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A Variational Characterization of Geodesics in Static Lorentzian Manifolds. Existence of Geodesics in Manifolds with Convex Boundary

Paolo Piccione

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A Variational Characterization of Geodesics in Static Lorentzian Manifolds. Existence of Geodesics in Manifolds with Convex Boundary.

Paolo Piccione*

Instituto de Matemática e Estatística
Departamento de Matemática
Universidade de São Paulo
Brazil

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Abstract

We study some global geometric properties of a static Lorentzian manifold Λ embedded in a differentiable manifold $\mathcal{M}$, with possibly non smooth boundary $\partial\Lambda$. We prove a variational principle for geodesics in static manifolds, and using this principle we establish the existence of geodesics that do not touch $\partial\Lambda$ and that join two fixed points of Λ. The results are obtained under a suitable completeness assumption for Λ, that generalizes the property of global hyperbolicity, and a weak convexity assumption on $\partial\Lambda$. Moreover, under a non triviality assumption on the topology of Λ, we also get a multiplicity result for geodesics in Λ joining two fixed points.

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1 Preliminaries and Statement of the Results

In this paper we consider the problem of the existence of geodesics that do not touch the boundary of a convex subset of a Lorentzian manifold, with a non necessarily smooth boundary. Some space–times of great physical relevance, like the Schwarzschild space–time and the Reissner–Nordström space–time (see [11]), provide examples of the Lorentzian structures that we will consider.

Here, by singularity, we mean loosely a geometrical or a metric singularity at the boundary points of a manifold. The two concepts of singularity may in some cases be considered equivalent, up to a change of coordinates. For instance, in the case of the Schwarzschild space–time $\mathcal{M}$, its boundary $\partial\mathcal{M}$ is smooth, but its metric cannot be smoothly extended to $\partial\mathcal{M}$. Nonetheless, if $\mathcal{M}_1$ denotes the *Kruskal* space–time (see [11]), there exists an injective isometry $j : \mathcal{M} \longmapsto \mathcal{M}_1$, but $j(\partial\mathcal{M})$ is not a smooth submanifold of $\mathcal{M}_1$. In this sense, we say that that $\mathcal{M}$ has singular boundary.

The geodesic problem in Lorentzian manifolds is much more delicate than in the Riemannian case, where the Hopf–Rinow theorem gives the equivalence of the metric and the geodesical completeness. These conditions also imply the existence of geodesics between any points. On the other hand, in the Lorentzian case there exist several inequivalent forms of singularity and incompleteness. To emphasize the difference between the Riemannian and the Lorentzian case, it suffices to observe that there exist compact manifolds which are neither geodesically complete nor geodesically connected (see for instance [2, 16, 17]).

In this paper, we will prove the existence of geodesics between two fixed events of static manifolds with boundary, under suitable completeness and convexity assumptions. The notion of convexity used in this paper is adapted to situations where the boundary is non necessarily smooth. The results are based on an intrinsic variational principle for geodesics in stationary manifolds (see Theorem 2.1) which generalizes the variational principles proven in [3, 8]. In these papers, the statement and the proof of the principle is made under an extra assumption on the topology of the manifold and it is used a (non canonical) spacetime splitting of the manifold.

We present an intrinsic approach to the problem of the existence of (at least) one geodesic between two fixed points of a Lorentzian manifold which is static with respect to a given timelike Killing vector field. To this purpose, we introduce a sort of *completeness* condition, which, in the case of a manifold $\mathcal{M}$ which is a warped product of $\mathbb{R}$ by a Riemannian manifold $\mathcal{M}_0$, generalizes in a non trivial way the notion of completeness for the underlying manifold $\mathcal{M}_0$ (see Definition 1.1 and Example 1.8).

The main existence and multiplicity results of the paper (Theorem 1.2, Corollary 1.3 and Theorem 1.4) are proven with techniques of Critical Point Theory. Consistently with the spirit of the paper, an effort was made to keep the proofs of the technical results free from non intrinsic (hence non canonical) choices; for instance, in no part of the paper the Nash's embedding Theorem was used. It should also be remarked that, as an element of novelty, the proof of the Palais–Smale condition of Proposition 3.3 is carried out without referring to any particular embedding of the Lorentzian manifold into a Euclidean space.

Moreover, a localization argument of the completeness assumption allows to extend the existence results to the case of Lorentzian manifolds given by non trivial fiber bundles over Riemannian manifolds (see Corollary 1.3).

Before stating the main results, we recall some basic notions of the Lorentzian geometry. The books [1, 11, 13, 14] are excellent references for a complete account of the theory and for all the background material assumed in this paper.

A Lorentzian manifold is a smooth, finite dimensional manifold Λ, equipped with a $(0,2)$–tensor g of index 1, i.e., for every $z \in \Lambda$, $g(z)[\cdot,\cdot]$ is a nondegenerate, symmetric bilinear for on the tangent space $T_z\Lambda$, such that the dimension of a maximal subspace of $T_z\Lambda$ on which g is negative definite is 1. The bilinear form $g(z)[\cdot,\cdot]$ on $T_z\Lambda$ will also be denoted by $\langle\cdot,\cdot\rangle$ in the rest of the article.

A vector $v \in T_z\Lambda$ is said *timelike* (resp. *lightlike, spacelike*) if $\langle v, v\rangle$ is negative (resp. *null, positive*); v is called causal if it is not spacelike.

A smooth curve $\gamma : (a,b) \subseteq I\!R \longmapsto \Lambda$ is called *timelike* (resp. *lightlike, spacelike, causal*) if $\dot\gamma(s)$ is timelike (resp. lightlike, spacelike, causal) for all $s \in (a,b)$. This classification is called the *causal character* of a tangent vector, or of a curve.

A Lorentzian manifold is said to be *time-oriented* if there exists a vector field Y on Λ such that $Y(z)$ is timelike for all $z \in \Lambda$. A timelike vector field defines the past and the future of a point z in Λ: a causal vector $v \in T_z\Lambda$ is said to be *future pointing* (resp. *past pointing*) if $\langle v, Y(z)\rangle$ is negative (resp. positive).

Moreover, a timelike vector Y field on Λ (or on an open subset of Λ) defines a Riemannian metric $g_{(\mathrm{R})}$ on Λ by setting:

$$g_{(\mathrm{R})}(x)[\zeta_1,\zeta_2] = \langle\zeta_1,\zeta_2\rangle_{(\mathrm{R})} = \langle\zeta_1,\zeta_2\rangle - 2\frac{\langle\zeta_1,Y(x)\rangle\langle\zeta_2,Y(x)\rangle}{\langle Y(x),Y(x)\rangle}, \qquad (1)$$

for every $x \in \Lambda$ and every $\zeta_1,\zeta_2 \in T_x\Lambda$. It is not difficult to see that, for every $\zeta \in T_z\Lambda$, it is:

$$\langle\zeta,\zeta\rangle_{(\mathrm{R})} \geq |\langle\zeta,\zeta\rangle|. \qquad (2)$$

A smooth curve $z : (a,b) \longmapsto \Lambda$ is a *geodesic* if it satisfies the differential equation:

$$\nabla_{\dot z}\dot z \equiv 0, \qquad (3)$$

where ∇ denotes the covariant derivative relative to the Levi–Civita connection of the metric tensor g. Given an absolutely continuous curve z and an absolutely continuous vector field ζ along z, whenever there is no danger of confusion we will denote by $\nabla_s\zeta$ the covariant derivative of ζ along z, defined for almost all s.

It is well known that if z is a geodesic in Λ, then there exists a constant E_z such that:

$$\langle\dot z(s),\dot z(s)\rangle = E_z, \quad \forall\, s, \qquad (4)$$

hence, the geodesics have a causal character. A geodesic z is said to be *timelike* (resp. *lightlike, spacelike*) if E_z is negative (resp. null, positive).

We recall that a vector field Y on Λ is a *Killing* vector field if the Lie derivative $L_Y g$ of the metric tensor g is everywhere vanishing. Equivalently, Y is a Killing vector field if and only if the stages of all its local flows are isometries, i.e., if the metric tensor g of Λ is invariant by the flow of Y.

In this paper, we will often use the following well known characterization of Killing vector fields (see [14], Proposition 9.25). If $\mathcal{X}(\Lambda)$ denotes the space of all C^1-vector fields on on Λ, then $Y \in \mathcal{X}(\Lambda)$ is Killing if and only if for every pair $W_1, W_2 \in \mathcal{X}(\Lambda)$ it is:

$$\langle \nabla_{W_1} Y, W_2 \rangle = -\langle \nabla_{W_2} Y, W_1 \rangle. \tag{5}$$

In particular, if $z :]a, b[\longmapsto \Lambda$ is an absolutely continuous curve and Y is Killing, then

$$\langle \dot z, \nabla_s Y(z) \rangle \equiv 0 \quad \text{a.e.} \tag{6}$$

This implies that, if Y is Killing, then for every geodesic z in Λ the quantity $\langle \dot z, Y(z) \rangle$ is constant. An open subset Λ of a Lorentzian manifold Λ is called *stationary* if it is the domain of a timelike Killing vector field Y; such a set Λ is said to be *static* with respect to Y if Y is integrable in the sense that the orthogonal distribution to Y is integrable. We recall that a distribution $\mathcal{D} \subset T\Lambda$ is said to be integrable if through every point $p \in \Lambda$ there exists a submanifold $\mathcal{N}_p$ of Λ such that $T_q\mathcal{N}_p = \mathcal{D}_q$ for all $q \in \mathcal{N}_p$. It is well known (see e.g. Proposition 12.30 of [14]) that the orthogonal distribution to Y is integrable if and only if Y is irrotational. This means that the curl of Y is null on the orthogonal distribution $Y^\perp$:

$$(\operatorname{curl} U_Y)[W_1, W_2] = \langle \nabla_{W_1} Y, W_2 \rangle - \langle \nabla_{W_2} Y, W_1 \rangle = 0 \quad \forall W_1, W_2 \perp Y. \tag{7}$$

If $\Phi : \Lambda \longmapsto \mathbb{R}$ is a C^1 function, $\nabla\Phi(x)$ will denote the Lorentzian gradient of Φ, defined by $\langle \nabla\Phi(x), v \rangle = \mathrm{d}\Phi(x)[v]$ for every $v \in T_x\Lambda$. If Φ is of class C^2, the *Hessian* $H^\Phi(x)$ is the bilinear form on $T_x\Lambda$ defined by:

$$H^\Phi(x)[v, v] = \frac{\mathrm{d}^2(\Phi \circ \gamma_v)}{\mathrm{d}s^2}(0), \tag{8}$$

where $v \in T_x\Lambda$ and γ_v is the unique geodesic in Λ satisfying $\gamma_v(0) = x$ and $\dot\gamma_v(0) = v$. We will denote by $\exp_z$ the exponential map from an open neighborhood of 0 in $T_z\Lambda$ to Λ.

A Lorentzian manifold Λ is said to be *geodesically connected* if given any two points $p, q \in \Lambda$ there exists at least one geodesic z in Λ with endpoints in p and q. In order to prove the existence of geodesics, we need a completeness condition that we are going to define in the following.

Let f denote the *action functional* on Λ:

$$f(z) = \frac{1}{2} \int_0^1 \langle \dot z, \dot z \rangle \, \mathrm{d}s,$$

defined on the set of all curves $z : [0,1] \longmapsto \Lambda$ of class C^1.

Let p and q be fixed points in Λ, and consider the set of C^1-curves in Λ joining p and q and such that $\langle \dot z, Y \rangle$ is constant:

$$\mathcal{C}_{p,q} = \mathcal{C}_{p,q}(\Lambda) = \Big\{ z \in C^1([0,1], \Lambda) : z(0) = p, \ z(1) = q, \ \langle \dot z, Y \rangle \equiv C_z \Big\}. \tag{9}$$

We denote by $\overline{\Lambda} = \Lambda \bigcup \partial\Lambda$ the closure of Λ in $\mathcal{M}$.

Definition 1.1. Let c be a real number. A subset $\mathcal{S} \subseteq \mathcal{C}_{p,q}$ is said to be c-precompact if every sequence $\{z_n\}_{n\in\mathbb{N}} \subset \mathcal{S}$ with $f(z_n) \le c$ has a uniformly convergent subsequence in $\overline{\Lambda}$. We say that the restriction of f to $\mathcal{C}_{p,q}$ is *pseudo-coercive* if $\mathcal{C}_{p,q}$ is c-precompact for all $c > \inf\limits_{\mathcal{C}_{p,q}} f$.

We remark here that, as it is shown in [9], if the pseudo-coercivity condition on the functional f holds for all pairs p and q, then this implies the *global hyperbolicity* of Λ (see [1]). Nevertheless, the global hyperbolicity alone does not in general imply the geodesical connectedness, even in the static case, as simple counterexamples show (see Example 1.10). And even more, it may happen that the set $\mathcal{C}_{p,q}$ is *not* c-precompact for any choice of c, but it is the countable union of open connected components which are c-precompact for all values of c (see Example 1.9).

On the other hand, even in non globally hyperbolic static Lorentzian manifolds, the set $\mathcal{C}_{p,q}$ may happen to be c-precompact for some choices of $c > \inf_{\mathcal{C}_{p,q}} f$ and for all pairs of points p and q (see Example 1.8).

We will assume in the rest of the paper that Λ is a Lorentzian manifold which is embedded as an open subset of a smooth differentiable manifold $\mathcal{M}$. Observe that we do not require that the boundary $\partial\Lambda$ of Λ in $\mathcal{M}$ is a smooth submanifold of $\mathcal{M}$. We need to introduce a suitable notion of convexity for Λ, in order to guarantee that the geodesics in Λ cannot touch $\partial\Lambda$, in a sense to be clarified. The notion of convexity used in this paper (compare with [4, 7, 8]) is given in terms of a condition for the Hessian near the boundary of Λ of a suitable function, as explained below.

For p, $q \in \mathcal{M}$ and $c \in I\!R$, we denote by $\Lambda_{p,q}$ and $\Lambda_{p,q,c}$ the following subsets of Λ:

$$\begin{aligned} \Lambda_{p,q} &= \Big\{\lambda \in \Lambda : \exists\, z \in \mathcal{C}_{p,q} \text{ and } \bar{s} \in [0,1] \text{ such that } \lambda = z(\bar{s})\Big\}; \\ \Lambda_{p,q,c} &= \Big\{\lambda \in \Lambda : \exists\, z \in \mathcal{C}_{p,q} \text{ with } f(z) \leq c \text{ and } \bar{s} \in [0,1] \\ &\qquad\qquad \text{such that } \lambda = z(\bar{s})\Big\}. \end{aligned} \tag{10}$$

Obviously, $\Lambda_{p,q} = \bigcup_{c \in I\!R} \Lambda_{p,q,c}$; moreover, if $c < \inf_{\mathcal{C}_{p,q}} f$, then $\Lambda_{p,q,c} = \emptyset$.

The following is a generalization of a result of Benci, Fortunato and Giannoni for the geodesical connectedness of static manifolds (see Theorem 1.8 of [3]):

Theorem 1.2. *Let Λ be a Lorentzian manifold, which is embedded as an open subset of a smooth differentiable manifold $\mathcal{M}$. Let Y be a smooth vector field on Λ which is timelike and Killing. Suppose that Λ is static with respect to Y, and that the following hypotheses are satisfied:*

(1) for a fixed pair of points p and q in Λ there exists a real constant $c > \inf_{\mathcal{C}_{p,q}} f$ such that $\mathcal{C}_{p,q}$ is c-precompact;

(2) there exist a positive constant $\mu = \mu(p,q,c)$ *such that:*

$$-\langle Y(z), Y(z)\rangle \leq \mu < +\infty \tag{11}$$

for all $z \in \Lambda_{p,q,c}$*;*

(3) there exists a C^2*-regular function* $\phi : \Lambda \longmapsto \mathbb{R}^+$ *satisfying:*

(a) $\lim_{z \to z_0 \in \partial\Lambda} \phi(z) = 0$*;*

(b) for every subset B *of* $\mathcal{M}$ *with* $B \cap \Lambda$ *bounded, there exists a neighborhood* U *of* $\partial\Lambda \cap B$ *and a positive constant* M *such that* $\langle \nabla\phi(z), \nabla\phi(z)\rangle \leq M$ *for all* $z \in U \cap \Lambda$*;*

(c) for every subset B *of* $\mathcal{M}$ *with* $B \cap \Lambda$ *bounded, there exists a neighborhood* U *of* $B \cap \partial\Lambda$ *and a positive constant* ρ *such that* $\langle H^\phi(z)v, v\rangle \leq \rho \cdot \|v\|_R^2 \cdot \phi(z)$*, for all* $z \in U \cap \Lambda$ *and all* $v \in T_z\mathcal{M}$*;*

(d) $\langle \nabla\phi, Y\rangle \equiv 0$ *in* Λ*.*

Then, there exists at least one geodesic in Λ *joining* p *and* q*. In particular, if (1) and (2) hold for all pairs* p *and* q*, then* Λ *is geodesically connected.*

Observe that in our Theorem, no assumption is made on the topology of the spacetime Λ. In particular, Λ is not assumed to be *standard*, as in the case of the previous works [3, 4, 7, 8, 13]. We would also like to point out that the assumptions (3c) and (3d) of Theorem 1.2 are weaker than the assumptions made in Theorem 1.8 of [3] and in [7]; moreover, the hypothesis (1) is more general than the completeness assumptions of [3].

The weak notion of convexity for Λ is given by the hypotheses (3a) and (3d) of Theorem 1.2.

The results of Theorem 1.2 can be refined as follows. We recall that a *homotopy class* $\mathcal{H}$ of $C_{p,q}$ is an equivalence class of curves in $C_{p,q}$ with respect to the C^1-homotopy equivalence (with fixed endpoints).

Corollary 1.3. *Suppose that Y is a complete vector field on Λ and that hypothesis (3) of Theorem 1.2 is satisfied. If the following are satisfied:*

(1)' for a fixed pair of points p and q in Λ there exists a homotopy class $\mathcal{H}$ of $C_{p,q}$ and a real constant $c > \inf_{\mathcal{H}} f$ such that $\mathcal{H}$ is c-precompact;

(2)' there exist a positive constant $\mu = \mu(p,q,c)$ such that the inequality $-\langle Y(z), Y(z)\rangle \leq \mu < +\infty$ holds for all $z \in \Lambda_{p,q,c} \cap \mathcal{H}$;

then, there exists one geodesic in the homotopy class $\mathcal{H}$. If the constant c of (1)' can be chosen negative, then the geodesic in $\mathcal{H}$ can be found timelike.

Some examples will be presented at the end of the section to discuss in more details the hypotheses of Theorem 1.2 and of Corollary 1.3.

Under a non triviality condition on the topology of Λ, there exist geodesics of arbitrarily large energy joining any pair of points in Λ. Indeed, we have the following multiplicity result:

Theorem 1.4. *Under the same hypotheses of Theorem 1.2, suppose that the inequality (11) holds in $\Lambda_{p,q}$. If f is pseudo-coercive on $C_{p,q}$ and if Λ is non-contractible, then there exists a sequence $\{z_n\}_{n\in\mathbb{N}}$ of spacelike geodesics between p and q in Λ such that:*

$$\lim_{n\to\infty} f(z_n) = +\infty. \tag{12}$$

The integrability assumption for the vector field Y is almost certainly unnecessary for the results proven in the paper, but it is used to avoid some technical complicacies in our proofs. More specifically, it has been used in the proof of part 6 of Theorem 2.1. The same proof can be obtained in a more general context, but using non intrinsic conditions on the coefficients of the metric with respect to a given (non canonical) coordinate system (see for instance [8, 9, 13]).

To motivate the relevance of the results of Theorem 1.2 and Corollary 1.3, we now discuss some examples. In the following, Example 1.5

and Example 1.6 serve only as a physical motivation for the problem of the geodesical connectedness of static spacetimes with singular boundary; it is presented a quick proof that these two example fit into the model of Theorem 1.2. It should be observed that the geodesical connectedness for the Schwarzschild and for the Reissner-Nordström spacetimes can be established using the results in [4, 7]. In the rest of the cases discussed, the emphasis is given to the differences between the results of the paper and the results contained in the above mentioned papers.

Example 1.5. the exterior Schwarzschild spacetime.
Let (r, θ, φ) be the polar coordinates in $\mathbb{R}^3$ and m be any positive number. Consider the subset $\mathcal{M}_0$ of $\mathbb{R}^3$ given by the exterior of the ball of radius $2m$ and centered in the origin. Let $\Lambda \subset \mathbb{R}^4$ be the Lorentzian manifold $\mathcal{M}_0 \times \mathbb{R} \subset \mathbb{R}^4$, with metric:

$$\mathrm{d}s^2 = \frac{1}{\beta(r)}\mathrm{d}r^2 + r^2\,(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2) - c^2\beta(r)\,\mathrm{d}t^2, \tag{13}$$

where

$$\beta(r) = 1 - \frac{2m}{r}$$

and c is the speed of light. For simplicity, we set $c = 1$. This solution of the Einstein equations is the model for the gravitational field produced by a static, spherically symmetric massive body.

In this manifold, the timelike vector field $Y = \frac{\partial}{\partial t}$ is Killing, because the coefficients of the metric (13) do not depend on the coordinate t. The Riemannian metric $g_{(\mathrm{R})}$ is obtained simply changing the sign in front of the last term of (13).

Moreover, we consider the function $\phi(r, \theta, \phi, t) = \phi(r) = \sqrt{\beta(r)}$. Such a function clearly satisfies the hypotheses (3a) and (3b) of Theorem 1.2. Also hypothesis (3e) is satisfied, since ϕ does not depend on the variable t.

As to the pseudo-coercivity of the actions functional, let $p = (x_0, t_0)$ and $q = (x_1, t_1)$ be any pair of points in $\mathcal{M}_0 \times \mathbb{R}$ and let c be any real number. A C^1-curve $z(s) = (x(s), t(s))$ joining p and q is in $\mathcal{C}_{p,q}$ if:

$$-\langle \dot z, Y\rangle = \beta(r)\cdot \dot t \equiv C_z \quad \text{(constant)}. \tag{14}$$

From (14), we have:

$$C_z = \Delta\cdot\left(\int_0^1 \frac{\mathrm{d}s}{\beta(r(s))}\right)^{-1}, \tag{15}$$

where

$$\Delta = t(1) - t(0) = t_1 - t_0,$$

and so:

$$\beta(r)\cdot \dot t \equiv \Delta\left(\int_0^1 \frac{\mathrm{d}s}{\beta}\right)^{-1}. \tag{16}$$

The boundedness condition $f(z) \le c$ is written as:

$$\int_0^1 \left[\frac{\dot r^2}{\beta} + r^2(\dot\theta^2 + \sin^2\theta\,\dot\varphi^2) - \beta \dot t^2\right] \mathrm{d}s \le c;$$

from (14) and (15) we get:

$$\int_0^1 \left[\frac{\dot r^2}{\beta} + r^2(\dot\theta^2 + \sin^2\theta\,\dot\varphi^2)\right] \mathrm{d}s \le c + \int_0^1 \beta(r)\,\dot t^2\,\mathrm{d}s = \tag{17}$$

$$= c + C_z\cdot\Delta = c + \Delta^2\left(\int_0^1 \frac{\mathrm{d}s}{\beta(r(s))}\right)^{-1}.$$

Since $0 < \beta < 1$, it is:

$$\left(\int_0^1 \frac{\mathrm{d}s}{\beta(r(s))}\right)^{-1} \le 1, \quad \text{and} \quad \int_0^1 \frac{\dot r^2}{\beta}\,\mathrm{d}s \ge \int_0^1 \dot r^2\,\mathrm{d}s,$$

hence, from (17) we deduce:

$$\int_0^1 \left[\dot r^2 + r^2(\dot\theta^2 + \sin^2\theta\,\dot\varphi^2)\right] \mathrm{d}s \le c + \Delta^2. \tag{18}$$

Now, the left hand side of (18) is the Euclidean energy of the spatial part of the curve z written in polar coordinates:

$$\int_0^1 \langle \dot{x}, \dot{x} \rangle \, \mathrm{d}s \leq c + \Delta^2. \tag{19}$$

By the completeness of the set $\{x \in I\!R^3 : \|x\| \geq 2m\}$ and the Ascoli–Arzelá Theorem, any sequence x_n satisfying (19) has a uniformly convergent subsequence. From (16), if x_n is uniformly convergent, then so is t_n, and $C_{p,q}$ is c-precompact for every $p, q \in \mathcal{M}$ and all $c \in I\!R$.

One easily computes $\nabla\phi = a(r) \cdot \frac{\partial}{\partial r}$, with

$$a(r) = \frac{1}{2}\sqrt{\beta(r)}\beta'(r) = \frac{m}{r^2}\sqrt{1 - \frac{2m}{r}}.$$

Hence, $\langle \nabla\varphi, \nabla\varphi \rangle = \frac{m^2}{r^4}$, which is bounded on every bounded subset of $\mathcal{M}_0$, so that also hypothesis (3c) is satisfied. Finally, using the same computations of [4], one proves that for every geodesic γ in Λ, if $\phi(\gamma(s))$ is small enough, then it is:

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}\phi(\gamma(s)) \leq \frac{\beta'}{2r}\langle \dot{\gamma}(s), \dot{\gamma}(s) \rangle \cdot \phi(\gamma(s)). \tag{20}$$

Thus, (2) and (20) imply that also the hypothesis (3d) is satisfied.

Example 1.6. the exterior Reissner–Nordström spacetime. In the notation of Example 1.5, let's consider the subset $\mathcal{M}_0$ of $I\!R^3$, given by:

$$\mathcal{M}_0 = \{(r, \theta, \varphi) : r > m + \sqrt{m^2 - e^2}\}, \quad (m^2 > e^2)$$

Here m and e represent respectively the mass and the electric charge of the spherically symmetric body responsible for the gravitational field. The Reissner–Nordström spacetime is the Lorentzian manifold $\Lambda = \mathcal{M}_0 \times I\!R \subset I\!R^4$ endowed with the metric:

$$\mathrm{d}s^2 = \frac{1}{\beta(r)}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\varphi^2\right) - \beta(r)\,\mathrm{d}t^2, \tag{21}$$

where

$$\beta(r) = 1 - \frac{2m}{r} + \frac{e^2}{m^2}.$$

We consider the function $\phi(r,\theta,\varphi,t) = \phi(r) = \sqrt{\beta(r)}$. Since ϕ does not depend on the variable t, hypothesis (3e) is satisfied.

The gradient of ϕ is easily computed as $\nabla\phi = \sqrt{\beta(r)}\beta'(r)\frac{\partial}{\partial r}$, and

$$\langle \nabla\phi, \nabla\phi \rangle = \frac{1}{2}\beta'(r)^2 = \left(\frac{mr - e^2}{r^3}\right)^2.$$

The same arguments of Example 1.5 show that, also in the Reissner–Nordström spacetime, for for all pair of points p and q in $\mathcal{M}$, the action functional f is pseudo-coercive on $\mathcal{C}_{p,q}$.

Finally, using the same computations of [4], one proves that also the hypothesis (3d) is satisfied and Theorem 1.2 holds for the exterior Reissner–Nordström spacetime. Further results on the geodesical connectedness of Reissner–Nordström type Lorentzian manifolds may be found in [7] and [18].

In the following easy example we show a non geodesically connected manifold for which the hypothesis (1) of Theorem 1.2 holds precisely for those points that are joined by a geodesic.

Example 1.7. Let $\mathcal{M} = \mathbb{R}^4 \setminus \{0\}$ be endowed with the Minkowski flat metric; we denote by x the 3-space variable and by t the time variable in $\mathcal{M}$. Let Y be the vector field $\frac{\partial}{\partial t}$; in this example we also denote by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ respectively the Euclidean inner product and norm in $\mathbb{R}^3$.

Given any two points $p = (x_0, t_0)$ and $q = (x_1, t_1)$ in $\mathcal{M}$; then any curve z in $\mathcal{C}_{p,q}$ is of the form $z(s) = (x(s), t_0 + (t_1 - t_0)s)$, where x is a curve in the 3-space joining x_0 and x_1. Hence:

$$\inf_{\mathcal{C}_{p,q}} f = \frac{1}{2}\left(\|x_1 - x_0\|^2 - (t_1 - t_0)^2\right). \tag{22}$$

If $\tilde{z}(s) = (\tilde{x}(s), t_0 + (t_1 - t_0)s)$ is a curve in $\mathbb{R}^4$ joining p and q and passing through the origin, then an easy calculation shows that:

$$f(\tilde{z}) \geq \frac{1}{2}\left(\left(\sqrt{\|x_0\|^2 + t_0^2} + \sqrt{\|x_1\|^2 + t_1^2}\right)^2 - 2(t_1 - t_0)^2\right) \equiv C(p,q). \tag{23}$$

By continuity, if z is any curve in $\mathcal{C}_{p,q}$ with $f(z) < C(p,q)$, then the image of z stays uniformly far from the origin. From the Ascoli-Arzelá's theorem, it follows immediately that, any sequence $\{z_n\} \subset \mathcal{C}_{p,q}$ satisfying $f(z_n) \leq c < C(p,q)$ is uniformly convergent in $\mathcal{M}$.

Observe that, from (22) and (23) it is easily obtained that the inequality $C(p,q) > \inf_{\mathcal{C}_{p,q}} f$ is equivalent to:

$$\sqrt{\|x_0\|^2 + t_0^2} \cdot \sqrt{\|x_1\|^2 + t_1^2} > -\langle x_1, x_0\rangle - t_1 t_0,$$

which holds precisely when the segment in $\mathbb{R}^4$ joining p and q does not contain the origin. So, in this case Theorem 1.2 gives exactly the set of points which are joined by one geodesic in $\mathcal{M}$.

The following example shows that, even in the case of a standard static manifold $\mathcal{M}$ which is a warped product $\mathcal{M}_0 \times \mathbb{R}$, with $\mathcal{M}_0$ a *non complete* Riemannian manifold, one can have the condition of c-precompactness satisfied for all pair of points p and q.

Example 1.8. Let $\mathcal{M}_0$ be the open hemisphere $x^2 + y^2 + z^2 = 1$, $z > 0$, with the induced Euclidean metric of $\mathbb{R}^3$ denoted by $\mathrm{d}l_0^2$, and let $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}$ be the standard static Lorentzian manifold with metric $\mathrm{d}l^2 = \mathrm{d}l_0^2 - \mathrm{d}t^2$. Here, t denotes the real variable in the second factor; clearly, the vector field $Y = \frac{\partial}{\partial t}$ is Killing and $\mathcal{M}$ is static with respect to Y. Observe that, since $\mathcal{M}_0$ is not complete, then $\mathcal{M}$ is not globally hyperbolic.

Given the points $p = (p_0, t_0)$ and (p_1, t_1) in $\mathcal{M}_0 \times \mathbb{R}$, a geodesic joining p and q in $\mathcal{M}$ is of the form $\gamma = (\gamma_0, t_*)$, where $\gamma_0 : [0,1] \longmapsto \mathcal{M}_0$ is a geodesic in $\mathcal{M}_0$ joining p_0 and p_1, and $t_*(s) = (1-s)t_0 + st_1$. Observe that any pair of points in $\mathcal{M}_0$ is joined by a *minimal* geodesic in $\mathcal{M}_0$; moreover, the set $\mathcal{C}_{p,q}$ consists precisely of curves of the form $z = (x, t_*)$, where x is curve joining p_0 and p_1 in $\mathcal{M}_0$.

Hence, we have:

$$\inf_{\mathcal{C}_{p,q}} f = \min_{\mathcal{C}_{p,q}} f = \frac{1}{2}\left(\operatorname{dist}(p_0,p_1)^2 - (t_1 - t_0)^2\right), \tag{24}$$

where $\operatorname{dist}(\cdot,\cdot)$ denotes the distance function on $\mathcal{M}_0$. Clearly, the following inequality holds:

$$\operatorname{dist}(p_0,p_1) < \min_{q_0 \in \partial\mathcal{M}_0} (\operatorname{dist}(p_0,q_0) + \operatorname{dist}(p_1,q_0)); \tag{25}$$

we set:

$$d = \frac{1}{2}\left(\operatorname{dist}(p_0,p_1) + \min_{q_0 \in \partial\mathcal{M}_0} (\operatorname{dist}(p_0,q_0) + \operatorname{dist}(p_1,q_0))\right),$$

and

$$c = \frac{1}{2}\left(d^2 - (t_1 - t_0)^2\right).$$

By (24), it is $c > \inf_{\mathcal{C}_{p,q}} f$. If $z_n = (x_n, t_*)$ is any sequence in $\mathcal{C}_{p,q}$ such that $f(z_n) \le c$, then

$$\int_0^1 \|\dot{x}_n(s)\|^2 \, ds \le 2c + (t_1 - t_0)^2,$$

hence, by the Ascoli–Arzelá's Theorem, x_n has a subsequence which is uniformly convergent to a curve x in $\overline{\mathcal{M}_0} = \mathcal{M}_0 \bigcup \partial\mathcal{M}_0$. By the choice of d and c, it follows that x_n stays uniformly far from $\partial\mathcal{M}_0$, hence x has image in $\mathcal{M}_0$. This implies that z_n has a uniformly convergent subsequence in $\mathcal{M}$, and $\mathcal{C}_{p,q}$ is c-precompact.

Example 1.8 may be easily generalized by taking $\mathcal{M}_0$ to be any non complete Riemannian manifold (possibly with singular boundary) having the property that any two of its points are joined by a minimal geodesic. Examples of this type can be constructed in such a way that the metric does not have a continuous extension to the boundary of

$\mathcal{M}_0$, like for instance in the case of Riemannian *orbifolds* (see for instance the examples in [5] and the references therein). In this case, the Riemannian manifold $\mathcal{M}_0$ cannot be smoothly embedded into a complete Riemannian manifold, and for this class of examples the results of [4, 7, 8] do *not* apply.

The following example shows a non standard static manifolds, for which Corollary 1.3 can be applied to obtain the geodesical connectedness by timelike geodesics.

Example 1.9. the Lorentzian Klein bottle. Consider the plane $\mathbb{R}^2$ with coordinates x and t endowed with the flat Minkowski metric $g = \mathrm{d}x^2 - \mathrm{d}t^2$; let $Y = \frac{\partial}{\partial t}$. Let ϕ_1 and ϕ_2 the two diffeomorfisms of $\mathbb{R}^2$ given by:

$$\phi_1(x,t) = (x, t+1), \quad \phi_2(x,t) = (x+1, 1-t).$$

It is easy to see that ϕ_1 and ϕ_2 preserve the metric g and the vector field Y. Hence, if G denotes the subgroup of the group of isometries of $(\mathbb{R}^2, g)$, the quotient $\mathcal{K} = \mathbb{R}^2/G$ (which is homeomorphic to the Klein bottle) inherits a static Lorentzian structure. Observe indeed that G acts properly discontinuously on $\mathbb{R}^2$, hence $\mathbb{R}^2$ is locally isometric to $\mathcal{K}$, which implies that Y projects to the quotient as a (timelike) integrable Killing vector field. Since $\mathcal{K}$ is compact, then Y is a complete vector field.

Let $\pi : \mathbb{R}^2 \longmapsto \mathcal{K}$ denote the quotient map; with a little abuse of notation we will denote by f the action functional for curves on both manifolds $\mathbb{R}^2$ and $\mathcal{K}$; clearly, if z is a curve in $\mathbb{R}^2$, then $f(z) = f(\pi \circ z)$. Given any two points p and q in $\mathcal{K}$, and any homotopy class $\mathcal{H}$ of $\mathcal{C}_{p,q}$, then, fixing a base point $p_0 \in \pi^{-1}(p) \in \mathbb{R}^2$, then there exists a unique $q_0 \in \pi^{-1}(q)$ such that every curve $z \in \mathcal{H}$ has a unique lift $\tilde{z} \in \mathcal{C}_{p_0,q_0}$; moreover, $f(z) = f(\tilde{z})$.

Since $\mathcal{C}_{p_0,q_0}$ is c-precompact in $(\mathbb{R}^2, g)$ for all c, it follows that every homotopy class $\mathcal{H}$ of $\mathcal{C}_{p,q}$ is c-precompact for all c. By Corollary 1.3, every homotopy class of $\mathcal{C}_{p,q}$ contains a geodesic and $\mathcal{K}$ is geodesically connected.

Observe that, in this example, the entire space $\mathcal{C}_{p,q}$ is not c-precompact for any choice of the points p and q and of the constant $c > \inf\limits_{\mathcal{C}_{p,q}} f$.

Observe also that, for all p and q in $\mathcal{K}$, one can find a homotopy class $\mathcal{H}$ of $C_{p,q}$ such that $\inf_{\mathcal{H}} f < 0$, which implies that any two points in $\mathcal{K}$ are joined by a timelike geodesic.

We conclude this section with an example of a non geodesically connected static and globally hyperbolic Lorentzian manifold.

Example 1.10. Let $\mathcal{M}$ be the 4-dimensional Minkowski space, with $Y = \frac{\partial}{\partial t}$ the timelike Killing vector field. Take any *non convex* open subset A of the spacelike surface $t = 0$, and consider the *Cauchy development* $D(A)$ of A, which is the set of points p in $\mathcal{M}$ such that every past or future pointing, inextendible causal curve through p meets A (see Definition 14.45 of [14]). The interior of $D(A)$ is non empty, as it contains A, and so by Theorem 14.38 of [14] it is a globally hyperbolic manifold. Nonetheless, it is not geodesically connected, because two points in A cannot joined by any geodesics contained in $D(A)$.

The paper is organized as follows. In section 2 we set up our variational framework, we state and prove a variational principle for geodesics in Λ, and we introduce a family of penalizing functionals. In section 3 we will prove the Palais–Smale condition for our functionals, which is the main technical tool used to get the existence and the multiplicity of critical points for our variational problems. In section 4 we will prove some uniform estimates on the critical points of the penalizing functionals, needed to pass from critical points of these functionals to geodesics in Λ. Finally, in section 5 we collect all the results and we will present a proof of Theorem 1.2 and of Corollary 1.3. The proof of Theorem 1.4 is only sketched, as it can be easily deduced from our framework using the same arguments as in [4].

2 The Variational Setup. A Variational Principle for Geodesics.

We use the Riemannian metric (1) to define intrinsically the main spaces of our functional framework. We denote by $\nabla^{(\mathrm{R})}$ the Levi–Civita connection of $g_{(\mathrm{R})}$ and by $\mathrm{dist}(\cdot,\cdot)$ the distance function on $\Lambda \times \Lambda$ induced by $g_{(\mathrm{R})}$. We assume that Y is a smooth vector field defined in $\mathcal{M}$ which is timelike in Λ.

Let p and q two arbitrarily fixed points in Λ. $\Omega^{1,2}_{p,q}$ will denote the space of $H^{1,2}$-curves in Λ joining p and q:

$$\Omega^{1,2}_{p,q} = \Big\{ z : [0,1] \longmapsto \Lambda \;\Big|\; z \text{ absolutely continuous}, \; z(0) = p, \quad z(1) = q, \int_0^1 \langle \dot z, \dot z\rangle_{(\mathrm{R})} \, \mathrm{d}s < +\infty \Big\}.$$

It is well known that $\Omega^{1,2}_{p,q}$ is an infinite dimensional Hilbert manifold (see [15]); for $z \in \Omega^{1,2}_{p,q}$ the tangent space $T_z\Omega^{1,2}_{p,q}$ may be identified with the space of $H^{1,2}$-vector fields along z:

$$T_z\Omega^{1,2}_{p,q} = \Big\{ \zeta : [0,1] \longmapsto T\Lambda \;\Big|\; \zeta(s) \in T_{z(s)}\Lambda, \; \zeta \text{ absolutely continuos}, \; \zeta(0) = \zeta(1) = 0, \int_0^1 \langle \nabla_s\zeta, \nabla_s\zeta\rangle_{(\mathrm{R})} \, \mathrm{d}s < +\infty \Big\}. \tag{26}$$

Observe that $T_z\Omega^{1,2}_{p,q}$ is a Hilbert space with respect to the norm:

$$\|\zeta\|_* = \left(\int_0^1 \langle \nabla^{(\mathrm{R})}_s \zeta, \nabla^{(\mathrm{R})}_s \zeta\rangle_{(\mathrm{R})} \, \mathrm{d}s \right)^{\frac{1}{2}}. \tag{27}$$

For $r \geq 1$, we will denote by $L^r([0,1], Tv)$ the set of all r-integrable vector valued functions on $[0,1]$ with values in $T\Lambda$:

$$L^r([0,1], T\mathcal{M}) = \Big\{ \zeta : [0,1] \longmapsto T\Lambda \text{ measurable} : \; \|\zeta\|_r \equiv \left(\int_0^1 \langle \zeta(s), \zeta(s)\rangle_{(\mathrm{R})}^{\frac{r}{2}} \, \mathrm{d}s \right)^{\frac{1}{r}} < +\infty \Big\}.$$

Similarly, one defines the space $L^\infty([0,1], T\Lambda)$ as the set of all measurable maps $\zeta : [0,1] \longmapsto T\Lambda$ for which

$$\|\zeta\|_\infty = \operatorname{ess\,sup} \sqrt{\langle \zeta(s), \zeta(s)\rangle_{(\mathrm{R})}} < +\infty.$$

We say that a sequence ζ_n in $L^r([0,1],\Lambda)$ tends to 0 if $\|\zeta_n\|_r$ converges to 0 as n goes to infinity.

The *action functional* f on $\Omega^{1,2}_{p,q}$ is defined by:

$$f(z) = \frac{1}{2}\int_0^1 \langle \dot{z}, \dot{z}\rangle \,\mathrm{d}s; \tag{28}$$

observe that, by (2), it is $|\langle \dot{z}, \dot{z}\rangle| \leq \langle \dot{z}, \dot{z}\rangle_{(\mathrm{R})}$, hence the integral in (28) makes sense for $z \in \Omega^{1,2}_{p,q}$. The action functional is smooth on $\Omega^{1,2}_{p,q}$, and its critical points are smooth curves that satisfy the equation (3), hence they are geodesics. The differential of f is given by:

$$f'(z)[\zeta] = \int_0^1 \langle \dot{z}, \nabla_s \zeta\rangle \,\mathrm{d}s, \tag{29}$$

for every $\zeta \in T_z\Omega^{1,2}_{p,q}$.

We denote by W the distribution on the manifold $\Omega^{1,2}_{p,q}$ consisting of vector fields parallel to the timelike vector field Y:

$$W = \left\{ (z,\zeta) \in T\Omega^{1,2}_{p,q} \;\middle|\; \zeta(s) \parallel Y(z(s)) \;\forall\, s \in [0,1] \right\}. \tag{30}$$

Since Y is smooth, it follows immediately that W is a smooth distribution on $\Omega^{1,2}_{p,q}$. We set $\Pi(z,\zeta) = z$ the projection of W onto $\Omega^{1,2}_{p,q}$, and for $z \in \Omega^{1,2}_{p,q}$, W_z will denote the subspace of $T_z\Omega^{1,2}_{p,q}$ given by $\Pi^{-1}(z)$.

Finally, we introduce the space $\mathcal{N}_{p,q}$ of curves z in $\Omega^{1,2}_{p,q}$ such that the derivative $f'(z)$ vanishes in the directions of W:

$$\mathcal{N}_{p,q} = \left\{ z \in \Omega^{1,2}_{p,q} \;\middle|\; f'(z)[\zeta] = 0 \,\forall\, \zeta \in W_z \right\}. \tag{31}$$

We will denote by J the restriction of the action functional f on the space $\mathcal{N}_{p,q}$:

$$J = f\big|_{\mathcal{N}_{p,q}}.$$

The space $\mathcal{N}_{p,q}$ and the functional J have been explicitly introduced in [9], but they appear in a hidden form in some previous works by Benci, Fortunato, Giannoni and Masiello (see [3, 4, 7, 8, 13]).

We state and prove the following variational principle for geodesics in static manifolds, that will be used in the rest of the paper.

Theorem 2.1. *Suppose that Λ is an open connected subset of $\mathcal{M}$, and Y is a smooth vector field on $\mathcal{M}$ such that Λ is static with respect to Y. Then*

1. $\mathcal{N}_{p,q} = \{z \in \Omega^{1,2}_{p,q} : \langle \dot z(s), Y(z(s))\rangle$ *is constant a.e. on* $[0,1]\}$;
2. $\mathcal{N}_{p,q}$ *is a* C^1*-submanifold of* $\Omega^{1,2}_{p,q}$, *and* J *is* C^1*-functional;*
3. *for every* $z \in \mathcal{N}_{p,q}$, *the tangent space* $T_z\mathcal{N}_{p,q}$ *is identified with the set:*

$$T_z\mathcal{N}_{p,q} = \{\zeta \in T_z\Omega^{1,2}_{p,q} : \langle \nabla_s\zeta, \dot z\rangle \text{ is constant a.e. on } [0,1]\}.$$

4. *if* Y *satisfies (11) on* Λ, *and if* Λ *satisfies the hypothesis (3b) of Theorem 1.2, then* $\mathcal{N}_{p,q}$ *is non empty;*
5. *a curve* z *in* $\Omega^{1,2}_{p,q}$ *is a geodesic joining* p *and* q *if and only if* $z \in \mathcal{N}_{p,q}$ *and* z *is a critical point for* J;
6. *if* $C_{p,q}$ *is c-precompact for some* $c > \inf\limits_{C_{p,q}} f$, *then* J *is bounded from below in* $\mathcal{N}_{p,q}$.

Remark 2.2. From part (1) of Theorem 2.1, using standard arguments in Sobolev spaces one sees that the set $C_{p,q}$ introduced in Section 1 is contained as a dense subset of $\mathcal{N}_{p,q}$. Hence, in the statements of Definition 1.1 and of Theorem 1.2, we can replace the space $C_{p,q}$ with $\mathcal{N}_{p,q}$. The reason for introducing the space $\mathcal{N}_{p,q}$ is that it is the natural space for obtaining the Palais–Smale compactness condition for the action functional. The details of this fact will be discussed in Section 3.

Proof of Theorem 2.1. For part (1), suppose that $z \in \Omega^{1,2}_{p,q}$ is such that the quantity $\langle \dot z, Y(z)\rangle = C_z$ is constant. Then, for every $\zeta \in T_z\Omega^{1,2}_{p,q}$ such that $(z,\zeta) \in W$, it is $\zeta(s) = \mu(s)\cdot Y(z(s))$, for some $\mu \in H^{1,2}([0,1], \mathbb{R})$ and $\mu(0) = \mu(1) = 0$. Then,

$$\begin{aligned} f'(z)[\zeta] &= \int_0^1 \langle \dot z, \nabla_s(\mu Y)\rangle\, ds = \int_0^1 (\mu'\langle \dot z, Y\rangle + \mu\langle \dot z, \nabla_s Y\rangle)\, ds = \\ &= \int_0^1 \mu'\langle \dot z, Y\rangle\, ds = C_z \int_0^1 \mu'\, ds = 0. \end{aligned}$$

Moreover, if $\int_0^1 \mu'\langle\dot z, Y\rangle\,\mathrm{d}s = 0$ for all $\mu \in H^{1,2}([0,1],\mathbb{R})$ with $\mu(0) = \mu(1) = 0$, then $\langle\dot z, Y\rangle$ must be constant, and part (1) is proven.

For part (2), consider the map:

$$F : \Omega^{1,2}_{p,q} \longmapsto L^2([0,1],\mathbb{R})$$

given by $z \longmapsto \langle\dot z, Y\rangle$. It is easily seen that F is smooth, and its Gateaux derivative is given by:

$$F'(z)[\zeta] = \langle\nabla_s\zeta, Y\rangle + \langle\dot z, \nabla_\zeta Y\rangle.$$

If $\mathcal{C}$ denotes the submanifold of $L^2([0,1],\mathbb{R})$ consisting of all the constant functions, clearly $\mathcal{N}_{p,q} = F^{-1}(\mathcal{C})$. By the Implicit function Theorem (see Proposition 3.II.2 of [12]), in order to prove that $\mathcal{N}_{p,q}$ is a C^1 submanifold of $\Omega^{1,2}_{p,q}$ it suffices to show that, for every $z \in \mathcal{N}_{p,q}$ and $h \in L^2([0,1],\mathbb{R})$ the equation in ζ:

$$F'(z)[\zeta] = h + \text{const.}, \tag{32}$$

can be solved. In order to prove this, we fix $z \in \mathcal{N}_{p,q}$ and $h \in L^2([0,1],\mathbb{R})$, and we consider the equation (32). We set

$$\zeta(s) = \mu(s)\cdot Y(z(s)),$$

for some $\mu \in H^{1,2}([0,1],\mathbb{R})$, with $\mu(0) = \mu(1) = 0$, so that $\zeta \in T_z\Omega^{1,2}_{p,q}$. Substituting ζ in (32), and considering that, since Y is Killing, it is:

$$\begin{aligned}\langle\dot z, \nabla_\zeta Y(z)\rangle &= \mu\cdot\langle\dot z, \nabla_Y Y(z)\rangle = \\ &= -\mu\cdot\langle Y(z), \nabla_s Y(z)\rangle,\end{aligned} \tag{33}$$

we obtain the equation:

$$\begin{aligned}\langle\nabla_s\zeta, Y(z)\rangle + \langle\dot z, \nabla_\zeta Y(z)\rangle &= \mu'\cdot\langle Y(z), Y(z)\rangle + \\ &\quad +\mu\cdot\langle\nabla_s Y(z), Y(z)\rangle - \mu\cdot\langle\nabla_s Y(z), Y(z)\rangle = \\ &= \mu'\cdot\langle Y(z), Y(z)\rangle = h + C,\end{aligned} \tag{34}$$

where C is a constant. Since $\langle Y, Y\rangle < 0$, we can always solve (34) for μ by setting:

$$\mu(s) = \int_0^s \frac{h(r) + C}{\langle Y(z(s)), Y(z(s))\rangle}\,\mathrm{d}r,$$

and clearly $\mu(0) = 0$. Moreover, choosing

$$C = -\int_0^1 \frac{h(s)}{\langle Y(z(s)), Y(z(s))\rangle}\, dr,$$

we also have $\mu(1) = 0$. So, $\mathcal{N}_{p,q}$ is a C^1 submanifold of $\Omega^{1,2}_{p,q}$, and J is the restriction to this submanifold of the smooth functional f. Hence, J is C^1 and part (2) is proven.

Part (3) follows immediately from part (2) and the Implicit Function Theorem.

For part (4), given any $z \in \Omega^{1,2}_{p,q}$ and denoting by ψ the flow of the vector field Y in Λ, we observe that the boundedness assumptions for Y and the completeness assumptions for Λ guarantee that $\psi(z,t)$ is defined for all values of $t \in \mathbb{R}$. Then, we set $\tilde{z}(s) = \psi(z(s), t(s))$ for function t to be determined in such a way that $\tilde{z}$ belongs to $\mathcal{N}_{p,q}$. Using the fact that Y is Killing and the map $\psi(\cdot, t)$ is an isometry for every t, it is checked easily that t needs to satisfy the differential equation:

$$\dot{t} \cdot \langle Y, Y\rangle + \langle \dot{z}, Y\rangle = C \text{ (constant)} \tag{35}$$

and the boundary conditions $t(0) = t(1) = 0$. Since $\langle Y, Y\rangle \neq 0$ everywhere, this problem admits a unique solution t for a unique value of the constant C that appears in (35), which proves part (4).

For part (5), observe that since Y is Killing, then by part (1), all the geodesics belong to $\mathcal{N}_{p,q}$. Since the geodesics are critical points of f, then they are also critical points of J in $\mathcal{N}_{p,q}$. Conversely, suppose that $z \in \mathcal{N}_{p,q}$ is a critical point for J. In order to show that z is a geodesic, we need to prove that z is also a critical point for f in $\Omega^{1,2}_{p,q}$. This follows immediately from the definition of $\mathcal{N}_{p,q}$, and the fact that the tangent space $T_z\Omega^{1,2}_{p,q}$ is spanned by the subspaces $T_z\mathcal{N}_{p,q}$ and W_z. In order to see this, from part (3), we need to show that every $\zeta \in T_z\Omega^{1,2}_{p,q}$ can be written as the sum:

$$\zeta = \zeta_1 + \mu \cdot Y,$$

where $\langle \nabla_s \zeta_1, \dot{z}\rangle$ is constant and $\mu \in H^{1,2}([0,1], \mathbb{R})$ is such that $\mu(0) = \mu(1) = 0$. By the same argument of the proof of part (4), we see that μ

has to satisfy a first order liner differential equation with an arbitrary constant on the right hand side, and two boundary conditions. Such a problem admits a unique solution, and we have proven part (5).

Finally, for part (6), let p and q be fixed and $c > \inf_{\mathcal{N}_{p,q}} f$ be such that $\mathcal{N}_{p,q}$ is c-precompact.

Let $z_n \in \mathcal{N}_{p,q}$ be a minimizing sequence for J. Then, eventually it is $J(z_n) = f(z_n) \leq c$, and by the c-precompactness, up to taking subsequences, we can assume that z_n is uniformly convergent to a curve z with image in $\overline{\Lambda}$.

First we observe that the quantity $C_{z_n} = \langle \dot{z}_n, Y \rangle$ is bounded, say:

$$|C_{z_n}| \leq D. \tag{36}$$

To see this it suffices to work in local coordinates and use the same result of [3] (see also Lemma 4.1 of [9] for details).

We consider the 1-form ω on Λ given by the dual of the vector field $Y \cdot \langle Y, Y \rangle^{-1}$:

$$\omega(x)[v] = \frac{\langle Y(x), v \rangle}{\langle Y(x), Y(x) \rangle}, \quad \forall x \in \Lambda,\ v \in T_x\Lambda.$$

A straightforward computation shows that ω is locally integrable, i.e. $d\omega \equiv 0$. Namely, denoting by $\xi(x)$ the vector field $Y(x)\langle Y(x), Y(x) \rangle^{-1}$, for all pairs $v_1, v_2 \in T_x\Lambda$, it is:

$$\begin{aligned} d\omega(x)[v_1, v_2] &= \langle \nabla_{v_1}\xi, v_2 \rangle - \langle \nabla_{v_2} Y, v_1 \rangle = \\ &= \frac{\langle \nabla_{v_1} Y, v_2 \rangle - \langle \nabla_{v_2} Y, v_1 \rangle}{\langle Y, Y \rangle} + \\ &\quad -2\frac{\langle Y, v_2 \rangle \langle \nabla_{v_1}, Y \rangle + \langle Y, v_1 \rangle \langle \nabla_{v_2}, Y \rangle}{\langle Y, Y \rangle^2}. \end{aligned} \tag{37}$$

If both v_1 and v_2 are orthogonal to Y, by the integrability of Y it follows that $d\omega(x)[v_1, v_2]$ vanishes (see (7)); if both v_1 and v_2 are multiple of Y, then it is easily seen that(37) vanishes because of the Killing property of Y:

$$\langle \nabla_Y Y, Y \rangle \equiv 0.$$

If v_1 is orthogonal to Y and v_2 is a multiple of Y, using the anti-symmetry of the the expression $\langle \nabla_{v_1} Y, v_2 \rangle$, one checks immediately from (37) that $d\omega(x)[v_1, v_2] = 0$. By the bilinearity and the anti-symmetry of $d\omega$ we get that $d\omega \equiv 0$ in Λ.

We go back now to the study of the minimizing sequence z_n and its uniform limit z. Since $z([0,1])$ is compact, we can find a finite family $\{\mathcal{U}_k\}_{k=1}^N$ of open subset in $\mathcal{M}$ and a finite sequence $0 = a_0 < a_1 < a_2 < \ldots < a_N = 1$ such that:

(a) $z([0,1]) \subset \bigcup\limits_{k=1}^{N} \mathcal{U}_k$;

(b) $z_n([a_{k-1}, a_k]) \subset \mathcal{U}_k$ for all $k = 1, 2, \ldots, N$ and for n sufficiently large;

(c) $\overline{\mathcal{U}}_k \cap \overline{\Lambda}$ is compact for all k;

(d) $\mathcal{U}_k \cap \Lambda$ is contractible in Λ, and, in particular, it is simply connected;

(e) by the property above and the local integrability of ω, there exist smooth functions $T_k : \mathcal{U}_k \longmapsto I\!R$ such that $dT_k = \omega$ in $\mathcal{U}_k$, i.e.:

$$\nabla T_k(x) = \frac{Y(x)}{\langle Y(x), Y(x) \rangle}, \quad \forall x \in \mathcal{U}_k, \ \forall k = -1, 2, \ldots, N; \qquad (38)$$

(f) $\sup\limits_{x_1, x_2 \in \mathcal{U}_k} |T_k(x_1) - T_k(x_2)| \leq 1$, for all k.

Hence, from (36) and the above properties, for n large enough we have:

$$
\begin{aligned}
J(z_n) &= \frac{1}{2}\int_0^1 \langle \dot z_n, \dot z_n\rangle \,\mathrm{d}s = \\
&= \frac{1}{2}\sum_{k=1}^{N}\int_{a_{k-1}}^{a_k}\left(\langle \dot z_n, \dot z_n\rangle_{(\mathrm{R})} + 2\,\frac{\langle \dot z_n, Y\rangle^2}{\langle Y, Y\rangle}\right)\mathrm{d}s \geq \\
&\geq \sum_{k=1}^{N}\int_{a_{k-1}}^{a_k}\frac{\langle \dot z_n, Y\rangle^2}{\langle Y, Y\rangle}\,\mathrm{d}s = C_{z_n}\sum_{k=1}^{N}\int_{a_{k-1}}^{a_k}\langle \dot z_n, \nabla T_k\rangle\,\mathrm{d}s = \quad (39)\\
&= C_{z_n}\sum_{k=1}^{N}\left(T_k(z_n(a_k)) - T_k(z_n(a_{k-1}))\right) \geq -ND > -\infty.
\end{aligned}
$$

This concludes the proof. □

Observe that the boundedness property of the functional J is crucial to obtain the existence of minimizers using standard techniques of Nonlinear Analysis. Observe also that the proof of the boundedness of J presented here relies heavily on the integrability assumption for Y, i.e. on the staticity of the metric of Λ. The non integrable case, that corresponds to stationary manifolds, is much more delicate to deal with; a complete reference for the standard stationary case is the book [13], the non standard case of a manifold with no boundary is studied in [9].

We isolate the following result for later use:

Lemma 2.3. *Suppose that Y is a complete vector field on Λ. Then, there exists a smooth open map $\mathcal{F}: \Omega^{1,2}_{p,q} \longmapsto \mathcal{N}_{p,q}$ which is a homotopy equivalence.*

Proof. Let $\psi : \Lambda \times I\!R \longmapsto \Lambda$ denote the flow of Y; since Y is Killing, then the map $p \longmapsto \psi(p, t_0)$ is an isometry of Λ for all $t_0 \in I\!R$. The map $\mathcal{F}$ is given by:

$$\mathcal{F}(z) = w,$$

with

$$w(s) = \psi(z(s), t_z(s))$$

where t_z is the solution of the initial value problem:

$$t_z' = \frac{C_z - \langle \dot{z}, Y(z)\rangle}{\langle Y(z), Y(z)\rangle}, \quad t_z(0) = 0, \tag{40}$$

and C_z is given by:

$$C_z = \left(\int_0^1 \frac{ds}{\langle Y(z), Y(z)\rangle}\right)^{-1} \cdot \int_0^1 \frac{\langle \dot{z}, Y(z)\rangle}{\langle Y(z), Y(z)\rangle}\, ds. \tag{41}$$

By the isometry property of ψ and the fact that Y is ψ-invariant one checks immediately that $\mathcal{F}$ is a well defined map between $\Omega_{p,q}^{1,2}$ and $\mathcal{N}_{p,q}$; moreover, it is easily established that $\mathcal{F}$ is the identity on $\mathcal{N}_{p,q}$. The smoothness of $\mathcal{F}$ follows immediately from the smooth dependence on z of the solution of the differential equation (40) and from the smooth dependence of C_z on z in (41). The fact that $\mathcal{F}$ is open can be easily checked using local coordinates, as in the proof of part (6) of Theorem 2.1. Finally, the map $H(t,z) : [0,1] \times \Omega_{p,q}^{1,2} \longmapsto \Omega_{p,q}^{1,2}$ given by:

$$H(r,z)(s) = \psi(z(s), r \cdot t_z(s))$$

is easily seen to give a smooth homotopy between $\mathcal{F}$ and the identity map on $\mathcal{N}_{p,q}$. □

In order to overcome the lack of compactness of the functional J, which is caused by the presence of the boundary $\partial\Lambda$, we introduce a family of *approximating* functionals J_ε, depending on a positive (small) parameters ε.

Let ϕ be the function introduced in the hypothesis of Theorem 1.2, and ε be a positive number. We define the *penalized functional* J_ε on $\mathcal{N}_{p,q}$ by:

$$J_\varepsilon(z) = J(z) + \varepsilon \cdot \int_0^1 \frac{ds}{\phi(z(s))^2}. \tag{42}$$

It is not too difficult to see that, for every $\varepsilon > 0$, the functional J_ε is differentiable on $\mathcal{N}_{p,q}$, and, for every $\zeta \in T_z\mathcal{N}_{p,q}$, the Gateaux derivative $J_\varepsilon'(z)[\zeta]$ is given by:

$$J_\varepsilon'(z)[\zeta] = J'(z)[\zeta] - 2\varepsilon \cdot \int_0^1 \frac{\langle \nabla\phi(z), \zeta\rangle}{\phi(z)^3}\, ds. \tag{43}$$

The critical points of the functional J_ε are smooth curves and they satisfy the following conservation law:

Proposition 2.4. *Let $\varepsilon > 0$ be fixed. If z_ε is a critical point for J_ε in $\mathcal{N}_{p,q}$, then z_ε is a curve of class C^2, and there exists a constant $E = E(z_\varepsilon)$ such that:*

$$\langle \dot z_\varepsilon, \dot z_\varepsilon \rangle - \frac{\varepsilon}{\phi(z_\varepsilon)^2} \equiv E. \tag{44}$$

Proof. If z_ε is a critical point for J_ε in $\mathcal{N}_{p,q}$, then z_ε satisfies:

$$\int_0^1 \langle \dot z_\varepsilon, \nabla_s \zeta \rangle \, \mathrm{d}s = 2\varepsilon \int_0^1 \langle \frac{\nabla \phi(z_\varepsilon)}{\phi(z_\varepsilon)^3}, \zeta \rangle, \tag{45}$$

for all $\zeta \in T_z\mathcal{N}_{p,q}$. We denote by F the vector field along z_ε given by the *covariant integral* of the C^1 vector field $\frac{2\varepsilon \nabla \phi(z_\varepsilon)}{\phi(z_\varepsilon)^3}$, i.e., F is uniquely defined by the equations $F(0) = 0$ and

$$\nabla_s F = 2\varepsilon \frac{\nabla \phi}{\phi^3}.$$

Then, integrating by parts the right hand side of (45), we obtain:

$$\int_0^1 \langle \dot z_\varepsilon + F, \zeta \rangle \, \mathrm{d}s \equiv 0, \quad \forall \zeta \in T_z\mathcal{N}_{p,q}. \tag{46}$$

Then, a classical *boot-strap* argument shows that the component of the vector field $\dot z_\varepsilon + F$ in $T_{z_\varepsilon}\mathcal{N}_{p,q}$ is of class C^1. Since F is of class C^1, then the component of $\dot z_\varepsilon$ in $T_{z_\varepsilon}\mathcal{N}_{p,q}$ is of class C^1. Moreover, since $z \in \mathcal{N}_{p,q}$, then by (31) it is:

$$f'(z_\varepsilon)[\zeta] = \int_0^1 \langle \dot z_\varepsilon, \nabla_s \zeta \rangle \, \mathrm{d}s = 0$$

for all $\zeta \in W_{z_\varepsilon}$, the same boot-strap argument shows that also the component of $\dot z_\varepsilon$ in W_{z_ε} is C^1.

Since $T_{z_\varepsilon}\Omega^{1,2}_{p,q} = W_{z_\varepsilon} \oplus T_{z_\varepsilon}\mathcal{N}_{p,q}$, it follows that $\dot z_\varepsilon$ is of class C^1.

Then, using (29) and integrating by parts in (43), one sees that the vector

$$-\nabla_s \dot{z}_\varepsilon - 2\varepsilon \frac{\nabla\phi(z_\varepsilon)}{\phi(z_\varepsilon)^3}$$

has null component on the space $T_{z_\varepsilon}\mathcal{N}_{p,q}$. Actually, since $\langle \nabla\phi, Y\rangle = 0$, arguing as in the proof of the regularity of z_ε, we see that z_ε satisfies the Euler–Lagrange equations:

$$-\nabla_s \dot{z}_\varepsilon - 2\varepsilon \frac{\nabla\phi(z_\varepsilon)}{\phi(z_\varepsilon)^3} = 0. \tag{47}$$

Multiplying (47) by $\dot{z}_\varepsilon$, we get:

$$-\langle \nabla_s \dot{z}_\varepsilon, \dot{z}_\varepsilon\rangle - 2\frac{\varepsilon}{\phi(z_\varepsilon)^3}\langle \nabla\phi(z_\varepsilon), \dot{z}_\varepsilon\rangle = -\frac{\mathrm{d}}{\mathrm{d}s}\big(\langle \dot{z}_\varepsilon, \dot{z}_\varepsilon\rangle - \frac{\varepsilon}{\phi(z)^2}\big) = 0,$$

which gives the thesis. □

3 The Palais–Smale Condition

We will assume henceforth that Λ, Y, ϕ are given and p, q and c are chosen in such a way that the hypotheses of Theorem 1.2 are satisfied.

We recall that if (X, h) is a Hilbertian manifold and $F : X \longmapsto \mathbb{R}$ is a C^1-functional on X, then F is said to satisfy the *Palais–Smale* condition at the level $c \in \mathbb{R}$ if every sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ satisfying:

(PS1) $\lim_{n\to\infty} F(x_n) = c$,

(PS2) $\lim_{n\to\infty} \|F'(x_n)\| = 0$,

has a subsequence converging in X. The norm $\|\cdot\|$ used in (PS2) is the operator norm of $F'(x_n)$ in the Hilbert space $T_{x_n}X$.

The Palais–Smale condition is an essential tool for studying the existence and the multiplicity of critical points for regular functionals. The main goal of this section is to prove that, for every $\varepsilon > 0$ and every

$c \in I\!R$, the functional J_ε satisfies the Palais–Smale condition at the level c in $\mathcal{N}_{p,q}$.

We start with a basic Lemma, whose proof is essentially contained in [10].

Lemma 3.1. *Let $\{z_n\}_{n\in I\!N}$ be a sequence in $\mathcal{N}_{p,q}$ for which there exists a sequence $s_n \in (0,1)$ with:*

$$\lim_{n\to\infty} \phi(z_n(s_n)) = 0.$$

If $\dot z_n$ is bounded in $L^2([0,1], T\Lambda)$, then

$$\lim_{n\to\infty} \int_0^1 \frac{\mathrm{d}s}{\phi(z_n(s))^2} = +\infty. \tag{48}$$

Proof. First of all, we observe that, since $\langle \nabla\phi, Y\rangle = 0$, it is easily computed that:

$$\langle \nabla\phi, \nabla\phi\rangle = \langle \nabla_{(R)}\phi, \nabla_{(R)}\phi\rangle_{(\mathrm{R})},$$

where $\nabla_{(R)}\phi$ denotes the gradient of ϕ with respect to the Riemannian structure $g_{(\mathrm{R})}$ on Λ. It follows that the hypothesis (3c) of Theorem 1.2 can be written as:

$$\langle \nabla_{(R)}\phi, \nabla_{(R)}\phi\rangle_{(\mathrm{R})} \le M. \tag{49}$$

Now, suppose that $\int_0^1 \langle \dot z, \dot z\rangle_{(\mathrm{R})}\,\mathrm{d}s \le \alpha$ and let $s > s_n$ be a point in $[0,1]$ (we can assume that s_n is uniformly far away from the endpoints 0 and 1). From (49), we have:

$$\phi(s) - \phi(s_n) = \int_{s_n}^{s} \langle \nabla_{(R)}\phi, \dot z\rangle_{(\mathrm{R})}\,\mathrm{d}s \le C_1\sqrt{s - s_n}, \tag{50}$$

with $C_1 = \sqrt{\alpha M}$. Then, there exists a constant C_2 such that, for $s > s_n$, it is:

$$\frac{1}{\phi(z_n(s))^2} \ge \frac{1}{4}\left(\frac{1}{C_2(s - s_n) + \phi(z_n(s_n))^2}\right), \tag{51}$$

and, integrating on $[s_n, 1]$, we get:

$$\int_{s_n}^{1} \frac{\mathrm{d}s}{\phi(s_n(s))^2} \geq \frac{1}{4C_2}\left[\log[c_2(1-s_n)+\phi(z_n(s_n))^2] - \log(\phi(z_n(s_n))^2).\right] \tag{52}$$

The thesis follows immediately from the fact that $1 - s_n$ is uniformly far from 0 and that $z_n(s_n))$ tends to 0. □

A crucial point for using the classical results of Palais–Smale functionals is the completeness of the manifold, or, more generally, of the sublevels of the functionals. This is proven in the next

Proposition 3.2. *For every $a \in I\!\!R$ and every $\varepsilon > 0$, the sublevel J_ε^a:*

$$J_\varepsilon^a = \{z \in \mathcal{N}_{p,q} : J_\varepsilon(z) \leq a\} \tag{53}$$

is a complete metric subspace of $\mathcal{N}_{p,q}$.

Proof. Let $\varepsilon > 0$ be fixed. We claim that the quantity:

$$\int_0^1 \langle \dot{z}, \dot{z}\rangle_{(\mathrm{R})}\, \mathrm{d}s$$

is bounded in J_ε^a. To prove this, we argue as in the proof of part (6) of Theorem 2.1, and we introduce a coordinate function t defined in a neighborhood of a curve $z \in J_\varepsilon^a$. Using the same notations as in the proof of Theorem 2.1 and recalling that

$$\frac{\langle \dot{z}, Y\rangle^2}{\langle Y, Y\rangle} = \langle \dot{z}, Y\rangle \cdot \langle \dot{z}, \nabla t\rangle = C_z \langle \dot{z}, \nabla t\rangle$$

we have:

$$\int_0^1 \langle \dot{z}, \dot{z}\rangle_{(\mathrm{R})}\, \mathrm{d}s = 2J(z) - 2\int_0^1 \frac{\langle \dot{z}, Y\rangle^2}{\langle Y, Y\rangle}\, \mathrm{d}s \leq J_\varepsilon(z) + C_z \Delta. \tag{54}$$

The proof of the Proposition follows directly from (36) and (54). □

We will use the following notation. If $z:[0,1]\longmapsto\mathcal{M}$ is an absolutely continuous curve and $\beta\in L^1([0,1],T\mathcal{M})$ is a vector field along z, then the *covariant integral* of β along z, denoted by $B=\int_z\beta$, is the (unique) vector field along z that satisfies the initial value problem:

$$\nabla_{\dot z}B=\beta,\quad B(0)=0.$$

We need the following elementary result, which holds generally in semi-Riemannian geometry:

Lemma 3.1. *Let K be a compact subset of Λ. Suppose that z is an absolutely continuous curve in K, with $\dot z\in L^1([0,1],T\Lambda)$, and that $\beta\in L^1([0,1],T\Lambda)$ is a vector field along z. Then, the covariant integral $B=\int_z\beta$ of β along z is in $L^\infty([0,1],T\Lambda)$, and there exists constant $M=M(K)$ such that:*

$$\|B\|_\infty\le\|\beta\|_1\cdot e^{M\cdot\|\dot z\|_1}. \tag{55}$$

Proof. Since K is covered by a finite number of charts, using local coordinates, we can assume that Λ is an open subset of $\mathbb{R}^N$. We denote by $|\cdot|$ the Euclidean norm. The vector field B is the solution of the initial value problem:

$$\frac{\mathrm{d}}{\mathrm{d}s}B=-\Gamma(z)[\dot z,B]+\beta,\quad B(0)=0, \tag{56}$$

where

$$\Gamma(z(s))[\cdot,\cdot]:T_{z(s)}\Lambda\times T_{z(s)}\Lambda\longmapsto T_{z(s)}\Lambda$$

is the bilinear map given by the Christoffel symbols $\{\Gamma^k_{ij}\}$ of the Lorentzian metric g.

Integrating (56) on $[0,s]$, we obtain:

$$B(s)=\int_0^s\beta\,\mathrm{d}r-\int_0^s\Gamma(z)[\dot z,B]\,\mathrm{d}r,$$

hence

$$|B(s)|\le\int_0^1|\beta|\,\mathrm{d}r+M\int_0^s|\dot z|\cdot|B|\,\mathrm{d}r, \tag{57}$$

where M is the maximum of the norm of the operator Γ on K. Applying Gronwall's Lemma to (57), we obtain:

$$|B(s)| \leq \int_0^1 |\beta| \, \mathrm{d}r \cdot e^{M \int_0^1 |\dot{z}| \, \mathrm{d}r},$$

which gives (55). □

Remark 3.2. Suppose that $\{z_n\}$ is a sequence of absolutely continuous curves having image in a fixed compact subset of Λ, and with $\|\dot{z}_n\|_1$ bounded. Suppose further that β_n is a sequence of vector fields along the z_n's that tends to 0 in $L^1([0,1], T\Lambda)$. From Lemma 3.1 and its proof it follows that the sequence $B_n = \int_{z_n} \beta_n$ converges to 0 in $L^\infty([0,1], T\Lambda)$.

We can now prove our main compactness result:

Proposition 3.3. *Under the hypotheses of Theorem 1.2, for every $\varepsilon > 0$ and $c \in \mathbb{R}$, the functional J_ε satisfies the Palais–Smale condition at the level c on $\mathcal{N}_{p,q}$.*

Proof. Let ε and c be fixed, and let $\{z_n\}_{n \in \mathbb{N}}$ be a Palais–Smale sequence for J_ε in $\mathcal{N}_{p,q}$, i.e., it satisfies:

$$\lim_{n \to \infty} J_\varepsilon(z_n) = c, \quad \text{and} \quad \lim_{n \to \infty} J'_\varepsilon(z_n) = 0.$$

We know that $\dot{z}_n$ is bounded in $L^2([0,1], T\Lambda)$, so, by Lemma 3.1, z_n stays far from $\partial\Lambda$. By hypothesis (3b) of Theorem 1.2, the z_n's have image in a complete Riemannian manifold. Since $z_n(0) = p$ is fixed, then z_n is equibounded and equicontinuous, so that, by Ascoli–Arzelá's theorem, up to passing to a subsequence, we can assume that z_n is uniformly convergent to some curve $z \in \Omega^{1,2}_{p,q}$. Moreover, since $\dot{z}_n$ is bounded in L^2, we can assume that the convergence of z_n to z is weak in $\Omega^{1,2}_{p,q}$.

To prove that the convergence of z_n is strong, we now use the condition $J'_\varepsilon(z_n) \to 0$:

$$\int_0^1 \left(\langle \dot{z}_n, \nabla_s \zeta_n \rangle - \frac{2\varepsilon}{\phi(z_n)^3} \langle \nabla\phi(z_n), \zeta_n \rangle \right) \mathrm{d}s \longmapsto 0, \tag{58}$$

for all $\zeta_n \in T_{z_n}\mathcal{N}_{p,q}$ bounded.

Using the same argument as in the proof of Proposition 2.4, since $z_n \in \mathcal{N}_{p,q}$, $\langle \nabla\phi, Y\rangle = 0$ and $T_{z_n}\Omega^{1,2}_{p,q} = T_{z_n}\mathcal{N}_{p,q} \oplus W_{z_n}$, then we can assume that (58) holds for every bounded sequence $\zeta_n \in T_{z_n}\Omega^{1,2}_{p,q}$.

Using the fact that ζ_n converges uniformly to 0, and the fact that $\phi(z_n)$ and $\nabla\phi(z_n)$ are uniformly convergent, we get that:

$$\lim_{n\to\infty}\int_0^1 \langle \frac{\nabla\phi(z_n)}{\phi(z_n)^3}, \zeta_n\rangle \, ds = 0,$$

hence, by (58), we have

$$\lim_{n\to\infty}\int_0^1 \langle \dot z_n, \nabla_s\zeta_n\rangle \, ds = 0. \tag{59}$$

Let's prove the following Lemma:

Lemma 3.3. *In the above notations, there exists a sequence α_n in $T_{z_n}\Omega^{1,2}_{p,q}$ that tends to 0 in $L^2([0,1], T\Lambda)$ and such that:*

$$\int_0^1 \langle \dot z_n, \nabla_{\dot z_n}\zeta_n\rangle \, ds = \int_0^1 \langle \alpha_n, \nabla_{\dot z_n}\zeta_n\rangle \, ds. \tag{60}$$

Proof. We denote by Θ_n the vector field along z_n which is the gradient $\nabla J(z_n)$ of the functional J with respect to the Hilbertian norm $\|\cdot\|_*$ defined by (27). By definition, we have:

$$\int_0^1 \langle \dot z_n, \nabla_{\dot z_n}\zeta_n\rangle \, ds = \int_0^1 \langle \nabla^{(\mathrm{R})}_{\dot z_n}\Theta_n, \nabla^{(\mathrm{R})}_{\dot z_n}\zeta_n\rangle_{(\mathrm{R})} \, ds,$$

and, by (59), the sequence of vector fields

$$A_n = \nabla^{(\mathrm{R})}_{\dot z_n}\Theta_n$$

goes to 0 in $L^2([0,1], T\Lambda)$.

Using the Christoffel symbols of the metric tensors g and $g_{(\mathrm{R})}$, we can express the Riemannian covariant covariant derivative $\nabla^{(\mathrm{R})}_{\dot z_n}\zeta_n$ in terms of the Lorentzian covariant derivative $\nabla_{\dot z_n}\zeta_n$. Then, we write

$$\int_0^1 \langle A_n, \nabla^{(\mathrm{R})}_{\dot z_n}\zeta_n\rangle_{(\mathrm{R})} \, ds = \int_0^1 \langle A_n, \nabla_{\dot z_n}\zeta_n + G(z_n)[\dot z_n][\zeta_n]\rangle_{(\mathrm{R})} \, ds, \tag{61}$$

where $G(z)[\zeta_1][\zeta_2]$ is a bilinear functions in the variables ζ_1, ζ_2 which is continuous in the first variable z. Using (1), it is immediately checked the existence of two sequences B_n and b_n going to 0 in $L^2([0,1], T\Lambda)$ such that:

$$\int_0^1 \langle A_n, \nabla^{(\mathrm{R})}_{\dot z_n}\zeta_n\rangle_{(\mathrm{R})}\,\mathrm{d}s = \int_0^1 \Big(\langle B_n, \nabla_{\dot z_n}\zeta_n\rangle + \langle b_n, \zeta_n\rangle\Big)\,\mathrm{d}s. \tag{62}$$

Now, it is:

$$\int_0^1 \langle b_n, \zeta_n\rangle\,\mathrm{d}s = -\int_0^1 \langle \textstyle\int_{z_n} b_n, \nabla_{\dot z_n}\zeta_n\rangle,$$

because $\zeta_n(0) = \zeta_n(1) = 0$. By Remark 3.2, it follows that $\int_{z_n} b_n$ tends to 0 uniformly, therefore (60) follows from (62). □

Going back to the proof of Theorem 3.3, we now consider the sequence of vector fields

$$\omega_n = \dot z_n - \alpha_n. \tag{63}$$

From (60) we get that ω_n is of class C^1 and that

$$\nabla_{\dot z_n}\omega_n = 0. \tag{64}$$

The next observation is that the L^2-norm $\|\omega_n\|_2$ of ω_n is bounded, because $\|\dot z_n\|_2$ is bounded and α_n tends to 0 in $L^2([0,1], T\Lambda)$. This implies, in particular, that, for some sequence $\{s_n\} \subset [0,1]$, the sequence $|\omega_n(s_n)|$ is bounded, say:

$$|\omega_n(s_n)| \le c_0, \quad \forall n \in \mathbb{N}. \tag{65}$$

Once again, Gronwall's Lemma applied to the differential equation (64) and the boundedness condition (65) gives the existence of $\gamma_0 > 0$ such that:

$$|\omega_n(s)| \le c_0 \cdot e^{\gamma_0 \int_0^1 |\dot z_n|\,dr}, \forall s \in [0,1].$$

It follows that ω_n is bounded in L^∞. From (63) it follows that $\dot z_n$ is bounded in L^2, and since $z_n(0)$ is fixed the sequence z_n is uniformly bounded.

Writing equation (64) in coordinates, it becomes:

$$\omega_n' + \Gamma(z_n)[\dot z_n, \omega_n] = 0, \tag{66}$$

where Γ is a continuous function in z_n (that can be expressed using the Christoffel symbols of g), which is linear in the arguments $\dot z_n$ and ω_n. From (66), we obtain that ω_n' is bounded in L^2, and thus ω_n is bounded in $H^{1,2}$.

It follows that a subsequence of ω_n still denoted by ω_n, is weakly convergent in $H^{1,2}$, and, in particular, ω_n is convergent in $L^2([0,1], T\Lambda)$.

Therefore, there exists a subsequence of z_n that tends to z strongly in $\Omega^{1,2}_{p,q}$.

We are now left with the proof that z belongs to $\mathcal{N}_{p,q}$. Since $\dot z_n$ tends to $\dot z$ in L^2, up to passing to a subsequence we can assume that $\dot z_n$ tends to $\dot z$ pointwise almost everywhere. So, $\langle \dot z_n, Y\rangle$ tends to $\langle \dot z, Y\rangle$ pointwise a.e., which implies that $\langle \dot z, Y\rangle$ is constant almost everywhere, and $z \in \mathcal{N}_{p,q}$. This concludes the proof. □

We have the following Corollary to Theorem 3.3

Corollary 3.4. *Under the hypotheses of Theorem 1.2, for $\varepsilon > 0$ small enough, the functional J_ε attains its minimum in $\mathcal{N}_{p,q}$. Moreover, if z_ε is a family of minimal points for J_ε, there exists two real constants A and B such that;*

$$A \le J_\varepsilon(z_\varepsilon) \le B, \quad \forall \varepsilon \in]0, \varepsilon_0[. \tag{67}$$

Proof. It is a classical argument in Critical Point Theory. Thanks to the Palais–Smale condition and the completeness of the sublevels of the functionals J_ε, if the infimum i_ε of J_ε on $\mathcal{N}_{p,q}$ weren't a critical value, then it would be possible to find a homotopy between the sublevels $J_\varepsilon^{i_\varepsilon-\delta}$ and $J_\varepsilon^{i_\varepsilon+\delta}$, where $\delta > 0$ is sufficiently small. This is clearly impossible, because, for every $\delta > 0$, $J_\varepsilon^{i_\varepsilon-\delta} = \emptyset$ while $J_\varepsilon^{i_\varepsilon+\delta} \ne \emptyset$.

If we set $A = \inf\limits_{\mathcal{N}_{p,q}} J$, then clearly $J_\varepsilon(z_\varepsilon) \ge J(z_\varepsilon) \ge A$. Moreover, since for all $z \in \mathcal{N}_{p,q}$ the map $\varepsilon \longmapsto J_\varepsilon(z)$ is increasing, hence also the map $\varepsilon \longmapsto \inf\limits_{\mathcal{N}_{p,q}} J_\varepsilon$ is increasing, so $J_\varepsilon(z_\varepsilon) \le J_{\varepsilon_0}(z_{\varepsilon_0})$. □

4 A Priori Estimates for the Penalized Functional

In this section we will assume that $\{z_\varepsilon\}_{\varepsilon>0}$ is any fixed family in $\mathcal{N}_{p,q}$, consisting of critical points for the functionals J_ε, and satisfying the boundedness condition (67). For instance, z_ε can be a minimal point for J_ε. We prove some estimates on z_ε, uniform in ε, with the aim of passing to the limit as ε tends to 0 to obtain a critical point for the functional J.

Proposition 4.1. *The family $\dot z_\varepsilon$ is bounded in $L^2([0,1],T\Lambda)$.*

Proof. It is:

$$\int_0^1 \langle \dot z_\varepsilon, \dot z_\varepsilon\rangle \,\mathrm{ds} \le 2J_\varepsilon(z_\varepsilon) \le 2A. \tag{68}$$

Moreover, from the proof of part (6) of Theorem 2.1, it follows that the constant $C_{z_\varepsilon} = \langle \dot z_\varepsilon, Y\rangle$ is bounded. Arguing as in that proof, we introduce a local coordinate function t around the image of z_ε, and we compute:

$$\int_0^1 \langle \dot z_\varepsilon, \dot z_\varepsilon\rangle_{(\mathrm{R})} \,\mathrm{ds} = 2J(z_\varepsilon) - 2C_{z_\varepsilon}\int_0^1 \langle \dot z_\varepsilon, \nabla t\rangle \,\mathrm{ds} = 2J(z_\varepsilon) - 2C_{z_\varepsilon}\Delta. \tag{69}$$

The thesis follows at once from (36), (68) and (69). □

The main result of the section is that the family z_ε stays far from the boundary of Λ, uniformly in ε:

Proposition 4.2. *There exists a positive constant $\overline{r}$ such that, for ε sufficiently small, it is:*

$$dist(z_\varepsilon(s), \partial\Lambda) \ge \overline{r} > 0, \quad \forall\, s \in [0,1]. \tag{70}$$

Proof. For every $\varepsilon > 0$, let $t_\varepsilon \in [0,1]$ be a minimal point for the function $\nu_\varepsilon(s) = \phi(z_\varepsilon(s))$. Then, it is $\nu_\varepsilon'(t_\varepsilon) = 0$ and:

$$\nu_\varepsilon'' = \langle H^\phi(z_\varepsilon)\dot z_\varepsilon, \dot z_\varepsilon\rangle + \langle \nabla\phi(z_\varepsilon), \nabla_s \dot z_\varepsilon\rangle. \tag{71}$$

Using Euler–Lagrange equation (47) and the assumptions (3c) and (3d) of Theorem 1.2, from (71) we get:

$$\nu_\varepsilon'' \le M\langle \dot z_\varepsilon, \dot z_\varepsilon\rangle_{(\mathrm{R})}\cdot \nu_\varepsilon - \frac{2\varepsilon}{\phi(z_\varepsilon)^3}\langle \nabla\phi, \nabla\phi\rangle \le M\langle \dot z_\varepsilon, \dot z_\varepsilon\rangle_{(\mathrm{R})}\cdot \nu_\varepsilon. \tag{72}$$

Observe indeed that the condition $\langle \nabla\phi, Y\rangle = 0$ implies that $\nabla\phi$ is spacelike. Since $\int_0^1 \langle \dot z_\varepsilon, \dot z_\varepsilon\rangle_{(\mathrm{R})}\,ds$ is bounded and $\nu_\varepsilon'(t_\varepsilon) = 0$, the Gronwall's Lemma says that, if $\nu_\varepsilon(t_\varepsilon)$ is not bounded away from 0, then ν_ε would converge to 0 uniformly on $[0,1]$. But this implies that $J_\varepsilon(z_\varepsilon)$ is not bounded from above, and that contradicts (67). This concludes the proof. □

5 Proof of the Main Results

In this section we put together the partial results obtained in the previous sections to prove Theorems 1.2 and 1.4 and Corollary 1.3.

Proof of Theorem 1.2. Let $\{z_\varepsilon\}$ be a family of critical points of J_ε in $\mathcal{N}_{p,q}$, satisfying (67). By Proposition 4.2 and the hypothesis (3b), the z_ε's lie in a complete metric subspace of Λ. Moreover, since $z_\varepsilon(0) \equiv p$ for all ε, Proposition 4.1 says that the family $\{z_\varepsilon\}$ is bounded and equicontinuous. By Ascoli–Arzelá's theorem, we can find a sequence ε_n converging to 0 and a curve $z \in \mathcal{N}_{p,q}$, such that z_{ε_n} converges to z uniformly, and by Proposition 4.1, we can assume that we have weak convergence in $H^{1,2}$.

Since z_{ε_n} is far from $\partial\Lambda$, then $\phi(z_{\varepsilon_n})$ is far from 0, hence

$$2\varepsilon_n \frac{\nabla\phi(z_{\varepsilon_n})}{\phi(z_{\varepsilon_n})}$$

is uniformly convergent to 0. Using the conservation law (44), we get that $\nabla_s \dot z_{\varepsilon_n}$ is uniformly convergent to 0. This implies that the limit curve z satisfies the equation $\nabla_s \dot z = 0$. Then, z is a geodesic between p and q, and the Theorem is proven. □

Proof of Corollary 1.3. To prove Corollary 1.3, it suffices to prove that every homotopy class $\mathcal{H}$ of $\mathcal{N}_{p,q}$ is an open connected component of $\mathcal{N}_{p,q}$. Using the same arguments of the proof of Theorem 1.2 it would then follow that the Palais–Smale condition, the completeness of the sublevels and the boundedness property for the penalized functionals J_ε hold in $\mathcal{H}$, yielding the existence of a critical point for J in $\mathcal{H}$. For the completeness of the sublevels, observe indeed that the uniform limit z of a sequence $\{z_n\}$ of curves belongs to the same homotopy class of all the z_n's with n sufficiently large.

It is well known that the homotopy classes are connected components of the manifold $\Omega^{1,2}_{p,q}$; moreover, they are open. Using the map $\mathcal{F}$ of Lemma 2.3 we can conclude that $\mathcal{H}$ is an open connected component of $\mathcal{N}_{p,q}$, and the Corollary is proven. □

The proof of the multiplicity result of Theorem 1.4 is based on the Theory of Ljusternik and Schnirelman. We recall that the Ljusternik–Schnirelman category $\mathrm{cat}_X(F)$ of a subset F of the topological space X is the minimal number (possibly infinite) of closed, contractible subsets of X that cover F.

A well known result by Fadell and Husseini (see [6]) states that, if Λ is non contractible, then there exists a sequence K_n of compact subsets of $\Omega^{1,2}_{p,q}$ such that:

$$\lim_{n\to\infty} \mathrm{cat}_{\Omega^{1,2}_{p,q}}(K_n) = +\infty. \tag{73}$$

Using the flow of the vector field Y (see Lemma 2.3) it is easy to see that, under the hypothesis of Theorem 1.4, the spaces $\Omega^{1,2}_{p,q}$ and $\mathcal{N}_{p,q}$ are homotopically equivalent, hence (73) is valid also in $\mathcal{N}_{p,q}$.

Now, using a well known *minimax* argument, for every $\varepsilon > 0$ and $m \in \mathbb{N}$, we define:

$$c^\varepsilon_m = \inf_{L\in\Gamma_m} \sup_{z\in L} J_\varepsilon(z),$$

where:

$$\Gamma_m = \Big\{ L \subset \mathcal{N}_{p,q} : L \text{ compact}, \ \mathrm{cat}_{\mathcal{N}_{p,q}}(L) \geq m \Big\}.$$

The c_m^ε's are critical values of J_ε.

Now, the same arguments used in [4] can be repeated *verbatim* in our case to prove that one can pass to the limit as $\varepsilon \downarrow 0$, obtaining an unbounded sequence c_m of critical values for J. We omit the proof of this fact, as it can be found in [4].

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