely known that T is minimal. Hereafter, we take all the notation of the proof of Theorem 2.2. Let $y_0 = 0$, $y_1 = 1 - \alpha$ and $y_2 = 1$. Let $\gamma = \{p_k\}_{k=1}^{\infty} = \{\alpha, T(\alpha), \ldots\}$ be the T-orbit of α, then γ is a dense orbit contained in $I \backslash \{y_0, y_1\}$. Let $\theta = \theta_0\theta_1 \ldots$ be the natural T-coding of α, then θ is a Sturmian word. Let us define the 2-PC f_T. Let G_k, $k \geq 1$, be the pairwise disjoint intervals of length $|G_k| = 2^{-k}$ defined by (7). Let $I_i = h^{-1}(J_i)$ for $i = 1, 2$, then $I_1 = [0, x_1)$, $I_2 = [x_1, 1)$, where $x_1 = h^{-1}(y_1)$. Let

$$\{m_k\}_{k \geq 1} = \{\ell \geq 1 : p_\ell \in J_1\} = \{\ell \geq 1 : \theta_{\ell-1} = 1\},$$

then $\widehat{J_1} = \cup_{k \geq 1}\{p_{m_k}\}$ is a dense subset of J_1 and $\widehat{I_1} = \cup_{k \geq 1} G_{m_k}$ is a dense subset of I_1. In this way, since $|G_{m_k}| = 2^{-m_k}$, we have that

$$x_1 = \sup I_1 = \sum_{k \geq 1} |G_{m_k}| = \sum_{k \geq 1} 2^{-m_k} = \sum_{\ell \geq 1} (2 - \theta_{\ell-1}) 2^{-\ell} = \frac{1}{2} \sum_{\ell \geq 0} (2 - \theta_\ell) 2^{-\ell} = 2 - \frac{1}{2} \sum_{\ell \geq 0} \theta_\ell 2^{-\ell}.$$

Since $T'(y) = 1$ for every $y \in I$, we have that the slope λ_i of f_T is $\frac{1}{2}$. In this way, we have that

$$f_T(x) = \begin{cases} \dfrac{1}{2}x + b_1 & \text{if} \quad x \in [0, x_1) \\ \dfrac{1}{2}x + b_2 & \text{if} \quad x \in [x_1, 1) \end{cases}.$$

Since $\frac{1}{2}x_1 + b_1 = 1$ and $\frac{1}{2}x_1 + b_2 = 0$, we conclude that

$$f_T(x) = \frac{1}{2}x + \delta, \quad \text{where} \quad \delta = \frac{1}{4} \sum_{\ell \geq 0} \theta_\ell 2^{-\ell}.$$

It is clear that

$$\delta = \frac{1}{4} \left(1 + \theta_0 + \sum_{\ell \geq 1} (\theta_\ell - 1) 2^{-\ell} \right), \tag{12}$$

thus $\{\theta_\ell - 1\}_{\ell \geq 1}$ is the binary expansion of $\sum_{\ell \geq 1} (\theta_\ell - 1) 2^{-\ell}$. In this case, the transcendence of δ follows from Ferenczi and Mauduit [13, Proposition 2] or Adamczewski and Cassaigne [1, Theorem 1] together with the fact that $w = (\theta_1 - 1)(\theta_2 - 2) \ldots$ is a Sturmian word.

Now let us consider the particular case in which $\alpha = 2 - \varphi$, where $\varphi = (1 + \sqrt{5})/2$ is the golden ratio. In this case, it is known that $(\theta_1 - 1)(\theta_2 - 1) \ldots$ is the Fibonacci word

$$\theta - 1 = 0100101001001010010100100101001001010010100100010100\ldots$$

The number

$$R = 1 - \sum_{\ell \geq 0} (\theta_\ell - 1) 2^{-(\ell+1)} = 0.7098034428612913146\ldots$$

is known in the mathematical literature as the rabbit constant. Notice that by (12), we have that

$$\delta = \frac{1}{4}\left(1+\theta_0+2\sum_{\ell\geq 1}(\theta_\ell-1)2^{-(\ell+1)}\right) = \frac{1}{4}\left(2+2\sum_{\ell\geq 0}(\theta_\ell-1)2^{-(\ell+1)}\right) = 1-\frac{R}{2}.$$

The transcendence of the rabbit constant was proved by Davison [9].

□

6. Proofs of Corollary 2.3 and Corollary 2.4

Proof of Corollary 2.3. Let $\theta=\theta_0\theta_1\ldots$ be a natural coding of an injective n-PC $f\colon I\to I$. By Theorem 2.1, there exist a m-IET $T\colon I\to I$ with $2\leq m\leq n$ and $q\geq 0$ such that the infinite word $\theta^*=\theta_q\theta_{q+1}\ldots$ is either periodic or isomorphic to the non ultimately periodic natural T-coding $\omega=\omega_0\omega_1\ldots$ of some point $y\in I$. For the sake of simplification, we will only consider the case in which T is an orientation-preserving m-IET with associated partition $I_1=[y_0,y_1)$, …, $I_m=[y_{m-1},y_m)$. Since ω is non ultimately periodic, there exist $r\geq 0$ and $y^*=T^r(y)$ whose T-orbit is regular, which means

$$O_T(y^*)=\big\{T^r(y),T^{r+1}(y),\ldots\big\}\subset I\backslash\{y_0,y_1,\ldots,y_{m-1}\}.$$

Because $O_T(y*)$ is regular, it is entirely contained in a minimal component of T. More specifically, there exist open intervals $A_1,\ldots,A_p$ with pairwise disjoint closures such that $O_T(y^*)$ is a dense subset of $A_1\cup\cdots\cup A_p$ and $T(A_1)\subset A_2$, …, $T(A_{p-1})\subset A_p$, $T(A_p)\subset A_1$, and T takes $I\backslash(\overline{A_1}\cup\cdots\cup\overline{A_p})$ into itself (see [3, 24]). Let μ be the normalized Lebesgue measure on $A_1\cup\cdots A_p$, then μ is T-invariant: $\mu\big(T^{-1}(B)\big)=\mu(B)$ for every Borel set $B\subset I$. Let $h\colon[0,1]\to[0,1]$ be the nondecreasing, continuous, surjective map defined by $h(t)=\mu\big([0,t]\big)$, $t\in I$. Notice that h is strictly increasing on $A_1\cup\cdots\cup A_p$ and constant on each of the finitely many connected components of $I\backslash A_1\cup\cdots\cup A_p$. Given $y,y'\in I_i$ with $h(y)=h(y')$, we claim that $h\big(T(y)\big)=h\big(T(y')\big)$. Without loss of generality, assume that $y\leq y'$, then $T(y)\leq T(y')$. Moreover, since $T|_{I_i}$ is a translation,

$$[y,y']=T^{-1}\left(\big[T(y),T(y')\big]\right).$$

Hence, since μ is non-atomic and T-invariant, we have that for any $y,y'\in I_i$,

$$h\big(T(y')\big)-h\big(T(y)\big)=\mu\Big(\big[T(y),T(y')\big]\Big)=\mu\Big(T^{-1}\big(\big[T(y),T(y')\big]\big)\Big)=h(y')-h(y), \tag{13}$$

which proves the claim.

We will use (13) to define an IET $E\colon I\to I$. Let

$$\mathcal{I}=\big\{1\leq i\leq m : I_i\cap(A_1\cup\cdots\cup A_p)\neq\emptyset\big\}.$$

Let $m'\leq m$ be the cardinality of $\mathcal{I}$, then we may write $\mathcal{I}=\{i_1,\ldots,i_{m'}\}$. The intervals $J_1=h\big(I_{i_1}\big)$, …, $J_{m'}=h\big(I_{i_{m'}}\big)$ form a partition of I into non-degenerate intervals with endpoints $0 = z_0 < z_1 < \cdots < z_{m'} = 1$ defined by $z_\ell = h(y_{i_\ell})$, $0 \le \ell \le m'$. Let $E{:}I \to I$ be the right-continuous map that at $z = h(y) \in I \backslash \{z_0, z_1, \ldots, z_{m'-1}\}$ takes the value

$$E\big(h(y)\big) = h\big(T(y)\big). \tag{14}$$

The map E is well-defined. In fact, if $y, y' \in I$ are such that $h(y) = h(y')$, then y, y' belong to the same connected component of $I \backslash (A_1 \cup \cdots \cup A_p)$. There is no discontinuity of T between y and y', otherwise $h(y)$ would belong to $\in \{z_1, \ldots, z_{m'-1}\}$. In this way, y, y' belong to the same interval I_i and (13) asserts that E is well-defined. Notice that, by definition, $E(z_\ell) = \lim_{\epsilon \to 0^+} E(z_\ell + \epsilon)$ for all $0 \le \ell \le m' - 1$.

Let us prove that $E|_{(z_{\ell-1}, z_\ell)}$, $1 \le \ell \le m'$, is a translation. If z, z' are two points in $(z_{\ell-1}, z_\ell)$, then there exist $y, y' \in \big(y_{i_{\ell-1}}, y_{i_\ell}\big)$ such that $z = h(y)$ and $z' = h(y')$. Now (13) and (14) yield

$$E(z') - E(z) = E\big(h(y')\big) - E\big(h(y)\big) = h\big(T(y')\big) - h\big(T(y)\big) = h(y') - h(y) = z' - z,$$

proving that $E|_{J_\ell}$ is a translation.

The map E is surjective. In fact, since h and T are surjective, given $z \in I$, there exists $y \in I$ such that $E\big(h(y)\big) = h\big(T(y)\big) = z$. To see that E is also injective, by the above, E takes each interval J_ℓ into its translate $E(J_\ell)$, which therefore has the same length, that is, $|E(J_\ell)| = |J_\ell|$. Since E is surjective, we have that

$$1 = \sum_{\ell=1}^{m'} |E(J_\ell)| \le \sum_{\ell=1}^{m'} |J_\ell| \le 1,$$

implying that no overlapping is possible for the intervals $E(J_1), \ldots, E(J_{m'})$. This proves that E is a m'-IET.

Becasuse $O_T(y^*)$ is a dense subset of $A_1 \cup \cdots \cup A_p$ and $h(A_1 \cup \cdots \cup A_p)$ is dense in I, we have that h takes the T-orbit $O_T(y^*)$ onto a dense E-orbit, thus E is topologically transitive. Moreover, if $\zeta = \zeta_0\zeta_1 \ldots$ is the natural T-coding of y^* and $\eta = \eta_0\eta_1 \ldots$ is the natural E-coding of $z^* = h(y^*)$, then $\zeta_k = i_\ell \in \{i_1, \ldots, i_{m'}\}$ if and only if $\eta_k = \ell \in \{1, \ldots, m'\}$, proving that ζ and η are isomorphic infinite words. To conclude the proof, we recall that $\theta_{q+r}\theta_{q+r+1} \ldots$ is isomorphic to ζ.

□

Lemma 6.1. *Let $\theta = \theta_0\theta_1 \ldots$ be an infinite word and $\theta^* = \theta_{q+1}\theta_{q+2} \ldots$ an infinite subword of θ, then there exist $k_0 \ge 1$ and $\beta \ge 0$ such that*

$$p_k(\theta) = p_k(\theta^*) + \beta \quad \textit{for every} \quad k \ge k_0.$$

Proof. For each $k \ge q + 1$, let

$$\begin{aligned} \mathcal{W}_k &= \{\theta_0\theta_1 \ldots \theta_{k-1}, \quad \theta_1\theta_2 \ldots \theta_k, \quad \ldots, \quad \theta_q\theta_{q+1} \ldots \theta_{q+k-1}\} \subset L_k(\theta) \\ \mathcal{W}_k^* &= \{\omega \in \mathcal{W}_k : \omega \notin L_k(\theta^*)\} \end{aligned}$$

Notice that $\mathcal{W}_k$ is formed by at most $q+1$ distinct finite words and $k \mapsto \#\mathcal{W}_k^*$ is a nondecreasing map, thus there exist $k_0 \geq 0$ and $\beta \leq q+1$ such that $\#\mathcal{W}_k = \beta$ for every $k \geq k_0$. Moreover, for every $k \geq k_0$, we have the disjoint union

$$L_k(\theta) = \mathcal{W}_k^* \cup L_k(\theta^*), \quad \text{thus} \quad p_k(\theta) = p_k(\theta^*) + \beta.$$

□

Lemma 6.2. *Let θ be a natural coding of a topologically transitive m-IET $T: I \to I$, then there exist $k_0 \geq 1$, $\alpha \in \{0, \ldots, m-1\}$ and $\beta \geq 1$ such that*

$$p_\theta(k) = k\alpha + \beta \quad \textit{for every} \quad k \geq k_0. \tag{15}$$

Moreover, if T is a standard m-IET, with $m \geq 2$, satisfying the i.d.o.c., then $\alpha = m-1$, $\beta = 1$ and $k_0 = 1$.

Proof. Let $T: I \to I$ be a topologically transitive m-IET and $\mathscr{P} = \{I_1, \ldots, I_m\}$ be the partition associated to T, then, since T^{-1} is also an IET, $T^{-k}(\mathscr{P})$ is a partition of I into intervals for every $k \geq 0$, implying that the members of the set

$$\mathscr{P}_k = \bigwedge_{\ell=0}^{k-1} T^{-\ell}(\mathscr{P}) = \left\{ I_{i_0} \cap T^{-1}\big(I_{i_1}\big) \cap \cdots \cap T^{-(k-1)}\big(I_{i_{k-1}}\big) : 1 \leq i_0, i_1, \ldots, i_{k-1} \leq m \right\}.$$

are pairwise disjoint intervals. Moreover, if θ is a natural coding of T, then the k-word $i_0 i_1 \ldots i_{k-1}$ occurs in θ if and only if the interval $J = I_{i_0} \cap T^{-1}\big(I_{i_1}\big) \cap \cdots \cap T^{-(k-1)}\big(I_{i_{k-1}}\big) \in \mathscr{P}_k$ is nom-empty.

Let θ be the natural T-coding of some point $x \in I$. If θ is (ultimately) periodic, then by the Morse-Hedlund Theorem, there exist $k_0 \geq 1$ and $\beta \geq 1$ such that $p_\theta(k) = \beta$ for every $k \geq k_0$, meaning that (15) holds with $\alpha = 0$. Hence, we may assume that θ is not (ultimately) periodic. In this case, there exists $q \geq 0$ such that the orbit $\{x^*, T(x^*), \ldots\}$ of $x^* = T^{q+1}(x)$ is a dense subset of $I \backslash \{x_0, x_1, \ldots, x_{m-1}\}$, where $0 = x_0 < x_1 < \cdots < x_m = 1$ are the endpoints of the partition $\mathscr{P}$. In this way, for each $k \geq 1$, $\{x^*, T(x^*), \ldots\}$ is contained in the union of the interiors of the intervals of $\mathscr{P}_k$. Hence, the k-word $i_0 i_1 \ldots i_{k-1}$ occurs in the natural T-coding θ^* of x^* if and only if the interval $J = I_{i_0} \cap T^{-1}\big(I_{i_1}\big) \cap \cdots \cap T^{-(k-1)}\big(I_{i_{k-1}}\big) \in \mathscr{P}_k$ has non-empty interior. Therefore, the number of such intervals J in $\mathscr{P}_k$ equals $p_k(\theta^*)$ and is related to the number of endpoints of the partition $\mathscr{P}_k$ as follows

$$p_k(\theta^*) = 1 + \sum_{\ell=0}^{k-1} m_\ell, \tag{16}$$

where $m_0 = m - 1$ and

$$m_\ell = \left\{T^{-\ell}(x_1), \ldots, T^{-\ell}(x_{m-1})\right\} \Big\backslash \bigcup_{p=0}^{\ell-1} \left\{T^{-p}(x_1), \ldots, T^{-p}(x_{m-1})\right\}$$

gives the number of new division points at the ℓ-th step towards the construction of $\mathscr{P}_k$. The map $\ell \mapsto m_\ell$ is a non-increasing, therefore there exist $k_0' \geq 0$ and $\alpha \geq 1$ such that

$m_\ell = \alpha$ for every $\ell \geq k_0'$. Notice that $\alpha \geq 1$ because, as θ^* is not (ultimately) periodic, $p_k(\theta^*) \to \infty$ as $k \to \infty$. Let $\beta_0, \beta_1, \ldots, \beta_{k_0-1} \geq 0$ be such that

$$m_\ell = \begin{cases} \alpha + \beta_\ell & \text{if} \quad \ell \in \{0, 1, \ldots, k_0' - 1\} \\ \alpha & \text{if} \quad \ell \geq k_0' \end{cases}. \tag{17}$$

By (16) and (17), we have that if $\beta' = 1 + \beta_0 + \beta_1 + \cdots + \beta_{k_0-1}$, then

$$p_k(\theta^*) = 1 + \sum_{\ell=0}^{k_0'-1} (\alpha + \beta_\ell) + \sum_{k_0'}^{k-1} \alpha = \alpha k + \beta' \quad \text{for all} \quad k \geq k_0' + 1.$$

By Lemma 6.1, there exist $k_0 \geq k_0' + 1$ and $\beta'' \geq 0$ such that

$$p_k(\theta) = p_k(\theta^*) + \beta'' = \alpha k + \underbrace{(\beta' + \beta'')}_{\beta} = \alpha k + \beta \quad \text{for all} \quad k \geq k_0.$$

Notice that if T satisfies the i.d.o.c., then $\theta^* = \theta$ and $m_\ell = m - 1$ for all $\ell \geq 0$, then (16) yields

$$p_k(\theta) = p_k(\theta^*) = (m-1)k + 1 \quad \text{for all} \quad k \geq 1,$$

implying that in this case (15) holds with $\alpha = m - 1$, $\beta = 1$ and $k_0 = 1$. $\square$

Proof of Corollary 2.4. Let $f: I \to I$ be an injective n-PC and $\theta = \theta_0\theta_1\ldots$ be the natural f-coding of $x \in I$. By Corollary 2.3, there exist $k \geq 0$ and a topologically transitive m-IET, with $2 \leq m \leq n$, such that the natural coding θ^* of $f^k(x)$ is either periodic or isomorphic to a non ultimately periodic natural coding of T. By Lemma 6.2, there exist $k_0 \geq 1$, $\alpha \in \{0, \ldots, m-1\}$ and $\beta \geq 1$ such that

$$p_k(\theta^*) = k\alpha + \beta \quad \text{for all} \quad k \geq k_0. \tag{18}$$

Notice that in the case in which θ^* is periodic, by the Morse-Hedlund Theorem, (18) holds with $\alpha = 0$. To conclude the proof of the item (i), apply Lemma 6.1. As for tye item (ii), we apply Theorem 2.2 together with Lemma 6.2. $\square$

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