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C^{ℓ}-G-Triviality of Newton Non Degenerate Map Germs, $G = R$, C and K

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C^ℓ-G-triviality of Newton non degenerate map germs, $G = \mathcal{R}$, $\mathcal{C}$ and $\mathcal{K}$.

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Resumo

Neste artigo são determinadas condições suficientes para a C^ℓ-G-trivialidade (G é um dos grupos de Mather $\mathcal{R}$, $\mathcal{C}$ or $\mathcal{K}$) de deformações de germes de aplicação $f_t : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ of type $f_t(x) = f(x) + th(x)$ que satisfazem uma condição de Newton não degeneração. Estas condições são determinadas em termos da filtração de Newton do germe h.

Abstract

We provide a suficient condition for the C^ℓ-G-triviality (G is one of Mather's groups $\mathcal{R}$, $\mathcal{C}$ or $\mathcal{K}$) of deformations of map germs $f_t : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ of type $f_t(x) = f(x) + th(x)$ which satisfy a non-degeneracy condition. This condition is given in terms of the Newton filtration of the map germ h.

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1 Introduction

The determinacy of topological triviality map germs is a fundamental subject in singularity theory, and many works are devoted to the characterization of finite determinacy and to estimating the order of determinacy, with respect to various equivalence relations. In particular, finding the accurate order of determinacy of a map germ is important for applications or practical problems as well as for pure mathematical theory.

In this paper we provide new estimates on the degree of C^ℓ-G-determinacy $0 \leq \ell < \infty$ G is one of Mather's groups $\mathcal{R}$, $\mathcal{C}$ or $\mathcal{K}$, of Newton non-degenerated map germs. We generalize previous results on weighted homogeneous map germs satisfying a convenient Lojasiewicz condition given by Ruas and Saia in [3]. The results give an explicit order such that the C^ℓ geometrical structure of a Newton non-degenerated homogeneous polynomial map germ is preserved after higher order perturbations. Our method consists of constructing controled vector fields based on control functions determined by a convenient Newton polyhedron.

2 The Newton filtration

We denote the ring of real analytic germs by $\mathcal{E}_n$. To construct a Newton polyhedron we fix an $n \times m$ matrix $A = (a_i^j)$, with $i = 1, ..., n$, $j = 1, ..., m$, $a^j = (a_1^j, \ldots, a_n^j) \in \mathbb{Q}_+^n$ and $m \geq n$, such that the first n colluns of A are of type $a^j = (0, ..., 0, a_j^j, 0, ..., 0)$ and $a_j^j > 0$, for all $j = 1, ..., n$..

When $n = 2$, for example we have: $A = \begin{pmatrix} a_1^1 & 0 & a_1^3 & \cdots & a_1^m \\ 0 & a_2^2 & a_2^3 & \cdots & a_2^m \end{pmatrix}$.

The support of A is denoted by $Supp(A) = \{a^j,\ j = 1, ..., m\}$, its Newton polyhedron, denoted by $\Gamma_+(A)$, is the convex hull in $\mathbb{R}^n$ of the set $Supp(A) + \mathbb{R}_+^n$ and the Newton diagram of A, denoted $\Gamma(A)$, is the union of the compact faces of $\Gamma_+(A)$.

For a fixed Newton polyhedron $\Gamma_+(A)$ and for each $w = (w_1, \ldots, w_n) \in \mathbb{R}_+^n$ we define:

Definition 2.1 *(a)* $\ell(w) = \min\{\langle w, k\rangle : k \in \Gamma_+(g)\}$, $\langle w, k\rangle = \sum_{i=1}^{n} w_i k_i$.

(b) $\Delta(w) = \{k \in \Gamma_+(g) : \langle w, k\rangle = \ell(w)\}$.

(c) Two vectors $a, a' \in \mathbb{R}_+^{n*}$ are equivalent if $\Delta(a) = \Delta(a')$.

A vector w is called a primitive integer if it is the vector with minimum length in $C(w) \cap (\mathbb{Z}_+^n - \{0\})$, where $C(w)$ is the half ray emanating from 0 passing through w.

Since each $(n-1)$-dimensional face Δ of $\Gamma(g)$ is associated to a primitive integer $w \in \mathbb{R}_+^{n*}$ such that $\Delta = \Delta(w)$, we fix $v^k = (v_1^k, ..., v_n^k)$, $k = 1, ..., r$, the set of primitive integers associated to the $(n-1)$-dimensional faces δ of $\Gamma_+(A)$ and denote by M the number $M = m.m.c.\{\ell(v^k)\}$.

Definition 2.2 *For any monomial $x^\alpha = x_1^{\alpha_1}, \dots, x_n^{\alpha_n}$ in $\mathcal{E}_n$, we define*

$$\varphi(\alpha) = \min_{k=1}^{r} \left\{ \frac{M}{\ell(v^k)} \langle \alpha, v^k \rangle \right\}.$$

For an analytic real germ $f : \mathbb{R}^n, 0 \to \mathbb{R}, 0$ we call

$$fil(f) := \inf\{\varphi(\alpha) \ / \ \alpha \in Supp(f)\}.$$

In order to construct the controlled vector fields we define a control function in terms of a fixed Newton polyhedron $\Gamma_+(A)$.

For any vector a^j of $\mathbb{Q}_+^n$ and $p \in \mathbb{R}_+$, we denote by p^j the vector $p^j = pa^j = (pa_1^j, pa_2^j, \dots, pa_n^j)$.

We choose the number p as large as needed to have $2p_i^j$ integer and define the smooth function $\rho(x)$, called *the control* of the Newton polyhedron $\Gamma_+(A)$:

$$\rho(x) = \left(\sum_{j=1}^{m} x^{2p^j} \right)^{\frac{1}{2p}} = \left(\sum_{j=1}^{m} x_1^{2p_1^j} x_2^{2p_2^j} \cdots x_n^{2p_n^j} \right)^{\frac{1}{2p}}. \tag{1}$$

Definition 2.3 ([6], p. 524) *An analytic function germ $f : \mathbb{R}^n, 0 \to \mathbb{R}, 0$ with Taylor series $f(x) = \sum_\nu c_\nu x^\nu$, onde $\nu = (\nu_1, ..., \nu_n)$ is an A-form of degree d if $f(x) = \sum_{\nu \in \Gamma(\rho^d)} c_\nu x^\nu$.*

Lemma 2.4 ([6], p. 524) *Let $f : \mathbb{R}^n, 0 \to \mathbb{R}, 0$ be a polynomial function germ with $supp(f) \in \Gamma_+(\rho^d)$. T hen there exists a constant $c_1 > 0$ and a neighbourhood of the origin such that $\|f(x)\| \leq c_1 \rho(x)^d$.*

In the sequel we shall consider the decomposition of any analytic function germ $f : \mathbb{R}^n, 0 \to \mathbb{R}, 0$ in A-forms

$$f(x) = H_d(x) + \cdots + H_\ell(x) + \cdots \quad , \ell \geq d$$

where each function germ H_i is an A-form of degree i.

Definition 2.5 ([6], p. 525) *We say that 0 is an A-isolated point of f if for each compact face γ of $\Gamma(\rho^d)$, the equation $f_\gamma(x) = 0$ does not have solution in $(\mathbb{R} - \{0\})^n$.*

Lemma 2.6 ([6], p. 525) *Let $f : \mathbb{R}^n, 0 \to \mathbb{R}, 0$ be an analytic function germ written as $f(x) = H_d(x) + \cdots H_\ell(x) + \cdots$. If 0 is A isolated for f then there exists a real $c_2 > 0$ such that $\|f(x)\| \geq c_2 \rho(x)^d$ for all x in a neighborhood of the origin.*

The next lemmas form the key tool to guarantee the class of differentiability of the controlled vector fields.

Lemma 2.7 *For any analytic function germ $h : \mathbb{R}^n, 0 \to \mathbb{R}, 0$ with $h \in int(\Gamma_+(\rho^d))$:*

$$\lim_{x \to 0} \frac{h(x)}{\rho(x)^d} = 0.$$

Proof: It is sufficient to prove that for all $a = (a_1, ..., a_n)$ in the interior of $\Gamma_+(\rho^d)$, we have $\lim_{x \to 0} x^a / \rho(x)^d = 0$.

Suppose that this does not occurs, hence there exists a constant $c > 0$ such that $\|x^a / \rho(x)^d\| \geq c$ for all x in a neighborhood of the origin. Therefore the origin 0 is in the closure of the set $X := \{x \in \mathbb{R}^n \; / \; \|x\| \geq c\rho(x)^d\}$.

As the set X semi-analytic we apply the Curve Selection Lemma to conclude that there exists an analytic curve $\lambda : (0, \epsilon] \to X$, with $\lambda(0) = 0$ such that

$$\lambda_1(t) \sim t^{\alpha_1}, ..., \lambda_n(t) \sim t^{\alpha_n}.$$

Therefore, since $\rho(\lambda(t))^d \leq \frac{1}{c} \|\lambda(t)^a\|$ we obtain

$$inf_j \{\langle da^j, \alpha \rangle\} \geq \langle a, \alpha \rangle.$$

But $\Delta(\alpha) := \{b \in \Gamma_+(\rho^d) \ / \ \langle b, \alpha \rangle = \ell(\alpha)\}$ is a face of $\Gamma_+(\rho^d)$ with $\ell(\alpha) := \min\{\langle c, \alpha \rangle \ / \ c \in \Gamma_+(\rho^d)\}$.

Since da^j, is one of the vertices of $\Delta(\alpha)$,

$$\langle da^j, \alpha \rangle = \ell(\alpha).$$

However,

$$\langle a, \alpha \rangle \leq \langle da^j, \alpha \rangle = \ell(\alpha)$$

hence $a \in \Gamma(\rho^d)$ and we obtain a contradiction to the hypothesis. ■

Lemma 2.8 *Let $h : \mathbb{R}^n, 0 \to \mathbb{R}, 0$ in $\mathcal{E}_n$ such that $fil(h) \geq fil(\rho^d) + \ell R + 1$. Then $\frac{h(x)}{\rho(x)^d}$ is differentiable of class C^ℓ.*

Proof: The proof is done by induction on ℓ.

For $\ell = 1$, we obtain $fil(h(x)) \geq fil(\rho(x)^d) + R + 1$.

Since $f(x) = \frac{h(x)}{\rho(x)^d}$ we have

$$\nabla f(x) = \frac{1}{\rho(x)^{2d}}(\rho^d.\nabla h - h.\nabla \rho^d)$$

and also that

$$\begin{aligned} fil(\rho^d.\nabla h - h.\nabla \rho^d) &\geq fil(h) + fil(\rho^d) - R \\ &\geq 2fil(\rho^d) + 1 \\ &= fil(\rho^{2d}) + 1. \end{aligned}$$

Applying the Lemma 2.7 we obtain $\lim\limits_{x \to 0} \frac{1}{\rho^{2d}}(\rho^d.\nabla h - h.\nabla \rho^d) = 0$, and ∇f is continuous. Therefore f is of class C^1.

Suppose now that any function $f(x) = \frac{h_1(x)}{\rho(x)^d}$ satisfying

$$fil(h_1) \geq fil(\rho^d) + (\ell - 1)R + 1,$$

is of class $C^{\ell-1}$. Let $f(x) = \frac{h_1(x)}{\rho(x)^d}$ with $fil(h_1) \geq fil(\rho^d) + \ell R + 1$. Hence $\nabla f(x) = \frac{H(x)}{\rho(x)^d}$, with $fil(H(x)) \geq fil(\rho^d) + (\ell - 1)R + 1$, is of class $C^{\ell-1}$ therefore f is of class C^ℓ. ■

Ruas and Saia in [3] determined conditions for the C^ℓ-G- triviality,$\ell \geq 0$, $G = \mathcal{R}$, $\mathcal{C}$ or $\mathcal{K}$, of weighted homogeneous map germs with isolated singularity, in terms of the weights and degrees. Here we generalize these results for the class of map germs that are A-forms.

Definition 2.9 *An analytic map germ $f : \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$; $f = (f_1, \dots, f_p)$ is anA-form of degree $d = (d_1, \dots, d_p)$ if each f_i is an A-form of degree d_i.*

The main idea is to choose, for each group $G = \mathcal{R}$, $\mathcal{C}$ or $\mathcal{K}$, a convenient function such that we can compare with a control function $\rho(x)$ associated to a Newton polyhedron $\Gamma_+(A)$. First we shall do it for the group $\mathcal{R}$,

3 The group $\mathcal{R}$

For each polynomial map germ $f : \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$ we denote $\alpha := m.m.c.\{s_I\}$ and call $N_{\mathcal{R}}f := \sum_I M_I^{2\alpha_I}$ where M_I, $I = \{i_1, ..., i_p\}$, is a minor of order p of the Jacobian matrix df, and $\alpha_I = \alpha/s_I$, with $s_I := fil(M_I)$.

We write $N_{\mathcal{R}}f = H_D + \cdots + H_L$, with $L > D$, and suppose that $N_{\mathcal{R}}f$ has 0 as $\Gamma_+(\rho^D)$-isolated point, we can apply the Lemmas (2.4) e (2.6) to obtain

$$N_{\mathcal{R}}f \leq \|H_D\| + \cdots + \|H_L\| \leq c_D\rho^D + \cdots + c_L\rho^L \leq (c_D + \cdots c_L)\rho^D$$

and

$$N_{\mathcal{R}}f \geq c\rho^D$$

Hence there exist constants c_1 and $c_2 > 0$ such that

$$c_1\rho^D \leq N_{\mathcal{R}}f \leq c_2\rho^D$$

If we consider now a deformation $f_t = f + t\theta$ of f with $fil(\theta_i) > fil(f_i)$ we define $N_{\mathcal{R}}f_t := \sum_I M_{t_I}^{2\alpha_I}$.

Lemma 3.1 *Suppose that $N_{\mathcal{R}}f = \sum_I M_I^2 = H_D + \cdots + H_L$ has 0 as a $\Gamma_+(\rho^D)$-isolated point for some Newton polyhedron $\Gamma_+(A)$. If $f_t = f + t\theta$ is a deformation of f with $fil(\theta_i) > fil(f_i)$, there exist constants c_1 and $c_2 > 0$ and a neighborhood V of 0 such that*

$$c_1\rho^D \leq N_{\mathcal{R}}f_t \leq c_2\rho^D$$

for all $x \in V$

Proof: Since $N_{\mathcal{R}}f_t = N_{\mathcal{R}}f + t\Theta$, with $fil(\Theta) > fil(N_{\mathcal{R}}f)$, we can write $N_{\mathcal{R}}f \leq N_{\mathcal{R}}f_t + ||\Theta||, \forall\, 0 \leq t \leq 1$.

There exist a constant $c_1 > 0$ such that $c_1\rho^D \leq N_{\mathcal{R}}f \leq N_{\mathcal{R}}f_t + ||\Theta||$, since $fil(\Theta) > fil(N_{\mathcal{R}}f)$ we have $\lim_{x\to 0} \Theta/\rho^D = 0$. Therefore

$$c_1'\rho^D \leq N_{\mathcal{R}}f_t.$$

On the other hand,

$$N_{\mathcal{R}}f_t \leq N_{\mathcal{R}}f + ||\Theta|| \leq c_2\rho^D + ||\Theta|| \leq (c_2 + c_3)\rho^D.$$

In order to show the main result of this section we fix a Newton polyhedron $\Gamma_+(A)$ with associated primitive integers v^j, and call

$$R = \max_j \max_i \{\frac{M}{\ell(v^j)} v_i^j\}, \text{ and } r = \min_j \min_i \{\frac{M}{\ell(v^j)} v_i^j\}.$$

Let $f_t(x) = f(x) + t\theta(x)$, with $\theta = (\theta_1, ..., \theta_p)$, be a deformation of a polynomial map germ $f : \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$. Suppose that $N_{\mathcal{R}}f := \sum_i M_I^{2\alpha_I}$ has 0 as a $\Gamma_+(\rho^D)$-isolated point.

We prove now the following:

Proposition 3.2 **(a)** *If $fil(\theta_i) \geq fil(f_i) + \ell R - r + 1$, then for $t \in [0,1]$, f_t is C^ℓ-$\mathcal{R}$-trivial for all $\ell \geq 1$;*

(b) *If $fil(\theta_i) \geq fil(f_i)$, then f_t is C^0-$\mathcal{R}$-trivial for all $t \in [0,1]$.*

Proof: We follow the proof given by Ruas and Saia for the weighted homogeneous case, [3].

(a) For each p minor M_{t_I} of df_t, we construct the vector field W_I defined by the cofactors of M_{t_I} :

$$W_I = \sum_{i=1}^{n} w_i \frac{\partial}{\partial x_i}, \text{ with } \begin{cases} w_i = 0, \text{ if } i \notin I \\ w_{i_m} = \sum_{j=1}^{p} N_{ji_m}(\frac{\partial f_t}{\partial t})_j, \text{ if } i_m \in I \end{cases}$$

Here N_{ji_m} denotes the $(p-1) \times (p-1)$ minor cofactor de of $\frac{\partial f_j}{\partial x_{i_m}}$, in df.

Hence $\frac{\partial f_t}{\partial t} M_{t_I} = df(W_I)$.

Now we consider the vector field $W_{\mathcal{R}} := \sum_I M_I^{2\alpha_I - 1} W_I$, hence $N_{\mathcal{R}} f_t \cdot \frac{\partial f_t}{\partial t} = df_t(W_{\mathcal{R}})$ and

$$\begin{aligned}
fil(W_{\mathcal{R}}) &= \min\{fil(M_I^{2\alpha_I - 1}) + fil(W_I)\} \\
&\geq \min\{2\alpha - fil(M_I) + fil(N_{ji_m}) + fil(\theta_j)\} \\
&\geq \min\{2\alpha - fil(M_I) + fil(M_I) - fil(\frac{\partial f_j}{\partial x_{i_m}}) + fil(\theta_j)\} \\
&\geq \min\{2\alpha - (fil(f_j) - r) + fil(\theta_j)\} \\
&\geq 2\alpha + \ell R + 1.
\end{aligned}$$

Finally we consider the vector field $V : \mathbb{R}^n \times \mathbb{R}, 0 \to \mathbb{R}^n \times \mathbb{R}, 0$:

$$V(x) = \frac{W_{\mathcal{R}}}{N_{\mathcal{R}} f_t}.$$

which is of class C^ℓ from Lemmas (2.7) and (2.8).

The C^ℓ-triviality, for small values of t, follows from the equation $\frac{\partial f_t}{\partial t}(x,t) = (df_t)_x(V(x,t))$, and a similar argument shows that the result follows for all $t = t_0, \forall t_0 \in [0,1]$.

(b) We have that

$$\begin{aligned}
fil(W_{\mathcal{R}}) &= fil(\sum_I M_I^{2\alpha_I - 1} W_I) \\
&\geq \min\{2\alpha - fil(M_I) + fil(M_I) - fil(\frac{\partial f_j}{\partial x_{i_m}}) + fil(\theta_j)\} \\
&\geq \min\{2\alpha - (fil(f_j) - r) + fil(f_j) + R - r\} \\
&= 2\alpha + r = fil(\rho^D) + r.
\end{aligned}$$

And $fil(W_{\mathcal{R}}) \geq fil(\rho^D) + r = fil(\rho^D \|x\|)$. Hence $\frac{W_{\mathcal{R}}}{\rho^D \|x\|}$ is limited, and

$$\|\frac{W_{\mathcal{R}}}{N_{\mathcal{R}} f_t}\| \leq c \|\frac{W_{\mathcal{R}}}{\rho^D}\| \leq c' \|x\|$$

is integrable. ■

3.1 Examples

Example 3.3 Here we show that the estimates obtained can not be improved.

Let $f(x,y) = (x^2+y^2)^2$, Kuiper in [5] showed that the deformation $f_t(x,y) = (x^2+y^2)^2 + tx^{5+p}$ is not C^{p+2}-trivial, $\forall\, p \geq 0$.

However, in this case we have $fil(x^{5+p}) \geq fil(f) + \ell R - r + 1 \Leftrightarrow 5+p \geq 4+\ell$, therefore $\ell \in \{1,2,...,p+1\}$ and f_t is C^{p+1}-trivial.

Example 3.4 Let $f : \mathbb{R}^2,0 \to \mathbb{R}^2,0$; $f(x,y) = (xy, x^{2b+2} - y^{2b} + x^{2r}y^{2s})$, with $r+s=b$, $r+2s=b+1$ and $r>s$.

The 2×2 minor of df is $M = -2((b+1)x^{2b+2} + by^{2b} + (r-s)x^{2r}y^{2s})$.

We remark that we can write $M = H_{2b(b+1)}$ with

$$H_{2b(b+1)} = -2((b+1)x^{2b+2} + by^{2b} + (r-s)x^{2r}y^{2s})$$

being a degree $2b(b+1)$ A-form.

If $br+(b+1)s < (b+1)b$, we consider the matrix $A = \begin{pmatrix} \frac{1}{b} & 0 & \frac{r}{(b+1)b} \\ 0 & \frac{1}{b+1} & \frac{s}{(b+1)b} \end{pmatrix}$,

with corresponding Newton polyhedron $\Gamma_+(\rho^{2b(b+1)})$ having two faces with vertices $\{(2b+2,0),(2r,2s),(0,2b)\}$.

The associate control function is $\rho(x,y) = (x^{2b+2} + y^{2b} + x^{2r}y^{2s})^{\frac{1}{2b(b+1)}}$.

Since $m.m.c\{\ell(v_1),\ell(v_2)\} = 2b(b+1)$, we obtain

$$\begin{aligned}
fil(xy) &= \min\{(b+1)\langle(1,1),(1,1)\rangle, b\langle(1,2),(1,1)\rangle\} = 2b+2\\
fil(y^{2b}) &= \min\{(b+1)\langle(1,1),(0,2b)\rangle, b\langle(1,2),(0,2b)\rangle\} = 2b(b+1)\\
fil(x^{2r}y^{2s}) &= \min\{(b+1)\langle(1,1),(2r,2s)\rangle, b\langle(1,2),(2r,2s)\rangle\} = 2b(b+1)\\
fil(x^{2b+2}) &= \min\{(b+1)\langle(1,1),(2b+2,0)\rangle, b\langle(1,2),(2b+2,0)\rangle\} = 2b(b+1).
\end{aligned}$$

Hence $fil(xy) = 2b+2$ e $fil(x^{2b+2} - y^{2b} + x^{2r}y^{2s}) = 2b(b+1)$, with $R = 2b$ and $r = b$.

In order to obtain the conditions of the Proposition 3.2, it is needed that

$$fil(\theta_i) \geq fil(f_i) + 2b\ell - b + 1$$

therefore

$$fil(\theta_1) \geq b + 2b\ell + 3$$

and

$$fil(\theta_2) \geq 2b^2 + b + 2b\ell + 1.$$

For example, if we consider $(\theta_1, \theta_2) = (x^5y^9, y^{4(b+1)})$, $fil(\theta_1) = 14b + 14$ and $fil(\theta_2) = 4b^2 + 8b + 4$, then if $b \geq 3$, the number $\ell = 6$ satisfies the above inequalities and the family

$$f_t(x, y) = f(x, y) + t(\theta_1, \theta_2) = (xy + tx^5y^9, x^{2b+2} - y^{2b} + x^{2r}y^{2s} + ty^{4(b+1)})$$

is C^6-$\mathcal{R}$-trivial.

Example 3.5 Let $f : \mathbb{R}^2, 0 \to \mathbb{R}, 0$, with $f(x, y) = y^7 + x^4y + x^9$. Then, $df = (4x^3y + 9x^8, 7y^6 + x^4)$.

Consider $A = \begin{pmatrix} 4 & 0 & 3 \\ 0 & 6 & 1 \end{pmatrix}$, with associate control $\rho(x, y) = (y^{12} + x^6y^2 + x^8)^{\frac{1}{2}}$ and minors $M_1 = 4x^3y + 9x^8 = H_1 + H_2$ and $M_2 = 7y^6 + x^4 = \widetilde{H}_1$.

Here we remark that

$$\begin{aligned} \{(x, y) \in \mathbb{R}^2 \ / \ N^*_{\mathcal{R}}f|_{\Delta_1}\} &= \{(x, y) \in \mathbb{R}^2 \ / \ (4x^3y)^{2\alpha_1} + (7y^6)^{2\alpha_2} = 0\} \subset \\ &\subset \{(x, y) \in \mathbb{R}^2 \ / \ xy = 0\}. \end{aligned}$$

and

$$\begin{aligned} \{(x, y) \in \mathbb{R}^2 \ / \ N^*_{\mathcal{R}}f|_{\Delta_2}\} &= \{(x, y) \in \mathbb{R}^2 \ / \ (4x^3y)^{2\alpha_1} + (x^4)^{2\alpha_2} = 0\} \subset \\ &\subset \{(x, y) \in \mathbb{R}^2 \ / \ xy = 0\}. \end{aligned}$$

Then $N^*_{\mathcal{R}}f$ has the origin as a $\Gamma_+(\rho^{36})$-isolated point.

Since $m.m.c.\{18, 4\} = 36$, we calculate

$$\varphi(x^ay^b) = \min\{2\langle(a, b), (5, 3)\rangle, 9\langle(a, b), (1, 1)\rangle\}.$$

Hence $fil(f) = \min\{\varphi(y^7), \varphi(x^4y), \varphi(x^9)\} = 42$, with

$$R = \max_j \ \max_i\{(36/\ell(v^j))v^j_i\} = 10 \text{ and} r = \min_j \ \min_i\{(36/\ell(v^j))v^j_i\} = 6.$$

If we consider $\theta = x^5y^7$, hence $fil(\theta) = 92$. Then, it is enough to have $\ell = 5$ to apply the Proposition (3.2), since $fil(\theta) \geq fil(f) + \ell R - r + 1$ and

$$92 \geq 42 + 10\ell - 6 + 1 \Rightarrow \ell \leq \frac{55}{10}.$$

Therefore, from the Proposition (3.2) we have that $f_t(x, y) = y^7 + x^4y + x^9 + tx^5y^7$ is C^5-$\mathcal{R}$-trivial.

Example 3.6 The modified Briançon-Speder example

Let $f : \mathbb{K}^3, 0 \to \mathbb{K}, 0$, $f(x, y, z) = x^{15} + xy^7 + z^5$. For the case $\mathbb{K} = \mathbb{C}$, the family $F(x, y, z, t) = x^{15} + xy^7 + z^5 + ty^6z$ is topologically, since it has sconstant Milnor number for all t. Briançon e Speder showed in [1] that the variety $F^{-1}(0)$ in $\mathbb{C}^4$ is not Whtney equisingular on the parametre space $0 \times \mathbb{C}$ at 0. A full description of all equisingular deformations of f is given in [4]. The variety $F^{-1}(0)$, defined by $F(x, y, z, t) = f(x, y, z) + tx^ay^bz^c$ is Whitney equisingular if, and only if, $x^ay^bz^c \in \Gamma_+(\{(15, 0, 0), (0, 8, 0), (0, 0, 5), (1, 7, 0)\})$.

Her we consider the analogoues question for the real family $F : \mathbb{R}^4, 0 \to \mathbb{R}, 0$; $F(x, y, z, t) = f(x, y, z) + tx^ay^bz^c$.

Ruas and Saia in [3] showed that if $a + 2b + 3c \geq 18$ the family is Whitney equisingular. This result was improved by Fernandes and Ruas in [2], they showed the Whitney equisingularity for the family if $a + 2b + 3c \geq 17$.

Here we see that $df = (15x^{14} + y^7, 7xy^6, 5z^4)$. Let $A = \begin{pmatrix} 14 & 0 & 0 & 1 \\ 0 & 7 & 0 & 6 \\ 0 & 0 & 4 & 0 \end{pmatrix}$

We see that $N^*_{\mathcal{R}}f = M_1^{2\alpha_1} + M_2^{2\alpha_2} + M_3^{2\alpha_3}$ has the origin as an $\Gamma_+(\rho^{84})$-isolated point.

Since $m.m.c.\{84, 28\} = 84$, we have

$$\begin{aligned} \varphi(a, b, c) &= \min\{\langle (a, b, c), (6, 13, 21)\rangle, 3\langle (a, b, c), (4, 4, 7)\rangle\} \\ &= \min\{6a + 13b + 21c, 12a + 12b + 21c\}. \end{aligned}$$

Then, $\varphi(x^{15}) = 90$, $\varphi(xy^7) = 96$ and $\varphi(z^5) = 105$. Hence $fil(f) = 90$. As $R = 21$ and $r = 6$, for the C^1-triviality of the family f_t we need

$$fil(x^ay^bz^c) \geq fil(f) + R - r,$$

$$\text{i.e.,} \quad \min\{6a + 13b + 21c, 12a + 12b + 21c\} \geq 90 + 21 - 6 = 105.$$

Since the C^1-triviality implies the Whitney equisingularity, we obtain that $F^{-1}(0)$ is Whitney-equisingular along the parameter space if

$$\min\{6a + 13b + 21c, 12a + 12b + 21c\} \geq 105.$$

4 The group $\mathcal{C}$

For each anlytic map germ $f : \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$ we denote $N_{\mathcal{C}}f := \sum_{i=1}^{p}(f_i)^{2\beta_i}$ where

$$\beta_i := \frac{mmc\{fil(f_j), j = 1, ..., p\}}{fil(f_i)}.$$

If we suppose that $N_{\mathcal{C}}f = H_D + \cdots + H_L$ has 0 as $\Gamma_+(\rho^D)$- isolated point, we apply the Lemmas (2.4) and (2.6) to obtain

$$N_{\mathcal{C}}f \leq ||H_D|| + \cdots + ||H_L|| \leq c_D\rho^D + \cdots + c_L\rho^L \leq (c_D + \cdots + c_L)\rho^D$$

and

$$N_{\mathcal{C}}f \geq c\rho^D$$

Hence there exist constants c_1 and $c_2 > 0$ such that $c_1\rho^D \leq N_{\mathcal{C}}f \leq c_2\rho^D$.

If we consider now a deformation $f_t = f + t\theta$ of f with $fil(\theta_i) > fil(f_i)$ we define $N_{\mathcal{C}}f_t := \sum_{i=1}^{p}(f_{ti})^{2\beta_i}$ where β_i is defined in $N_{\mathcal{C}}f$

Lemma 4.1 *Suppose that $N_{\mathcal{C}}f$ has 0 as a $\Gamma_+(\rho^D)$-isolated point for some Newton polyhedron $\Gamma(A)$. If $f_t = f + t\theta$ is a deformation of f with $fil(\theta_i) > fil(f_i)$, there exist constants c_1 and $c_2 > 0$ and a neighbourhood V of 0*

$$c_1\rho^{D_1} \leq N_{\mathcal{C}}f_t \leq c_2\rho^{D_1}.$$

for all $x \in V$

Proof: Since $N_{\mathcal{C}}f_t = N_{\mathcal{C}}^*f + t\Theta$, with $fil(\Theta) > fil(N_{\mathcal{C}}f)$.

We obtain $N_{\mathcal{C}}f \leq N_{\mathcal{C}}f_t + ||\Theta||$, for all t with $0 \leq t \leq 1$.

Since $N_{\mathcal{C}}f = H_{D_1} + \cdots + H_{D_r}$ has 0 as an $\Gamma_+(\rho^{D_1})$-isolated point, there exist constants c_1 and $c_2 > 0$ with $c_1\rho^{D_1} \leq N_{\mathcal{C}}f \leq c_2\rho^{D_1}$.

Therefore

$$c_1\rho^{D_1} \leq N_{\mathcal{C}}f \leq N_{\mathcal{C}}^*f_t + ||\Theta||$$

and as $fil(\Theta) > fil(N_{\mathcal{C}}f)$, $\lim_{x\to 0} \Theta/\rho^{D_1} = 0$ and this implies that

$$c_1\rho^{D_1} \leq N_{\mathcal{C}}f_t.$$

On the other hand, $N_{\mathcal{C}} f_t \leq N_{\mathcal{C}} f \leq c_2 \rho^{D_1}$, and the result follows.

In order to show the main result of this section we fix a Newton polyhedron $\Gamma_+(A)$ with associated primitive integers v^j, and call

$$R = \max_j \max_i \{ \frac{M}{\ell(v^j)} v_i^j \}, \text{ and } r = \min_j \min_i \{ \frac{M}{\ell(v^j)} v_i^j \}.$$

Let $f_t(x) = f(x) + t\theta(x)$, with $\theta = (\theta_1, ..., \theta_p)$, be a deformation of a polynomial map germ $f : \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$. Suppose that $N_{\mathcal{C}} f$ has 0 as an $\Gamma_+(\rho^D)$-isolated point.

We prove now the following:

Proposition 4.2 *Deformations $f_t = f + t\theta$ of f are C^ℓ-$\mathcal{C}$-trivial for all $t \in [0,1]$, if $fil(\theta_i) \geq d + \ell R + 1$, for all i and $\ell \geq 1$, with $d := \max\{fil(f_i)\}$.*

Proof: The C^ℓ-$\mathcal{C}$-triviality of f_t is shown by constructing map germs $V_i : \mathbb{R}^n \times \mathbb{R}, 0 \to \mathbb{R}^p \times \mathbb{R}, 0$; $V_i = (V_{i1}, ..., V_{ip})$ of class C^ℓ, with $V_{ij}(x, 0) = \delta_{ij}(x)$ in such a way that $\frac{\partial f_t}{\partial t} = \sum_{i=1}^p V_i(x,t)(f_{ti})$.

Since

$$\frac{\partial f_t}{\partial t} = \frac{\partial f_t}{\partial t} \cdot \left(\frac{\sum_{i=1}^p f_{ti}^{2\beta_i - 1} f_{ti}}{N_{\mathcal{C}} f_t} \right), \text{ we define}$$

$W_i(x,t) = \frac{\partial f_t}{\partial t} \cdot f_{ti}^{2\beta_i - 1}$. Hence $\frac{\partial f_t}{\partial t} = \sum_{i=1}^{p} (W_i(x,t)/N_{\mathcal{C}} f_t)(f_{ti})$ and

$$\begin{aligned} fil(W_i) &= \min_j \{ fil(f_i^{2\beta_i - 1}) + fil(\theta_j) \} \geq \\ &\geq 2B - d + d + \ell R + 1 = \\ &= 2B + \ell R + 1, \ \forall i, \end{aligned}$$

where $B := mmc\{fil(f_j), j = 1, ..., p\}$.

Let $V : \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}, 0 \to \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}, 0$ the vector field defined as: $(0, V_p, 0)$, where $V_p(x, y, t) = \sum_{i=1}^p (W_i(x,t)/N_{\mathcal{C}} f_t) y_i$.

Therefore follows by Lemma (2.8) that V is of class C^ℓ. Hence the result follows by integrating the vector field V. ■

Example 4.3 Let $f : \mathbb{R}^2, 0 \to \mathbb{R}^2, 0$; $f(x,y) = (xy + x^2y^2, x^{2(c+1)} + xy - y^{2c})$ with $c \geq 2$.

Consider $A = \begin{pmatrix} 2(c+1) & 0 & 1 \\ \\ 0 & 2c & 1 \end{pmatrix}$ with associate control function

$$\rho(x,y) = (y^{4c} + x^2y^2 + x^{4(c+1)})^{\frac{1}{2}}.$$

Here $m.m.c.\{\ell(v^1), \ell(v^2)\} = 2c(c+1)$, $R = 2c^2 + c$ and

$$\varphi(a,b) = \min\{(c+1)\langle(a,b),(2c-1,1)\rangle, c\langle(a,b),(1,2c+1)\rangle\}.$$

Since $N_{\mathcal{C}}^* f = f_1^{2\beta_1} + f_2^{2\beta_2}$, we obtain

$$\begin{aligned}\{(x,y) \in \mathbb{R}^2 \ / \ N_{\mathcal{C}}^* f|_{\Delta_1} = 0\} &= \{(x,y) \in \mathbb{R}^2 \ / \ [xy]^{2\beta_1} + [xy - y^{2c}]^{2\beta_2} = 0\} \\ &\subset \{(x,y) \in \mathbb{R}^2 \ / \ x \cdot y = 0\}\end{aligned}$$

and

$$\begin{aligned}\{(x,y) \in \mathbb{R}^2 \ / \ N_{\mathcal{C}}^* f|_{\Delta_2} = 0\} &= \{(x,y) \in \mathbb{R}^2 \ / \ [xy]^{2\beta_1} + [x^{2(c+1)} + xy]^{2\beta_2} = 0\} \\ &\subset \{(x,y) \in \mathbb{R}^2 \ / \ x \cdot y = 0\}\end{aligned}$$

hence $N_{\mathcal{C}}^* f$ has the origin as an $\Gamma_+(\rho^{2\beta_1})$-isolated point.

Fix $(\theta_1, \theta_2) = (x^c y^c, x^{c-1} y^{c+1})$, to obtain $fil(\theta_i) \geq max_j\{fil(f_j)\} + \ell R + 1$ it is necessary and sufficient that $\ell \in \{1, 2, ..., c-3, c-2\}$. Therefore, $f_t(x,y) = (xy + x^2y^2 + tx^c y^c, x^{2(c+1)} + xy - y^{2c} + tx^{c-1}y^{c+1})$ is $C^{(c-2)}$-$\mathcal{C}$-trivial, for $c \geq 3$.

5 The group $\mathcal{K}$

To define the control function for the group $\mathcal{K}$ we use the control functions $N_{\mathcal{R}} f$ and $N_{\mathcal{C}}$ and consider two numbers a and b such that $fil([N_{\mathcal{R}} f]^a) = fil([N_{\mathcal{C}} f]^b)$ with respect to some Newton polyhedron $\Gamma_+(A)$. Then we define $N_{\mathcal{K}} f = [N_{\mathcal{R}} f]^a + [N_{\mathcal{C}} f]^b$.

If we suppose that $N_{\mathcal{K}} f$ has 0 as a $\Gamma_+(\rho^D)$-isolated point, for all $t \in [0,1]$ we have the following:

Proposition 5.1 *Deformations $f_t = f + t\theta$ of f, with com $fil(\theta_i) \geq d + \ell R + 1, \forall i$, are C^ℓ-$\mathcal{K}$-triviais.*

Proof: Define $N^*_{\mathcal{K}} f_t := [N_{\mathcal{R}} f_t]^a + [N_{\mathcal{C}} f_t]^b$, then

$$\begin{aligned} N_{\mathcal{K}} f_t \cdot \frac{\partial f_t}{\partial t} &= [N_{\mathcal{R}} f_t]^a \cdot \frac{\partial f_t}{\partial t} + [N_{\mathcal{C}} f_t]^b \cdot \frac{\partial f_t}{\partial t} \\ &= [N_{\mathcal{R}} f_t]^{a-1} \cdot [df_t]_x(W_{\mathcal{R}}) + [N_{\mathcal{C}} f_t]^{b-1} \cdot \sum W_i(x,t)(f_{ti}) \\ &= [df_t]_x([N_{\mathcal{R}} f_t]^{a-1} W_{\mathcal{R}}) + \sum([N_{\mathcal{C}} f_t]^{b-1} W_i(x,t))(f_{ti}). \end{aligned}$$

Hence,

$$\frac{\partial f_t}{\partial t} = [df_t]_x(\xi) + \sum(\eta_i)(f_{ti})$$

with

$$\xi := \frac{[N_{\mathcal{R}} f_t]^{a-1} W_{\mathcal{R}}}{N_{\mathcal{K}} f_t} \quad \text{and} \quad \eta_i := \frac{[N_{\mathcal{C}} f_t]^{b-1} W_i(x,t)}{N_{\mathcal{K}} f_t}$$

Since

$$\begin{aligned} fil([N_{\mathcal{R}} f_t]^{a-1} W_{\mathcal{R}}) &\geq (a-1) \cdot 2\alpha + 2\alpha + \ell R + 1 \\ &= 2\alpha a + \ell R + 1 \text{ onde } \alpha := m.m.c.\{fil(M_I)\}; \end{aligned}$$

$$\begin{aligned} fil([N_{\mathcal{C}} f_t]^{b-1} W_i) &\geq (b-1) \cdot 2B + 2B + \ell R + 1 \\ &= 2Bb + \ell R + 1 \text{ onde } B := m.m.c.\{fil(f_i)\}; \end{aligned}$$

and

$$\begin{aligned} fil(N_{\mathcal{K}} f_t) &= fil([N_{\mathcal{R}} f_t]^a) = fil([N_{\mathcal{C}} f_t]^b) \\ &= 2\alpha a = 2Bb, \end{aligned}$$

we obtain that the vector fields ξ e $\eta = (\eta_1, ..., \eta_p)$ are of class C^ℓ and $f_t = f + t\theta$ is a C^ℓ-$\mathcal{K}$-trivial deformation of f. ■

Example 5.2 Let $f : \mathbb{R}^3, 0 \to \mathbb{R}^2, 0$, $f(x,y,z) = (x, xy + yz^2 \pm y^4)$. We have $df = \begin{bmatrix} 1 & 0 & 0 \\ y & x + z^2 \pm 4y^3 & 2yz \end{bmatrix}$, hence $M_{12} = x + z^2 \pm 4y^3$ and $M_{13} = 2yz$. Consider $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & 2 & 1 \end{pmatrix}$. Here $R = 6$ and $\varphi(a,b,c) = \{6a + 3b + 3c, 6a + 2b + 4c\}$, hence $fil(f_1) = 6$ e $fil(f_2) = 8$.

For $\Theta = (0, z^5)$ we have $fil(z^5) = 15$, hence $f_t(x,y,z) = f(x,y,z) + t(0, z^5)$ is C^1-$\mathcal{K}$-trivial.

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