RT-MAT 97-04

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Março/97

A VARIATIONAL THEORY FOR LIGHT RAYS IN STABLY CAUSAL LORENTZIAN MANIFOLDS: REGULARITY AND MULTIPLICITY RESULTS

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ABSTRACT. This paper is dedicated to the study of light rays joining an event p with a timelike curve γ, in a light–convex subset Λ of a stably causal Lorentzian manifold $\mathcal{M}$. We set up a functional framework, defined intrinsically, consisting of a family of manifolds $\mathcal{L}^+_{p,\gamma,\epsilon}$ and a positive functional Q defined on them. The critical points of Q on $\mathcal{L}^+_{p,\gamma,\epsilon}$ approach, as $\epsilon \to 0$, the lightlike, future pointing geodesics joining p and γ. We prove some regularity results, including the C^1–regularity of $\mathcal{L}^+_{p,\gamma,\epsilon}$, the C^2–regularity of Q on $\mathcal{L}^+_{p,\gamma,\epsilon}$ and the C^2–regularity of its critical points. Using them, we develop a Ljusternik–Schnirelman theory for light rays, obtaining some multiplicity results, depending on the topology of the space of all lightlike curves joining p and γ in Λ.

1. INTRODUCTION

This paper is dedicated to the study of light rays joining an event p with a timelike curve γ, in a light–convex subset Λ of a stably causal Lorentzian manifold $\mathcal{M}$. We will assume that $(\mathcal{M}, g)$ is a stably causal Lorentzian manifold, with Lorentzian metric tensor g. For the sake of simplicity, we will denote by $\langle \cdot, \cdot \rangle$ the bilinear form on $T_z\mathcal{M}$ given by $g(z)$. We refer to classical books as [BEE, HE, ON] for the main definitions and properties in Lorentzian geometry.

We recall that a Lorentzian manifold is said to be stably causal if it is causal, i.e. it does not contain closed causal curves, and if this property is preserved after small C^0–variation of the metric g. Equivalently (see [HE, Proposition 6.4.9]),

1991 *Mathematics Subject Classification.* 53 C 22, 53 C 80, 49 J 50.
Key words and phrases. Lorentzian Geometry, Fermat's Principle, General Relativity, Light Rays, Convex Sets.
[1]The author gratefully acknowkledges the support of FAPESP, Processo 96/2677-0, and of CAPES, Processo AEX1697/96-0.

$\mathcal{M}$ is stably causal if there exists a time function T on $\mathcal{M}$, i.e. a smooth function $T : \mathcal{M} \longmapsto \mathbb{R}$, satisfying:

$$\langle \nabla T(q), \nabla T(q) \rangle < 0,$$

for every $q \in \mathcal{M}$. Here, ∇T is the Lorentzian gradient of T, defined by $\mathrm{d}T(\cdot) = \langle \nabla T, \cdot \rangle$.

Let T be a fixed time function on $\mathcal{M}$. As we will see later (up to multiplying the metric by a conformal factor), it is not restrictive (for the study of light rays) to assume that ∇T is *normalized*, i.e. $\langle \nabla T(q), \nabla T(q) \rangle = -1$ for every q.

The vector field ∇T gives a time orientation on $\mathcal{M}$: a vector $v \in T_z\mathcal{M}$ is *future pointing* if $\langle v, \nabla T(z) \rangle > 0$. Observe that, by this definition, ∇T points towards the past.

We will study the light rays in $\mathcal{M}$, i.e., geodesics z satisfying $\langle \dot z(s), \dot z(s) \rangle = 0$ for all s, joining an event $p \in \mathcal{M}$ with a future pointing timelike curve $\gamma : \mathbb{R} \longmapsto \mathcal{M}$, which is assumed to be a closed embedding of $\mathbb{R}$ in $\mathcal{M}$. In particular, this implies that γ has no *endpoints* in $\mathcal{M}$, i.e. $\gamma(s)$ is eventually outside every compact subset of $\mathcal{M}$ for $s \to \pm\infty$. The curve γ is future pointing in the sense that $\dot\gamma(s)$ is future pointing for every s. In other words, $T(\gamma(s))$ is strictly increasing.

Let Λ is an open subset of $\mathcal{M}$ containing $\mathrm{supp}(\gamma) = \gamma(\mathbb{R})$, with the following properties:

(1) $\partial\Lambda$ is a smooth (C^2) submanifold of $\mathcal{M}$;

(2) $\partial\Lambda$ is timelike, i.e. for every $z \in \partial\Lambda$ the normal vector to $T_z\partial\Lambda$ is spacelike;

(3) $\partial\Lambda$ is light–convex, i.e. all the lightlike geodesics in $\overline{\Lambda} = \Lambda \bigcup \partial\Lambda$ with endpoints in Λ are entirely contained in Λ.

We introduce an auxiliary Riemannian structure on $\mathcal{M}$, related to the time function T, that will be used systematically throughout the paper. It is denoted by $\langle \cdot, \cdot \rangle_{(R)}$ and it is given by the following formula:

$$\langle \zeta, \zeta \rangle_{(R)_z} = \langle \zeta, \zeta \rangle + 2\langle \zeta, \nabla T(z) \rangle^2, \tag{1.0.1}$$

for $z \in \mathcal{M}$ and $\zeta \in T_z\mathcal{M}$. Observe that (1.0.1) clearly defines a smooth bilinear form on $T_z\mathcal{M}$; the (strict) positivity of $\langle \cdot, \cdot \rangle_{(R)}$ follows easily from the wrong way Schwartz's inequality. We denote by $\|\cdot\|_R$ the norm on $T_z\mathcal{M}$ induced by $\langle \cdot, \cdot \rangle_{(R)}$, and by $\nabla_s^{(R)}$ the covariant derivative induced by the Levi–Civita connection of $\langle \cdot, \cdot \rangle_{(R)}$.

The metric (1.0.1) allows to define *intrinsically* the Sobolev space

$$H^{1,2}([0,1], \Lambda) = \left\{ z \in \mathrm{Ac}([0,1], \Lambda) \mid \int_0^1 \|\dot z\|_R^2 \,\mathrm{d}s < +\infty \right\},$$

where $\mathrm{Ac}([0,1], \Lambda)$ is the set of absolutely continuous curves from $[0,1]$ to Λ. It is well known that $H^{1,2}([0,1], \Lambda)$ is a smooth Hilbert manifold (see [K]). Using local coordinates, it is not difficult to see that the space $H^{1,2}([0,1],\Lambda)$ does not depend on the choice of the time function T.

We introduce the following space:

$$\Omega^{1,2}_{p,\gamma} = \Omega^{1,2}_{p,\gamma}(\Lambda) = \{z \in H^{1,2}([0,1],\Lambda) \mid z(0) = p,\ z(1) \in \operatorname{supp}(\gamma)\}.$$

It is not difficult to see that $\Omega^{1,2}_{p,\gamma}$ is a smooth manifold and, for every $z \in \Omega^{1,2}_{p,\gamma}$, the tangent space $T_z\Omega^{1,2}_{p,\gamma}$ is identified with:

$$T_z\Omega^{1,2}_{p,\gamma} = \{\zeta \in H^{1,2}([0,1],T\Lambda) \mid \zeta(0) = 0,\ \zeta(1) \parallel \dot\gamma(\gamma^{-1}(z(1)))\}.$$

The *Arrival Time* functional on $\Omega^{1,2}_{p,\gamma}$, introduced by Perlick in [Pe1], is defined by:

$$\tau_{p,\gamma}(z) = \gamma^{-1}(z(1)).$$

The natural space for the search of the lightlike geodesics in Λ joining p and γ is given by:

$$\mathcal{L}^+_{p,\gamma} = \mathcal{L}^+_{p,\gamma}(\Lambda) = \{z \in \Omega^{1,2}_{p,\gamma} \mid \langle \dot z, \dot z\rangle = 0 \text{ and } \langle \dot z, \nabla T(z)\rangle \geq 0 \text{ almost everywhere.}\}$$

Remark 1.1. Observe that the definition of $\mathcal{L}^+_{p,\gamma}$ does not depend on the particular choice of a time function T, but only on the orientation of its gradient ∇T.

To determine the lightlike geodesics in $\mathcal{L}^+_{p,\gamma}$, we would like to look for the critical points of the functional:

$$Q(z) = \int_0^1 \langle \dot z, \nabla T(z)\rangle^2\, ds, \quad \text{on } \mathcal{L}^+_{p,\gamma}.$$

Indeed, it is known (see [AP]) that if z is a C^2 curve in $\mathcal{L}^+_{p,\gamma}$, with $\dot z(s) \neq 0$ for every s, which is a critical point for Q, then z is a lightlike pregeodesic in $\mathcal{M}$, parametrized in such a way that $\langle \dot z, \nabla T(z)\rangle$ is constant. The main properties of Q will be discussed in full details in the next sections.

Unfortunately, calculations in local coordinates show that $\mathcal{L}^+_{p,\gamma}$ fails to be a C^1 manifold precisely at those points z for which $\dot z$ is null in a subset of the interval $[0,1]$ having positive Lebesgue masure. (see [GM]). For this reason, in order to use the standard techniques of Critical Point Theory, we introduce in section 2 a family of approximating manifolds denoted by $\mathcal{L}^+_{p,\gamma,\epsilon}$ (see (2.0.2). They consist of future pointing timelike curves $z \in \Omega^{1,2}_{p,\gamma}$ such that $\langle \dot z, \dot z\rangle = -\epsilon^2$ almost everywhere. Some connections between the spaces $\mathcal{L}^+_{p,\gamma,\epsilon}$ and $\mathcal{L}^+_{p,\gamma}$ will be shown in section 6.

Due to the presence of the boundary of Λ, we need to study a functional Q_δ, penalizing Q, defined as follows:

$$Q_\delta(z) = Q(z) + \delta \int_0^1 \frac{ds}{\varphi(z(s))^2},$$

where φ is defined in section 3, and nearby $\partial\Lambda$ it measures the distance from $\partial\Lambda$.

Even though we are only able to prove that the manifold $\mathcal{L}^+_{p,\gamma,\epsilon}$ is of class C^1, in section 3 we will prove that Q_δ is of class C^2. This fact allows to use the classical (infinite dimensional) techniques of the Morse theory for the functional Q_δ on the approximating manifolds (see [GMP2]).

Unfortunately, we are not able to prove in general the C^2-regularity for the critical points of Q_δ (see section 4). For this reason, we are forced to change the time function T as shown in section 5. The construction of a new time function can be carried over thanks to the following compactness condition.

For $c \in \mathbb{R}$, we denote by $\tau^c_{p,\gamma}$ the c-sublevel of $\tau_{p,\gamma}$ in $\Omega^{1,2}_{p,\gamma}$:

$$\tau^c_{p,\gamma} = \left\{ z \in \Omega^{1,2}_{p,\gamma} \mid \tau_{p,\gamma}(z) \le c \right\}.$$

Definition 1.2. Let c be a real number. $\mathcal{L}^+_{p,\gamma}$ is said to be c-precompact if there exists a compact subset $K = K(c)$ of $\overline{\Lambda}$ such that $\operatorname{supp}(z) \subset K$ for every $z \in \mathcal{L}^+_{p,\gamma} \cap \tau^c_{p,\gamma}$.

The above condition has a crucial role in the proof of the multiplicity results presented in this paper, and also to develop an infinite dimensional Morse theory for light rays between p and γ (see [GMP2]). Notice that the multiplicity results and the Morse Theory are obtained intrinsically, without the use of local coordinates.

Remark 1.3. It should be emphasized that, in Definition 1.2, if we give the c-precompactness in Λ rather than $\overline{\Lambda}$, we would basically be in the globally hyperbolic case. Indeed, it can be shown that if the compact subset $K(c)$ is contained in Λ for every c, then there exists a globally hyperbolic manifold $\tilde{\Lambda} \supset \Lambda$, such that $z(s) \in \tilde{\Lambda}$ for every $z \in \mathcal{L}^+_{p,\gamma}$ and every s (see Lemma 5.9).

In our case, in order to cover a more general class of cases, we have to use a compactness condition weaker than the global hyperbolicity, due to the presence of the boundary $\partial\Lambda$. In a certain sense, it is the only responsible for the lack of completeness.

The multiplicity of light rays is given in terms of the *Ljusternik–Schnirelman* category $\operatorname{cat}(\mathcal{L}^+_{p,\gamma})$ of $\mathcal{L}^+_{p,\gamma}$, which is the minimal integer number k, possibly infinite, such that there exists k closed, contractible subsets of $\mathcal{L}^+_{p,\gamma}$ covering $\mathcal{L}^+_{p,\gamma}$.

Theorem 1.4. *Suppose that $\mathcal{L}^+_{p,\gamma}$ is non empty and c-precompact for any $c \in \mathbb{R}^+$. Then, there are at least* $\operatorname{cat}(\mathcal{L}^+_{p,\gamma})$ *future pointing, lightlike geodesics joining p and γ with support in Λ.*

Theorem 1.5. *Under the same hypothesis of Theorem 1.4, if* $\operatorname{cat}(\mathcal{L}^+_{p,\gamma}) = +\infty$, *then there exists a sequence z_n of future pointing, lightlike geodesics joining p and γ in Λ, such that:*

$$\lim_{n\to\infty} \tau_{p,\gamma}(z_n) = +\infty.$$

Examples where cat $(\mathcal{L}^+_{p,\gamma}) = +\infty$ can be found e.g. in [GM].

We obtain the multiplicity results using a *curve-shortening* method for the functional Q on the space $\mathcal{L}^+_{p,\gamma}$. This approach seems more convenient than the use of curves of maximal slope for the functional Q_δ on $\mathcal{L}^+_{p,\gamma}$, especially because of the presence of the boundary $\partial\Lambda$. However, the use of the penalized functional Q_δ, together with the study of the Palais–Smale sequences and the a priori estimates, will be used to get the existence of minimizers. Moreover, such techniques will be also used in [GMP2] to develop a Morse theory for light rays under nondegeneracy assumptions, working directly in an infinite dimensional space.

The mathematical formalism developed here has applications to astrophysics in the following sense. If we consider the case of a four dimensional Lorentzian manifold $\mathcal{M}$, then $\mathcal{M}$ can be interpreted as a space–time in the sense of general relativity, γ can be interpreted as the worldline of a light source, and p can be interpreted as an event where the observation takes place. The solutions of our variational problem are lightlike geodesics from p to γ which are future oriented with respect to our time function T. If we interpret this orientation as *past pointing*, any such lightlike geodesic can be interpreted as a light ray from γ to p. If our variational problem has more than one solution, an observer at p will see more than one image of the light source γ on his/her celestial sphere. In this situation, astrophysicists speak of the *gravitational lens effect*. Whenever $\partial\Lambda$ is not empty, the set $\mathcal{M}\setminus\overline{\Lambda}$, with Λ defined above, represents a non transparent deflector (modelled by a hole in $\mathcal{M}$). For a comprehensive exposition of theoretical and observational material on this phenomenon we refer to [SEF]. Our formalism can be used to investigate if in some space–time, or in some *classes* of space–times the gravitational lens effect takes place; moreover, some information on the number of images is provided. Notice that, the Ljusternik–Schnirelman theory does *not* need nondegeneracy assumptions, and this fact allows to cover cases where continuous images, like arcs or rings, are observed (see [SEF]).

Theorems 1.4 and 1.5 improve the results of [GM], where the globally hyperbolic case is considered. The multiplicity results and the Morse relations were announced in [GMP1].

Acknowledgment: The authors wish to thank Volker Perlick for very useful discussions on the topics.

2. Approximation with Regular Manifolds

In this section we assume that a time function T on $\mathcal{M}$ is chosen in such a way that γ is a vertical timelike curve, i.e.

$$\dot\gamma(s) = \lambda(s)\nabla T(\gamma(s)) \tag{2.0.1}$$

for some function $\lambda \in C^0(\mathbb{R},\mathbb{R})$. This choice is always possible in a stably causal Lorentzian manifold, as it will be proven in Section 5.

We introduce a family of approximating manifolds as follows. For $\varepsilon > 0$, we define $\mathcal{L}^+_{p,\gamma,\varepsilon}$ by:

$$\mathcal{L}^+_{p,\gamma,\varepsilon} = \{z \in \Omega^{1,2}_{p\gamma} \mid \langle \dot z, \dot z\rangle = -\varepsilon^2 \text{ almost everywhere}, \tag{2.0.2}$$
$$s \longmapsto T(z(s)) \text{ strictly increasing}\}.$$

In order to study the regularity properties of $\mathcal{L}^+_{p,\gamma,\varepsilon}$, we introduce the following map between functional spaces:

$$\Psi_\varepsilon : \Omega^{1,2}_{p,\gamma} \longmapsto L^2([0,1], I\!R)$$

$$\Psi_\varepsilon(z) = \sqrt{2}\langle \nabla T(z), \dot z\rangle - \sqrt{\varepsilon^2 + \langle \dot z, \dot z\rangle + 2\langle \nabla T(z), \dot z\rangle^2}$$

Lemma 2.1. $\mathcal{L}^+_{p,\gamma,\varepsilon} = \Psi_\varepsilon^{-1}(0)$.

Proof. If $z \in \mathcal{L}^+_{p,\gamma,\varepsilon}$, since $T(z(s))$ is strictly increasing, then $\langle \nabla T(z), \dot z\rangle \geq 0$ almost everywhere. Moreover $\langle \dot z, \dot z\rangle = -\varepsilon^2$ almost everywhere, which implies $\Psi_\varepsilon(z) = 0$ almost everywhere, i.e. $\Psi_\varepsilon(z) = 0$ in $L^2([0,1], I\!R)$. Conversely, if $z \in \Omega^{1,2}_{p,\gamma}$ is such that $\Psi_\varepsilon(z) = 0$ almost everywhere, then

$$\sqrt{2}\langle \nabla T(z), \dot z\rangle = \sqrt{\varepsilon^2 + \langle \dot z, \dot z\rangle + 2\langle \nabla T(z), \dot z\rangle^2} \geq \varepsilon > 0,$$

so that $\langle \dot z, \dot z\rangle = -\varepsilon^2$ a.e. Furthermore, by the wrong way Schwartz's inequality, it is:

$$\langle \nabla T(z), \dot z\rangle^2 \geq \langle \nabla T(z), \nabla T(z)\rangle\langle \dot z, \dot z\rangle = \varepsilon^2 > 0 \text{ a.e.},$$

so that $\langle \nabla T(z), \dot z\rangle > 0$ a.e. and $T(z(s))$ is strictly increasing. $\square$

Proposition 2.2. *Let $z \in \Omega^{1,2}_{p,\gamma}$ and $\zeta \in T_z\Omega^{1,2}_{p,\gamma}$. The map Ψ_ε is Gateaux differentiable at z in the direction ζ, and the Gateaux derivative $\Psi'_\varepsilon(z)[\zeta]$ is given by:*

$$\Psi'_\varepsilon(z)[\zeta] = \sqrt{2}\,\big(\langle H^T(z)\zeta, \dot z\rangle + \langle \nabla T(z), \nabla_s\zeta\rangle\big) + \\ - \frac{\langle \dot z, \nabla_s\zeta\rangle + 2\langle \nabla T(z), \dot z\rangle\big(\langle H^T(z)\zeta, \dot z\rangle + \langle \nabla T(z), \nabla_s\zeta\rangle\big)}{\sqrt{\varepsilon^2 + \langle \dot z, \dot z\rangle + 2\langle \nabla T(z), \dot z\rangle^2}},$$

where H^T denotes the Hessian of T with respect to the Lorentzian metric.

Remark 2.3. Observe that the functions $H^T(z)\zeta$ and $\nabla T(z)$ are L^∞. Moreover, it is:

$$\frac{\dot z}{\sqrt{\varepsilon^2 + \langle \dot z, \dot z\rangle + 2\langle \nabla T(z), \dot z\rangle^2}} = \frac{\dot z}{\sqrt{\varepsilon^2 + \langle \dot z, \dot z\rangle_{(R)}}},$$

hence

$$\frac{\|\dot z\|_R}{\sqrt{\varepsilon^2 + \langle \dot z, \dot z\rangle + 2\langle \nabla T(z), \dot z\rangle^2}} = \sqrt{\frac{\langle \dot z, \dot z\rangle_{(R)}}{\varepsilon^2 + \langle \dot z, \dot z\rangle_{(R)}}} \leq 1.$$

This implies that $\Psi'_\varepsilon(z)[\zeta] \in L^2([0,1], I\!R)$.

The proof of Proposition 2.2 is split into several Lemmas.

Lemma 2.4. *Let* $\psi_1 : \Omega^{1,2}_{p,\gamma} \longmapsto L^2([0,1], \mathbb{R})$ *be given by:*

$$\psi_1(z) = \langle \nabla T(z), \dot z \rangle.$$

Then, for every $z \in \Omega^{1,2}_{p,\gamma}$ *and every* $\zeta \in T_z\Omega^{1,2}_{p,\gamma}$, ψ_1 *is Gateaux differentiable at* z *in the direction* ζ, *and its Gateaux derivative is given by:*

$$\psi_1'(z)[\zeta] = \langle H^T(z)\zeta, \dot z\rangle + \langle \nabla T(z), \nabla_s \zeta\rangle. \tag{2.4.1}$$

Proof. We will prove that ψ_1 is differentiable in all the manifold $H^{1,2}([0,1], \mathcal{M})$, with differential given by (2.4.1), which will imply the Lemma.

We denote by $\exp_q(\cdot)$ the exponential map with respect to the Riemannian structure $\langle \cdot, \cdot \rangle_{(R)}$ of $\mathcal{M}$ defined in (1.0.1). For $\lambda \in (-\delta, \delta)$, $\delta > 0$ sufficiently small, and $s \in [0,1]$, we define the two-parameter map $u(\lambda, s)$ by:

$$u(\lambda, s) = \exp_{z(s)}(\lambda \cdot \zeta(s)),$$

which satisfies:

(1) $u(0, s) = \zeta(s)$,

(2) $\dfrac{\partial}{\partial \lambda}(0, s) = \zeta(s)$,

for every s and λ. By definition, we have:

$$(\psi_1'(z)[\zeta])(s) = \lim_{\lambda \to 0} \frac{\psi_1(u(\lambda, s)) - \psi_1(u(0, s))}{\lambda},$$

where the limit is taken in the sense of the L^2-norm.

Let $D \subset [0,1]$ be defined by:

$$D = \{ s \in [0,1] \mid z \text{ and } \zeta \text{ are differentiable at } s\};$$

this is a set of full measure in $[0,1]$. We denote by $\eta(z, v, \sigma)$ the geodesic flow around z with respect to the Riemannian structure of $\mathcal{M}$. We have:

$$u(\lambda, s) = \eta(z(s), \lambda\zeta(s), 1) = \eta(z(s), \zeta(s), \lambda). \tag{2.4.2}$$

Then, for every $s \in D$, it is:

$$\frac{\partial}{\partial s}\eta(\lambda, s) = \eta_1(z(s), \zeta(s), \lambda)\dot z(s) + \eta_2(z(s), \zeta(s), \lambda)\nabla_s\zeta(s),$$

where η_i, $i = 1, 2$, denotes the differential of η with respect to the i-th variable. Observe that, originally, η_2 is a map from $T_{\zeta(s)}T\mathcal{M}$ to $T\mathcal{M}$. This map is reduced, with the help of the Riemannian metric, to a mapfrom $T_{z(s)}\mathcal{M}$ to $T\mathcal{M}$.

It follows:

$$\psi_1(u(\lambda, s)) = \langle \nabla T(u(\lambda, s)), \eta_1(z(s), \zeta(s), \lambda)\dot z(s) + \eta_2(z(s), \zeta(s), \lambda)\nabla_s\zeta(s)\rangle.$$

If we fix $s \in D$ and we apply the mean value theorem to the map $\lambda \longmapsto \psi_1(u(\lambda,s))$, we obtain:

$$\begin{aligned}\frac{1}{\lambda}(\psi_1(u(\lambda,s)) - \psi_1(u(0,s))) =& \langle H^T(u(\theta,s)\frac{\partial u}{\partial\lambda}(\theta,s), \eta_1(z(s),\zeta(s),\theta)\dot z(s)\rangle + \\ &+ \langle H^T(u(\theta,s)\frac{\partial u}{\partial\lambda}(\theta,s), \eta_2(z(s),\zeta(s),\theta)\nabla_s\zeta(s)\rangle + \\ &+ \langle \nabla T(u(\theta,s)), \eta_{1\lambda}(z(s),\zeta(s),\theta)\dot z(s)\rangle + \\ &+ \langle \nabla T(u(\theta,s)), \eta_{2\lambda}(z(s),\zeta(s),\theta)\nabla_s\zeta(s)\rangle,\end{aligned} \tag{2.4.3}$$

where $\theta \in [0,\lambda]$. Now, by (2.4.2),

$$\frac{\partial u}{\partial\lambda}(\theta,s) = \eta_2(z(s),\theta\zeta(s),1)\zeta(s).$$

Moreover, the following identities hold:

(i) $\eta_2(z,0,1)\zeta = \zeta$,
(ii) $\eta_2(z,\zeta,0) = 0$,
(iii) $\eta_1(z,\zeta,0)w = w$,
(iv) $\eta_{2\lambda}(z,\zeta,0)w = w$,
(v) $\eta_{1\lambda}(z,\zeta,0) = 0$,

for every $z \in \mathcal{M}$ and every ζ, w in $T_z\mathcal{M}$. We can therefore take the L^2-limit in (2.4.3), and from the Lebesgue Theorem the proof is concluded. □

Lemma 2.5. *Let $\psi_\varepsilon : \Omega^{1,2}_{p,\gamma} \longmapsto L^2([0,1], I\!R)$ be the map given by:*

$$\psi_\varepsilon(z) = \sqrt{\varepsilon^2 + \langle \dot z, \dot z\rangle_{(R)}}.$$

Then, for every $z \in \Omega^{1,2}_{p,\gamma}$ and every $\zeta \in T_z\Omega^{1,2}_{p,\gamma}$, ψ_ε is Gateaux differentiable at z in the direction ζ, and its Gateaux derivative is given by:

$$\psi'_\varepsilon(z)[\zeta] = \frac{\langle \dot z, \nabla^{(R)}_s\zeta\rangle_{(R)}}{\sqrt{\varepsilon^2 + \langle \dot z, \dot z\rangle_{(R)}}}. \tag{2.5.1}$$

Remark 2.6. Observe that, since $\dfrac{\|\dot z\|_{(R)}}{\sqrt{\varepsilon^2 + \langle \dot z, \dot z\rangle_{(R)}}} < 1$, then (2.5.1) gives a well defined L^2 function.

Proof. For this proof, we will use an argument formally identical to the one used in the proof of Lemma 2.4 (showing the Gateaux-differentiability in all the manifold $H^{1,2}([0,1],\mathcal{M})$), with the only difference that we will use the Riemannian metric structure on $\mathcal{M}$. So, we will still denote by $\eta(z,\zeta,\lambda)$ the geodesic flow with respect to $\langle\cdot,\cdot\rangle_{(R)}$, and η_i will denote the partial derivative of η with respect to the i-th variable. As observed in the previous Lemma, also in this case η_2 is reduced to a map from $T_{z(s)}\mathcal{M}$ to $T\mathcal{M}$ using the Riemannian metric.

Denoting by $\exp_q^{(R)}$ the exponential map with respect to the Riemannian metric, and setting:

$$u(\lambda, s) = \exp^{(R)}_{z(s)}(\lambda \cdot \zeta(s)),$$

For every $s \in D$ we have:

$$\psi_\varepsilon(u(\lambda,s)) =$$
$$= \sqrt{\varepsilon^2 + \langle \eta_1(z,\zeta,\lambda)\,\dot z + \eta_2(z,\zeta,\lambda)\,\nabla_s^{(R)}\zeta\,,\eta_1(z,\zeta,\lambda)\,\dot z + \eta_2(z,\zeta,\lambda)\,\nabla_s^{(R)}\zeta\rangle_{(R)}},$$

and

$$\frac{\partial\psi_\varepsilon}{\partial\lambda}(u(\lambda,s)) =$$
$$= \frac{\langle \eta_1(z,\zeta,\lambda)\,\dot z + \eta_2(z,\zeta,\lambda)\,\nabla_s^{(R)}\zeta\,,\eta_{1\lambda}(z,\zeta,\lambda)\,\dot z + \eta_{2\lambda}(z,\zeta,\lambda)\,\nabla_s^{(R)}\zeta\rangle_{(R)}}{\sqrt{\varepsilon^2 + \langle \eta_1(z,\zeta,\lambda)\,\dot z + \eta_2(z,\zeta,\lambda)\,\nabla_s^{(R)}\zeta\,,\eta_1(z,\zeta,\lambda)\,\dot z + \eta_2(z,\zeta,\lambda)\,\nabla_s^{(R)}\zeta\rangle_{(R)}}}.$$

Since

$$\frac{\|\eta_1\,\dot z + \eta_2\,\nabla_s^{(R)}\zeta\,\|_{(R)}}{\sqrt{\varepsilon^2 + \langle \eta_1(z,\zeta,\lambda)\,\dot z + \eta_2(z,\zeta,\lambda)\,\nabla_s^{(R)}\zeta\,,\eta_1(z,\zeta,\lambda)\,\dot z + \eta_2(z,\zeta,\lambda)\,\nabla_s^{(R)}\zeta\rangle_{(R)}}} \le 1,$$

taking the limit for $\lambda \longmapsto 0$ and using the Lebesgue Dominated Convergence Theorem gives immediately the proof of the Lemma. □

We are now ready for the:

Proof of Proposition 2.2. Thanks to Lemma 2.4 and 2.5, all we need to check is that:

$$\langle \dot z\,,\nabla_s^{(R)}\zeta\rangle_{(R)} = \langle \dot z\,,\nabla_s\zeta\rangle + 2\langle\nabla T(z),\dot z\rangle\left(\langle H^T(z)\zeta\,,\dot z\rangle + \langle\nabla T(z)\,,\nabla_s\zeta\rangle\right).$$

This is a direct consequence of the definition of $\langle\cdot\,,\cdot\rangle_{(R)}$ in (1.0.1). □

Proposition 2.6. *Ψ_ε is a map of class C^1.*

Proof. From Lemma 2.4, it follows easily that ψ_1 is of class C^1. From Lemma 2.5, to prove the Proposition it will suffice to show that if z_n tends to z in $H^{1,2}$, then

$$\text{(2.6.1)}\qquad \frac{\dot z_n}{\sqrt{\varepsilon^2 + \langle \dot z_n\,,\dot z_n\rangle_{(R)}}} \longmapsto \frac{\dot z}{\sqrt{\varepsilon^2 + \langle \dot z\,,\dot z\rangle_{(R)}}} \quad \text{in } L^2, \text{ as } n \longmapsto \infty.$$

Suppose by contradiction that (2.6.1) does not hold. Up to passing to a subsequence, we can assume that there exists a positive constant $\delta_0 > 0$ such that:

$$\text{(2.6.2)}\qquad \int_0^1 \tilde{\mathrm{d}}^2_{\mathrm{R}}\left(\frac{\dot z_n(s)}{\sqrt{\varepsilon^2 + \langle \dot z_n(s),\dot z_n(s)\rangle_{(R)}}}, \frac{\dot z(s)}{\sqrt{\varepsilon^2 + \langle \dot z(s),\dot z(s)\rangle_{(R)}}}\right) ds \ge \delta_0,$$

for every $n \in \mathbb{N}$, where $\tilde{d}_R$ is the Riemannian distance induced by $\langle \cdot, \cdot \rangle_{(R)}$ on the tangent bundle $T\mathcal{M}$. Since $\dot{z}_n$ is L^2-convergent to $\dot{z}$, we can assume that $\dot{z}_n(s)$ converges to $\dot{z}(s)$ almost everywhere on $[0,1]$. Then also

$$\frac{\dot{z}_n(s)}{\sqrt{\varepsilon^2 + \langle \dot{z}_n(s), \dot{z}_n(s)\rangle_{(R)}}} \longmapsto \frac{\dot{z}(s)}{\sqrt{\varepsilon^2 + \langle \dot{z}(s), \dot{z}(s)\rangle_{(R)}}} \quad \text{a.e.,}$$

and since $\frac{\dot{z}_n}{\sqrt{\varepsilon^2 + \langle \dot{z}, \dot{z}\rangle_{(R)}}}$ is in L^∞, by the Lebesgue Dominated Convergence Theorem, we get:

$$\lim_{n\to\infty} \int_0^1 \tilde{d}_R^2 \left(\frac{\dot{z}_n(s)}{\sqrt{\varepsilon^2 + \langle \dot{z}_n(s), \dot{z}_n(s)\rangle_{(R)}}}, \frac{\dot{z}(s)}{\sqrt{\varepsilon^2 + \langle \dot{z}(s), \dot{z}(s)\rangle_{(R)}}} \right) ds = 0,$$

which contradicts (2.6.2) and proves the Proposition. □

We come now to the main result of the Section.

Theorem 2.7. *Let $\varepsilon > 0$ and suppose that $\mathcal{L}^+_{p,\gamma,\varepsilon}$ is non empty. Then, $\mathcal{L}^+_{p,\gamma,\varepsilon}$ is a C^1 submanifold of $\Omega^{1,2}_{p,\gamma}$.*

Proof. From Proposition 2.6 and the Implicit Function Theorem, all we need to prove is that, for every $z \in \mathcal{L}^+_{p,\gamma,\varepsilon}$ the differential:

$$\Psi'_\varepsilon(z) : T_z\Omega^{1,2}_{p,\gamma} \longmapsto T_{\Psi_\varepsilon(z)} L^2([0,1], \mathbb{R}) \simeq L^2([0,1], \mathbb{R})$$

is a surjective map. In order to do this, we fix $z \in \mathcal{L}^+_{p,\gamma,\varepsilon}$, $\phi \in L^2([0,1], \mathbb{R})$ and we consider the problem:

$$\langle \dot{z}, \nabla_s \zeta \rangle = \sqrt{2}\, \langle \dot{z}, \nabla T(z) \rangle\, \phi, \tag{2.7.1}$$

where $\zeta \in T_z\Omega^{1,2}_{p,\gamma}$ is the unknown. We look for a solution of (2.7.1) of the form:

$$\zeta(s) = \mu(s) \cdot \nabla T(z(s)).$$

The condition for such a ζ to belong to $T_z\Omega^{1,2}_{p,\gamma}$, since γ is vertical, is simply that:

$$\mu(0) = 0.$$

The equation (2.7.1) becomes:

$$\langle \dot{z}, \nabla_s \zeta \rangle = \mu' \langle \dot{z}, \nabla T(z) \rangle + \mu \langle \dot{z}, H^T(z)\dot{z} \rangle = \sqrt{2}\, \langle \dot{z}, \nabla T(z) \rangle\, \phi.$$

Since $\langle \dot{z}, \nabla T(z) \rangle \geq \varepsilon > 0$, to solve (2.7.1) we need to prove that the Cauchy linear problem:

$$\begin{cases} \mu' = -\dfrac{\langle \dot{z}, H^T(z)\dot{z} \rangle}{\langle \dot{z}, \nabla T(z) \rangle} \mu + \sqrt{2}\phi \\ \mu(0) = 0. \end{cases} \tag{2.7.2}$$

admits a solution $\mu \in H^{1,2}([0,1],\mathbb{R})$ whenever $\phi \in L^2([0,1],\mathbb{R})$. To prove this, we set

$$a = -\frac{\langle \dot z , H^T(z)\dot z\rangle}{\langle \dot z , \nabla T(z)\rangle}.$$

Since $z \in \mathcal{L}^+_{p,\gamma,\varepsilon}$, it is $\Psi_\varepsilon(z) = 0$ a.e., and

$$\langle \nabla T(z) , \dot z\rangle = \frac{1}{\sqrt{2}}\sqrt{\varepsilon^2 + \langle \dot z , \dot z\rangle_{(R)}},$$

so that:

$$a = -\sqrt{2}\frac{\langle H^T(z)\dot z , \dot z\rangle}{\sqrt{\varepsilon^2 + \langle \dot z , \dot z\rangle_{(R)}}}.$$

Thus, there exists a constant M, independent of s, such that:

$$|a(s)| \le M\|\dot z\|_R \quad \text{a.e.},$$

which implies that $a \in L^2([0,1],\mathbb{R})$. The solution of (2.7.2) can be written explicitly as:

$$\mu(s) = \sqrt{2}\, e^{\int_0^s a(\tau)\,d\tau}\left(\int_0^s \phi(r)e^{-\int_0^r a(\tau)\,d\tau}\,dr\right),$$

which is in $H^{1,2}([0,1],\mathbb{R})$, and this concludes the proof. □

Corollary 2.8. *For $z \in \mathcal{L}^+_{p,\gamma,\varepsilon}$, the tangent space $T_z\mathcal{L}^+_{p,\gamma,\varepsilon}$ is identified with the set:*

$$T_z\mathcal{L}^+_{p,\gamma,\varepsilon} = \{\zeta \in T_z\Omega^{1,2}_{p,\gamma} \mid \langle \dot z , \nabla_s\zeta\rangle = 0 \ \text{a.e.}\}.$$

Proof. By the Implicit Function Theorem, $T_z\mathcal{L}^+_{p,\gamma,\varepsilon} = \operatorname{Ker}\Psi'_\varepsilon(z)$. From Proposition 2.2, setting $\langle \dot z , \dot z\rangle = -\varepsilon^2$ and recalling that $\langle \nabla T(z) , \dot z\rangle > 0$, we have:

$$\Psi'_\varepsilon(z)[\zeta] = -\frac{\langle \dot z , \nabla_s\zeta\rangle}{\sqrt{2}\,\langle \nabla T(z) , \dot z\rangle},$$

for every $z \in \mathcal{L}^+_{p,\gamma,\varepsilon}$ and every $\zeta \in T_z\mathcal{L}^+_{p,\gamma,\varepsilon}$, which proves the thesis. □

Remark 2.9. The C^1 submanifold property of $\mathcal{L}^+_{p,\gamma,\varepsilon}$ can be established as well in the more general case of an arbitrary time orientable Lorentzian manifold. In that case, we have to replace ∇T with an arbitrary timelike vector field. For the following construction, however, we need the stable causality of $\mathcal{M}$.

3. The Penalized Functional and its Regularity

Also in this section, in order to use the results of section 2, we make the assumption that γ is vertical with respect to the given time function T.

Let Λ be the open subset introduced in Section 1 and d_R the distance function on $\mathcal{M}$ induced by the Riemannian structure (1.0.1).

Let T be a fixed time function on $\mathcal{M}$; we choose a C^2 function $\varphi : \mathcal{M} \longmapsto \mathbb{R}$ that satisfies the following properties:

$$\text{(3.0.1)} \qquad \begin{aligned} &\text{(a)} \quad \varphi > 0 \text{ on } \Lambda,\\ &\text{(b)} \quad \varphi < 0 \text{ on } \mathcal{M} \setminus \overline{\Lambda},\\ &\text{(c)} \quad \langle \nabla\varphi, \nabla\varphi \rangle > 0 \text{ on } \partial\Lambda. \end{aligned}$$

It is not difficult to prove that the existence of a function φ satisfying (a), (b) and (c) of (3.0.1) is implied by the fact that that Λ is open and $\partial\Lambda$ is smooth and timelike.

For instance, φ can be defined in a neighborhood U of $\partial\Lambda$ by:

$$\varphi(z) = \begin{cases} \mathrm{d}_R(z, \partial\Lambda) & \text{if } z \in \Lambda \cap U,\\ -\mathrm{d}_R(z, \partial\Lambda) & \text{if } z \in U \setminus \Lambda, \end{cases}$$

and then extended to $\mathcal{M}$ is a suitable way.

We introduce the following functionals on $\Omega^{1,2}_{p,\gamma}$:

$$Q, Q_\delta : \Omega^{1,2}_{p,\gamma} \longmapsto \mathbb{R}^+$$

$$\text{(3.0.2)} \qquad Q(z) = \int_0^1 \langle \dot z(s), \nabla T(z(s)) \rangle^2 \, ds,$$

$$\text{(3.0.3)} \qquad Q_\delta(z) = Q(\dot z) + \delta \int_0^1 \frac{ds}{\varphi(z(s))^2}.$$

These functionals are clearly differentiable on $\Omega^{1,2}_{p,\gamma}$; it is not difficult to see that, for every $z \in \Omega^{1,2}_{p,\gamma}$, the Gateaux differentials $Q'(z)$ and $Q'_\delta(z)$ are given by:

$$\text{(3.0.4)} \qquad Q'(z)[\zeta] = 2 \int_0^1 \langle \dot z, \nabla T(z) \rangle \left(\langle H^T(z)\, \dot z, \zeta \rangle + \langle \nabla T(z), \nabla_s \zeta \rangle \right) ds,$$

$$\text{(3.0.5)} \qquad Q'_\delta(z)[\zeta] = Q'(z)[\zeta] - 2\delta \int_0^1 \frac{\langle \nabla\varphi(z), \zeta \rangle}{\varphi(z)^3} ds,$$

for every $\zeta \in T_z \mathcal{L}^+_{p,\gamma,\epsilon}$.

The purpose of this section is to show that, even though the manifold $\mathcal{L}^+_{p,\gamma,\epsilon}$ is only C^1, the restriction of Q and Q_δ to $\mathcal{L}^+_{p,\gamma,\epsilon}$ has a regularity of higher order, in the sense explained as follows.

We consider the C^1 vector bundle W_ϵ over the manifold $\mathcal{L}^+_{p,\gamma,\epsilon}$, whose fiber $W_\epsilon(z)$ is given by the whole tangent space $T_z\Omega^{1,2}_{p,\gamma}$, $z \in \mathcal{L}^+_{p,\gamma,\epsilon}$. The elements of W_ϵ are pairs (z, ζ), where $z \in \mathcal{L}^+_{p,\gamma,\epsilon}$ and $\zeta \in T_z\Omega^{1,2}_{p,\gamma}$.

We will think of W_ϵ as a regular extension of the tangent bundle $T\mathcal{L}^+_{p,\gamma,\epsilon}$; they are related by the bundle map:

$$V : W_\epsilon \longmapsto T\mathcal{L}^+_{p,\gamma,\epsilon}$$

$$(z,\zeta) \longmapsto (z, V_\zeta),$$

where V_ζ is defined by:

$$V_\zeta(s) = \zeta(s) - \mu_{z,\zeta}(s)\nabla T(z(s)),$$

and $\mu_{z,\zeta}$ is the solution of the Cauchy problem:

$$\begin{cases} \mu' = -\dfrac{\langle H^T(z)\,\dot z\,,\dot z\rangle}{\langle \nabla T(z)\,,\dot z\rangle}\mu + \dfrac{\langle \nabla_s \zeta\,,\dot z\rangle}{\langle \nabla T(z)\,,\dot z\rangle}, \\ \mu(0) = 0. \end{cases} \tag{3.0.6}$$

We set:

$$a = -\frac{\langle H^T(z)\,\dot z\,,\dot z\rangle}{\langle \nabla T(z)\,,\dot z\rangle}. \tag{3.0.7}$$

Arguing as in the proof of Theorem 2.7, it is easy to see that:

$$|a| \le M\|\dot z\|_R,$$

so that $a \in L^2([0,1],\mathbb{R})$. Observe also that

$$\frac{\dot z}{\langle \nabla T(z)\,,\dot z\rangle} \in L^\infty([0,1],T\mathcal{M}).$$

Hence, (3.0.6) can be solved explicitly by setting:

$$\mu(s) = e^{\int_0^s a(\tau)d\tau}\left(\int_0^s \frac{\langle \nabla_r \zeta\,,\dot z\rangle}{\langle \nabla T(z)\,,\dot z\rangle} e^{-\int_0^r a(\tau)d\tau}\,dr\right). \tag{3.0.8}$$

Proposition 3.1. *V is a continuous map. Its restriction to the tangent bundle $T\mathcal{L}^+_{p,\gamma,\epsilon}$ is the identity map; in particular, for every $z \in \mathcal{L}^+_{p,\gamma,\epsilon}$, the map $\zeta \longmapsto V_\zeta$ is surjective.*

Proof. The continuity of V is immediately given by the continuous dependence on the data for the solution of the Cauchy problem (3.0.6). The second part of the thesis follows from the fact that, if $\zeta \in T_z\mathcal{L}^+_{p,\gamma,\epsilon}$, then from Corollary 2.8 it is $\langle \nabla_s\zeta\,,\dot z\rangle \equiv 0$, so that the solution of (3.0.6) is given by $\mu \equiv 0$, and $V_\zeta = \zeta$. □

It is not difficult to show that V is indeed C^1 considered as a map from W_ϵ to itself, with image in $T\mathcal{L}^+_{p,\gamma,\epsilon}$.

We are ready to state and prove our main regularity result:

Theorem 3.2. *For every $\delta \geq 0$, Q_δ is of class C^2 on $\mathcal{L}^+_{p,\gamma,\varepsilon}$, in the sense that the map $(z,\zeta) \longmapsto Q'(z)[V_\zeta]$ ($(z,\zeta) \longmapsto Q'_\delta(z)[V_\zeta]$) is of class C^1 on W_ε.*

Proof. From (3.0.4) and the definition of V_ζ, we have:

(3.2.1)
$$\begin{aligned}\frac{1}{2}Q'_\delta(z)[V_\zeta] = \int_0^1 \langle\nabla T(z),\dot z\rangle\Big(&\langle H^T(z)\dot z,\zeta-\mu\nabla T(z)\rangle+\\ &+\langle\nabla T(z),\nabla_s\zeta-\mu'\nabla T(z)-\mu H^T(z)\dot z\rangle+\\ &-\delta\langle\frac{\nabla\varphi(z)}{\varphi(z)^3},\zeta-\mu\nabla T(z)\rangle\Big)\,\mathrm{d}s\end{aligned}$$

Since $\langle\nabla T(z),\nabla T(z)\rangle$ is constant, then

$$\frac{\mathrm{d}}{\mathrm{d}s}\langle\nabla T(z),\nabla T(z)\rangle = 2\langle\nabla T(z),H^T(z)\dot z\rangle = 0,$$

so that, recalling the definition of μ (cf. (3.0.6)), (3.2.1) becomes:
(3.2.2)
$$\begin{aligned}\frac{1}{2}Q'_\delta(z)[V_\zeta] = \int_0^1 \Big(&\langle\nabla T(z),\dot z\rangle\langle H^T(z)\dot z,\zeta\rangle+\langle\nabla T(z),\dot z\rangle\langle\nabla T(z),\nabla_s\zeta\rangle+\\ &+\langle\nabla_s\zeta,\dot z\rangle-\mu\langle H^T(z)\dot z,\dot z\rangle+\\ &-\delta\langle\frac{\nabla\varphi(z)}{\varphi(z)^3},\zeta-\mu\nabla T(z)\rangle\Big)\,\mathrm{d}s\end{aligned}$$

Since on $\mathcal{L}^+_{p,\gamma,\varepsilon}$ it is

(3.2.3)
$$\sqrt{2}\langle\nabla T(z),\dot z\rangle = \sqrt{\varepsilon^2+\langle\dot z,\dot z\rangle_{(R)}},$$

substituting (3.0.8) and (3.2.3) in (3.2.2) and arguing as in Lemma 2.4 and Lemma 2.5 we obtain the thesis. □

Corollary 3.3. *For every local chart $\mathcal{U}$ on $\mathcal{L}^+_{p,\gamma,\varepsilon}$, the restriction of Q_δ to $\mathcal{U}$ is of class C^2.* □

Observe that, setting $\delta = 0$, the previous results remain true for the functional Q.

4. Euler–Lagrange Equation and Regularity for the Critical Points of Q_δ

In this section we will use a *boot strap* argument to prove that, under suitable assumptions on the time function T, the critical points of the functional Q and of the penalized functional Q_δ are C^2 curves in $\mathcal{M}$. The argument is well suited to be presented in small steps, and for the reader's convenience we break the main result into some Lemmas.

As customary we make the assumption that T is chosen in such a way that γ is vertical with respect to T; moreover, we also make the following two assumptions:

$$(4.0.1)\qquad \begin{cases} \langle \nabla T(q), \nabla \varphi(q) \rangle = 0 & \text{for every } q \text{ in a neighborhood of } \partial\Lambda, \\ \langle \nabla T(q), \nabla \varphi(q) \rangle \geq 0 & \text{for every } q \in \Lambda, \end{cases}$$

and

$$(4.0.2)\qquad H^T(z(s))[\zeta(s), \zeta(s)] \leq 0,$$

for every $z \in \mathcal{L}^+_{p,\gamma,\epsilon}$ and every causal vector field ζ along z. Here H^T denotes the Hessian of the time function T. It is not difficult to see that, since ∇T is normalized, the condition (4.0.2) holds for any vector field along z if it holds for causal vectors.

Even though this condition of negativity for H^T does not seem natural, we will discuss in the next section a way of producing, from a given time function, another one which satisfies the required property in a relevant region of the space. Also the assumption (4.0.1) will be discussed in the next section.

If z is a critical point of Q_δ in $\mathcal{L}^+_{p,\gamma,\epsilon}$, recalling (3.2.2), for every $\zeta \in T_z\mathcal{L}^+_{p,\gamma,\epsilon}$ it is:

(4.0.3)

$$\begin{aligned} 0 = &\int_0^1 \langle \nabla T(z), \dot z \rangle \langle H^T(z)\dot z, \zeta \rangle \, ds + \int_0^1 \langle \nabla T(z), \dot z \rangle \langle \nabla T(z), \nabla_s \zeta \rangle \, ds + \\ &+ \int_0^1 \langle \nabla_s \zeta, \dot z \rangle \, ds - \int_0^1 \mu_{z,\zeta} \langle H^T(z)\, \dot z, \dot z \rangle \, ds + \\ &- \delta \int_0^1 \langle \frac{\nabla \varphi(z)}{\varphi(z)^3}, \zeta \rangle \, ds + \delta \int_0^1 \mu_{z,\zeta} \langle \frac{\nabla \varphi(z)}{\varphi(z)^3}, \nabla T(z) \rangle \, ds. \end{aligned}$$

To simplify the notations, in the following computations we will drop the subscripts and we will denote by μ the function $\mu_{z,\zeta}$.

We have a preliminary Lemma:

Lemma 4.1. *Let α, β and γ be three functions in $L^1([0,1], \mathbb{R})$. Then:*

$$\int_0^1 \gamma(s) \left(\int_0^s \alpha(r) e^{\int_r^s \beta(\sigma)\, d\sigma} dr \right) ds = \int_0^1 \alpha(s) e^{-\int_0^s \beta(\sigma)\, d\sigma} \left(\int_s^1 \gamma(r) e^{\int_0^r \beta(\sigma)\, d\sigma}\, dr \right) ds.$$

Proof. It is an easy application of Fubini's Theorem. □

We set

$$L = L_\zeta(s) = \frac{\langle \nabla_s \zeta, \dot z \rangle}{\langle \nabla T(z), \dot z \rangle}.$$

Observe that L is linear in ζ. From (3.0.8), we have:

$$\mu(s) = \int_0^s L_\zeta(r) \exp\left[\int_r^s a(\sigma)\, d\sigma \right] dr,$$

where $a = a_z$ is the function defined in (3.0.7).

By Lemma 4.1 applied to the functions $\alpha = L_\zeta$, $\beta = a$ and $\gamma = -\langle H^T(z)\dot z, \dot z\rangle$, we have:

$$\int_0^1 \mu\left(\delta\langle\frac{\nabla\varphi(z)}{\varphi(z)^3}, \nabla T(z)\rangle - \langle H^T(z)\dot z, \dot z\rangle\right) ds =$$

$$= \int_0^1 \frac{\langle\nabla_s\zeta, \dot z\rangle}{\langle\nabla T(z), \dot z\rangle} e^{-\int_0^s a(\sigma)\,d\sigma}\left(\int_s^1 \mathcal{F}(z,\delta)\, e^{-\int_0^r a(\sigma)\,d\sigma}\, dr\right) ds,$$

where

$$\mathcal{F}(z,\delta) = \delta\langle\frac{\nabla\varphi(z)}{\varphi(z)^3}, \nabla T(z)\rangle - \langle H^T(z)\dot z, \dot z\rangle.$$

From (3.2.2), (4.0.1) and (4.0.3), it follows

(4.1.1)

$$\begin{aligned} 0 = &\int_0^1 \langle\dot z, \nabla_s\zeta\rangle\, ds + \\ &+ \int_0^1 e^{-\int_0^s a(\sigma)\,d\sigma}\left(\int_s^1 \mathcal{F}(z,\delta)\, e^{-\int_0^r a(\sigma)\,d\sigma}\, dr\right)\frac{\langle\dot z, \nabla_s\zeta\rangle}{\langle\nabla T(z), \dot z\rangle}\, ds + \\ &+ \int_0^1 \langle\nabla T(z), \dot z\rangle\langle\nabla T(z), \nabla_s\zeta\rangle\, ds + \int_0^1 \langle\nabla T(z), \dot z\rangle\langle H^T(z)\dot z, \zeta\rangle\, ds + \\ &- \int_0^1 \delta\langle\frac{\nabla\varphi(z)}{\varphi(z)^3}, \zeta\rangle\, ds. \end{aligned}$$

We denote by $\lambda_{\epsilon,\delta}$ the function:

$$\lambda_{\epsilon,\delta}(s) = e^{-\int_0^s a(\sigma)\,d\sigma}\left(\int_s^1 \mathcal{F}(z,\delta)\, e^{-\int_0^r a(\sigma)\,d\sigma}\, dr\right). \tag{4.1.2}$$

Observe that $\lambda_{\epsilon,\delta} \in H^{1,1}([0,1],\mathbb{R})$, and also that, due to the assumptions (4.0.1) and (4.0.2), it is:

$$\lambda_{\epsilon,\delta}(s) \geq 0. \tag{4.1.3}$$

From (4.1.1), integrating by parts, we obtain:

$$\dot z + \frac{\lambda_{\epsilon,\delta}(s)\,\dot z}{\langle\nabla T(z), \dot z\rangle} + \langle\nabla T(z), \dot z\rangle\nabla T(z) = h \in H^{1,1}([0,1], T\mathcal{M}). \tag{4.1.4}$$

An explicit analytic form of the function h of (4.1.4) may be given in terms of the Christoffel symbols of the Riemannian metric $\langle\cdot,\cdot\rangle_{(R)}$.

We are ready for our first regularity result:

Lemma 4.2. *$\langle\nabla T(z),\dot z\rangle$ is in $H^{1,1}([0,1],\mathbb{R})$.*

Proof. Let h be the function defined in (4.1.4). Since $h\in H^{1,1}$, also $\langle h,h\rangle\in H^{1,1}$. Recalling that $\langle\dot z,\dot z\rangle=-\varepsilon^2$ and that $\langle\nabla T(z),\nabla T(z)\rangle=-1$, we compute $\langle h,h\rangle$ as follows:

$$\langle h,h\rangle=-\varepsilon^2\left(1+\frac{\lambda_{\varepsilon,\delta}}{\langle\nabla T(z),\dot z\rangle}\right)^2+\left(1+\frac{2\lambda_{\varepsilon,\delta}}{\langle\nabla T(z),\dot z\rangle}\right)\langle\nabla T(z),\dot z\rangle^2. \tag{4.2.1}$$

Denoting by $\theta=\langle\nabla T(z),\dot z\rangle$ and $\rho=\langle h,h\rangle$, we write (4.2.1) as:

$$\chi(\theta,\rho,\lambda_{\varepsilon,\delta})=-\varepsilon^2\left(1+\frac{\lambda_{\varepsilon,\delta}}{\theta}\right)^2+\theta^2+2\lambda_{\varepsilon,\delta}\theta-\rho=0. \tag{4.2.2}$$

The derivative $\dfrac{\partial\chi}{\partial\theta}$ is:

$$\frac{\partial\chi}{\partial\theta}=2\varepsilon^2\frac{\lambda_{\varepsilon,\delta}}{\theta^2}\left(1+\frac{\lambda_{\varepsilon,\delta}}{\theta}\right)+2\theta+2\lambda_{\varepsilon,\delta}.$$

From (4.1.3) and the fact that $\theta>0$ it follows that $\dfrac{\partial\chi}{\partial\theta}>0$. Then, by the Implicit Function Theorem, (4.2.2) can be solved for θ locally, and θ is written locally as a C^1 function of $\lambda_{\varepsilon,\delta}$ and ρ. The conclusion follows from the fact that $\lambda_{\varepsilon,\delta}$ and ρ are of class $H^{1,1}$. □

The second step gives a regularity result for z:

Lemma 4.3. *$\dot z\in H^{1,1}([0,1],T\mathcal{M})$.*

Proof. From (4.1.4), since $\langle\nabla T(z),\dot z\rangle$ is in $H^{1,1}$, we have:

$$\left(1+\frac{\lambda_{\varepsilon,\delta}}{\langle\nabla T(z),\dot z\rangle}\right)\dot z=h_1,$$

where h_1 is in $H^{1,1}$. Since $H^{1,1}$ is closed with respect to products, it will suffice to show that

$$\frac{1}{\langle\nabla T(z),\dot z\rangle+\lambda_{\varepsilon,\delta}}\in H^{1,1}([0,1],\mathbb{R}). \tag{4.3.1}$$

This is easily established, since

$$\frac{\mathrm{d}}{\mathrm{d}s}\Big(\langle\nabla T(z),\dot z\rangle+\lambda_{\varepsilon,\delta}\Big)\in L^1([0,1],\mathbb{R}),$$

and

$$\Big(\langle\nabla T(z),\dot z\rangle+\lambda_{\varepsilon,\delta}\Big)^2\geq\langle\nabla T(z),\dot z\rangle^2=\varepsilon^2>0.$$

Then, (4.3.1) holds and the proof is finished. □

Proposition 4.4. *If $z \in \mathcal{L}^+_{p,\gamma,\epsilon}$ is a critical point for the functional Q_δ, then z is a curve of class C^2.*

Proof. From Lemma 4.3, $\dot z$ is in $H^{1,1}$ and in particular $\dot z$ is continuous. Then, from (4.1.2), $\lambda_{\epsilon,\delta}$ is a function of class C^1. Hence, the same integration by parts of (4.1.1) that gave (4.1.4) shows that the function h of (4.1.4) is also of class C^1. Moreover, the same argument of Lemma 4.2 shows that $\langle \nabla T(z), \dot z\rangle$ is of class C^1. Arguing as in Lemma 4.3, we get that $\dot z$ is of class C^1, and the Proposition is proven. □

Observe that the same argument can be repeated, and, by induction, one proves that z is a curve with the same regularity of the time function T.

Remark 4.5. By Proposition 4.4, integrating by parts in (4.1.1) allows to get the Euler-Lagrange equation satisfied by the critical points of Q_δ on $\mathcal{L}^+_{p,\gamma,\epsilon}$, which is given by:

$$\begin{aligned} &\nabla_s \dot z + \nabla_s\left(\frac{\lambda \dot z}{\langle \nabla T(z), \dot z\rangle}\right) + \nabla_s\left(\langle \nabla T(z), \dot z\rangle \nabla T(z)\right) + \\ &\qquad + \delta \frac{\nabla\varphi(z)}{\varphi(z)^3} - \langle \nabla T(z), \dot z\rangle H^T(z)\,\dot z = 0, \end{aligned} \tag{4.5.1}$$

where $\lambda = \lambda_{\epsilon,\delta}$ is the function defined in (4.1.2).

5. A New Time Function

In this section we present some results of technical nature, mentioned in the previous sections, that show how to modify a given time function on $\mathcal{M}$ to obtain some extra properties. We will be concerned particularly with the problem of verticalization of the timelike curve γ and with the assumptions (4.0.1) and (4.0.2) made at the beginning of section 4.

In the proof of Proposition 5.1 and 5.7, we have benefited of some ideas contained in Lemmas 2.3 and 2.5 of [U], proved in the globally hyperbolic case. Note that in [U], Lemmas 2.3 and 2.5 are proven (without the presence of $\partial\Lambda$) using a suitable coordinate system. Moreover, property (1) of Proposition 5.1 is proven only for a compact portion of the curve γ. For these reasons, we prefer to give a detailed proof of our next Propositions.

We recall that we are assuming the existence of a time function which is unbounded on the support of γ.

Proposition 5.1. *Let γ be a closed C^1 embedding from $\mathbb{R}$ into a stably causal Lorentzian manifold $\mathcal{M}$, with $\dot\gamma$ timelike everywhere. Then, there exists a time function T on $\mathcal{M}$ such that:*

(1) *γ is vertical with respect to T;*
(2) *there exists an open neighborhood $U_{\partial\Lambda}$ of $\partial\Lambda$ such that $\langle \nabla T(z), \nabla\varphi(z)\rangle = 0$ for every $z \in U_{\partial\Lambda}$, where φ is the function defined in (3.0.1)*
(3) $\sup\{T(\gamma(s)), s \in \mathbb{R}\} = +\infty$.

Proof. Let $\widetilde{T}$ be an arbitrary time function on $\mathcal{M}$ which satisfies (3). Denoting by $\exp_q$ the exponential map around the point $q \in \mathcal{M}$, we define a local time function T_1 in a connected neighborhood U_γ of $\mathrm{supp}(\gamma)$ as follows:

$$T_1(\exp_{\gamma(s)} \zeta) = \widetilde{T}(\gamma(s)),$$

for all $\zeta \in T_{\gamma(s)}\mathcal{M}$ with $\langle \zeta , \dot{\gamma}(s) \rangle = 0$. Clearly, γ is vertical with respect to T_1. Notice that T_1 is increasing on γ, so that T_1 and $\widetilde{T}$ are *equioriented* on U_γ, i.e.:

$$\text{(5.1.0)} \qquad \langle \nabla T_1(z) , \nabla \widetilde{T}(z) \rangle < 0 \quad \text{on} \quad U_\gamma.$$

Moreover, we define a local time function T_2 on a neighborhood $U_{\partial\Lambda}$ by setting:

$$\text{(5.1.1)} \qquad T_2(z) = \widetilde{T}(z) + \nu(z) \cdot \varphi(z),$$

where φ is given in (3.0.1), and ν is defined by:

$$\text{(5.1.2)} \qquad \nu(z) = -\frac{\langle \nabla \widetilde{T}(z) , \nabla \varphi(z) \rangle}{\langle \nabla \varphi(z) , \nabla \varphi(z) \rangle}.$$

Observe that, since $\partial\Lambda$ is timelike, then $\langle \nabla\varphi , \nabla\varphi \rangle > 0$ around $\partial\Lambda$, and (5.1.2) makes sense. Moreover, since $\varphi = 0$ on $\partial\Lambda$, for $z \in \partial\Lambda$ it is:

$$\begin{aligned} \langle \nabla T_2(z) , \nabla T_2(z) \rangle &= \langle \nabla \widetilde{T}(z) + \nu(z) \nabla \varphi(z) , \nabla \widetilde{T}(z) + \nu(z) \nabla \varphi(z) \rangle = \\ \text{(5.1.3)} \qquad &= \langle \nabla \widetilde{T}(z) , \nabla \widetilde{T}(z) \rangle - \frac{\langle \nabla \widetilde{T}(z) , \nabla \varphi(z) \rangle^2}{\langle \nabla \varphi(z) , \nabla \varphi(z) \rangle} < 0, \end{aligned}$$

so that (5.1.1) defines a time function in a connected neighborhood $U_{\partial\Lambda}$ of $\partial\Lambda$. Observe that, since $\langle \nabla T_2(z) , \nabla \varphi(z) \rangle = 0$ on $U_{\partial\Lambda}$, (5.1.3) implies that on $U_{\partial\Lambda}$ it is

$$\text{(5.1.4)} \qquad \langle \nabla T_2(z) , \nabla \widetilde{T}(z) \rangle = \langle \nabla T_2(z) , \nabla T_2(z) \rangle < 0.$$

This implies that ∇T_2 and $\nabla \widetilde{T}$ are equioriented on $U_{\partial\Lambda}$.

Clearly, we may assume $U_\gamma \cap U_{\partial\Lambda} = \emptyset$, so that, the function:

$$T_0(z) = \begin{cases} T_1(z) & \text{if } z \in U_\gamma, \\ T_2(z) & \text{if } z \in U_{\partial\Lambda}, \end{cases}$$

is a smooth time function on $U_\gamma \cup U_{\partial\Lambda}$. Observe that T_0 *coincides with* $\widetilde{T}$ *on* $\mathit{supp}(\gamma) \cup \partial\Lambda$. Moreover, from (5.1.0) and (5.1.4) it follows:

$$\text{(5.1.5)} \qquad \langle \nabla T_0(z) , \nabla \widetilde{T}(z) \rangle < 0 \quad \text{on} \quad U_\gamma \cup U_{\partial\Lambda}.$$

Let now K be a fixed compact set in $\mathcal{M}$.

Let $\alpha : \mathcal{M} \longmapsto [0,1]$ be a smooth function, with support in $U_\gamma \cup U_{\partial\Lambda}$, which is identically 1 in a neighborhood of $\operatorname{supp}(\gamma) \cup \partial\Lambda$. Then, a time function T on K that extends T_0 can be defined by:

$$T = \widetilde{T} + \alpha(z) \cdot (T_0 - \widetilde{T} - \beta(T_0 - \widetilde{T})), \tag{5.1.6}$$

where $\beta : \mathbb{R} \longmapsto \mathbb{R}$ is a smooth map satisfying:

a) $\beta \equiv 0$ in a neighborhood of 0;

b) $\beta'(t) \leq 1 + \sigma$;

c) $\beta(t) = t$ for $|t| \geq \sigma$,

and $\sigma = \sigma(K) > 0$ has been chosen sufficiently small, in such a way that the sign of $\langle \nabla T(z), \nabla T(z) \rangle$ on $U_\gamma \cup U_{\partial\Lambda}$ is the same as the sign of the quantity:

$$\begin{aligned} G = (1 - \alpha(1-\beta'))^2 \cdot \langle \nabla\widetilde{T}, \nabla\widetilde{T} \rangle + \alpha^2(1-\beta')^2 \cdot \langle \nabla T_0, \nabla T_0 \rangle + \\ + 2\alpha(1-\beta')[1 - \alpha(1-\beta')] \cdot \langle \nabla\widetilde{T}, \nabla T_0 \rangle. \end{aligned} \tag{5.1.7}$$

If the coefficient of $\langle \nabla\widetilde{T}, \nabla T_0 \rangle$ in the expression of G in (5.1.7) is nonnegative, then clearly $G < 0$ and we are done. If it is negative, which happens when $1 < \beta' < 1 + \sigma$, then, provided that σ is small enough, the sign of G coincides with the sign of its first term, and again $G < 0$.

Hence, the function T of (5.1.6) satisfies the hypothesis in K.

Since $\mathcal{M}$ is paracompact and γ is a closed embedding, there exists a sequence $(U_i)_{i\in\mathbb{N}}$ of open subsets of $\mathcal{M}$ having the following properties:

a) $\overline{U}_i$ is compact in $\mathcal{M}$;

b) $\gamma(s)$ is eventually outside of U_i;

c) $\overline{U}_i \subset U_{i+1}$, for every i;

d) $\bigcup_{i\in\mathbb{N}} U_i = \mathcal{M}$.

Set $K_0 = \overline{U}_0$, and, by induction, $K_i = \overline{U}_i \setminus U_{i-1}$; notice that $K_i \cap K_{i+2} = \emptyset$ for any $i \in \mathbb{N}$. In every K_i we can make the same construction as above. Notice that in such a proof, the relevant fact is that we change $\widetilde{T}$ only in a small neighborhood of $(\partial\Lambda \cup \gamma(\mathbb{R}))$, but it remains the same on $\partial\Lambda \cup \gamma(\mathbb{R})$.

We begin doing the proof above when $K = K_0$. Consider now K_1 and, up to *shrinking* the neighborhood of $\partial\Lambda \cup \gamma(\mathbb{R})$ (already chosen for K_0), we construct the function T on $K_0 \cup K_1$. Consider now K_2, since $K_0 \cap K_2 = \emptyset$, up to choosing a smaller neighborhood of $(\partial\Lambda \cup \gamma(\mathbb{R})) \cap K_1$ we can define the function T on $K_0 \cup K_1 \cup K_2$, so that it coincides with the previous one on K_0. An induction argument allows to define the required time function on $\mathcal{M}$, defining it on K_i without modifying the previous definition on K_{i-2} and we have done. □

We will now take care of the assumption (4.0.1). For every $c \geq 0$, we denote by T_0^c the strip:

$$T_0^c = \{q \in \mathcal{M} \mid 0 \leq T(q) \leq c\}.$$

Proposition 5.2. *Let Λ be an open subset of a stably causal Lorentzian manifold $\mathcal{M}$, with $\partial\Lambda$ smooth timelike submanifold of $\mathcal{M}$, and let γ be a timelike curve, which is a closed embedding of $\mathbb{R}$ in Λ. Let φ be a C^2 function on $\mathcal{M}$ that satisfies (a), (b) and (c) of (3.0.1), and let T be as in Proposition 5.1.*

Let $\mathcal{N} \subset \Lambda$ be a region such that $\mathcal{N} \cap T_0^c$ is precompact for every $c \geq 0$.

Then, there exists a smooth function φ_1 on $\mathcal{M}$, satisfying (a), (b) and (c) of (3.0.1), and: such that $\varphi_1 \equiv \varphi$ in a neighborhood of $\partial\Lambda$, and such that:

(5.2.1) $$\langle \nabla T(z), \nabla\varphi_1(z)\rangle = 0 \text{ on a neighborhood of } \partial\Lambda;$$

(5.2.2) $$\langle \nabla T(z), \nabla\varphi_1(z)\rangle \geq 0 \quad \text{on} \quad \Lambda \cap \mathcal{N}.$$

Proof. Let $U_{\partial\Lambda}$ be the neighborhood of $\partial\Lambda$ defined in Proposition 5.1. Thanks to the precompactness assumption on $\mathcal{N}$, we can find a smooth, positive real function χ, with $\chi' > 0$ everywhere, and such that the open set:

$$U = \{z \in \mathcal{M} \mid \chi(T_1(z)) \cdot \varphi(z) < 1\}$$

has the following properties:
(a) $\partial\Lambda \subset U$;
(b) the closure of U in $\mathcal{N}$ is included in $U_{\partial\Lambda}$.

Furthermore, let $\rho : \mathbb{R} \longmapsto [0,1]$ be a smooth function satisfying:

(1) $\rho(s) = 0$ if $s \leq \frac{1}{2}$;

(2) $\rho' > 0$ in $(\frac{1}{2}, 1)$;

(3) $\rho(s) = 1$ if $s \geq 1$.

Finally, we define:

$$\varphi_1(z) = \Big[1 - \rho(\varphi(z) \cdot \chi(T(z))\Big] \cdot \varphi(z) + \frac{1}{2}\frac{\rho(\varphi(z) \cdot \chi(T(z)))}{\chi(T(z))}.$$

Now, if $\varphi(z) \cdot \chi(T(z)) \leq \frac{1}{2}$, it is $\varphi_1(z) = \varphi(z)$. Hence, up to modifying φ_1 outside a neighborhood of Λ, we can assume that φ_1 satisfies (a), (b) and (c) of (3.0.1). By (2) of Proposition 5.1 and property (b), we deduce the existence of a neighborhod od $\partial\Lambda$ where (5.2.1) is satisfied. Clearly, (5.2.2) is satisfied if $\varphi(z) \cdot \chi(T(z)) \leq \frac{1}{2}$. If $\varphi(z) \cdot \chi(T(z)) \geq 1$, then $\rho \equiv 1$ and $\rho' \equiv 0$, so $\nabla\varphi_1 = -\frac{1}{2}\chi'\chi^{-2}\nabla T$, and $\langle \nabla\varphi_1(z), \nabla T(z)\rangle = \frac{1}{2}\chi'(T(z))\chi(T(z))^{-2} \geq 0$. If $\frac{1}{2} < \varphi(z) \cdot \chi(T(z)) < 1$, we compute $\nabla\varphi_1(z)$ as follows:

$$\nabla\varphi_1 = (1-\rho)\nabla\varphi - \rho'\varphi\Big[\chi\nabla\varphi + \varphi\chi'\nabla T\Big] + \frac{\chi\rho'\left[\chi\nabla\varphi + \varphi\chi'\nabla T\right] - \rho\chi'\nabla T}{2\chi^2}.$$

In this case, since $z \in U$, it is $\langle \nabla\varphi(z), \nabla T(z)\rangle = 0$, thus

$$\langle \nabla\varphi_1, \nabla T\rangle = \Big(-\rho'\varphi^2\chi' + \frac{\rho'\varphi\chi'}{2\chi} - \frac{\rho\chi'}{2\chi^2}\Big)\langle \nabla T, \nabla T\rangle =$$

$$= \Big(\rho'\varphi^2\chi' - \frac{\rho'\varphi\chi'}{2\chi} + \frac{\rho\chi'}{2\chi^2}\Big)\left(-\langle \nabla T, \nabla T\rangle\right) \tag{5.2.3}$$

Clearly, $\frac{\rho\chi'}{2\chi^2} \geq 0$. Moreover,

$$\rho'\varphi^2\chi' - \frac{\rho'\varphi\chi'}{2\chi} = \frac{\rho'\chi'\varphi}{\chi}\Big(\varphi\chi - \frac{1}{2}\Big) \geq 0,$$

so that, from (5.2.3), since $\varphi\chi > \frac{1}{2}$, it is $\langle \nabla\varphi_1(z), \nabla T_1(z)\rangle \geq 0$ on $\Lambda \cap \mathcal{N} \cap T_0^d$, and the proof is finished. □

Definition 5.3. Let T_1 be a time function on $\mathcal{M}$. A time function T_2 is said to be a *rescaling* of T_1 if there exists a C^2 map $\phi : \mathbb{R} \longmapsto \mathbb{R}$ such that:

(1) $\phi'(r) > 0$ for every $r \in \mathbb{R}$;

(2) $T_2(q) = \phi(T_1(q))$ for every $q \in \mathcal{M}$.

Remark 5.4. Observe that the results of Proposition 5.2 are invariant with respect to the rescaling of the time function.

We now have an easy Lemma that shows how the differential operators of covariant derivative, gradient and Hessian change by passing to a conformally equivalent metric. Its proof is not difficult and it will be only sketched.

Lemma 5.5. *Let $(\mathcal{M}, g_1)$ be any semi-Riemannian manifold, $\psi : \mathcal{M} \longmapsto \mathbb{R}^+$ a smooth map, and $g_2(z) = \psi(z)g_1(z)$ a semi-Riemannian metric on $\mathcal{M}$ conformally equivalent to g_1.*

Then, for every piecewise smooth curve z in $\mathcal{M}$, every piecewise smooth vector field ζ along z, with $\zeta(0) = \zeta(1) = 0$, and every smooth function $\theta : \mathcal{M} \longmapsto \mathbb{R}$, it is:

(1) $$\nabla_s^{(2)}\dot z = \nabla_s^{(1)}\dot z + \frac{1}{\psi(z(s))}g_1(z(s))[\nabla_1\psi(z(s)), \dot z(s)]\dot z(s) + \\ - \frac{1}{2\,\psi(z(s))}g_1(z(s))[\dot z(s), \dot z(s)]\,\nabla_1\psi(z(s));$$

(2) $\nabla_2\theta(p) = \psi(p)^{-1}\nabla_1\theta(p)$, *for every* $p \in \mathcal{M}$;

(3) $$H_2^\theta[\zeta,\zeta] = H_1^\theta[\zeta,\zeta] + \frac{1}{2\psi(z)}g_1(z)[\nabla_1\theta(z), \nabla_1\psi(z)]\cdot g_1(z)[\zeta,\zeta] + \\ - \frac{1}{\psi(z)}g_1(z)[\nabla_1\theta(z), \zeta]\cdot g_1(z)[\nabla_1\psi(z), \zeta],$$

where ∇_i, $\nabla_s^{(i)}$ and H_i denote respectively the gradient, the covariant derivative and the Hessian operator with respect to g_i, $i = 1, 2$.

Proof. The statements can be easily proven by direct computation, differentiating the action functional relative to the metric g_2 and, for part (3), using the weak equation satisfied by the g_2-geodesics. □

Remark 5.6. From part (2) of Lemma 5.5 it follows easily that the results of Proposition 5.2 continue to hold when we pass to a conformally equivalent metric on $\mathcal{M}$.

We now assume that T_1 is a time function on $\mathcal{M}$, and for $c \in \mathbb{R}^+$, we continue to denote by $(T_1)_0^c$ the *strip*:

$$(T_1)_0^c = \{q \in \mathcal{M} \mid 0 \leq T_1(q) \leq c\}.$$

Proposition 5.7. *Let $\mathcal{N} \subset \mathcal{M}$ be a region such that $\mathcal{N} \cap (T_1)_0^c$ is precompact in $\mathcal{M}$ for every $c \geq 0$. Then there exists a rescaling T_2 of T_1 and a Lorentzian metric g_2 on $\mathcal{M}$ which is conformally equivalent to g_1, such that:*

1. *$g_2(z)[\nabla_2 T_2(z), \nabla_2 T_2(z)] = \langle \nabla_2 T_2(z), \nabla_2 T_2(z) \rangle_2 = -1$ for all $z \in \mathcal{N}$;*
2. *$H_2^{T_2}(z)[\zeta, \zeta] \leq 0$ for every $z \in \mathcal{N}$ and every $\zeta \in T_z\mathcal{M}$, ζ causal;*
3. *If $\sup\{T_1(\gamma(s)), s \geq 0\} = +\infty$, then $\sup\{T_2(\gamma(s)), s \geq 0\} = +\infty$.*

Proof. Since we are to change conformally the metric g_1, we may assume without loss of generality that T_1 has normalized gradient with respect to g_1, i.e.:

$$g_1(z)[\nabla_1 T(z), \nabla_1 T(z)] = -1 \quad \forall z \in \mathcal{M}.$$

We set

$$g_2(z) = \Psi(T_1(z)) \cdot g_1(z), \quad (\Psi > 0)$$

and

$$T_2(z) = \phi(T_1(z)),$$

where ϕ, $\Psi : \mathbb{R} \longmapsto \mathbb{R}$ are smooth functions to be determined in such a way that (1) and (2) are satisfied.

The condition (1) is easily translated in terms of ϕ and Ψ; indeed, from Lemma 5.5, we have:

$$\begin{aligned} g_2(z)[\nabla_2 T_2(z), \nabla_2 T_2(z)] &= \Psi(T_1(z)) \cdot g_1(z)[\nabla_2 T_2(z), \nabla_2 T_2(z)] = \\ &= \frac{1}{\Psi(T_1(z))} g_1(z)[\nabla_1 T_2(z), \nabla_1 T_2(z)] = \\ &= \frac{\phi'(T_1(z))^2}{\Psi(T_1(z))} g_1(z)[\nabla_1 T_1(z), \nabla_1 T_1(z)] = -\frac{\phi'(T_1(z))^2}{\Psi(T_1(z))}. \end{aligned}$$

Hence, condition (1) becomes:

$$\Psi(r) = \phi'(r)^2, \quad \forall r \in \mathbb{R}. \tag{5.7.1}$$

Observe that $\phi'(r) \neq 0$ implies $\Psi(r) > 0$.

From part (3) of Lemma 5.5, we have:

$$
\begin{aligned}
H_2^{T_2}(z)[\zeta,\zeta] &= \\
&= H_1^{T_2}(z)[\zeta,\zeta] + \frac{1}{2\Psi(T_1(z))} g_1(z)[\zeta,\zeta] \cdot g_1(z)[\nabla_1(\phi(T_1(z)), \nabla_1 \Psi(T_1(z))] + \\
&\quad - \frac{1}{\Psi(T_1(z))} g_1(z)[\nabla_1 \phi(T_1(z)), \zeta] \cdot g_1[\nabla_1 \Psi(T_1(z)), \zeta] = \\
&= H_1^{T_2}(z)[\zeta,\zeta] + \frac{\phi'(T_1(z))\Psi'(T_1(z))}{2\Psi(T_1(z))} g_1(z)[\zeta,\zeta]\, g_1(z)[\nabla_1 T_1(z), \nabla_1 T_1(z)] + \\
&\quad - \frac{\phi'(T_1(z))\Psi'(T_1(z))}{\Psi(T_1(z))} g_1(z)[\nabla_1 T_1(z), \zeta]^2.
\end{aligned}
\tag{5.7.2}
$$

Substituting (5.7.1) in (5.7.2), we get

$$
\begin{aligned}
H_2^{T_2}(z)[\zeta,\zeta] = H_1^{T_2}(z)[\zeta,\zeta] + \phi''(T_1(z)) \cdot g_1(z)[\zeta,\zeta]\, g_1(z)[\nabla_1 T_1(z), \nabla_1 T_1(z)] + \\
- \phi''(T_1(z)) \cdot g_1(z)[\nabla_1 T_1(z), \zeta]^2.
\end{aligned}
\tag{5.7.3}
$$

Let $\gamma_\zeta : (-\delta,\delta) \longmapsto \mathcal{M}$ be a g_1–geodesic in $\mathcal{M}$ such that $\gamma_\zeta(0) = z$ and $\dot\gamma_\zeta(0) = \zeta$. We compute the Hessian $H_1^{T_2}(z)[\zeta,\zeta]$ as follows:

$$
\begin{aligned}
H_1^{T_2}(z)[\zeta,\zeta] &= \frac{d^2}{ds^2}\Big|_{s=0} T_2(\gamma_\zeta(s)) = \frac{d^2}{ds^2}\Big|_{s=0} \phi(T_1((\gamma_\zeta(s))) = \\
&= \frac{d}{ds}\Big|_{s=0} \phi'(T_1(\gamma_\zeta(s))) g_1(\gamma_\zeta(s))[\nabla_1 T_1(\gamma_\zeta(s)), \dot\gamma_\zeta(s))] = \\
&= \phi''(T_1(z))\, g_1(z)[\nabla_1 T_1(z), \zeta]^2 + \phi'(T_1(z))\, H_1^{T_1}(z)[\zeta,\zeta].
\end{aligned}
\tag{5.7.4}
$$

From (5.7.3) and (5.7.4), the condition (2) in the thesis becomes:

$$
\begin{aligned}
-\phi''(T_1(z)) \cdot g_1(z)[\zeta,\zeta] - g_1(z)[\nabla_1 T_1(z), \zeta]^2) + \\
+ \phi'(T_1(z)) \cdot H_1^{T_1}(z)[\zeta,\zeta] \le 0.
\end{aligned}
\tag{5.7.5}
$$

By the wrong way Schwartz's inequality, if $\zeta \neq 0$ is a causal vector field along z the coefficient of ϕ'' in (5.7.5) is non positive, and it is null only when $\nabla_1 T_1(z)$ is a multiple of ζ. But in this case, also $H_1^{T_1}(z)[\zeta,\zeta]$ is zero:

$$
H_1^{T_1}(z)[\nabla_1 T_1, \nabla_1 T_1] = \frac{1}{2}\frac{d}{ds} g_1(z)[\nabla_1 T_1(z), \nabla_1 T_1(z)] = \frac{d}{ds}(-1) = 0.
$$

Hence, we can define a continuous function $J : T\mathcal{M} \longmapsto I\!R$ by:

$$
J(z,\zeta) = -\frac{H_1^{T_1}(z)[\zeta,\zeta]}{g_1(z)[\zeta,\zeta] + g_1(z)[\nabla_1 T_1(z), \zeta]^2},
\tag{5.7.6}
$$

and (5.7.5) becomes:

$$\phi''(T_1(z)) - J(z,\zeta)\cdot\phi'(T_1(z)) \geq 0. \tag{5.7.7}$$

The function $J(z,\zeta)$ is homogeneous of degree 0 in ζ, so, by the precompactness, it follows that, for every $c \geq 0$, J is bounded on $(T_1)_0^c \cap \mathcal{N}$. Hence, there exists a continuous function $\mu : [0,+\infty) \longmapsto [0,+\infty)$ such that:

$$J(z,\zeta) \leq \mu(T_1(z))$$

for every $z \in \mathcal{N}$. Taking

$$\phi(t) = \int_0^t \exp\left(\int_0^r \mu(s)\,\mathrm{d}s\right)\mathrm{d}r, \tag{5.7.8}$$

then ϕ satisfies the differential equation $\phi'' - \mu\phi' = 0$. Since $\phi' > 0$, it is:

$$\phi''(T_1(z)) - \phi'(T_1(z))\,J(z,\zeta) \geq \phi''(T_1(z)) - \phi'(T_1(z))\,\mu(T_1(z)) = 0$$

for every $z \in \mathcal{N}$ and every $\zeta \in T_z\mathcal{M}$. Finally, since $\mu \geq 1$, $\lim_{t\to+\infty}\phi(t) = +\infty$, so also part (3) holds. □

Remark 5.8. In our case, Propositions 5.2 and 5.7 will be applied to the following set:

$$\mathcal{N}(p,\gamma) = \Big\{ q \in \Lambda \;\Big|\; \exists\, z \in \Omega^{1,2}_{p,\gamma}\,,\ \langle \dot z, \dot z\rangle \leq 0 \text{ a.e.}$$
$$\langle \dot z, \nabla T(z)\rangle \geq 0 \text{a.e. and } q \in \operatorname{supp}(z) \Big\}.$$

Indeed, in the definition of $\mathcal{N}(p,\gamma)$, the word 'causal' can be replaced with the word 'lightlike'. Thus, thanks to the c–precompactness, $\mathcal{N}(p,\gamma) \cap T^c$ is precompact in $\overline{\Lambda}$. This follows from our next Lemma.

Lemma 5.9. *Let $\mathcal{M}$ be an arbitrary Lorentzian manifold. Let p, q two arbitrarily fixed points of $\mathcal{M}$, and assume that there exists a causal curve from p to q of class $H^{1,2}$. Then, there exists a* lightlike *curve from p to q of class $H^{1,2}$. Furthermore, if $\mathcal{M}$ is time oriented and the causal curve is future (past) pointing, then the lightlike curve can be found future (past) pointing.*

The proof of Lemma 5.9 is omitted. It can be easily obtained using standard techniques in causal geometry, working in convex normal neighborhoods.

Remark 5.10. Notice that lightlike geodesics are independent on conformal changes of the metric, as a direct calculation shows. This fact allows to use Propositions 5.2 and 5.7 to study lightlike geodesics as critical points of the functional Q.

6. Relations Between $\mathcal{L}^+_{p,\gamma}$ and $\mathcal{L}^+_{p,\gamma,\varepsilon}$

In this section we will discuss the method of approximation of the non smooth manifold $\mathcal{L}^+_{p,\gamma}$ with the regular manifolds $\mathcal{L}^+_{p,\gamma,\varepsilon}$, assuming that:

$$\langle \nabla T(z), \nabla\varphi(z)\rangle = 0 \text{ on a neighborhood of } \partial\Lambda, \tag{6.0.1}$$

where φ is defined in (3.0.1).

The main result, which is stated in the following proposition, is concerned with the existence of transition functions between the Q–sublevels in $\mathcal{L}^+_{p,\gamma}$ and $\mathcal{L}^+_{p,\gamma,\varepsilon}$:

6.1 Proposition. *Suppose that $\mathcal{L}^+_{p,\gamma}$ is c-precompact for some $c > \inf\limits_{\mathcal{L}_{p,\gamma}} Q$. Then, there exists a positive number $\varepsilon_0 = \varepsilon_0(c) > 0$ and for every $\varepsilon \in (0, \varepsilon_0]$ there are two injective maps:*

$$\phi_\varepsilon : Q^c \cap \mathcal{L}_{p,\gamma} \longmapsto \mathcal{L}^+_{p,\gamma,\varepsilon},$$
$$\psi_\varepsilon : \mathcal{L}^+_{p,\gamma,\varepsilon} \longmapsto \mathcal{L}^+_{p,\gamma},$$

such that:

(1) *ϕ_ε and ψ_ε are continuous;*
(2) *for every $z \in \mathcal{L}^+_{p,\gamma,\varepsilon}$ such that $Q(\psi_\varepsilon(z)) \leq c$, it is $\phi_\varepsilon(\psi_\varepsilon(z)) = z$;*
(3) *for every $z \in Q^c \cap \mathcal{L}^+_{p,\gamma}$ it is $\psi_\varepsilon(\phi_\varepsilon(z)) = z$;*
(4) *there exists a positive constant $M = M(c)$ such that $d_1(\phi_\varepsilon(z), z) \leq M \cdot \varepsilon$ for every $z \in Q^c \cap \mathcal{L}^+_{p,\gamma}$, where d_1 denotes the distance induced by the Riemannian struture on $\Omega^{1,2}_{p,\gamma}$. In particular, $\lim\limits_{\varepsilon\to 0} \phi_\varepsilon(z) = z$ in $\Omega^{1,2}_{p,\gamma}$, for every $z \in Q^c \cap \mathcal{L}^+_{p,\gamma}$.*

Proof. We fix c and we find a compact subset K of $\overline{\Lambda}$ such that the support of every $z \in Q^c \cap \mathcal{L}_{p,\gamma}$ lies in K. Let δ be a positive number such that the flow $\Phi(s,q)$ of the vector field $-\nabla T$ is defined on $[-\delta,\delta] \times K$. By definition, the curve $\eta_q(s) = \Phi(s,q)$ is the maximal solution of the Cauchy problem:

$$\begin{cases} \dot\eta = -\nabla T(\eta), \\ \eta(0) = q. \end{cases}$$

Observe that T is strictly increasing on such a curve, namely:

$$\frac{d}{ds} T(\eta(s)) = \langle \nabla T(\eta(s)), \dot\eta(s)\rangle = -\langle \nabla T(\eta(s)), \nabla T(\eta(s))\rangle = 1 > 0.$$

For $z \in Q^c \cap \mathcal{L}_{p,\gamma}$, we define

$$z_\varepsilon(s) = \phi_\varepsilon(z)(s) = \Phi(\tau_{z,\varepsilon}(s), z(s)),$$

for some function $\tau_{z,\varepsilon}(s) = \tau(s)$ on $[0,1]$ and with values in $[0,\delta)$, to be determined in such a way that

$$\tau(0) = 0,$$

(which means that $z_\varepsilon(0) = p$),

$$\text{(6.1.1)} \qquad \langle \dot z_\varepsilon \, , \nabla T(z_\varepsilon) \rangle > 0,$$

and that

$$\langle \dot z_\varepsilon \, , \dot z_\varepsilon \rangle \equiv -\varepsilon^2.$$

Observe that any such curve automatically satisfies $z_\varepsilon(1) \in \operatorname{supp}(\gamma)$, since γ is an integral curve of ∇T and $\Phi(0, z(1)) = z(1) \in \operatorname{supp}(\gamma)$. Moreover $z_\varepsilon([0,1]) \subset \Lambda$, thanks to (6.0.1).

We compute $\dot z_\varepsilon$ as follows:

$$\dot z_\varepsilon = \Phi_q[\dot z] + \Phi_s[\dot \tau] = \Phi_q[\dot z] - \nabla T(z_\varepsilon)\,\dot\tau,$$

which gives:

$$\text{(6.1.2)} \qquad \langle \dot z_\varepsilon \, , \dot z_\varepsilon \rangle = -\dot\tau^2 - 2\dot\tau \langle \nabla T(z_\varepsilon) \, , \Phi_q[\dot z] \rangle + \langle \Phi_q[\dot z] \, , \Phi_q[\dot z] \rangle = -\varepsilon^2.$$

This is a quadratic equation on $\dot\tau$; observe that by the wrong way Schwartz inequality:

$$\langle \nabla T(z_\varepsilon) \, , \Phi_q[\dot z] \rangle^2 \geq -\langle \Phi_q[\dot z] \, , \Phi_q[\dot z] \rangle,$$

the discriminant Δ of the equation is strictly positive:

$$\text{(6.1.3)} \qquad \frac{\Delta}{4} = \langle \nabla T(z_\varepsilon) \, , \Phi_q[\dot z] \rangle^2 + \langle \Phi_q[\dot z] \, , \Phi_q[\dot z] \rangle + \varepsilon^2 \geq \varepsilon^2 > 0.$$

Take the solution $\dot\tau$ of (6.1.2) given by $\dot\tau$, given by:

$$\dot\tau = -\langle \nabla T(z_\varepsilon) \, , \Phi_q[\dot z] \rangle + \frac{1}{2}\sqrt{\Delta},$$

where Δ is given in (6.1.3). Notice that by this choice

$$\langle \dot z_\varepsilon \, , \nabla T(z_\varepsilon) \rangle = \langle \Phi_q[\dot z] - \nabla T(z_\varepsilon)\,\dot\tau \, , \nabla T(z_\varepsilon) \rangle > 0$$

and (6.1.1) is satisfied. Observe also that the coefficients of the equation (6.1.2) clearly depend continuously on ε. The function τ has to satisfy the Cauchy problem:

$$\text{(6.1.4)} \qquad \begin{cases} \dot\tau = -\langle \nabla T(\Phi) \, , \Phi_q[\dot z] \rangle + \dfrac{1}{2}\sqrt{\Delta}, \\ \tau(0) = 0. \end{cases}$$

Since for $\varepsilon = 0$ (6.1.4) has the null solution, which is defined on the whole real line, then for ε small enough (6.1.4) admits a unique solution defined on the interval $[0,1]$. The construction of the map ψ_ε is done in a similar fashion, considering the flow $\Psi(s,q)$ of the vector field ∇T, and setting:

$$\psi_\varepsilon(z)(s) = z^\varepsilon(s) = \Psi(\sigma(s), z(s)),$$

where $\sigma = \sigma_{z,\varepsilon}$ is to be determined with the conditions:

$$\sigma(0) = 0, \quad \langle \dot z^\varepsilon \, , \dot z^\varepsilon \rangle = 0, \quad \text{and} \quad \langle \dot z^\varepsilon \, , \nabla T(z^\varepsilon) \rangle \geq 0.$$

An argument similar to the previous case shows the existence and the continuity properties of such a map σ, which proves the first part of the Proposition.

Part (2) and (3) follows immediately from the construction of ϕ_ε and ψ_ε.

Part (4) follows from the Gronwall's Lemma. □

7. Some Compactness Results

In this section we will prove the completeness of the sublevels of the penalized functional Q_δ on the regular manifold $\mathcal{L}^+_{p,\gamma,\epsilon}$ and the Palais–Smale compactness condition for Q_δ. Thanks to Remark 5.8, from now on we can assume that T and φ satisfy the properties of Propositions 5.1, 5.2 and 5.7.

We recall the following definition:

Definition 7.1. Let X be a Hilbert manifold, and $I : X \longmapsto \mathbb{R}$ be a C^1 functional and c a real number. I is said to satisfy the Palais–Smale condition at the level c if every sequence x_n in X satisfying:

(PS1) $\lim_{n\to\infty} I(x_n) = c$,

(PS2) $\lim_{n\to\infty} I'(x_n) = 0$,

has a converging subsequence in X. A sequence x_n satisfying (PS1) and (PS2) will be called a Palais–Smale sequence.

Let $z \in \mathcal{L}_{p,\gamma,\epsilon}$; on the tangent space $T_z\mathcal{L}^+_{p,\gamma,\epsilon} = \{\zeta \in T_z\Omega^{1,2}_{p,\gamma} \mid \langle \dot z , \nabla_s \zeta\rangle = 0 \text{ a.e.}\}$ we introduce a Hilbertian norm, setting:

$$\langle \zeta , \zeta\rangle_1 = \int_0^1 \langle \nabla^{(R)}_s \zeta , \nabla^{(R)}_s \zeta\rangle_{(R)}\, ds. \tag{7.1.1}$$

We start with a technical Lemma that says that sequences contained in a sublevel $Q^c_\delta = \{z \in \mathcal{L}^+_{p,\gamma,\epsilon} \mid Q_\delta(z) \le c\}$ (with $c \in \mathbb{R}^+$) stay away from $\partial\Lambda$:

Lemma 7.2. *Let $(z_n)_{n\in\mathbb{N}}$ be a sequence in $\Omega^{1,2}_{p,\gamma}(\Lambda)$ that is weakly convergent to z in $H^{1,2}([0,1],\overline{\Lambda})$. If there exists $s \in [0,1]$ such that $z(s) \in \partial\Lambda$, then*

$$\lim_{n\to\infty} \int_0^1 \frac{ds}{\varphi(z_n(s))^2} = +\infty.$$

Proof. By the weak convergence in $H^{1,2}$, the sequence z_n is uniformly convergent to z, so that $\nabla_{(R)}\varphi(z_n(s))$ is bounded, i.e.

$$\|\nabla_{(R)}\varphi(z_n(s))\|_{(R)} \le c_1, \tag{7.2.1}$$

for some $c_1 > 0$. Moreover, since $\dot z_n$ is bounded in $L^2([0,1],T\mathcal{M})$, there exists a positive constant L_1 such that for every $n \in \mathbb{N}$ it is:

$$\int_0^1 \langle \dot z_n , \dot z_n\rangle_{(R)}\, ds \le L_1. \tag{7.2.2}$$

Assume that there exists a sequence s_n in $[0,1)$ such that $\lim_{n\to\infty} \varphi(z_n(s_n)) = 0$. Observe that by the uniform convergence of z_n and the fact that $z_n(1) \in \text{supp}(\gamma)$, which is far from $\partial\Lambda$, then s_n is bounded away from 1, say

$$1 - s_n \ge \eta_0 > 0. \tag{7.2.3}$$

From (7.2.1), (7.2.2) and the Hölder's inequality, for $s > s_n$ it follows:

$$\varphi(z_n(s)) - \varphi(z_n(s_n)) = \int_{s_n}^{s} \langle \nabla_{(n)} \varphi(z_n), \dot{z}_n \rangle_{(n)} \le c_1 L_1 \sqrt{s - s_n},$$

and since $\varphi > 0$ on Λ:

(7.2.4) $\varphi(z_n(s))^2 \le \left(c_1 L_1 \sqrt{s - s_n} + \varphi(z_n(s_n))\right)^2 \le 2c_1^2 L_1^2 (s - s_n) + 2\varphi(z_n(s_n))^2.$

Taking the inverse and integrating (7.2.4), we get:

$$\int_{s_n}^{1} \frac{ds}{\varphi(z_n(s))^2} \ge \frac{1}{2c_1^2 L_1^2} \log\left(\frac{c_1^2 L_1^2 (1 - s_n) + \varphi(z_n(s_n))^2}{\varphi(z_n(s_n))^2}\right). \tag{7.2.5}$$

Taking the limit for $n \longmapsto \infty$ in (7.2.5), recalling (7.2.3), we obtain the thesis. □

We will assume from now on that $\mathcal{L}^+_{p,\gamma}$ is c–precompact for every $c \in \mathbb{R}$. Observe that this condition does not depend on the choice of a parametrization for γ.

Notice that from the Hölder inequality we get:
(7.2.6)

$$Q_\delta(z) \ge Q(z) = \int_0^1 \langle \nabla T, \dot{z} \rangle^2 \, ds \ge \left(\int_0^1 \langle \nabla T, \dot{z} \rangle \, ds\right)^2 = (T(z(1)) - T(z(p)))^2,$$

for every $\delta \ge 0$ and for every $z \in \Omega^{1,2}_{p,\gamma}$. Since we are asuming that $\sup T(\gamma) = +\infty$, whenever $T(z(1))$ is bounded, $\tau_{p,\gamma}$ is bounded, too. Then for every $d \in \mathbb{R}^+$, there exists $m(d) > 0$, such that

$$Q^d_\delta \subseteq \tau^{m(d)}_{p,\gamma}, \quad \forall \delta \ge 0. \tag{7.2.7}$$

Lemma 7.3. *For every $c \in \mathbb{R}^+$, Q^c_δ is a complete metric subspace of $\mathcal{L}^+_{p,\gamma,\epsilon}$ with respect to the Hilbert stucture (7.1.1).*

Proof. Let $z_n \in Q^c_\delta$ be a Cauchy sequence. It suffices to show that there is a subsequence of z_n that converges in $\mathcal{L}^+_{p,\gamma,\epsilon}$. From (7.2.7), it is $z_n \in \tau^{m(c)}_{p,\gamma}$. From the $m(c)$–precompactness, it follows that there exists a compact subset $K = K(c)$ of $\overline{\Lambda}$ such that $\mathrm{supp}(z_n) \subset K$ for every $n \in \mathbb{N}$. Moreover, since (z_n) is a Cauchy sequence, there exists a constant $M > 0$ such that

$$\int_0^1 \langle \dot{z}_n, \dot{z}_n \rangle_{(n)} \, ds \le M. \tag{7.3.1}$$

From the Ascoli–Arzelà Theorem it follows that z_n has a subsequence (still denoted by z_n) that is uniformly convergent to a continuous curve z; from (7.3.1) it follows that $z \in H^{1,2}([0,1], \overline{\Lambda})$ and that we have the weak convergence in

$H^{1,2}([0,1],\mathcal{M})$. (By weak convergence in $H^{1,2}([0,1],\mathcal{M})$ we mean that z_n is uniformly convergent to z and that

$$\lim_{n\to\infty}\int_0^1 \langle \dot z_n \,, v_n\rangle_{(\mathbb{R})}\,ds = \int_0^1 \langle \dot z\,, v\rangle_{(\mathbb{R})}\,ds,$$

for every sequence $v_n \in L^2([0,1],T\mathcal{M})$ that is strongly convergent (in L^2) to v). Since $\int_0^1 \varphi(z_n)^{-2}\,ds \le \delta^{-1}Q_\delta(z_n)$ is bounded, Lemma 7.2 says that $z(s)\in\Lambda$ for every s, and therefore $z\in H^{1,2}([0,1],\Lambda)$. Since $\dot z_n$ is a Cauchy sequence in $L^2([0,1],\mathbb{R}^N)$ and the tangent bundle $T\mathcal{M}$ is locally trivial, the compactness of K and the completeness of $L^2([0,1],\mathbb{R}^N)$ give immediately the strong L^2-convergence of $\dot z_n$ to an element $w\in L^2([0,1],\mathbb{R}^N)$. But $\dot z_n$ tends to $\dot z$ weakly, so $w=\dot z$ and z_n is strongly convergent to z in $H^{1,2}([0,1],\Lambda)$. By continuity, we have:

$$\Psi_\varepsilon(z) = \lim_{n\to\infty}\Psi_\varepsilon(z_n) = 0,$$

where Ψ_ε is the map introduced in Section 2, whose locus of zeroes is the set $\mathcal{L}^+_{p,\gamma,\varepsilon}$. This implies that $z\in\mathcal{L}^+_{p,\gamma,\varepsilon}$ and the Lemma is proven. □

Most of the rest of this section will be devoted to the proof of the main compactness property for the functional Q_δ:

Proposition 7.4. *For every $c\in\mathbb{R}^+$, Q_δ satisfies the Palais–Smale condition at the level c on $\mathcal{L}^+_{p,\gamma,\varepsilon}$.*

Proof. Let z_n be a Palais–Smale sequence at the level c for Q_δ, i.e. z_n satisfies (PS1) and (PS2) of Definition 7.1. We assume without loss of generality that z_n is smooth and that $Q_\delta(z_n)\le c+1$, for every $n\in\mathbb{N}$.

We denote by Θ_n the gradient of $Q'_\delta(z_n)$ with respect to the Hilbertian norm (7.1.1) $\langle\cdot,\cdot\rangle_1$, which is the vector field in $T_{z_n}\mathcal{L}_{p,\gamma,\varepsilon}$ that satisfies:

$$Q'_\delta(z_n)[\zeta] = \langle\Theta_n\,,\zeta\rangle_1 = \int_0^1 \langle \nabla_s^{(\mathbb{R})}\Theta_n\,, \nabla_s^{(\mathbb{R})}\zeta\rangle_{(\mathbb{R})}\,ds, \tag{7.4.1}$$

for every $\zeta\in T_{z_n}\mathcal{L}_{p,\gamma,\varepsilon}$. By (PS2), it is:

$$\lim_{n\to\infty}\langle\Theta_n\,,\Theta_n\rangle_1 = \lim_{n\to\infty}\int_0^1 \langle \nabla_s^{(\mathbb{R})}\Theta_n\,, \nabla_s^{(\mathbb{R})}\Theta_n\rangle_{(\mathbb{R})}\,ds = 0,$$

i.e. $\nabla_s\Theta_n$ is strongly convergent to 0 in L^2. Using integration by parts (that can be done assuming z_n smooth) and the Riemannian metric (1.0.1), it is easy to prove the existence of a sequence of vector fields Δ_n in Λ, convergent to 0 in L^2, such that, for every $\zeta\in T_{z_n}\mathcal{L}^+_{p,\gamma,\varepsilon}$, it is:

$$\int_0^1 \langle \nabla_s^{(\mathbb{R})}\Delta_n\,, \nabla_s^{(\mathbb{R})}\zeta\rangle_{(\mathbb{R})}\,ds = \int_0^1 \langle \Delta_n\,, \nabla_s\zeta\rangle\,ds. \tag{7.4.2}$$

Now,

$$\int_0^1 \langle \dot z_n , \dot z_n \rangle_{(n)} \, ds = \int_0^1 \langle \dot z_n , \dot z + n \rangle \, ds + 2 \int_0^1 \langle \nabla T(z_n), \dot z_n \rangle^2 \, ds =$$

$$-\varepsilon^2 + 2Q(z_n) \le 2Q_\delta(z_n),$$

which is bounded. Then by Lemma 7.2, z_n has a subsequence, still denoted by z_n, which is weakly convergent to $z \in H^{1,2}([0,1], \Lambda)$.

Take $\zeta(s) \in T_{z_n} H^{1,2}([0,1], \mathcal{M})$, with $\zeta(0) = \zeta(1) = 0$, and set, as in Section 3,

$$V_\zeta(s) = \zeta(s) - \mu_{z_n,\zeta}(s) \cdot \nabla T(z_n(s)),$$

where $\mu = \mu_{z,\zeta}$ is the solution of the Cauchy problem (3.0.6). Since $V_\zeta \in T_{z_n} \mathcal{L}^+_{p,\gamma,\varepsilon}$, from (7.4.1) and (7.4.2) one computes easily the following formula:

$$Q'_\delta(z_n)[V_\zeta] = \int_0^1 \langle \Delta_n , \nabla_s \zeta - \mu' \, \nabla T(z_n) + \mu \, H^T(z_n) \, \dot z_n \rangle_{(n)} \, ds. \tag{7.4.3}$$

We need to prove that $\dot z_n$ is strongly convergent to $\dot z$ in $L^2([0,1], T\mathcal{M})$.

Set $\mu_n = \mu_{z_n,\zeta_n}$,

$$a = a_n(s) = -\frac{\langle H^T(z_n) \, \dot z_n , \dot z_n \rangle}{\langle \nabla T(z_n) , \dot z_n \rangle},$$

$$\mathcal{F}(z,\delta) = \delta \, \langle \frac{\nabla \varphi(z)}{\varphi(z)^3} , \nabla T(z) \rangle - \langle H^T(z) \, \dot z , \dot z \rangle \tag{7.4.4}$$

$$\lambda_n(s) = e^{-\int_0^s a_n(\sigma)\, d\sigma} \left(\int_s^1 \mathcal{F}(z_n, \delta) \, e^{-\int_0^r a_n(\sigma)\, d\sigma} \, dr \right).$$

Note that by Propositions 5.1 ad 5.2, $\mathcal{F}(z,\delta)$ is non-negative and it is uniformly bounded on the sublevels of Q_δ, and the same holds for λ_n.

From (3.2.2) and (7.4.3), it follows:

(7.4.5)

$$\int_0^1 \langle \dot z_n , \nabla_s \zeta \rangle \, ds + \int_0^1 \lambda_n \, \langle \frac{\dot z_n}{\langle \nabla T(z_n), \dot z_n \rangle} , \nabla_s \zeta \rangle \, ds +$$

$$+ \int_0^1 \langle \nabla T(z_n), \dot z_n \rangle \, \langle \nabla T(z_n), \nabla_s \zeta \rangle \, ds - \delta \int_0^1 \frac{1}{\varphi(z_n)^3} \langle \nabla \varphi(z_n), \zeta \rangle \, ds +$$

$$+ \int_0^1 \langle \nabla T(z_n), \dot z_n \rangle \langle H^T(z_n) \, \dot z_n , \zeta \rangle \, ds = \frac{1}{2} \int_0^1 \langle \Delta_n , \nabla_s \zeta \rangle \, ds +$$

$$+ \frac{1}{2} \int_0^1 \langle \Delta_n , \nabla T(z_n) \rangle \frac{\langle H^T(z_n) \, \dot z_n , \dot z_n \rangle}{\langle \nabla T(z_n), \dot z_n \rangle} \langle \nabla T(z_n), \dot z_n \rangle \, \mu_n \, ds +$$

$$- \frac{1}{2} \int_0^1 \langle \Delta_n , \nabla T(z_n) \rangle \, \langle \frac{\dot z_n}{\langle \nabla T(z_n), \dot z_n \rangle} , \nabla_s \zeta \rangle \, ds + \frac{1}{2} \int_0^1 \langle \Delta_n , H^T(z_n) \, \dot z_n \rangle \mu_n \, ds.$$

It is easy to check the following boundedness properties for the functions involved in our computations:

(a) a_n is bounded in $L^2([0,1],\mathbb{R})$;
(b) $\langle \Delta_n, \nabla T(z_n)\rangle \cdot a_n$ is convergent to 0 in $L^1([0,1],\mathbb{R})$;
(c) $e^{-\int_0^s a_n(\sigma)\,d\sigma}$ is bounded in $L^\infty([0,1],\mathbb{R})$;

and from these it follows that

$$c_n(s) = e^{-\int_0^s a_n(\sigma)\,d\sigma} \int_s^1 \left[\langle \Delta_n, \nabla T(z_n)\rangle \cdot a_n\right] e^{\int_0^r a_n(\sigma)\,d\sigma}\,dr$$

tends to 0 in $L^\infty([0,1],\mathbb{R})$. Moreover, we have:

(d) $\dfrac{\dot z_n}{\langle \nabla T(z_n), \dot z_n\rangle}$ is bounded in $L^\infty([0,1],T\mathcal{M})$.

Integrating by parts the terms in (7.4.5) that do *not* contain the covariant derivatives $\nabla_s\zeta$, it follows that we can write (7.4.5) as:

$$\int_0^1 \langle Y_n, \nabla_s\zeta\rangle\,ds = 0, \tag{7.4.6}$$

with

(7.4.7)

$$\begin{aligned} Y_n = {} & \dot z_n + \frac{\lambda_n}{\langle \nabla T(z_n), \dot z_n\rangle}\dot z_n + \langle \nabla T(z_n), \dot z_n\rangle \nabla T(z_n) - \frac{1}{2}\Delta_n + D_n \\ & - \frac{1}{2}\frac{\langle \Delta_n, \nabla T(z_n)\rangle}{\langle \nabla T(z_n), \dot z_n\rangle}\dot z_n + \delta \int_0^s \frac{\nabla\varphi(z_n)}{\varphi(z_n)^3}\,ds - \int_0^s \langle \nabla T(z_n), \dot z_n\rangle H^T(z_n)\,\dot z_n\,ds, \end{aligned}$$

where D_n is defined as the map such that:

$$\begin{aligned} \int_0^1 \langle D_n, \nabla_s\zeta\rangle\,ds = {} & \frac{1}{2}\int_0^1 \langle \Delta_n, H^T(z_n)\,\dot z_n\rangle \mu_n\,ds + \\ & + \frac{1}{2}\int_0^1 \langle \Delta_n, \nabla T(z_n)\rangle \frac{\langle H^T(z_n)\,\dot z_n, \dot z_n\rangle}{\langle \nabla T(z_n), \dot z_n\rangle}\langle \nabla T(z_n), \dot z_n\rangle \mu_n\,ds. \end{aligned}$$

A careful but straightforward check shows that, from the definition of μ_n and the boundedness properties of the functions involved, the sequence D_n is uniformly convergent to 0.

Observe that the last two integrals in (7.4.7) are *covariant integrals*, in the sense that, if V_1 is a vector field along z_n, then $\int_0^s V_1(z_n)\,ds$ denotes the unique vector field V_2 along z_n satisfying $V_2(0) = 0$ and $\nabla_s V_2 = V_1$. An explicit formula for V_2 may be given in terms of the Christoffel symbols of the Lorentzian metric g.

Notice that, by (7.4.6) we have $\nabla_s Y_n = 0$, so, by (7.4.7), Y_n is uniformly bounded, because its mean value is bounded.

Set:

$$H_n = -\delta \int_0^s \frac{\nabla\varphi(z_n)}{\varphi(z_n)^3}\,\mathrm{d}s + \int_0^s \langle \nabla T(z_n), \dot z_n\rangle\, H^T(z_n)\,\dot z_n\,\mathrm{d}s + Y_n,$$

and

$$h_n = \frac{1}{2}\Delta_n + \frac{1}{2}\frac{\langle \Delta_n, \nabla T(z_n)\rangle}{\langle \nabla T(z_n), \dot z_n\rangle}\,\dot z_n + D_n.$$

Then, we can write (7.4.7) shortly as:

$$\text{(7.4.8)}\qquad \dot z_n + \frac{\lambda_n}{\langle \nabla T(z_n), \dot z_n\rangle}\,\dot z_n + \langle \nabla T(z_n), \dot z_n\rangle\,\nabla T(z_n) = H_n + h_n.$$

Moreover, the following properties of H_n and h_n hold:

(1) h_n is convergent to 0 in $L^2([0,1], T\mathcal{M})$,

(2) $H_n(s) = \int_0^s B_n\,\mathrm{d}r$, with B_n bounded in $L^1([0,1], T\mathcal{M})$.

Passing to a subsequence, we can assume that B_n is convergent as a measure, hence H_n is pointwise convergent, with $\|H_n\|_\infty$ bounded. This implies that $\langle h_n, H_n\rangle$ is convergent to 0 in L^2. Moreover, there exists a positive constant K such that

$$|\langle h_n, h_n\rangle| \le K\cdot \|h_n\|^2_{(R)},$$

and so $\langle h_n, h_n\rangle$ converges to 0 in L^1. Moreover, up to passing to a subsequence, we have that h_n converges to 0 pointwise almost everywhere.

Using the same techniques as in the proof of Lemma 4.2, considering the product

$$\langle h_n + H_n, h_n + H_n\rangle$$

(see (7.4.8)) and using the Lebesgue's Dominated Convergence Theorem, from (7.4.8) one proves that $\langle \dot z_n, \nabla T(z_n)\rangle$ is pointwise convergent, and

$$|\langle \dot z_n, \nabla T(z_n)\rangle| \le \alpha_n + \beta_n,$$

with α_n convergent to 0 in L^1 and β_n bounded in L^∞. Passing to the Riemannian metric (1.0.1), this implies that $\dot z_n$ is pointwise convergent almost everywhere, and

$$\|\dot z_n\|_{(R)} \le \hat\alpha_n + \hat\beta_n,$$

with $\hat\alpha_n$ convergent to 0 in L^1, and $\hat\beta_n$ bounded in L^∞. Again, by Lebesgue's Theorem, this implies that $\dot z_n$ is convergent to $\dot z$ in L^2, and we are done. □

Remark 7.5. With the same arguments used in the proof of Proposition 7.4, it is not too difficult to prove the following uniform version of the Palais–Smale condition. Let $\delta > 0$ be fixed, ϵ_n be a positive, infinitesimal sequence, and $z_n \in \mathcal{L}_{p,\gamma,\epsilon_n}$ be such that:

(1) $\lim_{n\to\infty} Q_\delta(z_n) = c$,

(2) $\lim_{n\to\infty} Q'_\delta(z_n) = 0$.

Then, z_n has a subsequence that converges strongly in $H^{1,2}([0,1],\mathcal{M})$ to a curve $z \in \mathcal{L}^+_{p,\gamma}$.

At this juncture, we do not know whether the limit z of the sequence z_n above is a curve of class C^2, nor if it is $\dot z \neq 0$. Thus, we cannot conclude that z is a geodesic from p to γ. For this reason, we need to make an intermediate step and consider the critical points of Q_δ in $\mathcal{L}^+_{p,\gamma,\epsilon}$. In the next session we will discuss some a *priori* estimates for such critical points.

8. A Priori Estimates for the Critical Points of Q_δ

Let's consider the Euler–Lagrange equation for our variational problem, which is given by (4.5.1)

A direct computation of the derivatives in (4.5.1) shows that the differential equation satisfied by the critical points of Q_δ in $\mathcal{L}^+_{p,\gamma,\epsilon}$ is given by:

$$\nabla_s \dot z + \frac{\lambda'}{\langle \nabla T(z), \dot z\rangle}\dot z + \frac{\lambda}{\langle \nabla T(z), \dot z\rangle}\nabla_s \dot z - \lambda\frac{\langle H^T(z)\,\dot z, \dot z\rangle + \langle \nabla T(z), \nabla_s \dot z\rangle}{\langle \nabla T(z), \dot z\rangle^2}\dot z +$$

$$+\langle H^T(z)\,\dot z, \dot z\rangle \nabla T(z) + \langle \nabla T(z), \nabla_s \dot z\rangle \nabla T(z) + \frac{\delta}{\varphi(z)^3}\nabla\varphi(z) = 0. \tag{8.0.1}$$

Multiplying (8.0.1) by $\dot z$, and using the fact that $\langle \dot z, \dot z\rangle \equiv -\epsilon^2$, we obtain:

$$-\epsilon^2\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\lambda}{\langle \nabla T(z), \dot z\rangle}\right) + \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{2}\langle \nabla T(z), \dot z\rangle\right)^2 - \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\delta}{\varphi(z)^2}\right) = 0,$$

from which it follows a *conservation of energy* property for the critical points of Q_δ on $\mathcal{L}^+_{p,\gamma,\epsilon}$:

$$-\frac{\epsilon^2\lambda}{\langle \nabla T(z), \dot z\rangle} + \left(\frac{1}{2}\langle \nabla T(z), \dot z\rangle\right)^2 - \frac{\delta}{\varphi(z)^2} = E_{\epsilon,\delta} = \text{const.} \tag{8.0.2}$$

In this section, we assume that $\{z_{\epsilon,\delta}\}_{\epsilon,\delta>0}$ is a family of critical points for Q_δ in $\mathcal{L}^+_{p,\gamma,\epsilon}$, like for instance a family of minimal points, $C \in I\!\!R$ is a constant satisfying:

$$Q_\delta(z_{\epsilon,\delta}) \le C, \quad \forall\, \epsilon, \delta > 0 \text{ sufficiently small.} \tag{8.0.3}$$

and we will study the limit of $z_{\epsilon,\delta}$ when ϵ and δ tend to 0. Notice that (8.0.3) implies:

$$\int_0^1 \langle \nabla T(z_{\epsilon,\delta}), \dot z_{\epsilon,\delta}\rangle^2\,\mathrm{d}s \le C, \quad \text{and} \quad \int_0^1 \frac{\delta}{\varphi(z_{\epsilon,\delta})^2}\,\mathrm{d}s \le C. \tag{8.0.4}$$

From Proposition 4.4, $z_{\epsilon,\delta}$ is a curve of class C^2 on $\mathcal{M}$.

Remark 8.0. Define $\lambda(s) = \lambda_{\epsilon,\delta}(s)$ as in (7.4.4), with n replaced by ϵ, δ. Since the scalar product $\langle \nabla\varphi(q), \nabla T(q)\rangle$ is null in a neighborhood of $\partial\Lambda$, the quantity

$$\frac{\delta}{\varphi(z_{\epsilon,\delta})^3}\langle \nabla\varphi(z_{\epsilon,\delta}), \nabla T(z_{\epsilon,\delta})\rangle$$

is uniformly bounded on the sublevels of Q_δ, independently of δ and ϵ. This fact and (8.0.3) imply that also $\lambda = \lambda_{\epsilon,\delta}$ is uniformly bounded independently of δ and ϵ. Moreover by Proposition 5.2, it follows that $\lambda \ge 0$.

Lemma 8.1. *Let $z_{\varepsilon,\delta}$ be as in (8.0.3). Then, there exists a positive constant $L = L(C) > 0$, independent of δ and ε, such that:*

$$\left\| \frac{\delta}{\varphi(z_{\varepsilon,\delta})^3} \right\|_\infty \leq L, \qquad \forall\, \varepsilon,\, \delta > 0.$$

Proof. Let $s_0 = s_0(\varepsilon, \delta) \in [0,1]$ be a minimum point for the function $\rho(s) = \varphi(z_{\varepsilon,\delta}(s)) > 0$. We can assume that $s_0 \in (0,1)$, because if that weren't true for all $\delta \in (0, \delta_0]$, then the thesis of the Lemma would be trivially true. Then, it is

$$0 = \rho'(s_0) = \langle \nabla\varphi(z_{\varepsilon,\delta}(s_0)\,,\, \dot z_{\varepsilon,\delta}(s_0) \rangle,$$

and

$$0 \leq \rho''(s_0) = \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s_0} \langle \nabla\varphi(z_{\varepsilon,\delta}(s)\,,\, \dot z_{\varepsilon,\delta}(s) \rangle =$$

$$= \langle H^\varphi(z_{\varepsilon,\delta}(s_0))\, \dot z_{\varepsilon,\delta}(s_0)\,,\, \dot z_{\varepsilon,\delta}(s_0) \rangle + \langle \nabla\varphi(z_{\varepsilon,\delta}(s_0)\,,\, \nabla_s \dot z_{\varepsilon,\delta}(s_0) \rangle. \tag{8.1.1}$$

Obviously, we can assume that $z_{\varepsilon,\delta}(s_0)$ is close enough to $\partial\Lambda$ in such a way that, by (4.0.1), we have:

$$\langle \nabla\varphi(z_{\varepsilon,\delta}(s_0))\,,\, \nabla T(z_{\varepsilon,\delta}(s_0)) \rangle = 0. \tag{8.1.2}$$

Then, multiplying (8.0.1) by $\nabla\varphi(z_{\varepsilon,\delta}(s_0))$, we get:

$$\langle \nabla\varphi(z_{\varepsilon,\delta}(s_0))\,,\, \nabla_s \dot z_{\varepsilon,\delta}(s_0) \rangle + \frac{\lambda}{\langle \nabla T(z_{\varepsilon,\delta}(s_0))\,,\, \dot z_{\varepsilon,\delta}(s_0) \rangle} \langle \nabla_s \dot z_{\varepsilon,\delta}(s_0))\,,\, \nabla\varphi(z_{\varepsilon,\delta}(s_0)) \rangle +$$

$$+ \delta \cdot \frac{\langle \nabla\varphi(z_{\varepsilon,\delta}(s_0))\,,\, \nabla\varphi(z_{\varepsilon,\delta}(s_0)) \rangle}{\varphi(z_{\varepsilon,\delta}(s_0))^3} = 0. \tag{8.1.3}$$

From (8.1.1) and (8.1.3) it follows:

$$0 \leq \langle H^\varphi(z_{\varepsilon,\delta}(s_0))\, \dot z_{\varepsilon,\delta}(s_0)\,,\, \dot z_{\varepsilon,\delta}(s_0) \rangle +$$

$$- \frac{\delta}{\varphi(z_{\varepsilon,\delta}(s_0))^3} \langle \nabla\varphi(z_{\varepsilon,\delta}(s_0))\,,\, \nabla\varphi(z_{\varepsilon,\delta}(s_0)) \rangle \left(1 + \frac{\lambda}{\langle \nabla T(z_{\varepsilon,\delta}(s_0))\,,\, \dot z_{\varepsilon,\delta}(s_0) \rangle}\right)^{-1},$$

hence

$$\frac{\delta}{\varphi(z_{\varepsilon,\delta}(s_0))^3} \langle \nabla\varphi(z_{\varepsilon,\delta}(s_0))\,,\, \nabla\varphi(z_{\varepsilon,\delta}(s_0)) \rangle \leq$$

$$\leq \langle H^\varphi(z_{\varepsilon,\delta}(s_0))\, \dot z_{\varepsilon,\delta}(s_0)\,,\, \dot z_{\varepsilon,\delta}(s_0) \rangle \left(1 + \frac{\lambda}{\langle \nabla T(z_{\varepsilon,\delta}(s_0))\,,\, \dot z_{\varepsilon,\delta}(s_0) \rangle}\right). \tag{8.1.4}$$

If K_1 denotes the supremum of $\|H^\varphi\|$ on $\mathcal{N}(p,\gamma) \cap O_\delta^c$, recalling that:

$$\langle \nabla T(z_{\varepsilon,\delta}(s_0))\,,\, \dot z_{\varepsilon,\delta}(s_0) \rangle = \frac{1}{\sqrt{2}} \sqrt{\varepsilon^2 + \|\dot z_{\varepsilon,\delta}(s_0)\|^2_{(R)}},$$

from (8.1.4), we get:
(8.1.5)

$$\frac{\delta}{\varphi(z_{\epsilon,\delta}(s_0))^3}\langle\nabla\varphi(z_{\epsilon,\delta}(s_0)),\nabla\varphi(z_{\epsilon,\delta}(s_0))\rangle \le$$

$$\le K_1\,\|\dot z_{\epsilon,\delta}(s_0)\|^2_{(R)} + \sqrt{2}\,\lambda\, K_1\,\|\dot z_{\epsilon,\delta}(s_0)\|_{(R)} \le K_2(\|\dot z_{\epsilon,\delta}(s_0)\|^2_{(R)} + 1),$$

where K_2 is a constant independent of δ and ϵ; the last inequality depends on the fact that λ is uniformly bounded independently on δ and ϵ.

By the wrong way Schwartz's inequality $\langle\nabla T(z_{\epsilon,\delta}),\dot z_{\epsilon,\delta}\rangle \ge \epsilon$, recalling that $\lambda_{\epsilon,\delta}$ is uniformly bounded independently of ϵ and δ, we obtain the existence of a constant K_3 such that

$$\left|\epsilon^2\frac{\lambda}{\langle\nabla T(z_{\epsilon,\delta}),\dot z_{\epsilon,\delta}\rangle}\right| \le K_3\cdot\epsilon. \tag{8.1.6}$$

Integrating (8.0.2) gives that $E_{\epsilon,\delta}$ is uniformly bounded. Then, again (8.0.2) gives the existence of a constant K_4 such that

$$\langle\dot z_{\epsilon,\delta}\,,\dot z_{\epsilon,\delta}\rangle_{(R)} \le K_4 + \frac{\delta}{\varphi(z_{\epsilon,\delta})^2}, \tag{8.1.7}$$

and from (8.1.5) we get:

$$\frac{\delta}{\varphi(z_{\epsilon,\delta}(s_0))^3}\langle\nabla\varphi(z_{\epsilon,\delta}(s_0)),\nabla\varphi(z_{\epsilon,\delta}(s_0))\rangle \le K_5 + \frac{\delta}{\varphi(z_{\epsilon,\delta}(s_0))^2},$$

where K_5 is a positive constant independent on ϵ and δ. Since $z_{\epsilon,\delta}$ lies in a compact subset of $\overline{\Lambda}$, there exists a constant $\nu_0 > 0$ such that:

$$\langle\nabla\varphi(z_{\epsilon,\delta}(s_0)),\nabla\varphi(z_{\epsilon,\delta}(s_0))\rangle \ge \nu_0 > 0,$$

hence:

$$\nu_0\frac{\delta}{\varphi(z_{\epsilon,\delta}(s_0))^3} \le \langle\nabla\varphi(z_{\epsilon,\delta}(s_0)),\nabla\varphi(z_{\epsilon,\delta}(s_0))\rangle \le K_5 + \frac{\delta}{\varphi(z_{\epsilon,\delta}(s_0))^2}.$$

Finally, from this we conclude that there exists a positive constant K_6, independent of ϵ and δ, such that:

$$\frac{\delta}{\varphi(z_{\epsilon,\delta}(s_0))^3} \le K_6\left(1+\frac{\delta}{\varphi(z_{\epsilon,\delta}(s_0))^2}\right),$$

from which the proof follows. □

Lemma 8.2. *Let $z_{\epsilon,\delta}$ be as in (8.0.3). Then, the family $\{\dot z_{\epsilon,\delta}\}$ is bounded uniformly with respect to δ in $H^{1,\infty}([0,1],T\mathcal{M})$.*

Proof. As in (4.1.4), we have that the family:

$$\left(1+\frac{\lambda}{\langle\nabla T(z_{\epsilon,\delta}),\dot z_{\epsilon,\delta}\rangle}\right)\dot z_{\epsilon,\delta}+\langle\nabla T(z_{\epsilon,\delta}),\dot z_{\epsilon,\delta}\rangle\,\nabla T(z_{\epsilon,\delta})$$

is bounded in $H^{1,1}([0,1],T\mathcal{M})$. Keeping ϵ bounded away from 0 and arguing as in the proof of Lemma 4.2 one proves that $\langle\nabla T(z_{\epsilon,\delta}),\dot z_{\epsilon,\delta}\rangle$ is bounded in $H^{1,1}([0,1],\mathbb{R})$ uniformly with respect to δ, and from this fact it follows that $\dot z_{\epsilon,\delta}$ is bounded in $H^{1,1}$ uniformly with respect to δ.

Then, $\dot z_{\epsilon,\delta}$ is bounded in $H^{1,\infty}$ with respect to δ and we are done. □

In particular, from Lemma 8.2 it follows that $\dot z_{\epsilon,\delta}$ is bounded in $L^\infty([0,1],T\mathcal{M})$ with respect to δ. Hence, up to taking an infinitesimal sequence δ_n, there exists the strong limit:

$$\lim_{\delta\to 0} z_{\epsilon,\delta}=z_\epsilon \quad \text{in} \quad H^{1,2}([0,1],\mathcal{M}); \tag{8.2.1}$$

which is also a weak limit in $H^{2,2}([0,1],\mathcal{M})$. Moreover, recalling the function Ψ_ϵ defined in Section 2, we have:

$$\Psi_\epsilon(z_\epsilon)=\lim_{\delta\to 0}\Psi_\epsilon(z_{\epsilon,\delta})=\lim_{\delta\to 0}0=0,$$

from which it follows that $z_\epsilon\in\mathcal{L}^+_{p,\gamma,\epsilon}(\overline{\Lambda})$†.

Our next step is to show that z_ϵ satisfies a differential equation similar to (4.5.1). Let δ_n be an infinitesimal sequence such that (8.2.1) holds. From Lemma 8.1, since $\frac{\delta_n}{\varphi(z_{\epsilon,\delta_n})^3}$ is bounded in L^∞, then it is also bounded in L^2, and, up to passing to a subsequence, we can assume that $\frac{\delta_n}{\varphi(z_{\epsilon,\delta_n})^3}$ is weakly convergent in L^2, and we write:

$$\frac{\delta_n}{\varphi(z_{\epsilon,\delta_n})^3}\rightharpoonup \ell_\epsilon \quad \text{in } L^2([0,1],\mathbb{R}^+). \tag{8.2.2}$$

Since z_{ϵ,δ_n} tends to z_ϵ in $H^{1,2}$, taking the limit as $n\to\infty$ in (4.0.3), an integration by parts gives:

$$\nabla_s\dot z_\epsilon+\nabla_s\left(\frac{\lambda\,\dot z_\epsilon}{\langle\nabla T(z_\epsilon),\dot z_\epsilon\rangle}\right)+\nabla_s\big(\langle\nabla T(z_\epsilon),\dot z_\epsilon\rangle\,\nabla T(z_\epsilon)\big)+$$
$$+\ell_\epsilon\cdot\nabla\varphi(z_\epsilon)-\langle\nabla T(z_\epsilon),\dot z_\epsilon\rangle\,H^T(z_\epsilon)\,\dot z_\epsilon=0 \quad \text{a.e.,} \tag{8.2.3}$$

which is a sort of Euler–Lagrange equation satisfied (almost everywhere) by z_ϵ. Notice that

$$\ell_\epsilon(s)=0 \quad \text{if} \quad z_\epsilon(s)\notin\partial\Lambda. \tag{8.2.4}$$

Notice also that

$$z_{\epsilon,\delta_n}\to z_\epsilon \text{ in } H^{1,2} \quad \text{and} \quad Q_{\delta_n}(z_{\epsilon,\delta_n})\to Q(z_\epsilon).$$

We can also prove that z_ϵ satisfies a sort of *conservation of the energy property:*

† i.e., z may touch the boundary $\partial\Lambda$

Lemma 8.3. *For every $\varepsilon > 0$ there exists a constant E_ε such that:*

$$-\varepsilon^2 \frac{\lambda_\varepsilon}{\langle \nabla T(z_\varepsilon), \dot z_\varepsilon \rangle} + \frac{1}{2}\langle \nabla T(z_\varepsilon), \dot z_\varepsilon \rangle^2 = E_\varepsilon. \tag{8.3.1}$$

Moreover, $\{E_\varepsilon\}_{\varepsilon>0}$ is bounded (independently on ε).

Proof. Fix $s \in]0,1[$, if $z_\varepsilon(s) \in \partial\Lambda$, then s is a local minimum for $\varphi(z_\varepsilon(\cdot))$. Then, since z_ε is of class C^1, $\langle \nabla\varphi(z_\varepsilon(s)), \dot z_\varepsilon(s)\rangle$ is zero. Moreover, ℓ_ε is zero outside $\partial\Lambda$, and recalling that $\langle \dot z_\varepsilon, \dot z_\varepsilon\rangle = -\varepsilon$, if we multiply (8.2.3) by $\dot z_\varepsilon$, we get:

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(-\varepsilon^2 \frac{\lambda_\varepsilon}{\langle \nabla T(z_\varepsilon), \dot z_\varepsilon \rangle} + \frac{1}{2}\langle \nabla T(z_\varepsilon), \dot z_\varepsilon \rangle^2\right) = 0 \quad \text{a.e.},$$

which proves the first part of the Lemma. To obtain the boundedness of E_ε, we integrate (8.3.1) on $[0,1]$, and we get:

$$E_\varepsilon = -\varepsilon^2 \int_0^1 \frac{\lambda_\varepsilon}{\langle \nabla T(z_\varepsilon), \dot z_\varepsilon \rangle}\, \mathrm{d}s + \frac{1}{2} Q(z_\varepsilon). \tag{8.3.2}$$

By (8.1.6), the integral in (8.3.2) is bounded, while $Q(z_\varepsilon)$ is bounded by (8.0.3), and the proof is concluded. □

Lemma 8.4. *$\dot z_\varepsilon$ is uniformly bounded away from 0.*

Proof. By contradiction, assume that there exists a sequence ε_n tending to 0 such that $\dot z_{\varepsilon_n}(s_{\varepsilon_n}) \longmapsto 0$. Since as ε tends to 0 the quantity

$$-\varepsilon^2 \frac{\lambda_\varepsilon}{\langle \nabla T(z_\varepsilon)), \dot z_\varepsilon \rangle}$$

converges to 0 uniformly by (8.1.6), then E_{ε_n} must tend to 0, too. From Lemma 8.3 and the definition of $\mathcal{L}^+_{p,\gamma,\varepsilon}$, it follows that $\langle \dot z_{\varepsilon_n}, \dot z_{\varepsilon_n}\rangle_{(n)}$ is bounded. By the Tychonoff's theorem, up to passing to a subsequence, $\dot z_{\varepsilon_n}$ is pointwise convergent. Then, from the Lebesgue theorem, z_{ε_n} is strongly convergent to a curve $z \in \mathcal{L}^+_{p,\gamma}$ in $H^{1,2}$. From (8.3.1), it would then be:

$$0 = \lim_n E_{\varepsilon_n} = \langle \nabla T(z), \dot z \rangle,$$

which is an absurd, because $z \in \mathcal{L}^+_{p,\gamma}$ cannot be constant. □

Lemma 8.5. *Let $z_\varepsilon \in \mathcal{L}^+_{p,\gamma,\varepsilon}$ be such that $Q(z_\varepsilon) \le C$ and z_ε satisfies (8.2.3). Then, there exists $z \in \mathcal{L}^+_{p,\gamma}$ and infinitesimal sequence ε_n such that z_{ε_n} converges to z in $H^{1,2}$ (and $Q(z_{\varepsilon_n}) \to Q(z)$). The curve z is a lightlike, future pointing pregeodesic, parametrized by $\langle \nabla T(z), \dot z \rangle$ constant.*

Proof. Thanks to Lemma 8.4, arguing as in Lemma 8.2, one shows that $\dot z_\varepsilon$ is bounded in $H^{1,\infty}$ (uniformly in ε). Then, there exists z and ε_n as in the thesis, such that z_{ε_n} converges to z in $H^{1,2}$. Also, $\dot z$ is in $H^{1,\infty}$. Taking the limit in

(4.0.3) (see also (8.2.2) and (8.2.4)), we get the existence of $\ell \in L^2([0,1], I\!R^+)$ such that

$$\nabla_s \dot z + \nabla_s \left(\frac{\lambda \dot z}{(\nabla T(z), \dot z)} \right) + \nabla_s ((\nabla T(z), \dot z) \nabla T(z)) + \\ + \ell \cdot \nabla \varphi(z) - (\nabla T(z), \dot z) H^T(z) \dot z = 0 \quad \text{a.e.}$$

An easy contradiction argument shows that the light–convexity of $\partial\Lambda$ implies:

$$\langle H^\varphi(w) \zeta, \zeta \rangle \le 0,$$

for any $w \in \partial\Lambda$ and $\zeta \in T_w \partial\Lambda$, ζ lightlike.

Then, since z is a lightlike curve,

$$\langle H^\varphi(z(s)) \dot z(s), \dot z(s) \rangle \le 0,$$

for every s such that $z(s) \in \partial\Lambda$. Hence, arguing as in the proof of Lemma 8.1 (see also (8.1.1) and (8.1.4)), we see that $\ell(s) \le 0$ for almost every s such that $z(s) \in \partial\Lambda$. But $\ell \ge 0$ almost everywhere, and it is null if $z(s) \notin \partial\Lambda$. Then, $\ell = 0$ almost everywhere, and z is a critical point of class C^2 for Q on $\mathcal{L}^+_{p,\gamma}(\overline{\Lambda})$† in the Gateaux sense. By Lemma 8.4, $\dot z \ne 0$ everywhere, and from [AP, Theorem 1.2] z is a pregeodesic parametrized in such a way that $(\nabla T(z), \dot z)$ constant. Finally, the light-convexity of $\partial\Lambda$ shows that $z([0,1]) \subset \Lambda$, so z is a pregeodesic on $\mathcal{L}^+_{p,\gamma}(\Lambda)$, having $(\nabla T(z), \dot z)$ constant. □

Remark 8.6. To get the existence of minimizers for Q_δ on $\mathcal{L}^+_{p,\gamma,\epsilon}$ we needed to deal with minimizing Palais–Smale sequences. This is due to the fact that $\mathcal{L}^+_{p,\gamma,\epsilon}$ is closed with respect to the strong convergence, but not with respect to the weak $H^{1,2}$–convergence. In any case, a classical argument of critical point theory shows that every C^1-functional defined on a complete C^1-manifold, satisfying the Palais–Smale condition and bounded from below, attains its minimum value.

9. A Shortening Method for the Functional Q. Deformation Lemmas on $\mathcal{L}^+_{p,\gamma}$

The main goal of this section is to prove a series of *deformation Lemmas* for the sublevel of the functional Q on $\mathcal{L}^+_{p,\gamma}$, needed for the Ljusternik–Schnirelman Theory. These results are well known in the case of functionals satisfying the Palais–Smale condition on a regular manifold, but, in our setup, the lack of regularity of the manifold $\mathcal{L}^+_{p,\gamma}$ does not allow the use of the general theory.

For this reason, we will now discuss a different approach to the deformation Lemmas, based on a shortening method for the functional Q on $\mathcal{L}^+_{p,\gamma}$, that resembles the well known method of shortening geodesics in Riemannian manifolds.

For $c \in I\!R^+$, we introduce the following set:

$$\mathcal{N}(p,\gamma,c) = \Big\{ q \in \Lambda; \Big| \; \exists \, z \in \Omega^{1,2}_{p,\gamma} \text{ such that } \langle \dot z, \dot z \rangle \le 0 \text{ a.e.} \\ \langle \dot z, \nabla T(z) \rangle \ge 0 \text{ a.e.}, \quad \text{with } Q(z) \le c, \; q \in \operatorname{supp}(z) \Big\}. \tag{9.0.1}$$

† i.e., z may touch the boundary $\partial\Lambda$

From the c-precompactness of $\mathcal{L}^+_{p,\gamma}$, the set $\mathcal{N}(p,\gamma,c)$ is precompact in $\overline{\Lambda}$ for every $c \in \mathbb{R}^+$.

We fix a time function T on $\mathcal{M}$ satisfying the conditions of the Propositions 5.1, 5.2 and 5.7 (see Remark 5.8). For $q \in \mathcal{M}$, we denote by γ_q the maximal integral line of ∇T through q. We have the following

Lemma 9.1. *If $q \in \Lambda$, then $supp(\gamma_q) \subset \Lambda$.*

Proof. If φ is the function as in (4.0.1), it is:

$$\frac{\mathrm{d}}{\mathrm{d}s}\varphi(\gamma_q(s)) = \langle\nabla\varphi(\gamma_q(s)), \dot{\gamma}_q(s)\rangle = \langle\nabla\varphi(\gamma_q(s)), \nabla T(\gamma_q(s))\rangle. \tag{9.1.1}$$

By (4.0.1), the last term in (9.1.1) is null in a neighborhood of $\partial\Lambda$. Hence, γ_q cannot cross $\partial\Lambda$ and $\mathrm{supp}(\gamma_q) \subset \Lambda$. □

Proposition 9.2. *For every $c \in \mathbb{R}^+$ there exists a positive number $\rho = \rho(c) > 0$ such that for every $z \in Q^c \cap \mathcal{L}^+_{p,\gamma}$ and for every $q_1, q_2 \in$ $supp(z)$ with $\mathrm{d}_R(q_1,q_2) \leq \rho$, there exists a unique lightlike pre-geodesic, future pointing, joining q_1 and γ_{q_2}, and that minimizes Q in $\mathcal{L}^+_{q_1,\gamma_{q_2}}(\Lambda)$.*

Proof. Let $c \in \mathbb{R}^+$ be fixed such that $Q^c \cap \mathcal{L}^+_{p,\gamma}$ is non empty. By Lemma 9.1 and standard results in the theory of ordinary differential equations, it is easily seen that there exists a positive number $\epsilon = \epsilon(c)$ such that, for every $\epsilon \in (0, \epsilon(c)]$, for every $z \in \mathcal{L}^+_{p,\gamma}$ and every $q_1, q_2 \in$ $\mathrm{supp}(z)$, the manifold $\mathcal{L}^+_{q_1,\gamma_{q_2},\epsilon}$ is not empty (see also Proposition 6.1 its proof). Hence, by the c–precompactness and Lemma 5.9, there exists at least one minimizer $z_{\epsilon,\delta}$ for Q_δ on $\mathcal{L}^+_{q_1,\gamma_{q_2},\epsilon}$. Thanks to the results of Section 8, we can pass to the limit as $\epsilon, \delta \to 0$ to obtain that $z_{\epsilon,\delta}$ is convergent in $H^{1,2}$ to a curve $z \in \mathcal{L}^+_{q_1,\gamma_{q_2}}$ which minimizes Q on $\mathcal{L}^+_{q_1,\gamma_{q_2}}(\Lambda)$. Moreover z is of class C^2 and $\dot{z}$ is does not vanish anywhere. Finally, if ρ is the minimum injectivity radius on $\mathcal{N}(p,\gamma,c)$, the exponential map ensures the uniqueness of such a minimizer. Indeed, assume that z_1 and z_2 are minimizers for Q. Since the quantity $\langle\dot{z}, \nabla T(z)\rangle$ is constant on critical points, we have:

$$Q(z_i) = \int_0^1 \langle\dot{z}_i, \nabla T(z_i)\rangle^2\,\mathrm{d}s = \left[T(z_i(1)) - T(q_1)\right]^2, \quad i = 1,2.$$

Then, since $Q(z_1) = Q(z_2)$, it would be $T(z_1(1)) = T(z_2(1))$. But $T(\gamma_{q_2}(s)$ is monotone, hence this is in contradiction with the definition of ρ. □

Lemma 9.3. *Let $\bar{c} = \min_{\mathcal{L}^+_{p,\gamma}(\Lambda)} Q$ and $\rho : [\bar{c}, +\infty) \longmapsto \mathbb{R}^+$ be the function of Proposition 9.2. There exists a continuous function $\nu : [\bar{c}, +\infty) \longmapsto \mathbb{R}^+$ such that, for every pair $a, b \in [0,1]$ with $|b-a| < \nu(c)$, and every $z \in Q^c \cap \mathcal{L}^+_{p,\gamma}(\Lambda)$ it is $\mathrm{d}_R(z(a), z(b)) < \delta(c)$.*

Proof. By Hölder's inequality:

$$\mathrm{d}_{\mathbb{R}}(z(a), z(b)) \leq \int_a^b \sqrt{\langle \dot z(s), \dot z(s)\rangle_{(\mathbb{R})}}\, \mathrm{d}s \leq \sqrt{|b-a|}\left(\int_0^1 \langle \dot z(s), \dot z(s)\rangle_{(\mathbb{R})}\right)^{\frac12} =$$

$$= \sqrt{|b-a|}\left(\int_0^1 (\langle \dot z(s), \dot z(s)\rangle + 2\langle \dot z(s), \nabla T(z(s))\rangle^2)\, \mathrm{d}s\right)^{\frac12} =$$

$$= \sqrt{2\,|b-a|\,Q(z)} \leq \sqrt{2\,c\,|b-a|}.$$

This shows that it is possible to get the continuous function ν. □

We fix once and for all a continuous function ν as in Lemma 9.3, and for every $c \in [\bar c, +\infty)$, we define $N(c) \in \mathbb{N}$ to be the maximum natural number such that:

$$N(c) \leq (\nu(c))^{-1}.$$

We define a first *shortening operator* $S_1 : \mathcal{L}^+_{p,\gamma} \longmapsto \mathcal{L}^+_{p,\gamma}$ inductively, as follows.

For $z \in \mathcal{L}^+_{p,\gamma}$, we set $s_i = i \cdot N(Q(z))^{-1}$, $i = 0, 1, \ldots, N(Q(z))$. Moreover, if $s_{N(Q(z))} < 1$, which happens precisely when $\nu(c)^{-1}$ is not an integer, we also set $s_{N(Q(z))+1} = 1$.

The map S_1 is uniquely determined by the conditions:

(a) the restriction of $S_1(z)$ on the interval $[0, s_1]$ coincides with the unique lightlike, future pointing pre-geodesic joining p and $\gamma_{z(s_1)}$ that minimizes Q. If $s_{N(Q(z))} < 1$, then $S(z)$ is defined analogously on the interval $[s_{N(Q(z))}, 1]$;

(b) for $i \in \{1, \ldots, N(Q(z)) - 1\}$ the restriction of $S_1(z)$ on the interval $[s_i, s_{i+1}]$ gives the unique lightlike, future pointing pre-geodesic joining $S_1(z)(s_i)$ and $\gamma_{z(s_{i+1})}$ that minimizes Q;

(c) $S_1(z)$ is parametrized in such a way that $\langle \nabla T(S_1(z)), \dot S_1(z)\rangle$ is constant almost everywhere on $[0, 1]$.

The map S_1 is well defined thanks to Proposition 9.2.

In a similar fashion, we define the second shortening operator $S_2 : \mathcal{L}^+_{p,\gamma} \longmapsto \mathcal{L}^+_{p,\gamma}$, by setting $t_0 = 0$, $t_i = s_1/2 + (i-1)\cdot N(Q(z))^{-1}$, $i = 1, \ldots, N(Q(z))$, and defining recursively the restriction of $S_2(z)$ on $[t_{i-1}, t_i]$ to be the unique lightlike, future pointing pre-geodesic joining $S_2(z)(t_{i-1})$ and $\gamma_{z(t_i)}$, that minimizes Q. The above modifications in the definition of S_2 need to be done in case $\nu(c)^{-1}$ is not an integer.

Finally, we define the operator S as the composition:

$$S : \mathcal{L}^+_{p,\gamma} \longmapsto \mathcal{L}^+_{p,\gamma} \qquad \text{(9.3.1)}$$
$$S = S_2 \circ S_1.$$

Thanks to Lemma 9.1 and Proposition 9.2, a tedious but straightforward computation shows that S is a continuous map; moreover, by construction, $Q(S(z)) \leq$

$Q(z)$ for every z, where the equality holds only if z is a lightlike, future pointing pre-geodesic joining p and γ, with $\langle \nabla T(z), \dot z \rangle$ constant. The reader should observe that the continuity of S_1 and S_2 on $\mathcal{L}^+_{p,\gamma}$ depend on the modifications made to the their definitions in the case $\nu(c)^{-1}$ is not an integer. It is also important to observe that the shortening operator S can be defined on all the lightlike curves with support in $\mathcal{N}(p,\gamma)$, with endpoint lying on a timelike vertical curve.

Definition 9.4. A curve $z \in \mathcal{L}^+_{p,\gamma}$ is called a *critical point* for Q if z is a pre-geodesic in Λ, with $\langle \dot z(s), \nabla T(z(s)) \rangle$ constant. A real number c is called a *critical value* for Q on $\mathcal{L}^+_{p,\gamma}$ if there exists a critical point z for Q on $\mathcal{L}^+_{p,\gamma}$ such that $Q(z) = c$.

In the rest of this section we will denote by $\mathcal{K}$ the set of all critical points for Q on $\mathcal{L}^+_{p,\gamma}$:

$$\mathcal{K} = \left\{ z \in \mathcal{L}^+_{p,\gamma} \;\middle|\; z \text{ is a critical point for } Q \text{ on } \mathcal{L}^+_{p,\gamma} \right\}. \tag{9.4.1}$$

A real number c is called a *critical value* for Q on $\mathcal{L}^+_{p,\gamma}$ if there exists $z \in \mathcal{K}$ such that $Q(z) = c$. A real number $c \in \mathbb{R}^+$ which is not a critical value is called *regular value* for Q on $\mathcal{L}^+_{p,\gamma}$.

Before proving the deformation Lemmas, we need to show that the set of critical points of Q at any (fixed) sublevel is compact in $\mathcal{L}^+_{p,\gamma}$:

Lemma 9.5. *For every $c \in \mathbb{R}^+$, the set $\mathcal{K} \cap Q^c \cap \mathcal{L}^+_{p,\gamma}$ is compact.*

Proof. The geodesics are regular points of $\mathcal{L}^+_{p,\gamma}$, since they have non zero derivative, and Q is differentiable at those points. If z_n is a sequence of geodesics in $\mathcal{L}^+_{p,\gamma}$, with $Q(z_n) \le c$, then, since $Q'(z_n) = 0$, one can repeat *verbatim* the same proof as in Proposition 7.4, setting $\Theta_n = 0$ in (7.4.3), to conclude that z_n has a converging subsequence. Indeed, from the light-convexity asumption, it follows also that a sequence of critical points of Q on $\mathcal{L}^+_{p,\gamma}$ can not approach the boundary. □

Lemma 9.6. *Let $c \in \mathbb{R}$ be a regular value for Q on $\mathcal{L}^+_{p,\gamma}$. Then, there exists $\sigma > 0$ and a homotopy $\eta \in C^0([0,1] \times Q^{c+1} \cap \mathcal{L}^+_{p,\gamma}, Q^{c+1} \cap \mathcal{L}^+_{p,\gamma})$ such that:*

(1) $\eta(0,z) = z$ *for every* $z \in Q^{c+1} \cap \mathcal{L}^+_{p,\gamma}$,
(2) $\eta(1, Q^{c+\sigma} \cap \mathcal{L}^+_{p,\gamma}) \subset Q^{c-\sigma} \cap \mathcal{L}^+_{p,\gamma}$.

Proof. For $n > 0$, set $S^n = S \circ S \circ \cdots \circ S$ (n times). Since $\mathcal{L}^+_{p,\gamma}$ is c-precompact, the classical method of shortening geodesics (see [Mi]) shows that there exists $n = n(c) \in \mathbb{N}$ and $\sigma = \sigma(c) > 0$ such that:

$$Q(S^n(z)) \le c - \sigma, \quad \forall z \in Q^{c+\sigma} \cap \mathcal{L}^+_{p,\gamma}. \tag{9.6.1}$$

The homotopy searched is then obtained by setting:

$$\eta(t,z) = S^n\left(z|_{[1-t,1]}\right),$$

where $z|_{[1-t,1]}$ is the restriction of z on the interval $[1-t, 1]$. The continuity of η is easily established using local coordinates. Part (2) of the thesis follows from (9.6.1). □

In an absolutely similar fashion, we can prove the following:

Lemma 9.7. *Let c be a critical value for Q on $\mathcal{L}^+_{p,\gamma}$, and U be an open neighborhood of $\mathcal{K} \cap Q^{-1}(c) \cap \mathcal{L}^+_{p,\gamma}$. Then there exists $\sigma > 0$ and a homotopy $\eta \in C^0([0,1] \times Q^{c+1} \cap \mathcal{L}^+_{p,\gamma}, Q^{c+1} \cap \mathcal{L}^+_{p,\gamma})$ such that:*

(1) $\eta(0,z) = z$ *for every* $z \in Q^{c+1} \cap \mathcal{L}^+_{p,\gamma}$,

(2) $\eta(1, Q^{c+\sigma} \cap \mathcal{L}^+_{p,\gamma} \setminus U) \subset Q^{c-\sigma} \cap \mathcal{L}^+_{p,\gamma}$. □

In the last deformation Lemma we show that, if $\mathcal{K}$ is bounded, then the whole space $\mathcal{L}^+_{p,\gamma}$ can be continuously retracted to a sublevel of Q:

Lemma 9.8. *Let c be such that $\mathcal{K} \cap \{z \in \mathcal{L}^+_{p,\gamma} \mid Q(z) \geq c\} = \emptyset$. Then, there exists a homotopy $\eta \in C^0([0,1] \times \mathcal{L}^+_{p,\gamma}, \mathcal{L}^+_{p,\gamma})$ such that:*

(1) $\eta(0,z) = z$ *for every* $z \in \mathcal{L}^+_{p,\gamma}$,

(2) $\eta(1, \mathcal{L}^+_{p,\gamma}) \subset Q^{c+1} \cap \mathcal{L}^+_{p,\gamma}$.

Proof. Let $\tilde{\eta}(t,z) : [0,+\infty) \times \mathcal{L}^+_{p,\gamma} \longmapsto \mathcal{L}^+_{p,\gamma}$ be any continuous flow that *interpolates* the discrete flow $(n,z) \longmapsto S^n(z)$, in the sense that $\tilde{\eta}$ is a continuous map that satisfies:

(1) for every $z \in \mathcal{L}^+_{p,\gamma}$, the function $t \longmapsto Q(\tilde{\eta}(t,z))$ is non increasing,

(2) $\tilde{\eta}(n,z) = S^n(z)$, for every $n \in \mathbb{N}$ and every $z \in \mathcal{L}^+_{p,\gamma}$.

Such an interpolating map can be easily constructed using the same techniques of the geodesic shortening method of [Mi]. By Lemmas 9.3 and 9.6, for every $z \in \mathcal{L}^+_{p,\gamma}$ there exists a positive number $\bar{t}(z)$ such that $Q(\tilde{\eta}(t,z)) \leq c+1$ for every $t \geq \bar{t}(z)$. Using a partition of unity on $\mathcal{L}^+_{p,\gamma}$, the function $\bar{t} : \mathcal{L}^+_{p,\gamma} \longmapsto [0,+\infty)$ can be chosen continuous. The homotopy wanted is then given by $\eta(t,z) = \tilde{\eta}(t \cdot \bar{t}(z), z)$. □

10. Ljusternik–Schnirelman Theory and Multiplicity of Light Rays

In this section we will make use of the Deformation Lemmas proven in Section 5 to build a Ljusternik–Schnirelman theory for light rays joining p and γ. Let $\mathcal{K}$ be as in (9.4.1).

The first result in an easy consequence of Lemma 9.8:

Lemma 10.1. *Let $c \in \mathbb{R}^+$ be such that $\mathcal{K} \cap \{z \in \mathcal{L}^+_{p,\gamma} \mid Q(z) \geq c\} = \emptyset$. Then, it is:*

$$\underset{\mathcal{L}^+_{p,\gamma}}{\mathrm{cat}}\,(\mathcal{L}^+_{p,\gamma}) = \underset{\mathcal{L}^+_{p,\gamma}}{\mathrm{cat}}\,(Q^{c+1} \cap \mathcal{L}^+_{p,\gamma}).$$

Proof. The Ljusternik–Schnirelman category is invariant by homotopies and monotone by inclusion. Hence, from Lemma 9.8, it follows:

$$\operatorname*{cat}_{\mathcal{L}^+_{p,\gamma}}(\mathcal{L}^+_{p,\gamma}) \geq \operatorname*{cat}_{\mathcal{L}^+_{p,\gamma}}(Q^{c+1}\cap\mathcal{L}^+_{p,\gamma}) \geq \operatorname*{cat}_{\mathcal{L}^+_{p,\gamma}}(\eta(1,\mathcal{L}^+_{p,\gamma})) = \operatorname*{cat}_{\mathcal{L}^+_{p,\gamma}}(\mathcal{L}^+_{p,\gamma}). \quad \square$$

We prove now that all the sublevels of Q have finite category:

Proposition 10.2. *For every $c \in \mathbb{R}^+$, it is*

$$\operatorname*{cat}_{\mathcal{L}^+_{p,\gamma}}(Q^c\cap\mathcal{L}^+_{p,\gamma}) < +\infty.$$

Proof. By contradiction, suppose that there exists a number $c \in \mathbb{R}$ such that

$$\operatorname*{cat}_{\mathcal{L}_{p,\gamma}}(Q^c\cap\mathcal{L}^+_{p,\gamma}) = +\infty,$$

and let $\bar{c} \geq 0$ be defined by:

$$\bar{c} = \inf\{c \in \mathbb{R} \mid \operatorname*{cat}_{\mathcal{L}^+_{p,\gamma}}(Q^c\cap\mathcal{L}^+_{p,\gamma}) = +\infty\}.$$

It is shown in Lemma 9.5 that $\mathcal{K}\cap Q^{\bar{c}}\cap\mathcal{L}^+_{p,\gamma}$ is compact, so that it can be covered by a finite number of open contractible balls B_i, $i = 1,\dots,k$. Let $U = \bigcup_{i=1}^{k} B_i$; Lemma 9.7 implies that there exists $\sigma > 0$ such that

$$\operatorname*{cat}_{\mathcal{L}^+_{p,\gamma}}((Q^{\bar{c}+\sigma}\cap\mathcal{L}^+_{p,\gamma})\setminus U) \leq \operatorname*{cat}_{\mathcal{L}^+_{p,\gamma}}(Q^{\bar{c}-\sigma}\cap\mathcal{L}^+_{p,\gamma}). \tag{10.2.1}$$

But by definition of $\bar{c}$, it is

$$\begin{aligned}\operatorname*{cat}_{\mathcal{L}^+_{p,\gamma}}((Q^{\bar{c}+\sigma}\cap\mathcal{L}^+_{p,\gamma})\setminus U) &\geq \operatorname*{cat}_{\mathcal{L}^+_{p,\gamma}}(Q^{\bar{c}+\sigma}\cap\mathcal{L}^+_{p,\gamma}) - \operatorname*{cat}_{\mathcal{L}^+_{p,\gamma}}(U) \geq \\ &\geq \operatorname*{cat}_{\mathcal{L}^+_{p,\gamma}}(Q^{\bar{c}+\sigma}\cap\mathcal{L}^+_{p,\gamma}) - k = +\infty.\end{aligned}$$

But, by the definition of $\bar{c}$,

$$\operatorname*{cat}_{\mathcal{L}^+_{p,\gamma}}(Q^{\bar{c}-\sigma}\cap\mathcal{L}^+_{p,\gamma}) < +\infty,$$

which contradicts (10.2.1) and proves the Proposition. $\square$

In the next Proposition we prove a non smooth version of a classical *minimax* argument of the Ljusternik–Schnirelman theory:

Proposition 10.3. *There exist at least* $\operatorname{cat}_{\mathcal{L}^+_{p,\gamma}}(\mathcal{L}^+_{p,\gamma})$ *critical points of* Q *on* $\mathcal{L}^+_{p,\gamma}$.

Proof. If Q has infinitely many critical points on $\mathcal{L}^+_{p,\gamma}$, then the proof is done. Assume that $\mathcal{K}$ is finite and define

$$\bar{c} = \max_{z \in \mathcal{K}} Q(z).$$

By Lemma 10.1, there exists $d > \bar{c}$ such that

$$\operatorname{cat}_{\mathcal{L}^+_{p,\gamma}}(\mathcal{L}^+_{p,\gamma}) = \operatorname{cat}_{\mathcal{L}^+_{p,\gamma}}(Q^d \cap \mathcal{L}^+_{p,\gamma}) = m \in \mathbb{N} \setminus \{0\}.$$

For every $k \in \{1, 2, \dots, m\}$, let

$$\Gamma_k = \{B \subset (Q^d_\delta \cap \mathcal{L}^+_{p,\gamma}) \mid \operatorname{cat}_{\mathcal{L}^+_{p,\gamma}}(B) \geq k\},$$

and define:

$$c_k = \inf_{B \in \Gamma_k} \sup_{z \in B} Q(z).$$

Clearly, the c_k's are well defined, since $c_k \in [0, d]$ and The c_k's are critical values for Q on $\mathcal{L}^+_{p,\gamma}$. To prove this, assume by contradiction that, for some k, c_k is not a critical value for Q on $\mathcal{L}^+_{p,\gamma}$. Then by Lemma 9.6, there would exist $\sigma > 0$ and a homotopy $\eta : [0,1] \times \mathcal{L}^+_{p,\gamma} \longmapsto \mathcal{L}^+_{p,\gamma}$, such that

$$\eta(0, Q^{c_k+\sigma} \cap \mathcal{L}^+_{p,\gamma}) = Q^{c_k+\sigma} \cap \mathcal{L}^+_{p,\gamma} \tag{10.3.1}$$

and

$$\eta(1, Q^{c_k+\sigma} \cap \mathcal{L}^+_{p,\gamma}) \subset Q^{c_k-\sigma} \cap \mathcal{L}^+_{p,\gamma}. \tag{10.3.2}$$

Moreover, by definition of c_k, there would exist a $B \in \Gamma_k$ such that

$$\sup_{z \in B} Q(z) \leq c_k + \sigma,$$

so that $B \subseteq Q^{c_k+\sigma} \cap \mathcal{L}^+_{p,\gamma}$. Denoting by B' the set $\eta(1, B)$, from (10.3.1) and (10.3.2) it follows that $\operatorname{cat}_{\mathcal{L}^+_{p,\gamma}}(B') = \operatorname{cat}_{\mathcal{L}^+_{p,\gamma}}(B) \geq k$, so that $B' \in \Gamma_k$. Moreover, since $B' \subseteq Q^{c_k-\sigma} \cap \mathcal{L}^+_{p,\gamma}$, one has

$$\sup_{z \in B'} Q(z) \leq c_k - \sigma,$$

which contradicts the minimality of c_k.

If the c_k's are distinct, then we have at least m critical points of Q on $\mathcal{L}^+_{p,\gamma}$, and we are done. If for some $k \in \{1, \dots, m-1\}$ it is $c_k = c_{k+1}$, then Lemma 9.6 and a classical argument in critical point theory (see e.g. [MW]) show that there are infinitely many critical points at the level c_k, and the Proposition is proven. □

We are ready to prove the multiplicity results of Theorem 1.4 and Theorem 1.5:

Proof of Theorem 1.4. From [AP, Theorem 1.2], critical points of Q on $\mathcal{L}^+_{p,\gamma}$ correspond to light-like geodesics, up to a reparametrization. The proof is finished with the observation that this correspondence is one-to-one, since the reparametrizations needed for passing from a critical point of Q to a geodesic and vice versa are uniquely determined. □

Proof of Theorem 1.5. If $\operatorname{cat}_{\mathcal{L}^+_{p,\gamma}}(\mathcal{L}^+_{p,\gamma}) = +\infty$, then Q has arbitrarily large critical values on $\mathcal{L}^+_{p,\gamma}$. Indeed, if Q didn't have critical values in the half line $[d, +\infty)$, then, by Lemma 10.1 and Proposition 10.2, it would be $\operatorname{cat}_{\mathcal{L}^+_{p,\gamma}}(\mathcal{L}^+_{p,\gamma}) =$

$\operatorname{cat}_{\mathcal{L}^+_{p,\gamma}}(\mathcal{L}^+_{p,\gamma} \cap Q^{d+1}) < +\infty$, which is a contradiction.

It follows that there exists a sequence $\lambda_n \geq 0$ of critical values of Q on $\mathcal{L}^+_{p,\gamma}$ such that $\lim_{n\to\infty} \lambda_n = +\infty$, and a sequence $\tilde{z}_n \in \mathcal{L}^+_{p,\gamma}$ of critical points of Q, with $Q(\tilde{z}_n) = \lambda_n$. From [AP, Theorem 1.2], for every $n \in \mathbb{N}$ there exists a reparametrization z_n of $\tilde{z}_n$ which is a light-like geodesic in $\mathcal{M}$.

Since $T(z(1)) - T(p)$ is invariant by reparametrization and $\langle \dot{z}_n, \nabla T(z_n) \rangle$ is constant, it follows that $T(z_n(1)) - T(p) = \lambda_n$. Therefore $T(z_n(1)) \to +\infty$ and since $T(\gamma(s))$ is strictly increasing, the same happens for $\tau_{p,\gamma}(z_n)$. □

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