UNIVERSIDADE DE SÃO PAULO

Instituto de Ciências Matemáticas e de Computação

ISSN 0103-2577

STABILITY OF FUNCTIONAL DIFFERENTIAL EQUATIONS WITH VARIABLE IMPULSIVE PERTURBATIONS VIA GENERALIZED ORDINARY DIFFERENTIAL EQUATIONS

S. AFONSO
E. BONOTTO
M. FEDERSON
L. GIMENES

Nº 337

NOTAS DO ICMC

SÉRIE MATEMÁTICA

São Carlos – SP
Ago./2010

Resumo

Neste trabalho, consideraremos uma classe de equações diferenciais funcionais com retardamento e impulsos em tempo variável que pode ser identificada, de maneira biunívoca, com uma certa classe de equações diferenciais ordinárias generalizadas e estabeleceremos resultados de estabilidade uniforme e estabilidade uniforme assintótica das soluções dessas equações através da teoria das equações diferenciais ordinárias generalizadas, usando também funcionais de Lyapunov.

STABILITY OF FUNCTIONAL DIFFERENTIAL EQUATIONS WITH VARIABLE IMPULSIVE PERTURBATIONS VIA GENERALIZED ORDINARY DIFFERENTIAL EQUATIONS

S. AFONSO, E. BONOTTO, M. FEDERSON, AND L. GIMENES

ABSTRACT. We consider a class of functional differential equations with variable impulses and we establish new stability results which encompass those from [5] and other papers. We discuss the variational stability and variational asymptotic stability of the zero solution of a class of generalized ordinary differential equations where our impulsive functional differential equations can be embedded and we apply that theory to obtain our results, also using Lyapunov functionals.

1. INTRODUCTION

Let X be a Banach space and $I \subset \mathbb{R}$ be any interval of the real line. We denote by $G^{-}(I, X)$ the space of left continuous regulated functions $f : I \to X$, that is, $G^{-}(I, X)$ is the set of all functions $f : I \to X$ such that, for every compact interval $[a, b] \subset I$, $f(t-) = f(t)$ for each $t \in (a, b]$ and the right limit $f(t+)$ exists for each $t \in [a, b)$, where

$$f(t-) = \lim_{\rho \to 0^-} f(t + \rho) \quad \text{and} \quad f(t+) = \lim_{\rho \to 0^+} f(t + \rho).$$

The space $G^{-}(I, X)$ is a Banach space when endowed with the usual supremum norm.

We write $C(I, X)$ to denote the space of continuous functions $f : I \to X$. When I is a compact interval, we consider the Banach space $C(I, X)$ equipped with the norm induced by $G^{-}(I, X)$.

Consider $\mathbb{R}_+ = \{z \in \mathbb{R} : z \geq 0\}$. We say that $b : \mathbb{R}_+ \to \mathbb{R}_+$ is a function of *Hahn class*, if b is monotone increasing and $b(0) = 0$.

Let $r > 0$. Given a function $y : \mathbb{R} \to \mathbb{R}^n$, we consider $y_t \in G^{-}([-r, 0], \mathbb{R}^n)$ defined by

$$y_t(\theta) = y(t + \theta), \quad \theta \in [-r, 0],\ t \in \mathbb{R}.$$

Then for $t_0 \geq 0$ and a function $y \in G^{-}([t_0 - r, +\infty), \mathbb{R}^n)$, we have $y_t \in G^{-}([-r, 0], \mathbb{R}^n)$ for all $t \in [t_0, +\infty)$.

We consider the following retarded functional differential equation with variable moments of impulse action

$$\begin{cases} \dot{y}(t) = f(y_t, t), & t \neq \tau_k(y(t)), \\ \Delta y(t) = I_k(y(t)), & t = \tau_k(y(t)), \quad k = 1, 2, \ldots, \end{cases} \tag{1}$$

where $f(\varphi, t)$ maps $G^{-}([-r,0],\mathbb{R}^n) \times \mathbb{R}$ to $\mathbb{R}^n$, and for $k = 1, 2, \ldots$, I_k maps $\mathbb{R}^n$ to itself, τ_k maps $\mathbb{R}^n$ to $\mathbb{R}$, and $\Delta y\,(t) := y\,(t+) - y\,(t-) = y\,(t+) - y\,(t)$, for any $t \in \mathbb{R}$.

Given $t_0 \geq 0$ and an inicial function $\phi \in G^{-}([-r,0],\mathbb{R}^n)$, the initial value problem corresponding to equation (1) has the form

$$\begin{cases} \dot{y}\,(t) = f\,(y_t, t)\,, & t \neq \tau_k(y(t)), \quad t \geq t_0, \\ \Delta y\,(t) = I_k\,(y\,(t))\,, & t = \tau_k(y(t)), \quad k = 1, 2, \ldots, \\ y_{t_0} = \phi. \end{cases} \tag{2}$$

Assume $\tau_0(x) \equiv t_0$, for all $x \in \mathbb{R}^n$. For each $k = 1, 2, \ldots$, consider the set

$$S_k = \{(t, x) \in [t_0, +\infty) \times \mathbb{R}^n : t = \tau_k(x)\}.$$

By $m(\tau_k)$ we denote the number of times at which the integral curves of system (2) meet the hypersurface S_k, $k = 1, 2, \ldots$. By t_k^i we denote the i^{th} moment of time at which the integral curves of system (2) meet the hypersurface S_k, with $i = 1, \ldots, m(\tau_k)$ and $k = 1, 2, \ldots$.

Throughout this paper, we shall consider the following conditions:

(C1) $\tau_k \in C(\mathbb{R}^n, (t_0, +\infty))$, $k = 1, 2, \ldots$;
(C2) $t_0 < \tau_1(x) < \tau_2(x) < \ldots$, for each $x \in \mathbb{R}^n$;
(C3) $\tau_k(x) \longrightarrow +\infty$ as $k \longrightarrow +\infty$ uniformly on $x \in \mathbb{R}^n$;
(C4) The integral curves of system (2) meet successively each hypersurface $S_1, S_2, \ldots$ only a finite number of times;
(C5) $t_k^i < t_k^{i+1}$, $i = 1, \ldots, m(\tau_k) - 1$, for all $k = 1, 2, \ldots$.

It is clear that system (2) is equivalent to the "integral" equation

$$\begin{cases} y\,(t) = y(t_0) + \displaystyle\int_{t_0}^{t} f\,(y_s, s)\,ds + \sum_{\substack{t_0 < t_k^i < t, \\ i = 1, \ldots, m(\tau_k)}} I_k(y(t_k^i)) \\ y_{t_0} = \phi, \end{cases}$$

when the integral exists in some sense.

Let $PC_1 \subset G^{-}([t_0 - r, +\infty), \mathbb{R}^n)$ be an open set (in the topology of locally uniform convergence in $G^{-}([t_0 - r, +\infty), \mathbb{R}^n)$) with the following property: if y is an element of PC_1 and $\bar{t} \in [t_0, +\infty)$, then $\bar{y}$ given by

$$\bar{y}\,(t) = \begin{cases} y\,(t)\,, & t_0 - r \leq t \leq \bar{t}, \\ y\,(\bar{t})\,, & \bar{t} < t < +\infty, \end{cases}$$

is also an element of PC_1. In particular, any open ball in $G^{-}([t_0 - r, +\infty), \mathbb{R}^n)$ has this property.

We assume that $f : G^{-}([-r,0],\mathbb{R}^n) \times [t_0, +\infty) \longrightarrow \mathbb{R}^n$ is such that for every $y \in G^{-}([t_0 - r, +\infty), \mathbb{R}^n)$, the mapping $t \longmapsto f\,(y_t, t)$ is locally Lebesgue integrable on $t \in [t_0, +\infty)$. Moreover, we assume:

(A) There is a locally Lebesgue integrable function $M : [t_0, +\infty) \to \mathbb{R}$ such that for all $x \in PC_1$ and all $u_1, u_2 \in [t_0, +\infty)$,

$$\left| \int_{u_1}^{u_2} f(x_s, s)\, ds \right| \leq \int_{u_1}^{u_2} M(s)\, ds;$$

(B) There is a locally Lebesgue integrable function $L : [t_0, +\infty) \to \mathbb{R}$ such that for all $x, y \in PC_1$ and all $u_1, u_2 \in [t_0, +\infty)$,

$$\left| \int_{u_1}^{u_2} [f(x_s, s) - f(y_s, s)]\, ds \right| \leq \int_{u_1}^{u_2} L(s) \|x_s - y_s\|\, ds.$$

For the impulse operators $I_k : \mathbb{R}^n \to \mathbb{R}^n$, $k = 1, 2, \ldots$, we assume:

(A') There is a constant $K_1 > 0$ such that for all $k = 1, 2, \ldots$ and all $x \in \mathbb{R}^n$,

$$|I_k(x)| \leq K_1;$$

(B') There is a constant $K_2 > 0$ such that for all $k = 1, 2, \ldots$ and all $x, y \in \mathbb{R}^n$,

$$|I_k(x) - I_k(y)| \leq K_2 |x - y|.$$

It is possible to prove that, under the conditions above, system (2) is equivalent to a system of generalized ordinary differential equations which takes values in an abstract space. A proof of this fact follows the ideas of Theorems 3.4 and 3.5 from [4]. Local existence and uniqueness of solutions are guaranteed by [3], Theorem 2.1.

In the present paper, we consider system (1) and assume $f(0, t) \equiv 0$ and $I_k(0) = 0$, for $k = 1, 2, \ldots$, so that $y \equiv 0$ is a solution of (1). We also assume that conditions (A), (B), (A') and (B') are fulfilled.

The results we obtain here generalize many others. For example, our Theorem 4.4 improves Theorem 1 from [16]. In the absence of impulses, Theorem 4.4 generalizes [9], Theorem 5.4.1, for instance. Theorem 4.5 in the sequel improves [16], Theorem 3, [17], Theorem 1, and also [13], Theorem 3.2. In the absence of impulses, Theorem 4.5 generalizes [9], Theorem 5.4.2, for instance.

In literature, the usual requirement for f is that $f(\psi, t)$ is continuous in ψ. In the present paper, we require that the indefinite integral of f satisfies Carathéodory- and Lipschitz-type conditions given by conditions (A) and (B). Also the mapping $t \mapsto f(y_t, t)$ does not need to be piecewise continuous (see [13], [16] and [17] for instance). We require local Lebesgue integrability instead.

We start by describing our setting of retarded equations subject to impulse effects at variable times. Then we consider the corresponding class of generalized ordinary differential equations and we review the stability theory for such a class of generalized equations. By means of Lyapunov functionals and Lypaunov functions satisfying weak Krasovskiĭ-type conditions and because impulsive retarded differential equations can be regarded as generalized ordinary differential equations, we are able to obtain our main results.

2. Generalized ordinary differential equations

Let X be a Banach space and consider the set $\Omega = O \times [t_0, +\infty)$, where $O \subset X$ is an open set. Assume that $G : \Omega \to X$ is a given X-valued function defined for all $(x,t) \in \Omega$.

Having the concept of Kurzweil integrability in mind (see, e.g., [12], [18] or the Appendix), we recall the concept of generalized ordinary differential equation (see [4] or [18]).

Definition 2.1. *A function $x : [\alpha, \beta] \to X$ is called* a solution of the generalized ordinary differential equation

$$\frac{dx}{d\tau} = DG(x,t) \tag{3}$$

in the interval $[\alpha,\beta] \subset [t_0, +\infty)$ if $(x(t), t) \in \Omega$ for all $t \in [\alpha, \beta]$ and if the equality

$$x(v) - x(\gamma) = \int_{\gamma}^{v} DG(x(\tau), t) \tag{4}$$

holds for every γ, $v \in [\alpha, \beta]$.

Given an initial condition $(\widetilde{x}, t_0) \in \Omega$, a solution of the initial value problem for equation (3) is given as follows.

Definition 2.2. *A function $x : [\alpha, \beta] \to X$ is a* solution of the generalized ordinary differential equation (3) with the initial condition $x(t_0) = \widetilde{x}$, in the interval $[\alpha, \beta] \subset [t_0, +\infty)$, *if $t_0 \in [\alpha, \beta]$, $(x(t), t) \in \Omega$ for all $t \in [\alpha, \beta]$ and if the equality*

$$x(v) - \widetilde{x} = \int_{t_0}^{v} DG(x(\tau), t) \tag{5}$$

holds for every $v \in [\alpha, \beta]$.

Now we define a special class of functions $G : \Omega \to X$ for which we can derive interesting properties of the solutions of (3).

Definition 2.3. *A function $G : \Omega \to X$ belongs to the class $\mathcal{F}(\Omega, h)$, if there exists a nondecreasing function $h : [t_0, +\infty) \to \mathbb{R}$ such that*

$$\|G(x, s_2) - G(x, s_1)\| \leq |h(s_2) - h(s_1)| \tag{6}$$

for all (x, s_2), $(x, s_1) \in \Omega$ and

$$\|G(x, s_2) - G(x, s_1) - G(y, s_2) + G(y, s_1)\| \leq \|x - y\| |h(s_2) - h(s_1)| \tag{7}$$

for all (x, s_2), (x, s_1), (y, s_2), $(y, s_1) \in \Omega$.

Assume that $G : \Omega \to X$ satisfies condition (6) and let $\mathrm{var}_{\alpha}^{\beta}(x)$ denote the variation of a function $x : [\alpha, \beta] \to X$. If $[\alpha, \beta] \subset [t_0, +\infty)$ and $x : [\alpha, \beta] \to X$ is a solution of (3), then

$$\|x(s_1) - x(s_2)\| \leq |h(s_2) - h(s_1)| \tag{8}$$

for all $s_1, s_2 \in [\alpha, \beta]$, and hence x is of bounded variation on $[\alpha, \beta]$ with

$$\mathrm{var}_{\alpha}^{\beta} x \leq h(\beta) - h(\alpha) < +\infty. \tag{9}$$

Furthermore, every point in $[\alpha, \beta]$ at which the function h is continuous is a continuity point of the solution $x : [\alpha, \beta] \to X$ and we have

$$x(\sigma+) - x(\sigma) = G(x(\sigma), \sigma+) - G(x(\sigma), \sigma), \quad \text{for } \sigma \in [\alpha, \beta)$$

and

$$x(\sigma) - x(\sigma-) = G(x(\sigma), \sigma) - G(x(\sigma), \sigma-), \quad \text{for } \sigma \in (\alpha, \beta],$$

where

$$G(x, \sigma+) = \lim_{s \to \sigma+} G(x, s), \quad \text{for } \sigma \in [\alpha, \beta)$$

and

$$G(x, \sigma-) = \lim_{s \to \sigma-} G(x, s), \quad \text{for } \sigma \in (\alpha, \beta].$$

For proofs of the above statements, see [18], Lemmas 3.10 and 3.12.

Now we present a result on the existence of the integral involved in the definition of the solution of the generalized ordinary differential equation (3) (see Definition 2.1). This result is a particular case of Corollary 3.16 from [18].

Lemma 2.1. *Let $G \in \mathcal{F}(\Omega, h)$. Suppose $[\alpha, \beta] \subset [t_0, +\infty)$, $x : [\alpha, \beta] \to X$ is of bounded variation on $[\alpha, \beta]$ and $(x(s), s) \in \Omega$ for every $s \in [\alpha, \beta]$. Then the integral $\int_\alpha^\beta DG(x(\tau), t)$ exists and the function $s \mapsto \int_\alpha^s DG(x(\tau), t) \in X$, $s \in [\alpha, \beta]$, is of bounded variation.*

The next result we mention for generalized ordinary differential equations (we write generalized ODEs, for short) with righthand sides in $\mathcal{F}(\Omega, h)$ concerns the existence of a local solution. For a proof of this fact, see [4], Theorem 2.15.

Theorem 2.1 (Local existence and uniqueness). *Let $G : \Omega \to X$ belong to the class $\mathcal{F}(\Omega, h)$, where the function h is nondecreasing and left continuous. If for every $(\widetilde{x}, t_0) \in \Omega$ such that for $\widetilde{x}_+ = \widetilde{x} + G(\widetilde{x}, t_0+) - G(\widetilde{x}, t_0)$ we have $(\widetilde{x}_+, t_0) \in \Omega$, then there exists $\Delta > 0$ such that there exists a unique solution $x : [t_0, t_0 + \Delta] \to X$ of generalized ordinary differential equation (3) for which $x(t_0) = \widetilde{x}$.*

3. Some concepts of stability for GODE's

In this section, $(X, \|\cdot\|)$ is a Banach space and we set $\Omega = B_c \times [t_0, +\infty)$, where $B_c = \{y \in X : \|y\| < c\}$, with $c > 0$ and $t_0 \geq 0$. We also assume that $G \in \mathcal{F}(\Omega, h)$, where $h : [t_0, +\infty) \to \mathbb{R}$ is a left continuous nondecreasing function, and $G(0, t) - G(0, s) = 0$, for $t, s \geq t_0$. Then for every $[\gamma, v] \subset [t_0, +\infty)$, we have

$$\int_\gamma^v DG(0, t) = G(0, v) - G(0, \gamma) = 0.$$

Thus $x \equiv 0$ is a solution of the generalized ODE (3) on $[t_0, +\infty)$. Note also that, by (8), every solution of (3) is continuous from the left. Due to (9), it is natural to measure the distance between two solutions by the variation norm.

The next stability concepts were introduced by Š. Schwabik in [19] (see also [18]) and are based on the variation of the solutions of (3) around $x \equiv 0$.

Definition 3.1. *The trivial solution $x \equiv 0$ of (3) is said to be*

(*i*) Variationally stable, *if for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that if $\overline{x} : [\gamma, v] \to B_c$, $t_0 \leq \gamma < v < +\infty$, is a function of bounded variation on $[\gamma, v]$ such that*

$$\|\overline{x}(\gamma)\| < \delta$$

and

$$var_{\gamma}^{v}\left(\overline{x}(s) - \int_{\gamma}^{s} DG(\overline{x}(\tau), t)\right) < \delta,$$

then

$$\|\overline{x}(t)\| < \varepsilon, \quad t \in [\gamma, v].$$

(*ii*) Variationally attracting, *if there exists $\delta_0 > 0$ and for every $\varepsilon > 0$, there exist $T = T(\varepsilon) \geq 0$ and $\rho = \rho(\varepsilon) > 0$ such that if $\overline{x} : [\gamma, v] \to B_c$, $t_0 \leq \gamma < v < +\infty$, is a function of bounded variation on $[\gamma, v]$ such that*

$$\|\overline{x}(\gamma)\| < \delta_0$$

and

$$var_{\gamma}^{v}\left(\overline{x}(s) - \int_{\gamma}^{s} DG(\overline{x}(\tau), t)\right) < \rho,$$

then

$$\|\overline{x}(t)\| < \varepsilon, \quad t \in [\gamma, v] \cap [\gamma + T, +\infty), \ \gamma \geq t_0.$$

(*iii*) Variationally asymptotically stable, *if it is variationally stable and variationally a-ttracting.*

In the sequel, we turn our attention to direct Lyapunov-type theorems for equation (3). Such results are borrowed from [18], Theorems 10.13 and 10.14 (see also [19]).

Theorem 3.1. *Let $V : [t_0, +\infty) \times \overline{B_\rho} \to \mathbb{R}$, where $\overline{B_\rho} = \{y \in X : \|y\| \leq \rho\}$, $0 < \rho < c$, be such that $V(\cdot, x) : [t_0, +\infty) \to \mathbb{R}$ is left continuous on $(t_0, +\infty)$ for $x \in X$ and the following conditions hold:*

(i) $V(t, 0) = 0$, $t \in [t_0, +\infty)$;

(ii) *There is a constant $K > 0$ such that*

$$|V(t, z) - V(t, y)| \leq K\|z - y\|, \quad t \in [t_0, +\infty), \ z, y \in \overline{B_\rho};$$

(iii) *There is a function $b : [0, +\infty) \to \mathbb{R}$ of Hahn class such that*

$$V(t, z) \geq b(\|z\|), \quad (t, z) \in [t_0, +\infty) \times \overline{B_\rho};$$

(iv) *For all solutions* $x : [\gamma, v] \to B_\rho$ *of* (3), *with* $t_0 \le \gamma < v < +\infty$, *we have*

$$\limsup_{\eta \to 0^+} \frac{V(t+\eta, x(t+\eta)) - V(t, x(t))}{\eta} \le 0,$$

that is, the right derivative of V *along* $x(t)$ *is non positive.*

Then the trivial solution $x \equiv 0$ *of* (3) *is variationally stable.*

Theorem 3.2. *Let* $V : [t_0, +\infty) \times \overline{B_\rho} \to \mathbb{R}$, *where* $\overline{B_\rho} = \{y \in X : \|y\| \le \rho\}$, $0 < \rho < c$, *satisfies conditions (i) to (iii) from Theorem 3.1. Suppose there is a continuous function* $\Phi : X \to \mathbb{R}$, *with* $\Phi(0) = 0$ *and* $\Phi(x) > 0$ *for* $x \neq 0$, *such that for every solution* $x : [\gamma, v] \to B_\rho$, $[\gamma, v] \subset [t_0, +\infty)$, *of* (3), *we have*

$$\limsup_{\eta \to 0^+} \frac{V(t+\eta, x(t+\eta)) - V(t, x(t))}{\eta} \le -\Phi(x(t)), \ t \in [\gamma, v). \tag{10}$$

Then the trivial solution $x \equiv 0$ *of* (3) *is variationally asymptotically stable.*

4. Stability of impulsive RFDEs

Now we turn our attention to retarded functional differential equations (we write RFDEs) with variable impulses of type (1). We want to establish stability theorems for these equations by means of generalized ODEs.

Let $\sigma \ge 0$ and consider the impulsive RFDE (2), where $f : G^-([-r, 0], \mathbb{R}^n) \times [t_0, t_0 + \sigma] \to \mathbb{R}^n$, and for every $y \in G^-([t_0 - r, t_0 + \sigma], \mathbb{R}^n)$, $t \mapsto f(y_t, t)$ is locally Lebesgue integrable on $t \in [t_0, t_0 + \sigma]$. If there is a function $y \in G^-([t_0 - r, t_0 + \sigma], \mathbb{R}^n)$ such that

(i) $\dot{y}(t) = f(y_t, t)$, for almost every $t \in [t_0, t_0 + \sigma] \setminus \{s \in [t_0, t_0 + \sigma] : s = \tau_k(y(s)), k = 1, 2, \ldots\}$;

(ii) $y(t+) = y(t) + I_k(y(t))$, $t = \tau_k(y(t)) \in [t_0, t_0 + \sigma]$, $k = 1, 2, \ldots$;

(iii) $y_{t_0} = \phi$,

then y is called a *solution* of (2) on $[t_0 - r, t_0 + \sigma]$ with initial condition (ϕ, t_0).

Given $y \in PC_1$ and $t \in [t_0, +\infty)$, we define

$$F(y, t)(\vartheta) = \begin{cases} 0, & t_0 - r \le \vartheta \le t_0, \\ \int_{t_0}^{\vartheta} f(y_s, s)\, ds, & t_0 \le \vartheta \le t < +\infty, \\ \int_{t_0}^{t} f(y_s, s)\, ds, & t_0 \le t \le \vartheta < +\infty, \end{cases} \tag{11}$$

and

$$J(y, t)(\vartheta) = \sum_{k=1}^{+\infty} \sum_{i=1}^{m(\tau_k)} H_k^i(t) H_k^i(\vartheta) I_k(y(t_k^i)), \tag{12}$$

where $\vartheta \in [t_0 - r, +\infty)$ and H_k^i denotes the left continuous Heavyside function concentrated at t_k^i, that is,

$$H_k^i(t) = \begin{cases} 0, & \text{for } t_0 \le t \le t_k^i, \\ 1, & \text{for } t > t_k^i. \end{cases}$$

Let $G : PC_1 \times [t_0, +\infty) \to G^-([t_0 - r, +\infty), \mathbb{R}^n)$ be defined by

$$G(y,t) = F(y,t) + J(y,t) \tag{13}$$

for $y \in PC_1$ and $t \in [t_0, +\infty)$. Then, for $s_1, s_2 \in [t_0, +\infty)$ and $x, y \in PC_1$ we have

$$\|G(x,s_2) - G(x,s_1)\| \leq |h(s_2) - h(s_1)| \tag{14}$$

and

$$\|G(x,s_2) - G(x,s_1) - G(y,s_2) - G(y,s_1)\| \leq \|x - y\| |h(s_2) - h(s_1)|, \tag{15}$$

where

$$h(t) = \int_{t_0}^{t} [M(s) + L(s)] ds + \max(K_1, K_2) \sum_{k=1}^{+\infty} \sum_{i=1}^{m(\tau_k)} H_k^i(t), \quad t \in [t_0, +\infty).$$

Note that h is a nondecreasing real function which is continuous from the left at every point, continuous at $t \neq t_k^i$ and $h(t_k^i+)$ exists for $k = 1, 2, \ldots$, and $i = 1, 2, \ldots, m(\tau_k)$. For details, see [4].

By (14) and (15), it is clear that the function G defined by (13) belongs to the class $\mathcal{F}(\Omega, h)$, with $\Omega = PC_1 \times [t_0, +\infty)$.

Consider the generalized ordinary differential equation

$$\frac{dx}{d\tau} = DG(x,t), \tag{16}$$

where G is given by (13). The next result gives a one-to-one relation between the solution of the impulsive RFDE (2) and the solution of the generalized ODE (16) with initial condition described in terms of the initial condition of (2). A proof of it follows as in [4], Theorems 3.4 and 3.5, with obvious adaptations.

Theorem 4.1 (Correspondence of equations).

(i) *Consider system* (2), *where* $f : G^-([-r,0], \mathbb{R}^n) \times [t_0, t_0 + \sigma] \to \mathbb{R}^n$ *is such that for every* $y \in G^-([-r,0], \mathbb{R}^n)$, $t \mapsto f(y_t, t)$ *is locally Lebesgue integrable over* $[t_0, t_0 + \sigma]$ *and conditions* (A), (B), (A′), (B′) *are fulfilled. Let* $y(t)$ *be the solution of the impulsive RFDE* (2) *on* $[t_0 - r, t_0 + \sigma]$. *Given* $t \in [t_0, t_0 + \sigma]$, *let*

$$x(t)(\vartheta) = \begin{cases} y(\vartheta), & t_0 - r \leq \vartheta \leq t, \\ y(t), & t \leq \vartheta \leq t_0 + \sigma. \end{cases}$$

Then $x(t) \in G^-([t_0 - r, t_0 + \sigma], \mathbb{R}^n)$ *and* x *is a solution of* (16) *on* $[t_0, t_0 + \sigma]$, *with* G *given by* (13).

(ii) *Reciprocally, let* $x(t)$ *be a solution of* (16), *on* $[t_0, t_0 + \sigma]$, *with* G *given by* (13), *satisfying the initial condition*

$$x(t_0)(\vartheta) = \begin{cases} \phi(\vartheta - t_0), & t_0 - r \leq \vartheta \leq t_0, \\ x(t_0)(t_0), & t_0 \leq \vartheta \leq t_0 + \sigma. \end{cases}$$

For $\vartheta \in [t_0 - r, t_0 + \sigma]$, *define*

$$y(\vartheta) = \begin{cases} x(t_0)(\vartheta), & t_0 - r \le \vartheta \le t_0, \\ x(\vartheta)(\vartheta), & t_0 \le \vartheta \le t_0 + \sigma. \end{cases}$$

Then y *is a solution of* (2) *on* $[t_0 - r, t_0 + \sigma]$.

By Theorem 2.1, for $\widetilde{x} \in PC_1$ the condition

$$\widetilde{x}+ = \widetilde{x} + G(\widetilde{x}, t_0+) - G(\widetilde{x}, t_0) \in PC_1,$$

is needed, since it assures that the solution of the initial value problem for the generalized ODE (16) does not jump out of the set PC_1 immediately after the moment t_0. However, in our setting, G is given by (13) and hence $G(\widetilde{x}, t_0+) - G(\widetilde{x}, t_0) = 0$, since $t_0 < t_k^i, i = 1, \ldots, m(\tau_k), k = 1, 2, \ldots$. Thus t_0 is not a moment of impulse.

In the next lines, we assume that

$$f(0, t) = 0 \text{ for all } t \text{ and } I_k(0) = 0, \; k = 1, 2, \ldots.$$

This implies that the function $y \equiv 0$ is a solution of system (1) on any interval contained in $[t_0, +\infty)$. We also consider the set $E_c = \{\psi \in G^-([-r, 0], \mathbb{R}^n) : \|\psi\| < c\}$, $c > 0$.

We recall some classic concepts of stability.

Definition 4.1. *The trivial solution of* (1) *is said to be*

(i) Stable, *if for any* $t_0 \ge 0$, $\varepsilon > 0$, *there exists* $\delta = \delta(\varepsilon, t_0) > 0$ *such that if* $\phi \in E_c$ *and* $\overline{y} : [t_0 - r, v] \to \mathbb{R}^n$ *is solution of* (1) *on* $[t_0, v]$ *such that* $\overline{y}_{t_0} = \phi$ *and*

$$\|\phi\| < \delta,$$

then

$$\|\overline{y}_t(t_0, \phi)\| < \varepsilon, \quad t \in [t_0, v].$$

(ii) Uniformly stable, *if the number* δ *in item* (i) *is independent of* t_0.

(iii) Uniformly asymptotically stable, *if there exists* $\delta_0 > 0$ *and for every* $\varepsilon > 0$, *there exists* $T = T(\varepsilon) \ge 0$ *such that if* $\phi \in E_c$ *and* $\overline{y} : [t_0 - r, v] \to \mathbb{R}^n$ *is solution of* (1) *on* $[t_0, v]$ *such that* $\overline{y}_{t_0} = \phi$ *and*

$$\|\phi\| < \delta_0,$$

then

$$\|\overline{y}_t(t_0, \phi)\| < \varepsilon, \quad t \in [t_0, v] \cap [t_0 + T, +\infty).$$

We will apply Theorem 4.1 combined with Theorems 3.1 and 3.2 to obtain stability results for problem (1) under conditions (C_1) to (C_5), (A), (B), (A') and (B').

Given $t \ge t_0$ and a function $\psi \in G^-([-r, 0], \mathbb{R}^n)$, consider equation (1) with initial condition $y_t = \psi$. This initial value problem admits a unique local solution $y : [t - r, v] \to \mathbb{R}^n$ with $[t - r, v] \subset [t - r, +\infty)$ (see [3], Theorem 2.1). Then, by Theorem 4.1(i), we can find a solution $x : [t, v] \to G^-([t, v], \mathbb{R}^n)$ of the generalized ODE (16), with initial condition $x(t) = \widetilde{x}$, where $\widetilde{x}(\tau) = \psi(\tau - t)$, $t - r \le \tau \le t$, and $\widetilde{x}(\tau) = \psi(0)$, $\tau \ge t$. Then $x(t)(t + \theta) = y(t + \theta)$ for all

$\theta \in [-r, 0]$ and, hence, $(x(t))_t = y_t$. In this case, we write $y_{t+\eta} = y_{t+\eta}(t, \psi)$ for every $\eta \geq 0$. Then for $U : [t_0, +\infty) \times G^-([-r, 0], \mathbb{R}^n) \to \mathbb{R}$, we define

$$D^+U(t,\psi) = \limsup_{\eta \to 0+} \frac{U(t+\eta, y_{t+\eta}(t,\psi)) - U(t, y_t(t,\psi))}{\eta}, \quad t \geq t_0.$$

On the other hand, given $t \geq t_0$, if $\widetilde{x} \in G^-([t-r, +\infty), \mathbb{R}^n)$ is such that $\widetilde{x}(\tau) = \psi(\tau - t)$, $t-r \leq \tau \leq t$, and $\widetilde{x}(\tau) = \psi(0)$, $\tau \geq t$, there exists a unique solution $x : [t, \overline{v}] \to G^-([t, \overline{v}], \mathbb{R}^n)$ of the generalized ODE (16) such that $x(t) = \widetilde{x}$, with $[t, \overline{v}] \subset [t_0, +\infty)$. By Theorem 4.1(*ii*), there is a solution $y : [t-r, \overline{v}] \to \mathbb{R}^n$ of (1) which satisfies $y_t = \psi$ and is described in terms of x. In this case, we write $x_\psi(t)$ instead of $x(t)$ and we have $y_t(t, \psi) = (x_\psi(t))_t = \psi$. Consequently, $(t, x_\psi(t)) \mapsto (t, y_t(t, \psi))$ is a one-to-one mapping, and we can define a functional $V : [t_0, +\infty) \times G^-([t_0 - r, +\infty), \mathbb{R}^n) \to \mathbb{R}$ by

$$V(t, x_\psi(t)) = U(t, y_t(t, \psi)). \tag{17}$$

Then we have

$$D^+U(t,\psi) = \limsup_{\eta \to 0+} \frac{V(t+\eta, x_\psi(t+\eta)) - V(t, x_\psi(t))}{\eta}. \tag{18}$$

Remark 4.1. *With the previous notation, given $t \geq t_0$, we have $\|y_t(t,\psi)\| = \|x_\psi(t)\|$, since*

$$\begin{aligned} \|y_t(t,\psi)\| &= \|y_t\| = \sup_{-r\leq\theta\leq 0} |y(t+\theta)| = \sup_{t-r\leq\tau\leq t} |y(\tau)| = \sup_{t-r\leq\tau\leq t} |x_\psi(t)(\tau)| \\ &= \sup_{t-r\leq\tau<+\infty} |x_\psi(t)(\tau)| = \|x_\psi(t)\|, \end{aligned}$$

where we used Theorem 4.1(ii) to obtain the fourth equality.

In the sequel, we consider the sets $\overline{E}_\rho = \{y \in G^-([-r, 0], \mathbb{R}^n) : \|y\| \leq \rho\}$ and $\overline{B}_\rho = \{x \in G^-([t_0 - r, +\infty), \mathbb{R}^n) : \|x\| \leq \rho\}$, with $0 < \rho < c$.

Lemma 4.1. *Consider the impulsive RFDE (1), where $f : G^-([-r, 0], \mathbb{R}^n) \times [t_0, +\infty) \to \mathbb{R}^n$ is such that for every $y \in G^-([-r, 0], \mathbb{R}^n)$, $t \mapsto f(y_t, t)$ is locally Lebesgue integrable over $[t_0, +\infty)$ and conditions (A), (B), (A′), (B′) are fulfilled. Assume that $U : [t_0, +\infty) \times \overline{E}_\rho \to \mathbb{R}$ satisfies the conditions:*

(i) *$U(t, 0) = 0$, $t \in [t_0, +\infty)$;*
(ii) *There exists a constant $K > 0$ such that*

$$|U(t,\psi) - U(t,\overline{\psi})| \leq K\|\psi - \overline{\psi}\|, \quad t \in [t_0, +\infty), \quad \psi, \overline{\psi} \in \overline{E}_\rho.$$

Then the function $V : [t_0, +\infty) \times \overline{B}_\rho \to \mathbb{R}$ defined by (17) satisfies $V(t, 0) = 0$ for all $t \in [t_0, +\infty)$, and

$$|V(t, z) - V(t, \overline{z})| \leq K\|z - \overline{z}\|,$$

for $t \geq t_0$ and $z, \overline{z} \in \overline{B}_\rho$.

Proof. Given $t \geq t_0$ and $\psi, \overline{\psi} \in \overline{E}_\rho$, let $y, \overline{y}, \widehat{y} : [t-r, +\infty) \to \mathbb{R}^n$ be solutions of equation (1) such that $y_t = \psi$, $\overline{y}_t = \overline{\psi}$ and $\widehat{y}_t = 0$. Suppose $x, \overline{x}, \widehat{x}$ are solutions on $[t, +\infty)$ of the generalized ODE (16) given by Theorem 4.1(i) with respect to y,$\overline{y}$ and $\widehat{y}$ respectively. Then $(x(t))_t = y_t = \psi$, $(\overline{x}(t))_t = \overline{y}_t = \overline{\psi}$ and $(\widehat{x}(t))_t = \widehat{y}_t = 0$. By Remark 4.1, $x_\psi(t), x_{\overline{\psi}}(t) \in \overline{B}_\rho$.

Since f satisfies (A) and (B) and I_k satisfies (A') and (B') for $k = 1, 2, \ldots$, then the function G in equation (16) belongs to $\mathcal{F}(\Omega, h)$.

Let $V : [t_0, +\infty) \times \overline{B}_\rho \to \mathbb{R}$ be given by (17). By condition (i), we have

$$0 = U(t, 0) = U(t, \widehat{y}_t(t, 0)) = V(t, \widehat{x}(t)) = V(t, 0),$$

since $\widehat{x}(t)(\tau) = 0$ for all τ (see Theorem 4.1(i)), that is, $\widehat{x}(t) \equiv 0$.

By condition (ii), we have

$$|V(t, x_\psi(t)) - V(t, \overline{x}_{\overline{\psi}}(t))| = |U(t, y_t(t, \psi)) - U(t, \overline{y}_t(t, \overline{\psi}))| = |U(t, \psi) - U(t, \overline{\psi})|.$$

Then by Remark 4.1, we obtain

$$|V(t, x_\psi(t)) - V(t, \overline{x}_{\overline{\psi}}(t))| \leq K\|\psi - \overline{\psi}\| = K\|x_\psi(t) - \overline{x}_{\overline{\psi}}(t)\|. \tag{19}$$

It is clear that given $t \geq t_0$ and $z, \overline{z} \in \overline{B}_\rho$, there exist solutions x and $\overline{x}$ of the generalized ODE (16) and $\psi, \overline{\psi} \in G^-([-r, 0], \mathbb{R}^n)$ such that $z = x_\psi(t)$, $(x_\psi(t))_t = y_t(t, \psi)$, $\overline{z} = \overline{x}_{\overline{\psi}}(t)$ and $(\overline{x}_{\overline{\psi}}(t))_t = \overline{y}_t(t, \overline{\psi})$, by Remark 4.1.

Since

$$\|\psi\| = \|y_t(t, \psi)\| = \|x_\psi(t)\| = \|z\| \leq \rho$$

and

$$\|\overline{\psi}\| = \|\overline{y}_t(t, \overline{\psi})\| = \|\overline{x}_{\overline{\psi}}(t)\| = \|\overline{z}\| \leq \rho,$$

it follows by (19) that

$$|V(t, z) - V(t, \overline{z})| \leq K\|z - \overline{z}\|,$$

and the result follows. □

Remark 4.2. *The next two results, namely Theorem 4.2 and Theorem 4.3, are Lyapunov-type theorems for the impulsive RFDE* (1) *and they are analogous to Theorems 4.8 and 4.9, from* [6]. *Such results from* [6] *concern the uniform stability and the uniform asymptotic stability for a class of RFDEs with pre-assigned moments of impulse action. Our results concerns the uniform stability and the uniform asymptotic stability for a class of retarded differential equations with variable moments of impulse action. It is important to note that, in* [6], *the authors consider a functional* $U : [0, +\infty) \times G^-([-r, 0], \mathbb{R}^n) \to \mathbb{R}$ *with respect to the impulsive RFDE* (1) *and a functional* $V : [0, +\infty) \times G^-([-r, +\infty), \mathbb{R}^n) \to \mathbb{R}$ *with respect to the generalized ODE* (16) *and* U *and* V *are related by the equality*

$$V(t, x) = U(t, x_t),$$

for $t \geq 0$ *and* $x \in G^-([-r, +\infty), \mathbb{R}^n)$. *In the present paper, we relate* U *and* V *by* (17). *Also, in* [6], *it was assumed that*

$$U(t, y_t) \geq b(\|y\|), \quad t \in [-r, +\infty), y \in \overline{B}_\rho, \tag{20}$$

where b is a function of Hahn class. Here, we replace condition (20) *by the weaker condition*

$$U(t,\psi) \geq b(\|\psi\|), \quad t \in [t_0 - r, +\infty), \psi \in \overline{E}_\rho, \tag{21}$$

where $t_0 \geq 0$. The proofs of Theorems 4.2 and 4.3 were carried out following the ideas of the proofs of Theorems 4.8 and 4.9 from [6], *respectively. We include these proofs using* (21) *instead of* (20) *for the sake of completeness and self-containedness of the paper.*

Theorem 4.2. *Consider the impulsive RFDE* (1)*. Suppose conditions* (A), (B), (A'), (B') *are fulfilled. Let $U : [t_0, +\infty) \times \overline{E}_\rho \to \mathbb{R}$ be left continuous on $(t_0, +\infty)$ and assume the next conditions hold:*

(i) $U(t, 0) = 0$, $t \in [t_0, +\infty)$;

(ii) *There is a constant $K > 0$ such that*

$$|U(t,\psi) - U(t,\overline{\psi})| \leq K\|\psi - \overline{\psi}\|, \quad t \in [t_0, +\infty), \quad \psi, \overline{\psi} \in \overline{E}_\rho;$$

(iii) *There is a function $b : \mathbb{R}_+ \to \mathbb{R}_+$ of Hahn class such that*

$$U(t,\psi) \geq b(\|\psi\|),$$

for all $t \geq t_0$ and all $\psi \in \overline{E}_\rho$;

(iv) *The inequality*

$$D^+U(t,\psi) \leq 0$$

holds for each $t \geq t_0$ and each $\psi \in \overline{E}_\rho$.

Then the trivial solution $y \equiv 0$ of (1) *is uniformly stable.*

Proof. Since f satisfies (A) and (B) and I_k satisfies (A') and (B') for $k = 1, 2, \ldots$, the function G in equation (16) belongs to $\mathcal{F}(\Omega, h)$.

Let $V : [t_0, +\infty) \times \overline{B}_\rho \to \mathbb{R}$ be given by (17). By Lemma 4.1,

$$V(t, 0) = 0, \qquad \text{for } \; t \in [t_0, +\infty)$$

and

$$|V(t,z) - V(t,\overline{z})| \leq K\|z - \overline{z}\|, \quad \text{for } t \in [t_0, +\infty) \text{ and } z, \overline{z} \in \overline{B}_\rho.$$

By Remark 4.1 and condition (iii), given $t \geq t_0$, we have

$$b(\|x_\psi(t)\|) = b(\|y_t\|) \leq U(t, y_t(t,\psi)) = V(t, x_\psi(t))$$

for the solution y of (1) satisfying $y_t = \psi$. Then by previous arguments (see Lemma 4.1), we have

$$V(t,z) \geq b(z), \quad z \in \overline{B}_\rho.$$

Finally, condition (iv) above clearly implies condition (iv) from Theorem 3.1 and, therefore, the hypotheses of Theorem 3.1 are fulfilled. Hence the solution $x \equiv 0$ of the generalized ODE (16) is variationally stable. Thus if for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that if $\overline{x} : [\gamma, v] \to B_\rho$, $t_0 \leq \gamma < v < +\infty$, is a function of bounded variation on $[\gamma, v]$ such that

$$\|\overline{x}(\gamma)\| < \delta$$

and

$$\text{var}_{\gamma}^{v}\left(\overline{x}(s)-\int_{\gamma}^{s}DG(\overline{x}(\tau),t)\right)<\delta,$$

then

$$\|\overline{x}(t)\|<\varepsilon,\quad t\in[\gamma,v]. \tag{22}$$

Let $\phi\in\overline{E}_{\rho}$ and $\overline{y}:[t_0-r,+\infty)\to\mathbb{R}^n$ be a solution of (1) with $\overline{y}_{t_0}=\phi$. Suppose

$$\|\phi\|<\delta. \tag{23}$$

We want to prove that

$$\|\overline{y}_t(t_0,\phi)\|<\varepsilon,\quad t\in[t_0,+\infty). \tag{24}$$

Let us denote $\overline{y}_t=\overline{y}_t(t_0,\phi)$ and define

$$\overline{x}\,(t)\,(\tau)=\begin{cases}\overline{y}\,(\tau)\,, & t_0-r\le\tau\le t,\\ \overline{y}\,(t)\,, & \tau\ge t.\end{cases} \tag{25}$$

By Theorem 4.1, $\overline{x}(t)$ is a solution on $[t_0,+\infty)$ of the generalized ODE (16) satisfying the initial condition $\overline{x}(t_0)=\widetilde{x}$, where

$$\widetilde{x}\,(\tau)=\begin{cases}\phi\,(\tau-t_0)\,, & t_0-r\le\tau\le t_0,\\ \phi(0), & \tau\ge 0.\end{cases} \tag{26}$$

Moreover, $\overline{x}$ is of bounded variation on $[t_0,+\infty)$.

By (26) and (23), we have

$$\|\overline{x}(t_0)\|=\sup_{t_0-r\le\tau<+\infty}|\widetilde{x}(\tau)|=\|\phi\|<\delta. \tag{27}$$

Besides,

$$\text{var}_{t_0}^{v}\left(\overline{x}(s)-\int_{t_0}^{s}DG(\overline{x}(\tau),t)\right)=0<\delta. \tag{28}$$

Therefore (22) holds, that is, $\|\overline{x}(t)\|<\varepsilon$ for all $t\in[t_0,v]$, where $v\in[t_0,+\infty)$. In particular, $\|\overline{x}(v)\|<\varepsilon$. Hence (25) implies that for any $t\in[t_0,v]$, we have

$$\begin{aligned}\|\overline{y}_t(t_0,\phi)\| &= \|\overline{y}_t\|=\sup_{-r\le\theta\le 0}|\overline{y}(t+\theta)|\le\sup_{t_0-r\le\tau\le v}|\overline{y}(\tau)|\\ &= \sup_{t_0-r\le\tau\le v}|\overline{x}(v)(\tau)|=\sup_{t_0-r\le\tau<+\infty}|\overline{x}(v)(\tau)|\\ &= \|\overline{x}(v)\|<\varepsilon.\end{aligned} \tag{29}$$

Thus (24) holds and the proof is complete. □

Theorem 4.3. *Consider the impulsive RFDE* (1), *where conditions* (A), (B), (A'), (B') *are fulfilled. Assume that* $U:[t_0,+\infty)\times\overline{E_\rho}\to\mathbb{R}$ *satisfies conditions* (*i*) *to* (*iii*) *from Theorem 4.2. Suppose there is a continuous function* $\Lambda:\mathbb{R}_+\to\mathbb{R}_+$ *satisfying* $\Lambda(0)=0$ *and* $\Lambda(x)>0$ *if* $x\neq 0$, *such that for every* $\psi\in\overline{E}_\rho$,

$$D^+U(t,\psi)\le-\Lambda(\|\psi\|),\quad t\ge t_0. \tag{30}$$

Then the trivial solution $y \equiv 0$ of (1) *is uniformly asymptotically stable.*

Proof. We assume the notation of the previous theorem.

Suppose $V : [t_0, +\infty) \times \overline{B}_\rho \to \mathbb{R}$ is given by (17). Then the hypotheses of Theorem 4.2 are fulfilled.

Let $\Phi : \overline{B}_\rho \to \mathbb{R}$ be defined by $\Phi(z) = \Lambda(\|z\|)$, for $z \in \overline{B}_\rho$. Then Φ is continuous, $\Phi(0) = 0$ and $\Phi(z) > 0$ whenever $z \neq 0$.

Assume that $x : [t, +\infty) \to \overline{B}_\rho$ is a solution of (16) such that $(x(t))_t = \psi$, where $t \in [t_0, +\infty)$ and $\psi \in \overline{E}_\rho$, and suppose $y : [t - r, +\infty) \to \mathbb{R}^n$ is the solution of (1) given by Theorem 12(ii) such that $y_t = \psi$. By (30), we have

$$\limsup_{\eta \to 0+} \frac{V(t+\eta, x_\psi(t+\eta)) - V(t, x_\psi(t))}{\eta} = D^+U(t, y_t(t,\psi)) =$$

$$= D^+U(t,\psi) \leq -\Lambda(\|\psi\|) = -\Lambda(\|y_t\|).$$

But

$$\|y_t\| = \|x_\psi(t)\|,$$

by Remark 4.1. Therefore,

$$\limsup_{\eta \to 0+} \frac{V(t+\eta, x_\psi(t+\eta)) - V(t, x_\psi(t))}{\eta} \leq -\Lambda(\|y_t\|) = -\Lambda(\|x_\psi(t)\|) = -\Phi(x_\psi(t))$$

and the hypotheses of Theorem 3.2 are satisfied. Hence $x \equiv 0$ is variationally asymptotically stable, that is, there exists $\delta_0 > 0$ and for every $\varepsilon > 0$, there exist $T = T(\varepsilon) \geq 0$ and $\rho = \rho(\varepsilon) > 0$ such that if $\overline{x} : [\gamma, v] \to B_c$, $t_0 \leq \gamma < v < +\infty$, is a function of bounded variation on $[\gamma, v]$ such that

$$\|\overline{x}(\gamma)\| < \delta_0 \tag{31}$$

and

$$\operatorname{var}_\gamma^v \left(\overline{x}(s) - \int_\gamma^s DG(\overline{x}(\tau), t) \right) < \rho, \tag{32}$$

then

$$\|\overline{x}(t)\| < \varepsilon, \quad t \in [\gamma, v] \cap [\gamma + T, +\infty),\ \gamma \geq t_0. \tag{33}$$

Given $\varepsilon > 0$, let $\delta_0 > 0$ and $T = T(\varepsilon)$ be as above. Let $\phi \in E_c$, and $\overline{y} : [t_0 - r, +\infty) \to \mathbb{R}^n$ be the solution of (1) such that $\overline{y}_{t_0} = \phi$ and assume

$$\|\phi\| < \delta_0. \tag{34}$$

We want to prove that

$$\|\overline{y}_t(t_0, \phi)\| < \varepsilon, \quad t \in [t_0 + T, +\infty). \tag{35}$$

But this is immediate by the proof of Theorem 4.2. By (34), we obtain (31) as in (27). Also, as in Theorem 4.2, we have (28) and hence (32) follows. Finally (35) holds, since we have (29) as in Theorem 4.2 and because of (33). □

Now, we consider Lyapunov functions $U : [t_0 - r, +\infty) \times \mathbb{R}^n \to \mathbb{R}$ and establish new stability results for the trivial solution of the impulsive RFDE (1). We define the derivative of U along the solutions of (1) by

$$D^+U(t, y(t)) = \limsup_{\eta \to 0+} \frac{U(t+\eta, y(t+\eta, t, \psi)) - U(t, y(t, t, \psi))}{\eta}, \quad t \geq t_0,$$

where $y(s, t, \psi)$ is the solution of (1) which satisfies $y_t = \psi$, with $\psi \in G^-([-r, 0], \mathbb{R}^n)$. Note that we can write $D^+U(t, \psi(0))$ instead of $D^+U(t, y(t))$, since given an initial function $\psi \in G^-([-r, 0], \mathbb{R}^n)$ and $t \geq t_0$, there exists a unique solution of (1) which satisfies $y_t = \psi$ and, therefore, $y(t) = \psi(0)$ (see [3], Theorem 2.1).

Theorem 4.4. *Consider the impulsive RFDE (1). Assume that conditions (A), (B), (A′), (B′) are fulfilled. Suppose $U : [t_0 - r, +\infty) \times \mathbb{R}^n \to \mathbb{R}$ is left continuous on $[t_0 - r, +\infty)$, the limits*

$$U(t-, y(t-)) = \lim_{s \to t^-} U(s, y(s)), \quad t \in [t_0 - r, +\infty)$$

and

$$U(t+, y(t+)) = \lim_{s \to t+} U(s, y(s)), \quad t \in [t_0 - r, +\infty),$$

exist with $U(t-, y(t-)) = U(t, y(t))$ satisfied, where $y \in G^-([t_0 - r, +\infty), \mathbb{R}^n)$. Suppose U fulfills the following conditions:

(i) *$U(t, 0) = 0$, $t \in [t_0 - r, +\infty)$;*

(ii) *For each $a > 0$, there is a constant $K_a > 0$ such that*

$$|U(t, x) - U(t, y)| \leq K_a \|x - y\|, \qquad t \in [t_0 - r, +\infty) \text{ and } x, y \in \mathcal{B}_a,$$

where $\mathcal{B}_a = \{z \in \mathbb{R}^n : \|z\| < a\}$;

(iii) *There is a function $b : \mathbb{R}_+ \to \mathbb{R}_+$ of Hahn class such that*

$$U(t, y(t)) \geq b(\|y_t\|)$$

for any $y \in G^-([t_0 - r, +\infty), \mathbb{R}^n)$, $t \in [t_0, +\infty)$;

(iv) *There is a function $\Lambda : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$D^+U(t, \psi(0)) \leq -\Lambda(|\psi(0)|) \qquad \text{if } U(t+\theta, \psi(\theta)) \leq U(t, \psi(0))$$

for $t \in [t_0, +\infty)$, $\theta \in [-r, 0]$ and $\psi \in G^-([-r, 0], \mathbb{R}^n)$.

Then the trivial solution $y \equiv 0$ of (1) is uniformly stable.

Proof. For $s \geq t_0$ and $\xi \in \overline{E}_\rho$, let us define

$$\overline{U}(s, \xi) = \sup_{\theta \in [-r, 0]} U(s+\theta, \xi(\theta)). \tag{36}$$

It is clear that $\overline{U}(s, 0) = 0$ for all $s \geq t_0$.

Given $t \geq t_0$ and $\psi \in \overline{E}_\rho$, consider $y : [t - r, +\infty) \to \mathbb{R}^n$ a solution of (1) such that $y_t = \psi$. Note that by (8), the solution x of (3) corresponding to (1) is continuous from the left and so is y. This fact can be easily seen by the relations between the solutions y and x given in Theorem 4.1. Therefore the assumptions of the present theorem imply that the function $U(s, y(s))$ is continuous from the left for s in the interval where the solution y is defined.

By (36), we have

$$\overline{U}(t, \psi) = \overline{U}(t, y_t) = \sup_{\theta \in [-r,0]} U(t + \theta, y(t + \theta)). \tag{37}$$

We will show that $D^+\overline{U}(t, \psi) = D^+\overline{U}(t, y_t) \leq 0$.

Let

$$R(t) = \{U(t + \theta, y(t + \theta)) : \theta \in [-r, 0]\}.$$

We will consider two cases: when $\overline{U}(t, y_t)$ belongs to $R(t)$ and otherwise.

Suppose $\overline{U}(t, y_t)$ belongs to $R(t)$. Then there is a $\theta_0 \in [-r, 0]$ such that $U(t + \theta_0, y(t + \theta_0)) = \overline{U}(t, y_t)$. If $\theta_0 = 0$, then $\overline{U}(t, y_t) = U(t, y(t))$ which implies, by condition (iv), that $D^+\overline{U}(t, y_t) \leq 0$. If $\theta_0 < 0$, then $U(t + \theta, y(t + \theta)) < U(t + \theta_0, y(t + \theta_0))$ for all $\theta_0 < \theta \leq 0$ by (37) and by the choice of θ_0. Then for all $h > 0$ sufficiently small with $h < |\theta_0|$, we have

$$\begin{aligned} \overline{U}(t + h, y_{t+h}) &= \sup_{\theta \in [-r,0]} U(t + h + \theta, y(t + h + \theta)) \\ &= U(t + \theta_0, y(t + \theta_0)) = \overline{U}(t, y_t) \end{aligned}$$

and hence $D^+\overline{U}(t, y_t) = 0$.

Now, we consider the case where $\overline{U}(t, y_t)$ does not belong to $R(t)$. In this case,

$$\overline{U}(t, y_t) > U(t + \theta, y(t + \theta)), \qquad \theta \in [-r, 0],$$

and there is a convergent sequence $\{\theta_n\}_{n \in \mathbb{N}}$ in $[-r, 0]$, with $\overline{\theta} = \lim_{n \to +\infty} \theta_n$, such that

$$\overline{U}(t, y_t) = \lim_{n \to +\infty} U(t + \theta_n, y(t + \theta_n)).$$

Suppose there are countably many θ_{n_k}'s such that $\theta_{n_k} < \overline{\theta}$. Then

$$\overline{U}(t, y_t) = \lim_{k \to +\infty} U(t + \theta_{n_k}, y(t + \theta_{n_k})) = U(t + \overline{\theta}, y(t + \overline{\theta}))$$

which is a contradiction, since $\overline{\theta} \in [-r, 0]$ and in such a case $\overline{U}(t, y_t) \in R(t)$. This reasoning also shows that $\overline{\theta} < 0$. Therefore we may assume that $\overline{\theta} < \theta_n < 0$, for all n, and hence

$$\overline{U}(t, y_t) = U((t + \overline{\theta})+, y((t + \overline{\theta})+)).$$

Thus, for all $h > 0$ sufficiently small with $h < |\overline{\theta}|$, we have

$$\begin{aligned} \overline{U}(t + h, y_{t+h}) &= \sup_{\theta \in [-r,0]} U(t + h + \theta, y(t + h + \theta)) \\ &= U((t + \overline{\theta})+, y((t + \overline{\theta})+)) = \overline{U}(t, y_t). \end{aligned}$$

Hence $D^+\overline{U}(t, y_t) = 0$.

Now, we assert that $\overline{U}$ satisfies condition (ii) of Theorem 4.2. Indeed. Consider $t \geq t_0$. Given $\widehat{\psi}, \overline{\psi} \in \overline{E_\rho}$, let $\widehat{y}$, $\overline{y}$ be solutions of (1) such that $\widehat{y}_t = \widehat{\psi}$, $\overline{y}_t = \overline{\psi}$. With the above notation, we have

$$\overline{U}(t,\widehat{\psi}) = \overline{U}(t,\widehat{y}_t) = U(t+\theta_{\widehat{\psi}}, \widehat{y}(t+\theta_{\widehat{\psi}})) \quad \text{or}$$

$$\overline{U}(t,\widehat{\psi}) = \overline{U}(t,\widehat{y}_t) = U((t+\overline{\theta}_{\widehat{\psi}})+, \widehat{y}((t+\overline{\theta}_{\widehat{\psi}})+))$$

and

$$\overline{U}(t,\overline{\psi}) = \overline{U}(t,\overline{y}_t) = U(t+\theta_{\overline{\psi}}, \overline{y}(t+\theta_{\overline{\psi}})) \quad \text{or}$$

$$\overline{U}(t,\overline{\psi}) = \overline{U}(t,\overline{y}_t) = U((t+\overline{\theta}_{\overline{\psi}})+, \overline{y}((t+\overline{\theta}_{\overline{\psi}})+)),$$

where $\theta_{\widehat{\psi}}$, $\theta_{\overline{\psi}}$ and $\overline{\theta}_{\widehat{\psi}}$, $\overline{\theta}_{\overline{\psi}}$ correspond respectively to θ_0 and $\overline{\theta}$ for the functions $\widehat{\psi}, \overline{\psi} \in \overline{E_\rho}$.

Consider $\overline{\mathcal{B}}_\rho = \{x \in \mathbb{R}^n : \|x\| \le \rho\}$ and $\mathcal{B}_c = \{x \in \mathbb{R}^n : \|x\| < c\}$, where $\rho < c$. Since $U((t+\overline{\theta}_{\widehat{\psi}})+, \widehat{y}((t+\overline{\theta}_{\widehat{\psi}})+))$ and $U((t+\overline{\theta}_{\overline{\psi}})+, \overline{y}((t+\overline{\theta}_{\overline{\psi}})+))$ exist, condition (ii) implies

$$\begin{aligned}
|\overline{U}(t,\widehat{\psi}) - \overline{U}(t,\overline{\psi})| &= |\overline{U}(t,\widehat{y}_t) - \overline{U}(t,\overline{y}_t)| \\
&= |\sup_{\theta\in[-r,0]} U(t+\theta,\widehat{y}(t+\theta)) - \sup_{\theta\in[-r,0]} U(t+\theta,\overline{y}(t+\theta))| \\
&\le \sup_{\theta\in[-r,0]} |U(t+\theta,\widehat{y}(t+\theta)) - U(t+\theta,\overline{y}(t+\theta))| \\
&\le K_c \sup_{\theta\in[-r,0]} \|\widehat{y}(t+\theta) - \overline{y}(t+\theta)\| \\
&= K_c\|\widehat{y} - \overline{y}\| = K_c\|\widehat{\psi} - \overline{\psi}\|,
\end{aligned}$$

since $\widehat{y}(t+\theta), \overline{y}(t+\theta) \in \overline{\mathcal{B}}_\rho$, for all $\theta \in [-r,0]$, and $\overline{\mathcal{B}}_\rho \subset \mathcal{B}_c$. Furthermore, we have

$$\overline{U}(t,\psi) = \overline{U}(t,y_t) \ge U(t,y(t)) \ge b(\|y_t\|) = b(\|\psi\|),$$

by the definition of $\overline{U}$ and by condition (iii).

Thus all conditions of Theorem 4.2 are satisfied for $\overline{U}$ and hence the solution $\overline{y} \equiv 0$ of (1) is uniformly stable. □

Theorem 4.5. *Consider the impulsive RFDE* (1). *Assume that conditions* (A), (B), (A'), (B') *are fulfilled. Let* $U : [t_0 - r, +\infty) \times \mathbb{R}^n \to \mathbb{R}$ *be a left continuous function on* $(t_0 - r, +\infty)$ *and assume that the limits*

$$U(t-, y(t-)) = \lim_{s\to t^-} U(s, y(s)), \quad t \in [t_0 - r, +\infty),$$

and

$$U(t+, y(t+)) = \lim_{s\to t^+} U(s, y(s)), \quad t \in [t_0 - r, +\infty),$$

exist with $U(t-, y(t-)) = U(t, y(t))$ *satisfied, where* $y \in G^-([t_0 - r, +\infty), \mathbb{R}^n)$. *Suppose conditions* (i), (ii) *and* (iii) *of Theorem 4.4 are satisfied and there is a function* $d : \mathbb{R}_+ \to \mathbb{R}_+$ *of Hahn class such that for every solution* y *of* (1), *we have*

$$\sup_{\theta\in[-r,0]} U(s+\theta, y(s+\theta)) \le d(\|y(s)\|), \tag{38}$$

where $s \geq t_0$, with $d(\bar{t}) \geq b(\bar{t})$, for every $\bar{t} \geq 0$. Assume, in addition, that there is a function $\Lambda : \mathbb{R}_+ \to \mathbb{R}_+$ which satisfies $\Lambda(0) = 0$ and $\Lambda(x) > 0$, if $x \neq 0$, and there is a continuous nondecreasing function $p(s) > s$ for $s > 0$ such that

$$D^+U(t,\psi(0)) \leq -\Lambda(|\psi(0)|) \quad \text{if } U(t+\theta,\psi(\theta)) < p(U(t,\psi(0))), \tag{39}$$

for $\theta \in [-r,0]$, $t \in [t_0,+\infty)$ and $\psi \in G^-([-r,0],\mathbb{R}^n)$. Then the trivial solution $y \equiv 0$ of (1) is uniformly asymptotically stable.

Proof. This proof follows some ideas presented in the proof of Theorem 5.4.2 in [9].

If we define

$$\overline{U}(s,\xi) = \sup_{\theta\in[-r,0]} U(s+\theta,\xi(\theta)), \tag{40}$$

for $s \geq t_0$ and $\xi \in \overline{E}_\rho$, then the trivial solution $y \equiv 0$ is uniformly stable, by repeating the arguments used in the proof of Theorem 4.4.

Let $t_0 \geq 0$ and $\phi \in E_\rho$. Let $\overline{y} : [t_0 - r, +\infty) \to \mathbb{R}^n$ be a solution of (1) which satisfies $\overline{y}_{t_0} = \phi$. We write $\overline{y}(t) = \overline{y}(t,t_0,\phi)$ to denote this solution.

Set $\overline{y}_t = \overline{y}_t(t_0,\phi)$ for $t \geq t_0$, and let $\varepsilon > 0$. Since the solution $y \equiv 0$ is uniformly stable, there is $\delta > 0$ such that if $\|\phi\| < \delta$, then $\|\overline{y}_t(t_0,\phi)\| < \varepsilon$. Note that $\|\phi\| < \delta$ implies $U(t,\overline{y}(t)) < d(\varepsilon)$, for every $t \in [t_0,+\infty)$, since

$$U(t,\overline{y}(t)) \leq \sup_{\theta\in[-r,0]} U(t+\theta,\overline{y}(t+\theta)) \leq d(\|\overline{y}(t)\|) \leq d(\|\overline{y}_t\|) < d(\varepsilon), \tag{41}$$

where d is an increasing function.

Suppose $0 < \eta \leq \varepsilon$ is arbitrary. We will show that there exists a number $T = T(\varepsilon,\eta) > 0$ such that $\|\phi\| < \delta$ implies $\|\overline{y}_t\| \leq \eta$, for all $t \in [t_0+T,+\infty)$. This will be true if we show that $U(t,\overline{y}(t)) \leq b(\eta)$, for all $t \in [t_0+T,+\infty)$, where b is given by condition (iii) of Theorem 4.4.

At first, let us find the number T. By the properties of the function $p(s)$, there exists a number $\alpha > 0$ such that $p(s) - s \geq \alpha$ for $b(\eta) \leq s \leq d(\varepsilon)$ (note that $b(\eta) \leq b(\varepsilon) \leq d(\varepsilon)$).

Let $\mathcal{K}$ be the first positive integer such that $b(\eta) + \mathcal{K}\alpha > d(\varepsilon)$. Since $b(\eta) \leq d(\varepsilon)$, we have $d^{-1}(b(\eta)) \leq \varepsilon$. Let

$$\beta = \inf_{d^{-1}(b(\eta))\leq s\leq\varepsilon} \Lambda(s) > 0$$

and define

$$T := \frac{\mathcal{K}d(\varepsilon)}{\beta}.$$

Now, we will show that $U(t,\overline{y}(t)) \leq b(\eta)$, for all $t \in [t_0 + \frac{d(\varepsilon)}{\beta},+\infty)$. We assert that $U(t,\overline{y}(t)) \leq b(\eta)+(\mathcal{K}-1)\alpha$, for all $t \in [t_0+\frac{d(\varepsilon)}{\beta},+\infty)$. Indeed. Assume that $b(\eta)+(\mathcal{K}-1)\alpha < U(t,\overline{y}(t))$, for $t \in [t_0+\frac{d(\varepsilon)}{\beta},+\infty)$. By the choice of $\mathcal{K}$ and (41), we have

$$b(\eta) \leq b(\eta) + (\mathcal{K}-1)\alpha < U(t,\overline{y}(t)) < d(\varepsilon) \tag{42}$$

and

$$p(U(t,\overline{y}(t)) \geq U(t,\overline{y}(t)) + \alpha > b(\eta) + \mathcal{K}\alpha > d(\varepsilon) > U(t+\theta, \overline{y}(t+\theta)),$$

for $t_0 \leq t \leq t_0 + \frac{d(\varepsilon)}{\beta}$ and $\theta \in [-r, 0]$. Note that (41) and (42) imply

$$d^{-1}(b(\eta)) < d^{-1}(d(\|\overline{y}(t)\|)) < d^{-1}(d(\varepsilon)),$$

that is,

$$d^{-1}(b(\eta)) < \|\overline{y}(t)\| < \varepsilon,$$

where $t_0 \leq t \leq t_0 + \frac{d(\varepsilon)}{\beta}$. Consequently, by (39), we get

$$D^+U(t,\overline{y}(t)) \leq -\Lambda(\|\overline{y}(t)\|) \leq -\beta, \quad t_0 \leq t \leq t_0 + \frac{d(\varepsilon)}{\beta}.$$

Thus,

$$U(t_1, \overline{y}(t_1)) \leq U(t_0, \overline{y}(t_0)) - \beta(t_1 - t_0) < d(\varepsilon) - \beta(t_1 - t_0)$$

and $U(t_1, \overline{y}(t_1)) < 0$, where $t_1 = t_0 + \frac{d(\varepsilon)}{\beta}$, which is a contradiction, since we have the positiveness of U. Hence,

$$U(t, \overline{y}(t)) \leq b(\eta) + (\mathcal{K} - 1)\alpha, \quad t = t_0 + \frac{d(\varepsilon)}{\beta}.$$

Note that, when $U(t, \overline{y}(t)) = b(\eta) + (\mathcal{K} - 1)\alpha$, we have $D^+U(t, \overline{y}(t)) \leq 0$, because of (39), since $b(\eta) \leq U(t, \overline{y}(t)) = b(\eta) + (\mathcal{K} - 1)\alpha < d(\varepsilon)$. Hence

$$p(U(t,\overline{y}(t)) \geq U(t,\overline{y}(t)) + \alpha = b(\eta) + \mathcal{K}\alpha > d(\varepsilon) > U(t+\theta, \overline{y}(t+\theta)),$$

for $\theta \in [-r, 0]$.

Now, suppose there exists $\overline{t} > t_0 + \frac{d(\varepsilon)}{\beta}$ such that $U(\overline{t}, \overline{y}(\overline{t})) > b(\eta) + (\mathcal{K} - 1)\alpha$. Then $D^+U(t, \overline{y}(t)) > 0$, for t such that $U(t, \overline{y}(t)) = b(\eta) + (\mathcal{K} - 1)\alpha$, which is a contradiction. It is important to note that if $\overline{t} = t_k^i$, the same contradiction applies.

Let $\overline{t}_n = \frac{nd(\varepsilon)}{\beta}$, $n = 1, \cdots, \mathcal{K}$, $\overline{t}_0 = 0$ and assume that for some integer $N \geq 1$ and for t satisfying $\overline{t}_{n-1} \leq t - t_0 \leq \overline{t}_n$, we have

$$b(\eta) + (\mathcal{K} - N)\alpha < U(t, \overline{y}(t)) \leq b(\eta) + (\mathcal{K} - N + 1)\alpha.$$

Using the previous arguments, we get

$$D^+U(t, \overline{y}(t)) \leq -\beta, \qquad \overline{t}_{n-1} \leq t - t_0 \leq \overline{t}_n$$

and

$$U(t, \overline{y}(t)) \leq U(t_0 + \overline{t}_{n-1}, \overline{y}(t_0 + \overline{t}_{n-1})) - \beta(t - t_0 - \overline{t}_{n-1}) < d(\varepsilon) - \beta(t - t_0 - \overline{t}_{n-1}).$$

Thus $U(t, \overline{y}(t)) < 0$, whenever $t = t_0 + \overline{t}_n$. Analogously, one can prove that $U(t, \overline{y}(t)) \leq b(\eta) + (\mathcal{K} - N)\alpha$, for $t \geq t_0 + \overline{t}_n$. For $N = \mathcal{K}$, we have $U(t, \overline{y}(t)) \leq b(\eta)$ for all $t \geq t_0 + \frac{\mathcal{K}d(\varepsilon)}{\beta}$.

Finally, since

$$b(\|\overline{y}_t\|) \leq U(t, \overline{y}(t)) \leq b(\eta),$$

and b is an increasing function, we have $\|\overline{y}_t\| \leq \eta$, for all $t \geq t_0 + \frac{\mathcal{K}d(\varepsilon)}{\beta}$, and the proof is complete. □

5. Final comments and remarks

Consider the framework presented in Section 1.

Many problems in physics, mechanics, electronics, biology, economics, medicine, pharmacokinetics and several other sciences can be modelled as special cases of system

$$\begin{cases} \dot{y}(t) = f(y_t, y, t), \ t \neq \tau_k(y(t)), \quad t \geq t_0, \\ \Delta y(t_k) = I_k(y(t_k)), \ t = \tau_k(y(t)), \quad k = 1, 2, \ldots, \\ f(0, 0, t) \equiv 0, \ I_k(0) = 0, \ k = 1, 2, \ldots \\ y_{t_0} = \phi \end{cases} \tag{43}$$

and yet stability results similar to Theorems 4.4 and 4.5 hold. Such results can be easily accomplished. Indeed, in case of system (43), it is enough to replace (11) by

$$F(y, t)(\vartheta) = \begin{cases} 0, \quad t_0 - r \leq \vartheta \leq t_0, \\ \int_{t_0}^{\vartheta} f(y_s, y(s), s)\, ds, \quad t_0 \leq \vartheta \leq t < +\infty, \\ \int_{t_0}^{t} f(y_s, y(s), s)\, ds, \quad t_0 \leq t \leq \vartheta < +\infty, \end{cases}$$

and to consider equation (12) so that equations (43) and (3) are equivalent for G given by (13). The proof of this fact follows as in [4], Theorems 3.4 and 3.5 applying the observations in the last section of [4].

In the particular non impulsive case (i.e., when $I_k(x) \equiv 0$), there are a number of problems modelled by particular forms of $\dot{y}(t) = f(y_t, y, t)$. In population dynamics, for instance, we can mention the well-known Lasota-Wazewska model

$$N'(t) = -\mu N(t) + pe^{-\sigma N(t-r)},$$

the Nicholson's blowflies equation

$$N'(t) = -\delta N(t) + pN(t-r)e^{-aN(t-r)}$$

and the Nazarenko's equation

$$x'(t) = -px(t) + \frac{qx(t)}{\sigma + x^n(t-r)}.$$

See, e.g., [1], [7], [8], [11], [15]. However all such problems can be considered as being subject to variable moments of impulse effects (or controls). Therefore it is important to have stability results for such equations.

One can consider, further, that the delay in equation (43) is variable, that is $r(t)$ is a function of t satisfying $r'(t) < \Delta < 1$. In this case, we consider $G^-([-r(t), 0], \mathbb{R}^n)$ instead of

$G^-([-r,0],\mathbb{R}^n)$ and all other appropriate changes. Also $F(y,t)$ is given by

$$F(y,t)(\vartheta)=\begin{cases}0, & t_0-r(t)\leq\vartheta\leq t_0,\\ \int_{t_0}^{\vartheta} f(y_s,y(s),s)\,ds, & t_0\leq\vartheta\leq t<+\infty,\\ \int_{t_0}^{t} f(y_s,y(s),s)\,ds, & t_0\leq t\leq\vartheta<+\infty.\end{cases}$$

In the particular case of the differential-difference system $\dot{y}(t)=f(y(t-r(t)),y(t),t)$, $t\neq t_k$, we define

$$F(y,t)(\vartheta)=\begin{cases}0, & t_0-r(t)\leq\vartheta\leq t_0,\\ \int_{t_0}^{\vartheta} f(y(s-r(s)),y(s),s)\,ds, & t_0\leq\vartheta\leq t<+\infty,\\ \int_{t_0}^{t} f(y(s-r(s)),y(s),s)\,ds, & t_0\leq t\leq\vartheta<+\infty.\end{cases}$$

Then extensions to the case where several delays are present can be obtained similarly.

Consider the general delayed neural network

$$\dot{y}_i=-c_iy_i(t)+\sum_{j=1}^{n}a_{ij}g_j(y_j(t))+\sum_{j=1}^{n}b_{ij}f(y_j(t-r_{ij}(t)))+H_i(t),\quad i=1,2,\ldots,n, \tag{44}$$

where $c_i>0$, $0\leq r_{ij}(t)\leq r$. Equation (44) describes the evolution process of the neural networks, where n corresponds to the number of units in the neural networks, y_i corresponds to the state variable, $f_j(x_j),g_j(x_j)$ are activation functions of the neurons, c_i is the neuron changing time constant, a_{ij},b_{ij} are the weights of the neuron interconnections, H_i is the internal bias, and $r_{ij}(t)$ is the transmission delay. In particular, equation (44) encompasses models as the Hopfield neural network, bidirectional neural networks, cellular neural networks, recurrent neural networks, etc. See, e.g., [2], [10], [14], [20] and [21].

By appropriate transformations, equation (44) can be formulated in a form like $\dot{y}(t)=f(y(t-r_1(t)),y(t-r_2(t)),y(t-r_3(t)),y(t),t)$, but with several delays $r_{ij}(t)$, and similar results as Theorems 4.4 and 4.5 hold for this kind of system when it undergoes abrupt changes at fixed moments or variable moments as in (43).

6. Appendix

In this part of our paper, we present the concept of integrability Kurzweil.

A *tagged division* of a compact interval $[a,b]\subset\mathbb{R}$ is a finite collection

$$\{(\tau_i,[s_{i-1},s_i]):i=1,2,\ldots,k\},$$

where $a=s_0\leq s_1\leq\ldots\leq s_k=b$ is a division of $[a,b]$ and $\tau_i\in[s_{i-1},s_i]$, $i=1,2,\ldots,k$.

A *gauge* on $[a,b]$ is any function $\delta:[a,b]\to(0,+\infty)$. Given a gauge δ on $[a,b]$, a tagged division $d=(\tau_i,[s_{i-1},s_i])$ of $[a,b]$ is *δ-fine* if, for every i,

$$[s_{i-1},s_i]\subset\{t\in[a,b]:|t-\tau_i|<\delta(\tau_i)\}.$$

Let X be a Banach space. Now, we define the type of integration which belongs to Jaroslav Kurzweil.

Definition 6.1. *A function* $U(\tau, t) : [a,b] \times [a,b] \to X$ *is* Kurzweil integrable *over* $[a,b]$, *if there is a unique element* $I \in X$ *such that given* $\varepsilon > 0$, *there is a gauge* δ *of* $[a,b]$ *such that for every* δ*-fine tagged division* $d = (\tau_i, [s_{i-1}, s_i])$ *of* $[a,b]$, *we have*

$$\|S(U,d) - I\| < \varepsilon,$$

where $S(U,d) = \sum_i [U(\tau_i, s_i) - U(\tau_i, s_{i-1})]$. *In this case, we write* $I = \int_a^b DU(\tau, t)$ *and use the convention* $\int_a^b DU(\tau,t) = -\int_b^a DU(\tau,t)$, *whenever* $b < a$.

The Kurzweil integral was described extensively in Chapter I of [18] for the case $X = \mathbb{R}^n$ (see Definition 1.2 in [18]).

It worths mentioning that the Kurzweil integral is linear, additive on disjoint intervals and emcompasses the known Perron-Stieljs integral as well as its improper integrals. For more properties, the reader may want to consult [18].

References

[1] J. F. M. Al-Omari; S. A. Gourley, Dynamics of a stage-structured population model incorporating a state-dependent maturation delay. *Nonlinear Anal. Real World Appl.* 6(1), (2005), 13-33.

[2] J. Cao; J. Wang, Global exponential stability and periodicity of recurrent neural networks with time delays. *IEEE Trans. Circuits Syst. I Regul. Pap.* 52(5), (2005), 920-931.

[3] M. Federson; J. B. Godoy, New continuous dependence results for impulsive functional differential equations, preprint.

[4] M. Federson; Š. Schwabik, Generalized ODEs approach to impulsive retarded differential equations, *Differential and Integral Equations* 19(11), (2006), 1201-1234.

[5] M. Federson; Š. Schwabik, Stability for retarded differential equations, *Ukr. Math. Journal* 60, (2008), 107-126.

[6] M. Federson; Š. Schwabik, A new approach to impulsive retarded differential equations: stability results, *Functional Differential Equations* 16(4), (2009), 583-607.

[7] K. Gopalsamy, *Stability and oscillations in delay differential equations of population dynamics.* Mathematics and its Applications, 74. Kluwer Academic Publishers Group, Dordrecht, 1992.

[8] I. Györi; G. Ladas, *Oscillation theory of delay differential equations. With applications.* Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1991.

[9] J. K. Hale; S. M. Verduyn Lunel, *Introduction to functional differential equations.* Applied Mathematical Sciences, 99. Springer-Verlag, New York, 1993.

[10] J. J. Hopfield, Neural networks and physical systems with emergent collective computational abilities. *Proc. Nat. Acad. Sci. U.S.A.* 79(8), (1982), 2554-2558.

[11] Y. Kuang, *Delay differential equations with applications in population dynamics.* Mathematics in Science and Engineering, 191. Academic Press, Inc., Boston, MA, 1993. Kurzweil 1957

[12] J. Kurzweil, Generalized ordinary differential equations and continuous dependence on a parameter, *Czechoslovak Math. J.* 7(82) (1957), 418-448.

[13] Liu, Xinzhi; G. Ballinger, Uniform asymptotic stability of impulsive delay differential equations. *Comput. Math. Appl.* 41(7-8), (2001), 903-915.

[14] S. Mohamad, Global exponential stability in continuous-time and discrete-time delayed bidirectional neural networks. *Phys. D* 159(3-4), (2001), 233–251.

[15] S. H. Saker; S. Agarwal, Oscillation and global attractivity in a nonlinear delay periodic model of population dynamics. *Appl. Anal.* 81(4), (2002), 787-799.

[16] J. H. Shen, Razumikhin techniques in impulsive functional-differential equations. *Nonlinear Anal.* 36(1) (1999), 119-130.

[17] Jianhua Shen; Jurang Yan, Razumikhin type stability theorems for impulsive functional-differential equations. *Nonlinear Anal.* 33 (5), (1998), 519-531.

[18] Š. Schwabik, *Generalized Ordinary Differential Equations*, World Scientific,Singapore, Series in Real Anal., vol. 5, 1992.

[19] Š. Schwabik, Variational stability for generalized ordinary differential equations, *Časopis Pěst. Mat.* 109(4), (1984), 389-420.

[20] Xu, Daoyi; Zhao, Hongyong; Zhu, Hong, Global dynamics of Hopfield neural networks involving variable delays. *Comput. Math. Appl.* 42(1-2), (2001), 39-45.

[21] Zhou, Dongming; Cao, Jinde, Globally exponential stability conditions for cellular neural networks with time-varying delays. *Appl. Math. Comput.* 131(2-3), (2002), 487-496.

(S. Afonso) Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo-Campus de São Carlos, Caixa Postal 668, 13560-970, São Carlos SP, Brazil
E-mail address: suzmaria@icmc.usp.br

(E. Bonotto) Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo-Campus de São Carlos, Caixa Postal 668, 13560-970, São Carlos SP, Brazil
E-mail address: ebonotto@icmc.usp.br

(M. Federson) Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo-Campus de São Carlos, Caixa Postal 668, 13560-970, São Carlos SP, Brazil
E-mail address: federson@icmc.usp.br

(L. Gimenes) Departamento de Matemática, Universidade Estadual de Maringá, 87020-900, Maringá-PR, Brazil
E-mail address: lpgarantes@uem.br

NOTAS DO ICMC

SÉRIE MATEMÁTICA

336/10 AFONSO, S.; BONOTTO, E.; FEDERSON, M.; GIMENES, L. Boundedness of solutions of retarded functional differential equations with variable impulses via generalized ordinary differential equations.

335/10 ARAGÃO-COSTA, E.R.; CARABALLO, T.; CARVALHO, A.N.; LANGA, J.A. – Continuity of Lyapunov functions ando f energy level for a generalized gradient semigroup.

334/10 ARAGÃO-COSTA, E.R.; CARABALLO, T.; CARVALHO, A.N.; LANGA, J.A. – Stability of gradient semigroups under perturbations.

333/10 MA, T.F.; NARCISO, V. – Global attractor for a model of extensible beam with nonlinear damping and source terms.

332/10 CARVALHO, A.N.; CHOLEWA, J.W.; DLOTKO, T. – Equi-exponential attraction and Rate of convergence of attractors for singularly perturbed evolution equations.

331/10 FEDERSON, M.; GODOY, J.B. – Averaging for impulsive functional differential equations: a new approach.

330/10 MORGADO, M. F. Z.; SAIA, M. J. – Lê numbers of pham-Brieskorn arrangements.

329/10 GRULHA JÚNIOR, N.G. – Stability of the Euler obstruction of f on free divisors.

328/10 AHMED, I.; RUAS, M.A.S. – Invariants of relative right and contact equivalences.

327/10 CARO, N.; LEVCOVITZ, D. – On theorem of Stafford.