Lecture Notes in Mathematics

Dan Henry

Geometric Theory of Semilinear Parabolic Equations

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Introduction

In 1971, I read the beautiful paper of Kato and Fujita [32] on the Navier-Stokes equation and was delighted to find that, properly viewed, it looked like an ordinary differential equation, and the analysis proceeded in ways familiar for ODEs. This is perhaps no surprise to people in partial differential equations, but my training was in ordinary and functional differential equations, and my attempts to read PDEs usually became bogged down in technicalities.

Many PDE problems can be written as ODEs in Banach spaces, involving unbounded operators. Rewritten as Volterra integral equations, unbounded operators no longer appear (in parabolic problems) and the analysis is entirely analogous to the ODE case. One works exclusively with strong solutions, which often turn out to be classical solutions. The major technical differences with the ODE case are that we work in two spaces (or more) and Gronwall's inequality must be modifed to cover

$$0 \leq u(t) \leq at^{-\alpha} + b \int_0^t (t-s)^{-\beta} u(s)ds$$

with $\alpha < 1$, $\beta < 1$, when a, b, α, β are nonnegative constants.

In the geometric or qualitative theory of differential equations, the goal is to describe the geometry of the flow, and questions of stability predominate. (See the example of Sec. 1.1.) Even for ODEs, such questions rapidly become difficult, but many of the general results available for ODEs (for example, stability by the linear approximation) may be proved as easily for parabolic PDEs, once the necessary machinery has been constructed. Generally, we use information from the linear approximation, and so theorems on differentiability of solutions are important.

What is assumed of the reader? Linear functional analysis in Banach spaces is the main tool, and we use basic calculus in Banach spaces (continuity, Frechet differentiability, contraction maps and the implicit function theorem); some of these ideas are reviewed in Sec. 1.2. Knowledge of the ODE results here generalized would undoubtedly be helpful, but is probably not essential: with a grounding in modern analysis, $\mathbb{R}^n$ looks much the same as any other Banach space. The only hard results needed about elliptic boundary value problems are quoted in Sec. 1.2, namely solvability in L^p and C^α of the Dirichlet and mixed problems for the Laplacian. Maximum principle arguments are used in several examples, and are reviewed in Exercises 5-11 of Section 3.3.

Results are listed serially (except for exercises) so Theorem 3.4.2 (the second main result of the fourth section of Chapter 3) is followed by Corollaries 3.4.3, 3.4.4 and Lemma 3.4.5. Many exercises are scattered through these pages, of widely varying difficulty. Some of these, marked with an asterisk, are used frequently or significantly in later work. Some of the exercises describe other approaches - often better approaches - to problems described in earlier theorems; it is easier to insert an exercise than to rewrite the section. (For example Ex. 3, Sec. 4.2 and Ex. 10-11, Sec. 5.1.) In any case, the exercises are an important part of the work. Dismiss them at your peril!

The study of PDEs as evolution equations in infinite dimensional spaces has become fairly common, and beside the works cited in the bibliography, R. H. Martin's Nonlinear Operators and Differential Equations in Banach Spaces (Wiley, 1976) and F. Browder's Nonlinear Operators and Nonlinear Equations of Evolution in Banach Spaces (Am. Math. Soc., 1976) should be mentioned, both for their own interest and for the fact that their intersection with the present work is virtually empty. Such is the size of our subject!

The work described here began in a seminar at the University of Kentucky in 1971, and a version was typed in 1974. While visiting Northwestern University (1974-5), I gave a course covering some of this material and made fairly extensive corrections and revisions, along with some new material especially gradient flows (Sec. 5.3 and Thm. 6.1.9-10). Sections 7.5 and 7.6 and Chapters 9 and 10 were developed while visiting Brown University and were exposed there in 1979. I am grateful to colleagues at each of these institutions for encouragement and advice, and above all for providing a critical audience which forced me to clarify my thoughts. Special thanks to Jack Hale for encouragement - and an occasional boot in the pants - when enthusiasm flagged, and to Kate MacDougall who typed the final version despite my delaying tactics.

Chapter 1

Preliminaries

1.1 What is geometric theory?

The geometric or qualitative theory of ordinary differential equations was effectively initiated by Poincaré and Liapunov, and concerns itself with questions of the existence of special solutions (equilibrium points, periodic solutions, almost-periodic solutions, etc.) or collections of solutions (invariant manifolds), and the stability or instability of these - including their behavior under "small" changes in the equation. Global questions are also asked and sometimes answered: Starting from an "arbitrary" initial value, what can be said of the solution as time goes to infinity? Is the system as a whole structurally stable? (See, for example, [11, 18, 37, 61, 62, 68, 76, 79, 89]).

Some of these questions have also been posed and answered for functional differential equations [39,40,41], and this theory frequently guides our study of semilinear parabolic PDEs. J. K. Hale and K. Meyer suggest [42] that the most important features to generalize from ODEs are the variation of constants formula, the decomposition of the state space into subspaces invariant with respect to the linearized equation, and exponential bounds for solutions of the linearized equation in these subspaces. With these tools, the Fundamental Trick of Calculus (linearization) can be applied in ways familiar in ODE theory. Aside from occasional work with Liapunov functions and maximum principle arguments, the approach is almost always linearization and contraction maps.

It may be thought that the results must then all be trivial. In a way, perhaps they are: they are rarely surprising, piece-by-piece, either in hypotheses, conclusions, or methods of proof. Yet these tools have not, it seems, been systematically developed or systematically applied to this subject.

"But surely", someone objects, "you have not solved the time dependent Navier-Stokes equations in three dimensions. Else it would be emblazoned across the night sky in words of fire - or at least in the AMS Notices." True. The methods and results here apply to the Navier-Stokes equations, but do not by any means establish the existence of global solutions (existing for all positive time) for arbitrary initial data (restricted only by smoothness and compatability conditions). I would only claim that there are many other interesting questions to investigate. "Is existence something to boast about?" (L. C. Young).

Consider an example, the jewel of my collection. Chafee and Infante [14] studied a class of problems including

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + au - bu^3, \quad (0 < x < \pi,\ t > 0)$$

$$u(0,t) = 0, \quad u(\pi,t) = 0,$$

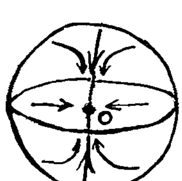
where a,b are positive constants. In this case, the initial value problem is well-posed in the Sobolev space $H_0^1(0,\pi)$, and the solution exists for all positive time. Further, as $t \to +\infty$, $u(\cdot,t)$ converges in $H_0^1(0,\pi)$ to some equilibrium ϕ,

$$\frac{d^2\phi}{dx^2}(x) + a\phi(x) - b\phi^3(x) = 0, \quad 0 < x < \pi$$

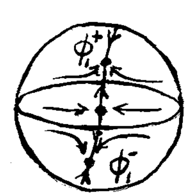
$$\phi(0) = 0, \quad \phi(\pi) = 0.$$

Chafee and Infante prove there are only a finite number of such equilibria - precisely $2n+1$ if $n^2 < a \leq (n+1)^2$ for some integer $n \geq 0$. If $0 < a \leq 1$, the zero solution is thus globally asymptotically stable. If $a > 1$, the zero solution is unstable, as are all other equilibrium points except for two, denoted ϕ_1^+, ϕ_1^-, characterized by the fact $\phi_1^+(x) > 0 > \phi_1^-(x)$ for all $0 < x < \pi$. These solutions are asymptotically stable, and we prove (in Sec. 5.3) that the domain of attraction of $\{\phi_1^+,\phi_1^-\}$ is an open dense set in $H_0^1(0,\pi)$.

We also show (Sec. 6.3) that there is a neighborhood of the origin in $H_0^1(0,\pi)$ which is positively invariant when $|a-1|$ is sufficiently small, and which is split into two open sets (domains of attraction of ϕ_1^+,ϕ_1^-) by the stable manifold for the zero solution, for small $a-1 > 0$. Thus we justify the following pictures of the flow: for a close to 1,



$a \leq 1$

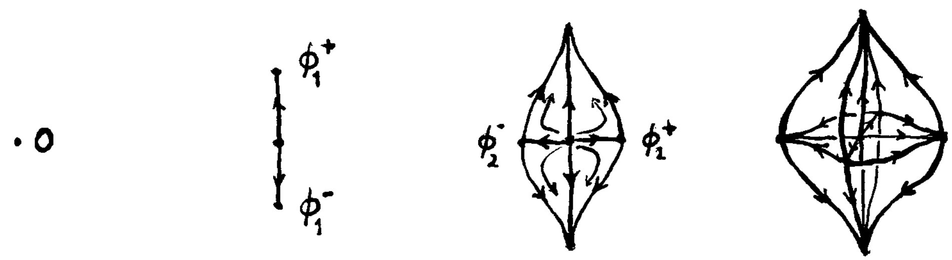


$a > 1$

If B is any sufficiently large ball about 0 in $H_0^1(0,\pi)$ and $u(t,B)$ is the set of points at time t reached by solutions u with initial value in B, then

$$K = \bigcap_{t \geq 0} u(t,B)$$

is the maximal bounded invariant set. K is compact, connected and finite dimensional, the union of the unstable manifolds of the equilibrium points, and we have the following pictures of K:



$0 \leq a \leq 1$ (0-dim.) $\quad$ $1 < a \leq 4$ (1-dim.) $\quad$ $4 < a \leq 9$ (2-dim.) $\quad$ $9 < a \leq 16$ (3-dim.)

When $n^2 < a < (n+1)^2$, K is n-dimensional, the closure of the unstable manifold of the origin.

Most examples cannot be treated in such detail, in part because of the difficulty of understanding the equilibrium problem, a nonlinear elliptic equation. But there are some other examples (in particular, [107], Ch. 10 and Ex. 11 of Sec. 6.1) which admit fairly complete discussion. We may hope our stock of examples will increase and put more meat on the bones of the theory.

1.2 Basic facts and notation

In this section, we collect some results and notations which will be used throughout these pages.

1.2.1 Gronwall's inequality

T.H. Gronwall showed that, if a,b are nonnegative constants and

$$0 \le u(t) \le a + b \int_0^t u(s)ds$$

for all $0 \le t < T$, then $u(t) \le ae^{bt}$ for $0 \le t < T$.

We need a different form, to discuss the weakly singular Volterra integral equations encountered below.

Assume a,b,α,β are nonnegative constants, with $\alpha < 1$, $\beta < 1$, and $0 < T < \infty$; there exists a constant $M = M(b,\alpha,\beta,T) < \infty$ so that for any integrable function $u: [0,T] \to R$ satisfying

$$0 \le u(t) \le at^{-\alpha} + b \int_0^t (t-s)^{-\beta}u(s)ds$$

for a.e. t in $[0,T]$, we have

$$0 \le u(t) \le a\, Mt^{-\alpha}, \text{ a.e. on } 0 < t \le T.$$

This is a special case of Theorem 7.1.1. The proof is an elementary iteration argument, followed by Lebesgue's dominated convergence theorem, and the reader is referred to Theorem 7.1.1.

1.2.2 Notation for linear operators and spaces

All Banach spaces are <u>real</u>, with the customary caveat that spectral theory is done in the complexification. The (real) dual of a Banach space X is denoted by X^*, and the value of $y \in X^*$ on $x \in X$ is written $\langle x,y\rangle$.

$\mathscr{L}(X,Y)$ is the space of continuous linear operators with domain X and range in Y, where X,Y are Banach spaces.

For a linear operator L,

$D(L)$ = domain of L;

$R(L)$ = range of L;

$N(L)$ = null space of L;

$\rho(L)$ = resolvent set of L,

$\sigma(L)$ = spectrum of $L = P\sigma(L) \cup C\sigma(L) \cup R\sigma(L)$,

$P\sigma(L)$ = point spectrum (eigenvalues of L),

$C\sigma(L)$ = continuous spectrum

$R\sigma(L)$ = residual spectrum;

$\sigma_e(L)$ = essential spectrum of L (see references in section 5.4)

$r(L)$ = spectral radius of L, when L is continuous.

We sometimes write "Re $\sigma(L) > k$" to mean the real part of λ = Re λ is greater than k whenever $\lambda \in \sigma(L)$.

We also occasionally write "Δ_D" for a differential operator which is the Laplace operator $(\Delta = \sum_{j=1}^n \partial^2/\partial x_j^2)$ on smooth functions which vanish at the boundary of the (implied) region in R^n. The precise domain is usually clear in context.

A projection E is a bounded linear operator on a space X to itself, satisfying $E^2 = E$. Any projection determines a decomposition of the space into a sum of subspaces $X = X_1 \oplus X_2$ where $X_1 = R(E)$, $X_2 = N(E)$; we say E projects X onto X_1, along X_2.

$C(S,X)$ = bounded continuous functions from a metric space S to a Banach space X with norm

$$\|f\|_{C(S,X)} = \sup\{\|f(s)\|_X \mid s \in S\}$$

$C_{unif}(S,X)$ = bounded uniformly continuous functions from S to X with the "sup" norm.

$C^\nu(S,X)$ = the space of $[\nu]$ times continuously differentiable functions from S, an open subset of a Banach space, to another Banach space X, $(\nu \geq 0)$, where the $[\nu]$-order derivative satisfies a Hölder condition with exponent $\nu - [\nu]$, if ν is not an integer. The norm is

$$\|f\|_{C^\nu(S,X)} = \sum_{k=0}^{[\nu]} \sup_S \|D^k f\| + \sup_{x\neq y} \frac{\|D^{[\nu]}f(x) - D^{[\nu]}f(y)\|}{\|x-y\|^\alpha}$$

where $\alpha = \nu - [\nu]$. (The last term is omitted when ν is an integer.)

$C^k_{Lip}(S,X)$ = functions in $C^k(S,X)$ whose k-derivative satisfies a Lipschitz condition (exponent 1).

$C^k_0(S,X)$ = the linear space of function in $C^k(S,X)$ having support compact in S.

$L^p(\Omega,X)$ = p^{th} power integrable functions from a measure space Ω into a Banach space X,

$$\|f\|_{L^p(\Omega,X)} = \{\int_\Omega \|f(t)\|^p_X \, dt\}^{1/p}$$

for $1 \leq p < \infty$.

$L^\infty(\Omega,X)$ = strongly measurable essentially bounded functions from Ω to X,

$$\|f\|_{L^\infty(\Omega,X)} = \text{ess.sup}\{\|f(t)\|_X \mid t \in \Omega\}.$$

$W^{k,p}(\Omega,X)$ = Sobolev space of $f \in L^p(\Omega,X)$ which have distributional derivatives of order $\leq k$ all p^{th} power integrable. (Here Ω is an open set in R^n).

$$\|f\|_{W^{k,p}(\Omega,X)} = \{\int_\Omega \sum_{j=0}^{k} \|f^{(j)}(t)\|^p \, dt\}^{1/p}.$$

$H^k(\Omega,X)$ = $W^{k,2}(\Omega,X)$, which is a Hilbert space when X is a Hilbert space.

$W^{k,p}_0(\Omega,X)$ = closure of $C^k_0(\Omega,X)$ in the norm of $W^{k,p}(\Omega,X)$; we frequently use the equivalent norm

$$\|f\|_{W^{k,p}_0(\Omega,X)} = \{\int_\Omega \|f^{(k)}(t)\|^p \, dt\}^{1/p}.$$

$H^k_0(\Omega,X)$ = $W^{k,2}_0(\Omega,X)$.

In general, if the range is not specified, it is taken to be the real line R; exceptions are hopefully clear in context.

1.2.3 Sobolev Imbedding Theorem ([92, 97, 8, 23])

Suppose Ω is an open set in R^n such that its boundary $\partial\Omega$ is minimally smooth ([97], p. 181), for example, $\Omega = R^n$ or Ω is a bounded domain whose boundary is C^1. Then we have the continuous imbeddings:

$$W^{k,p}(\Omega) \subset L^q(\Omega) \quad \text{if} \quad 1/p \geq 1/q \geq 1/p - k/n > 0;$$

$$W^{k,p}(\Omega) \subset C(\Omega) \quad \text{if} \quad kp > n.$$

(In the second case, the intention is that every function f in $W^{k,p}(\Omega)$ has a continuous representative $\hat{f}$, $\hat{f} = f$ almost everywhere in Ω.) If $kp = n$ and Ω is bounded, then $W^{k,p}(\Omega) \subset L^r(\Omega)$ for any r, $p \leq r < \infty$.

Further if $0 \leq \nu < k-n/p$, then $W^{k,p}(\Omega) \subset C^\nu(\Omega)$ with continuous imbedding. (Here we use the continuous representative of a function in $W^{k,p}(\Omega)$, and claim it is actually a member of $C^\nu(\Omega)$.) Proof of slightly weaker (but adequate) results of this kind, along with a version of the Nirenberg-Gagliardo inequality, is given in section 1.6.

1.2.4 Some elliptic boundary value problems

We list a few regular boundary value problems for the Laplacian which are used in examples and exercises below. For the general formulation, see [1, 2, 8, 28, 29, 30, 81] for the L^2-theory, [2, 8, 91] for the L^p-theory, [2, 8, 29] for the C^α-theory, and [2,8] for everything. Fichera [28] discusses elliptic systems in the L^2-setting, including the equations of linear elasticity. Ladyzhenskaya [65] covers the Stokes problem of viscous flow.

Dirichlet problem. Suppose Ω is a bounded smooth (C^2) domain in R^n, and consider the problem: given $f\colon \Omega \to R$, find u so that

$$\sum_{j=1}^{n} \partial^2 u/\partial x_j^2 = \Delta u = f \quad \text{in} \quad \Omega, \quad u\big|_{\partial\Omega} = 0.$$

(i) If $f \in L^p(\Omega)$ $(1 < p < \infty)$, there exists a unique solution $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$.

(ii) If $f \in C^\alpha(\Omega)$, $0 < \alpha < 1$, and $\partial\Omega \in C^{2+\alpha}$, there exists a unique solution $u \in C^{2+\alpha}(\Omega)$.

Mixed problem. Ω as above, with $a \in C^1(\partial\Omega, R)$ $a(x) \geq 0$ on $\partial\Omega$, $a(x) > 0$ somewhere; the problem is to find u such that

$$\Delta u = f \quad \text{in} \quad \Omega$$

$$\frac{\partial u}{\partial n} + a(x)u = 0 \quad \text{on} \quad \partial\Omega.$$

(i) If $f \in L^p(\Omega)$ $(1 < p < \infty)$ there exists a unique solution $u \in W^{2,p}(\Omega)$

(ii) If $f \in C^\alpha(\Omega)$ $(0 < \alpha < 1)$ there exists a unique solution $u \in C^{2+\alpha}(\Omega)$.

1.2.5 Polynomials, derivatives and analyticity ([22, 49])

If X, Y are Banach spaces, an n-linear map from X to Y is a map f from the n-fold product $X^n = X \times \ldots \times X$ into Y such that $x_k \to f(x_1,\ldots,x_n)$ is linear for each fixed $(x_1,\ldots,x_{k-1}, x_{k+1},\ldots,x_n)$, for $k = 1,2,\ldots,n$. A *homogeneous polynomial* (of degree n) from X to Y is a map of the form $x \to f(x,x,\ldots,x): X \to Y$, for some n-linear map f. (Special cases: $n = 0$, constant; $n = 1$, linear.) A homogeneous polynomial g (of degree n) is continuous if and only if it is bounded: for some constant M,

$$\|g(x)\|_Y \leq M \|x\|_X^n, \text{ all } x \in X.$$

For such a polynomial we sometimes write $g(x^n)$, rather than $g(x)$, and its norm is

$$\|g\| = \sup\{\|g(x^n)\|_Y \mid \|x\|_X \leq 1\}$$

so

$$\|g(x^n)\|_Y \leq \|g\| \|x\|_X^n \text{ for all } x \in X.$$

A map $f: X \to Y$ is *differentiable* at a point $a \in X$ if there exists a continuous linear map $f'(a): X \to Y$ so that

$$\|f(x) - f(a) - f'(a)(x-a)\|_Y = o(\|x-a\|_X) \text{ as } x \to a.$$

In this case, $f'(a) \in \mathcal{L}(X,Y)$ is called the (Frechet) derivative at a. Derivatives are sometimes also written $Df(a)$ or $f_x(a)$. The map is continuously differentiable on an open set $U \subset X$ if it is differentiable at each point of U and if $x \mapsto f'(x): U \to \mathcal{L}(X,Y)$ is continuous. We say f is twice differentiable at a if f is differentiable at each point in a neighborhood of a and $x \mapsto f'(x) \in \mathcal{L}(X,Y)$ is differentiable at a; and so on.

When f is n-times continuously differentiable in an open set $U \subset X$, we may consider its k^{th} derivative $f^{(k)}(x)$ $(1 \leq k \leq n)$ to be a continuous k^{th} order homogeneous polynomial for each $x \in U$, determined by the Taylor formula

$$f(x+h) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(x)(h^k) + o(\|h\|_X^n)$$

as $h \to 0$, for each $x \in U$. Here we set $f^{(0)}(x) = f(x)$.

A map $f: U \subset X \to Y$, U open, is *analytic* in U if f is infinitely often differentiable at each point of U and if, for each $x \in U$, there exists $\delta = \delta(x) > 0$ so that whenever $\|h\|_X \leq \delta$,

$$f(x+h) = \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(x)(h^k),$$

the series converging in Y-norm uniformly in $\|h\|_X \leq \delta$.

An equivalent condition is that, for some $\delta > 0$, there exists a constant $M = M(x,\delta)$ so that

$$\frac{1}{k!} \|f^{(k)}(x)\| \delta^k \leq M < \infty, \quad \text{all} \quad k \geq 0.$$

(This definition is equivalent to that used in Hille and Phillips [49], according to [49, Th. 3.17.1].)

Example. If f is a continuous homogeneous n^{th} order polynomial map from X and Y, then f is analytic. Indeed, $x \to f^{(k)}(x)$ is a continuous $(n-k)$-order polynomial from X to the space of k^{th} order polynomials from X to Y $(k = 0,1,\ldots,n)$ and $f^{(k)}(x) = 0$ if $k > n$.

Another example. If $X = Y = C[0,1]$, $f(x)(t) = \sin x(t)$ for $0 \leq t \leq 1$; then

$$f^{(k)}(x)(h^k)(t) = g_k(x(t))(h(t))^k \quad k = 0,1,2,\ldots$$

where $g_k(z) = (d/dz)^k \sin z$, so $\|f^{(k)}(x)\| \leq 1$ for all $k \geq 0$ and $\|f^{(k)}(x) - f^{(k)}(y)\| \leq \sup_{0 \leq t \leq 1} |g_k(x(t)) - g_k(y(t))| \leq \|x-y\|_{C[0,1]}$. Thus $f: X \to Y$ is analytic.

Exercise 1. If $X = Y = L^2(0,1)$, $f(x)(t) = \sin x(t)$, $0 \leq t \leq 1$, then f is Lipschitz continuous:

$$\|f(x) - f(y)\|_{L^2} \leq \|x - y\|_{L^2(0,1)}$$

However, f is nowhere differentiable. (Considered as a map from $H^1(0,1)$ to $H^1(0,1)$, f is analytic.)

Converse Taylor Theorem (see [ ] for proof).

Assume X,Y are Banach spaces, U is open in X and $f: U \to Y$ satisfies, for each $x_0 \in U$,

$$f(x+h) = \sum_{k=0}^{m} a_k(x)h^k + o(\|h\|^m) \quad \text{as} \quad h \to 0$$

uniformly for $\|x-x_0\| \leq \|h\|$, where $a_k(x)$ is a bounded k^{th} order homogeneous polynomial map from X to Y. Then f is m-times continuously differentiable and its k^{th} derivative is

$$f^{(k)}(x)(h_1,\ldots,h_k) = \sum_{\{i_1,\ldots,i_k\}=\{1,\ldots,k\}} a_k(x)h_{i_1}\ldots h_{i_k}$$

(sum over permutations),

$$= k!\ a_k(x)h_1\ldots h_k \quad \text{if} \quad a_k(x) \text{ is symmetric.}$$

A related result, with a similar proof, is: if $0 < \theta \leq 1$, $U = X$ and there is a constant B with

$$\|f(x+h) - \sum_{k=0}^{m} a_k(x)h^k\| \leq B\|h\|^{m+\theta}$$

for all x,h in X, where $a_k(x)$ is a bounded homogeneous k^{th} order polynomial (for each k,x). Then f is C^m and the m^{th} derivative satisfies $\|f^{(m)}(x+h) - f^{(m)}(x)\| \leq N_m B\|h\|^{\theta}$, where N_m depends only on m. (A direct proof in the case m = 1 shows $N_1 = 6$; see ex. 1, sec. 9.1.) The inequality needs only to hold on a dense set if f is continuous.

Exercise 2. If $1 \leq p < \infty$, $\Omega \subseteq \mathbb{R}^n$, prove $f \to \int_{\Omega} |f(x)|^p dx$ is C^p uniformly on bounded sets of $L_p(\Omega, \mathbb{R})$ when p is not an integer. If p is an odd integer the function is C^{p-1}_{Lip}, and when p is an even integer, it is analytic (a polynomial). Hint: first examine $t \to |t|^p: \mathbb{R} \to \mathbb{R}$.

1.2.6 Contraction maps with parameters; implicit function theorem

The Banach contraction mapping theorem is well-known.

Theorem (Banach). Let (S,d) be a complete metric space and let $T: S \to S$ be a contraction, i.e. there exists $\theta < 1$ such that

$$d(T(x), T(y)) \le \theta\, d(x,y) \quad \text{for all } x,y \text{ in } S.$$

Then there exists a unique fixed point of T in S: $a \in S$, $T(a) = a$. Also, for any $b \in S$, if $T^n(b) = T(T^{n-1}(b))$ is the n-fold composition, then $T^n(b) \to a$ as $n \to \infty$, in fact,

$$d(T^n(b),a) \le \theta^n\, d(b,a).$$

We need more detailed information about the dependence of the fixed point on various parameters. If (S,d) is a complete metric space and (Λ,ℓ) is another metric space, we say $T: S \times \Lambda \to S$ is a *uniform contraction* if there exists $\theta < 1$ so that

$$d(T(x,\lambda), T(y,\lambda)) \le \theta d(x,y)$$

for all x,y in S and all $\lambda \in \Lambda$.

For each λ, there exists a unique fixed point $g(\lambda) \in S$; we study the smoothness of $\lambda \to g(\lambda)$.

Exercise 3. If $T: S \times \Lambda \to S$ is a uniform contraction as above and if $\lambda \to T(x,\lambda)$ is continuous on Λ for each $x \in S$, then $\lambda \to g(\lambda)$: $\Lambda \to S$ is continuous.

The following is essentially the formulation of Hale [32, Th. 3.2].

Theorem. Let U,V be open sets in Banach spaces X,Y, and let $\overline{U}$ denote the closure of U (a complete metric space). Suppose $T: \overline{U} \times V \to \overline{U}$ is a uniform contraction on $\overline{U}$, and let $g(y)$ denote the unique fixed point of $T(\cdot,y)$ for each $y \in V$.

If $T \in C^k(\overline{U} \times V,X)$ $(0 \le k < \infty)$, i.e. the partial Frechet derivatives up to order k exist in $U \times V$ and extend continuously to $\overline{U} \times V$, then $y \to g(y)$ is in $C^k(V,X)$.

If $(x,y) \to T(x,y)$ is analytic from $U_1 \times V$ into X, where U_1 is some neighborhood of $\overline{U}$, then $y \to g(y)$ is analytic from V to X.

Proof. The case $k = 0$ is a simple exercise; consider $k = 1$. Since $\|T(x_1,y) - T(x_2,y)\| \le \theta\|x_1-x_2\|$, it follows that the derivative with respect to x, $T_x(x,y)$ has $\|T_x(x,y)\| \le \theta < 1$ on $U \times V$. Formally, differentiating $g(y) = T(g(y),y)$ we see $g'(y)$ should be the solution M of

$$M - T_x(g(y),y)M = T_y(g(y),y).$$

But $\|T_x(g(y),y)\| \leq \theta < 1$, so this equation has a unique continuous solution $M(y)$; we must show $\|g(y+\eta) - g(y) - M(y)\eta\| = o(\|\eta\|)$ as $\eta \to 0$.

Let $\gamma = g(y+\eta) - g(y)$; then $T(g(y) + \gamma, y + \eta) - T(g(y),y) = \gamma$, so

$$(I-T_x(g(y),y))\gamma = T_y(g(y),y)\eta + \Delta$$

where $\Delta = T(g(y) + \gamma, y+\eta) - T(g(y),y) - T_x(g(y),y)\gamma - T_y(g(y),y)\eta$. Since T is C^1, it follows $\|\Delta(\gamma,\eta)\| = o(\|\gamma\| + \|\eta\|)$ as $\|\gamma\| + \|\eta\| \to 0$, i.e. as $\eta \to 0$ (since $\gamma(\eta) \to 0$ by continuity of g). Thus for any $\varepsilon > 0$, with $\|\eta\|$ small enough, $\|\Delta(\eta)\| \leq \varepsilon(\|\gamma(\eta)\| + \|\eta\|)$, so

$$\|\gamma(\eta)\| \leq \frac{1}{1-\theta}\|T_y(g(y),y)\eta\| + \frac{\varepsilon}{1-\theta}(\|\gamma(\eta)\| + \|\eta\|)$$

which implies $\gamma(\eta) = O(\|\eta\|)$ as $\eta \to 0$. It follows that

$$(I-T_x(g(y),y))(\gamma(\eta) - M\eta) = \Delta(\eta,\gamma(\eta)) = o(\|\eta\|)$$

so $\|\gamma(\eta) - M(y)\eta\| = o(\|\eta\|)$, i.e. $M(y) = g'(y)$.

For $k > 1$, proceed by induction. If the result holds for $(k-1)$, then when $T \in C^k$, we have $g \in C^{k-1}$, at least, and

$$(I-T_x(g(y),y))g'(y) = T_y(g(y))y)$$

which shows $g' \in C^{k-1}$, so $g \in C^k$.

For the analytic case, observe that for any $y \in V$, there exists a *complex* neighborhood of $(g(y),y)$ on which T is analytic and a uniform contraction. The argument above proves differentiability of the fixed point in this complex neighborhood, hence analyticity [49, Th. 3.17.1]. Alternatively, one can obtain the result using power series and the method of majorants.

Exercise 4. Suppose T is a uniform contraction and for some α, $0 < \alpha \leq 1$, $\|T(x,y_1) - T(x,y_2)\| \leq M(x)\|y_1 - y_2\|^\alpha$ where $M(x)$ is bounded in a neighborhood of any $x \in \overline{U}$. Then $\|g(y_1) - g(y_2)\| = O(\|y_1-y_2\|^\alpha)$ as $y_1 \to y_2$. If, in fact, $T \in C^\nu(U \times V,X)$ for some $\nu > 0$ and T is a uniform contraction, then the fixed point g $g \in C^\nu(V,X)$.

Implicit Function Theorem.

Let X, Y, Z be Banach spaces and U, V open sets in X, Y respectively. Assume $F: U \times V \to Z$ is continuously differentiable, that $(x_0, y_0) \in U \times V$, $F(x_0, y_0) = 0$, and the partial derivative $F_x(x_0, y_0) \in \mathscr{L}(X, Z)$ has a continuous inverse.

Then there exists a neighborhood of (x_0, y_0), $U_1 \times V_1 \subset U \times V$, and a function $f: V_1 \to U_1$, $f(y_0) = x_0$, such that for $(x,y) \in U_1 \times V_1$, $F(x,y) = 0$ if and only if $x = f(y)$. Thus $F(f(y),y) = 0$ for all $y \in V_1$.

If $F: U \times V \to Z$ is C^k $(1 \le k \le \infty)$ or analytic near (x_0, y_0), then f is C^k or analytic, respectively, near y_0.

Proof. Let $L = (F_x(x_0, y_0))^{-1} \in \mathscr{L}(Z, X)$ and define

$$G(x,y) = x - LF(x,y).$$

Then G is C^1 from a neighborhood of (x_0, y_0) to X (or C^k or analytic) with $G(x_0, y_0) = x_0$, $G_x(x_0, y_0) = 0$ and $\|G_x(x,y)\| \le \theta < 1$ in a neighborhood of (x_0, y_0). Now the result follows from the contraction mapping theorem.

1.2.7 The grasshopper's guide

We may distinguish between the ants, who read page n before reading page $(n+1)$, and the grasshoppers who skim and skip until something of interest appears and only then attempt to trace its logical ancestry. For the sake of the grasshoppers, herewith is a listing of certain basics from Chapter 1-4.

Chapter 1

- 1.3.1 definition of sectorial operators
- 1.3.2 definition of analytic semigroups
- 1.3.4 A sectorial $\Rightarrow \{e^{-At}, t \ge 0\}$ analytic
- 1.4.1 definition of fractional powers
- 1.4.2 $A^\alpha A^\beta = A^{\alpha+\beta}$
- 1.4.3 estimates of $A^\alpha e^{-At}$
- 1.4.7, 1.4.8 definitions and properties of $X^\alpha = D(A^\alpha)$.
- 1.5.2, 1.5.3 invariant subspaces and exponential bounds
- 1.6.1 imbedding theorem $X^\alpha \subset W^{k,q}(\Omega)$.

Chapter 2 Examples of parabolic problems.

Chapter 3

3.2.2 existence and uniqueness for $dx/dt + Ax = f(t)$

3.3.3, 3.3.4 existence and uniqueness for $dx/dt + Ax = f(t,x)$

3.3.6 compactness of bounded orbits

3.3; exercises 5-11: maximum principle arguments

3.4.1, 3.4.2 continuous and differentiable dependence

3.5.2 smoothing action of the equation

3.6, 3.7, 3.8 examples.

Chapter 4

4.1.1, 4.1.3 definition of dynamical systems and Liapunov functions

4.1.4, 4.2.1 asymptotic stability and Liapunov functions

4.3.3, 4.3.4 stability and invariant sets.

1.3 Sectorial operators and analytic semigroups

We begin with an example, then generalize. Consider the heat equation

$$\frac{\partial u}{\partial t} = K \frac{\partial^2 u}{\partial x^2} \qquad (t > 0,\ 0 < x < \ell)$$

with boundary conditions $u(0,t) = 0$, $u(\ell,t) = 0$ for $t > 0$; here K is a positive constant, the specific conductivity, and $u(x,t)$ is the temperature at position x, at time t. Define the linear operator A by

$$A\,\phi(x) = -K \frac{d^2\phi}{dx^2}(x), \quad 0 < x < \ell,$$

whenever ϕ is a smooth function on $[0,\ell]$ with $\phi(0) = 0$, $\phi(\ell) = 0$. For such functions ϕ and ψ,

$$(A\phi,\phi) = -K \int_0^\ell \phi''(x)\phi(x)dx = K \int_0^\ell (\phi'(x))^2 dx \geq 0,$$

and

$$(A\phi,\psi) = -K \int_0^\ell \phi''(x)\psi(x)dx = (\phi,A\psi)$$

so we may (and shall) consider A to be extended to a self-adjoint densely defined linear operator in $L^2(0,\ell)$, using Friedrichs theorem [82, 103]. In this case,

$$D(A) = \{\phi \in L^2(0,\ell) \mid A\phi \in L^2(0,\ell)\} = H_0^1(0,\ell) \cap H^2(0,\ell),$$

and the spectrum $\sigma(A)$ consists of the simple eigenvalues $\lambda_n = (k\pi^2/\ell^2)n^2$, $(n = 1,2,3,\ldots)$ with corresponding eigenfunctions $\phi_n(x) = (2/\ell)^{\frac{1}{2}}\sin(n\pi x/\ell)$

$$A\phi_n = \lambda_n\phi_n, \quad (\phi_n,\phi_m) = \delta_{nm} \quad \text{for} \quad m,n = 1,2,3,\ldots$$

where $\delta_{nm} = 0$ if $n \neq m$, $\delta_{nm} = 1$ if $n = m$.

The heat equation, together with the boundary conditions, may then be written (formally) as a differential equation in a Banach space,

$$\frac{du}{dt} + Au = 0,$$

and it is tempting to express the solution in the form

$$u(t) = e^{-At}u(0), \qquad t > 0.$$

In fact, if the initial value $u(x,0)$ for the solution of the heat equation is a smooth function which vanishes at $x = 0$ and $x = \ell$, the solution may be found by separation of variables:

$$u(x,t) = \sum_{n=1}^{\infty} e^{-\lambda_n t} E_n(u(\cdot,0))(x)$$

where E_n is the projection on the n^{th} eigenfunction,

$$E_n(\psi)(x) = \phi_n(x)\ (\phi_n,\psi).$$

But this solution is precisely $e^{-At}u(\cdot,0)$, where the exponential is defined in the usual sense for functions of a self adjoint operator, using the spectral decomposition of A.

Other useful functions of A may be similarly defined, for example

$$A^{\alpha}\psi = \sum_{n=1}^{\infty} \lambda_n^{\alpha} E_n(\psi)$$

$$(\lambda I - A)^{-1}\psi = \sum_{n=1}^{\infty} (\lambda-\lambda_n)^{-1}E_n(\psi) \quad \text{if} \quad \lambda \neq \lambda_n, \text{ all } n.$$

Using the fact $||\psi||^2 = \int_0^{\ell} \psi^2(x)dx = \sum_{n=1}^{\infty} (\phi_n,\psi)^2$ it follows easily that for any $\alpha \geq 0$

$$D(A^{\alpha}) = \{\psi \in L^2(0,\ell): \sum_{n=1}^{\infty} \lambda_n^{2\alpha}(\phi_n,\psi)^2 < \infty\}$$

and that

$$\|A^{\alpha}e^{-At}\psi\| < \max_{n\geq 1} \{\lambda_n^{\alpha}e^{-\lambda_n t}\} \cdot \|\psi\| \leq b_{\alpha}(t)\ \|\psi\|,$$

$$\begin{aligned} b_{\alpha}(t) &= (te/\alpha)^{-\alpha} \quad \text{for} \quad 0 < t \leq \alpha/\lambda_1, \\ &= \lambda_1^{\alpha}e^{-\lambda_1 t} \quad \text{for} \quad t \geq \alpha/\lambda_1. \end{aligned}$$

Thus $R(e^{-At}) \subset D(A^{\alpha})$ for every $\alpha \geq 0$, whenever $t > 0$, and $\|A^{\alpha}e^{-At}\| = O(t^{-\alpha})$ as $t \to 0$.

Exercises.

(1) $A^{-\alpha}$ is bounded on $L^2(0,\ell)$ if $\alpha \geq 0$.

(2) $A^{-\alpha}$ is compact on $L^2(0,\ell)$ if $\alpha > 0$.

Hint: let $A_N^{-\alpha}\psi = \sum_{n=1}^{N} \lambda_n^{-\alpha}E_n\psi$ and show $\|A^{-\alpha}-A_N^{-\alpha}\| \to 0$ as $N \to \infty$.

(3) $D(A^{\frac{1}{2}}) = H_0^1(0,\ell)$

Hint: $(A\psi,\psi) = \|A^{\frac{1}{2}}\psi\|^2 = K\int_0^1 (\psi'(x))^2 dx.$

(4) $A^{\frac{1}{2}}u \neq du/dx$, in general, but $\frac{d}{dx}A^{-\frac{1}{2}}$ and $A^{-\frac{1}{2}}\frac{d}{dx}$ are bounded in the $L^2(0,\ell)$-norm on $H_0^1(0,\ell)$.

(5) $\|(\lambda-A)^{-1}\| \leq \max_n \frac{1}{|\lambda-\lambda_n|} \leq \csc\phi\cdot|\lambda|^{-1}$ for $\lambda \neq 0$, $\phi \leq |\arg\lambda| \leq \pi$ $(0 < \phi < \pi/2)$.

The generalization is the class of sectorial operators.

Definition 1.3.1. We call a linear operator A in a Banach space X a *sectorial operator* if it is a closed densely defined operator such that, for some ϕ in $(0, \pi/2)$ and some $M \geq 1$ and real a, the sector

$$S_{a,\phi} = \{\lambda \mid \phi \leq |\arg(\lambda-a)| \leq \pi, \quad \lambda \neq a\}$$

is in the resolvent set of A and

$$\|(\lambda-A)^{-1}\| \leq M/|\lambda-a| \quad \text{for all} \quad \lambda \in S_{a,\phi}.$$

Note: the angle opening of the section $S_{a,\phi}$ is $2\pi-2\phi > \pi$.

Remark. Several equivalent formulations (due to Yosida) are presented in the appendix to Hoppenstadt's paper [51].

Examples.

(1) If A is a bounded linear operator on a Banach space, then A is sectorial.

(2) If A is a self adjoint densely defined operator in a Hilbert space, and if A is bounded below, then A is sectorial.

(3) If A is sectorial in X, B is sectorial in Y, then $A \times B$, is sectorial in $X \times Y$, where $(A \times B)(x,y) = (Ax,By)$ for $x \in D(A)$, $y \in D(B)$.

(4) If $Au(x) = -\Delta u(x)$, $x \in \Omega$, when $u \in C_0^2(\Omega)$ $(\Omega \subset \mathbb{R}^n)$, and A is the closure in $L_p(\Omega)$ of $-\Delta \mid C_0^2(\Omega)$ $(1 \leq p < \infty)$ then A is sectorial (see sec. 1.6) if its resolvent set meets the left half-plane.

(5) Many other elliptic boundary-value problems define sectorial operators, as shown by Friedman [30].

(6) If A is sectorial in X and B is a linear operator with $D(B) \supset D(A)$ and for all $x \in D(A)$ $\|Bx\| \leq \varepsilon\|Ax\| + K(\varepsilon)\|x\|$ (for sufficiently small $\varepsilon > 0$) then $A+B$ is sectorial (see Th. 1.3.2).

(7) An operator on a Hilbert space which is m-sectorial in the sense of Kato [56] is also sectorial under our definition. (See ex. 6).

Theorem 1.3.2. Suppose A is a sectorial operator and $\|A(\lambda-A)^{-1}\| \leq C$ for $|\arg \lambda| \geq \phi_0$, $|\lambda| \geq R_0$ for some positive constants R_0, C, and $\phi_0 < \pi/2$. Suppose also B is a linear operator with $D(B) \supset D(A)$ with $\|Bx\| \leq \varepsilon\|Ax\| + K\|x\|$ for all $x \in D(A)$ and ε, K are positive constants with $\varepsilon C < 1$. Then $A+B$ is sectorial.

Proof. $\|B(\lambda-A)^{-1}\| \leq \varepsilon\|A(\lambda-A)^{-1}\| + K\|(\lambda-A)^{-1}\| \leq \varepsilon C + K(1+C)/|\lambda|$ for $|\arg \lambda| \geq \phi_0$, $|\lambda| \geq R_0$ so

$$\|\{\lambda-(A+B)\}^{-1}\| = \|(\lambda-A)^{-1}\{I-B(\lambda-A)^{-1}\}^{-1}\| \leq \frac{1+C}{|\lambda|}\{1-\varepsilon C-\frac{K(1+C)}{|\lambda|}\}^{-1} \leq \frac{\text{Constant}}{|\lambda|}$$

for $|\arg \lambda| \geq \phi_0$ and $|\lambda|$ sufficiently large. It follows from this that $A+B$ is sectorial.

Remark. If A is also self adjoint, it suffices that $\varepsilon < 1$.

<u>Exercise 6</u>. Suppose T is a linear operator in a Hilbert space which is m-sectorial in the sense of T. Kato ([56, p. 280]), i.e. T (or $T+aI$ for some real a) satisfies

(i) The numerical range $\{(Tu,u) \mid u \in D(T)\}$ is contained in a sector $|\arg \lambda| \le \theta$, for some $\theta < \pi/2$;

(ii) $T^{-1} \in \mathscr{L}(H,H)$ $(0 \in \rho(T))$.

Prove T is sectorial in our sense.

<u>Hint</u>: First let $u \in D(T)$, $\arg \lambda = \phi$ with $|\phi| > \theta$, and $f = \lambda u - Tu$; show

$$|\lambda| \, \|u\| \sin(|\phi|-\theta) \le (\cos \theta + \sin \theta) \|f\|$$

so $|\lambda| \, \|u\| \le M \|(\lambda-T)u\|$ when $|\arg \lambda| \ge \theta' > \theta$. Then show that whenever $\mu \in \rho(T)$, $|\arg \mu| \ge \theta$, we have $\lambda \in \rho(T)$ for $|\lambda-\mu| < \frac{1}{M} |\mu|$, hence prove $|\arg \lambda| \ge \theta'$ implies $\lambda \in \rho(T)$.

<u>Exercise 7</u>. For $u \in C^2[0,1]$, and $v,w \in \mathbb{R}$, define $A(u,v,w) = (-d^2u/dx^2, 0, 0)$ provided $u_x(0) = v$ and $u_x(1) + u(1) = w$. Extend A to be a closed operator in $L^2(0,1) \times \mathbb{R} \times \mathbb{R}$ and prove A is sectorial.

<u>Hint</u>: if $(\lambda-A)(u,v,w) = (f,g,h) \in L^2 \times \mathbb{R} \times \mathbb{R}$, $\lambda \neq 0$ then $v = g\lambda^{-1}$, $w = h\lambda^{-1}$, and $u_{xx} + \lambda u = f$ with the boundary conditions. But if $z(x) \equiv u(x) - xv + 2v-w$ then z is in the domain of a certain self-adjoint positive definite operator B in $L^2(0,1)$: $Bz = -d^2z/dx^2$, $z_x\big|_0 = 0$, $(z_x+z)\big|_1 = 0$.

<u>Definition 1.3.3</u>. An *analytic semigroup* on a Banach space X is a family of continuous linear operators on X, $\{T(t)\}_{t \ge 0}$, satisfying

(i) $T(0) = I$, $T(t)T(s) = T(t+s)$ for $t \ge 0$, $s \ge 0$

(ii) $T(t)x \to x$ as $t \to 0+$, for each $x \in X$

(iii) $t \to T(t)x$ is real analytic on $0 < t < \infty$ for each $x \in X$.

The infinitesimal generator L of this semigroup is defined by $Lx = \lim_{t \to 0+} \frac{1}{t}(T(t)x - x)$, its domain $D(L)$ consisting of all $x \in X$ for which this limit (in X) exists. We usually write $T(t) = e^{Lt}$.

<u>Theorem 1.3.4</u>. If A is a sectorial operator, then $-A$ is the infinitesimal generator of an analytic semigroup $\{e^{-tA}\}_{t \ge 0}$, where

$$e^{-At} = \frac{1}{2\pi i}\int_{\Gamma}(\lambda+A)^{-1}\, e^{\lambda t} d\lambda,$$

where Γ is a contour in $\rho(-A)$ with $\arg\lambda \to \pm\theta$ as $|\lambda| \to \infty$ for some θ in $(\pi/2,\pi)$.

Further e^{-At} can be continued analytically into a sector $\{t \neq 0: |\arg t| < \varepsilon\}$ containing the positive real axis, and if $\mathrm{Re}\,\sigma(A) > a$, i.e. if $\mathrm{Re}\,\lambda > a$ whenever $\lambda \in \sigma(A)$, then for $t > 0$

$$\|e^{-At}\| \le Ce^{-at},\ \|Ae^{-At}\| \le \frac{C}{t}\, e^{-at}$$

for some constant C.

Finally $\frac{d}{dt}\, e^{-At} = -Ae^{-At}$ for $t > 0$.

Proof. Without loss of generality, assume $a = 0$ and $\|(\lambda+A)^{-1}\| \le M/|\lambda|+\delta$ for $|\pi-\arg\lambda| \ge \phi$ for some constants $\delta > 0$, $M > 0$ and $\phi \in (0,\pi/2)$; otherwise replace A by $A-aI$.

Choose θ in $\pi/2 < \theta < \pi-\phi$. Define e^{-At} by the above integral, and note that the integral converges absolutely if $t > 0$. By Cauchy's theorem, the integral is unchanged when the contour Γ is shifted to the right a small distance: call the shifted contour Γ'. Then for $t > 0$, $s > 0$

$$\begin{aligned} e^{-At}e^{-As} &= (2\pi i)^{-2}\int_{\Gamma}\int_{\Gamma'} e^{\lambda t}(\lambda I+A)^{-1}e^{\mu s}(\mu I+A)^{-1}d\mu d\lambda \\ &= (2\pi i)^{-2}\int_{\Gamma}\int_{\Gamma'} e^{\lambda t+\mu s}(\mu-\lambda)^{-1}\{(\lambda I+A)^{-1}-(\mu I+A)^{-1}\}d\mu d\lambda, \end{aligned}$$

using the resolvent identity. But for $\lambda \in \Gamma$, $\mu \in \Gamma'$,

$$\int_{\Gamma} e^{\lambda t}(\mu-\lambda)^{-1}d\lambda = 0,\quad \int_{\Gamma'} e^{\mu s}(\mu-\lambda)^{-1}d\mu = 2\pi i\ e^{\lambda s}$$

so $e^{-At}e^{-As} = (2\pi i)^{-1}\int_{\Gamma} e^{\lambda(t+s)}(\lambda I+A)^{-1}d\lambda = e^{-A(t+s)}$, and $\{e^{-At}\}_{t\ge 0}$ is a semigroup. In fact, for $0 < \varepsilon < \theta-\pi/2$, the integral converges uniformly in any compact set of $\{|\arg t| < \varepsilon\}$, so the semigroup is analytic there.

Also, putting $\mu = \lambda t$ in the integral (with $t > 0$)

$$\|e^{-At}\| = \Big\|\frac{1}{2\pi i}\int_{\Gamma} e^{\mu}\Big(\frac{\mu}{t} + A\Big)^{-1}\frac{d\mu}{t}\Big\| \le \frac{M}{2\pi}\int_{\Gamma} |e^{\mu}|\,\frac{|d\mu|}{|\mu|}$$

and

$$||Ae^{-At}|| \leq \frac{1}{2\pi}\frac{M}{\delta}\int_\Gamma |e^\mu|\frac{|d\mu|}{|\mu|}\cdot\frac{1}{t} = \frac{\text{Constant}}{t}.$$

We now prove $e^{-At}x \to x$ as $t \to 0+$ for each $x \in X$. It suffices to prove this for $x \in D(A)$, a dense set, since $||e^{-At}|| \leq C$ for all $t \geq 0$. If $x \in D(A)$ and $t > 0$,

$$e^{-At}x-x = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}[(\lambda I+A)^{-1} - \lambda^{-1}]x d\lambda = -\frac{1}{2\pi i}\int_\Gamma \lambda^{-1}e^{\lambda t}A(\lambda I+A)^{-1}x d\lambda$$

so $||e^{-At}x-x|| \leq \text{Constant}\,||Ax||\,t$.

Thus $\{e^{-At}\}_{t\geq 0}$ is a strongly continuous semigroup which extends to an analytic semigroup in $|\arg t| < \varepsilon$. If $x \in D(A)$, $t > 0$, then

$$\frac{d}{dt}e^{-At}x + Ae^{-At}x = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}(\lambda+A)(\lambda+A)^{-1}x\, d\lambda = 0.$$

Thus if $x \in D(A)$, as $t \to 0+$

$$\frac{1}{t}(e^{-At}x-x) = -\frac{1}{t}\int_0^t e^{-As}Ax\, ds \to -Ax,$$

so $-A$ is contained in the generator G of the semigroup.

To see that $-A$ actually is the generator, define for $\lambda \geq 0$

$$R(\lambda)x = \int_0^\infty e^{-\lambda t}e^{-At}x\, dt.$$

For any x, $e^{-At}x \in D(A)$ for $t > 0$, and for $\delta > 0$

$$A\int_\delta^\infty e^{-\lambda t}e^{-At}x\, dt = e^{-\lambda\delta}\, e^{-A\delta}x - \lambda\int_\delta^\infty e^{-\lambda t}e^{-At}x\, dt.$$

By closedness of A, it follows that $R(\lambda)x \in D(A) \subset D(G)$ for every $\lambda \geq 0$, $x \in X$. But, if $x \in D(G)$, then $e^{-At}x \in D(G)$ for all $t \geq 0$ and $G\, e^{-At}x = \frac{d}{dt}e^{-At}x = e^{-At}Gx$, and a similar argument shows

$$R(\lambda)(\lambda-G)x = x \quad \text{for} \quad x \in D(G).$$

Thus $D(G) \subset$ range of $R(\lambda) \subset D(A)$, hence $-A = G$ as claimed.

Remark. The converse is also true: if $-A$ generates an analytic semigroup, then A is sectorial [30, 51].

Exercises.

(7)* If A is a sectorial operator and $\operatorname{Re}\lambda > a$ whenever $\lambda \in \sigma(A)$, then there exist $M > 0$, $0 < \phi < \pi/2$ such that $\|(\lambda-A)^{-1}\| \le M/|\lambda-a|$ whenever $|\arg(\lambda-a)| > \phi$.

(8)* If A is sectorial and m is any positive integer, then for every $t > 0$, $R(e^{-At}) \subset D(A^m)$; thus $D(A^m)$ is dense in X for every $m \ge 1$.

(9) If $\{e^{-At}, t \ge 0\}$ is a strongly continuous semigroup (satisfies (i) and (ii) of def. 1.3.3) such that $\|e^{-At}\| \le C$, $\|Ae^{-At}\| \le Ct^{-1}$ for $0 < t \le 1$, then $\{e^{-At}, t \ge 0\}$ is an analytic semigroup.

Hint: $\|\frac{d^m}{dt^m} e^{-At}\| = \|(-A)^m e^{-At}\| \le C^m m^m t^{-m}$ for $0 < t \le 1$ and $m = 0,1,2,\ldots$.

(10)* If A is a bounded linear operator, then e^{-At} as defined above extends to a group of linear operators

$$e^{-At}\, e^{-As} = e^{-A(t+s)} \quad \text{for} \quad -\infty < s,t < \infty$$

and $e^{-At} = \sum_{n=0}^{\infty} (-At)^n/n!$.

(11) If A is a self adjoint positive definite operator in a Hilbert space with spectral representation $A = \int_0^\infty \lambda\, dE_\lambda$, then e^{-At}, as defined above, is

$$e^{-At} = \int_0^\infty e^{-\lambda t} dE_\lambda, \qquad t \ge 0.$$

(12) Prove that, if A is sectorial, $t > 0$ and $x \in X$, then $(I + \frac{t}{n}A)^{-n}x \to e^{-At}x$ as $n \to +\infty$. This result is used in 3.3, exercise 6.

Hint: Show $(I + \frac{t}{n} A)^{-n}x = \frac{1}{2\pi i}\int_\Gamma (\lambda+A)^{-1}x(1 - \frac{\lambda t}{n})^{-n} d\lambda$, and apply Lebesgue's dominated convergence theorem.

(13) Let $X = C[-\infty,\infty]$, the bounded continuous functions from $\mathbb{R}$ to $\mathbb{R}$ with the sup-norm. Let $Au(x) = -d^2u/dx^2$ for $u \in D(A) = \{u \in C^2(\mathbb{R}) \mid u,u',u'' \in X\}$, and let X_1 = closure of $D(A)$ in X. (Note: $X_1 \ne X$). If $\lambda = m^2$, $\operatorname{Re} m > 0$, calculate

$$(\lambda+A)^{-1}f(x) = \frac{1}{2m}\int_{-\infty}^{\infty} e^{-m|x-\xi|} f(\xi)\, d\xi,$$

for $f \in X_1$. Prove A is sectorial in X_1, $\sigma(A) = [0,+\infty)$, and for any complex $\lambda \neq 0$ with $|\arg \lambda| \leq \theta < \pi$, and any $f \in X_1$,

$$\|(\lambda+A)^{-1}f\|_X \leq |\lambda|^{-1}\sec \theta/2 \, \|f\|_X.$$

Show that, for any $\phi \in X_1$ (i.e., any uniformly continuous $\phi \in X$) if $u(x,t) = (e^{-At}\phi)(x)$ $(t > 0,\ -\infty < x < \infty)$ then u, $\frac{\partial u}{\partial t}$, $\frac{\partial^2 u}{\partial x^2}$ are continuous and $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ $(t > 0,\ -\infty < x < \infty)$. Also $u(x,t) \to \phi(x)$ as $t \to 0+$, uniformly in x. Can you prove this is the only solution of

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad (t > 0)$$

$$u(0+,x) = \phi(x), \quad -\infty < x < \infty.$$

(14) Prove existence of a solution $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ $(0 < x < 1,\ t > 0)$ $u_x(0,t) = v(t)$, $u_x(1,t) + u(1,t) = w(t)$, $\frac{dv}{dt} = \alpha v + \beta w$, $\frac{dw}{dt} = \gamma v + \delta w + \int_0^1 u(x,t)dx$ given $u(\cdot,0) \in L^2(0,1)$ and real $v(0)$, $w(0)$. (Cf. ex. 7 above.)

1.4 Fractional powers of operators

Definition 1.4.1. Suppose A is a sectorial operator and $\operatorname{Re} \sigma(A) > 0$; then for any $\alpha > 0$

$$A^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-At} dt.$$

Examples

(1) If A is a positive scalar $(X = \mathbb{R}^1)$, then $A^{-\alpha}$ as defined is the usual $(-\alpha)$ power of A.

(2) If A is a positive definite, self adjoint operator in a Hilbert space with spectral representation $A = \int_0^\infty \lambda \, dE(\lambda)$, then $A^{-\alpha} = \int_0^\infty \lambda^{-\alpha} dE(\lambda)$.

(3) If $A = I+B$ where $\|B\| < 1$, then $A^{-\alpha}$ as defined above agrees with the usual power series representation: $(I+B)^{-\alpha} = \sum_{n=0}^\infty \binom{-\alpha}{n} B^n$,

where $\binom{-\alpha}{n} = (-1)^n \frac{\Gamma(\alpha+n)}{n!\ \Gamma(\alpha)} = (-1)^n \frac{\alpha(\alpha+1)\ldots(\alpha+n-1)}{n!}$.

(4) A^{-1} (the case $\alpha = 1$) is theinverse of A.

Theorem 1.4.2. If A is a sectorial operator in X with $\operatorname{Re}\sigma(A) > 0$, then for any $\alpha > 0$, $A^{-\alpha}$ is a bounded linear operator on X which is one-one and satisfies $A^{-\alpha}A^{-\beta} = A^{-(\alpha+\beta)}$ whenever $\alpha > 0$, $\beta > 0$. Also, for $0 < \alpha < 1$,

$$A^{-\alpha} = \frac{\sin \pi\alpha}{\pi} \int_0^\infty \lambda^{-\alpha}(\lambda+A)^{-1}d\lambda.$$

Proof. For some $\delta > 0$, $\operatorname{Re}\sigma(A) > \delta$, so by Th. 1.3.4, $\|e^{-At}\| \le Ce^{-\delta t}$ for $t > 0$. Thus $\|A^{-\alpha}x\| \le \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1}Ce^{-\delta t}dt\ \|x\|$, and $A^{-\alpha}$ is bounded when $\alpha > 0$. Also for $\alpha > 0$, $\beta > 0$

$$\begin{aligned} A^{-\alpha}A^{-\beta} &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^\infty \int_0^\infty t^{\alpha-1}s^{\beta-1}e^{-A(t+s)}dsdt \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^\infty du \int_0^t t^{\alpha-1}(u-t)^{\beta-1}e^{-Au}dt \\ &= A^{-(\alpha+\beta)}, \text{ using } \int_0^1 z^{\alpha-1}(1-z)^{\beta-1}dz = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}. \end{aligned}$$

Also if $A^{-\alpha}x = 0$ for some $\alpha > 0$, then for integer $n > \alpha$, $A^{-n}x = A^{-(n-\alpha)}A^{-\alpha}x = 0$; but A^{-1} is one-one, so $A^{-n} = n^{th}$ power of A^{-1} is also one-one, so $x = 0$.

Finally $(\lambda+A)^{-1} = \int_0^\infty e^{-At}e^{-\lambda t}dt$ for $\lambda \ge 0$, so

$$\begin{aligned} \int_0^\infty \lambda^{-\alpha}(\lambda+A)^{-1}d\lambda &= \int_0^\infty e^{-At}\left(\int_0^\infty \lambda^{-\alpha}e^{-\lambda t}d\lambda\right)dt \\ &= \int_0^\infty e^{-At}t^{\alpha-1}\Gamma(1-\alpha)dt = \frac{\pi}{\sin \pi\alpha} A^{-\alpha}, \end{aligned}$$

using the fact $\Gamma(\alpha)\Gamma(1-\alpha) = \pi/\sin \pi\alpha$ for $0 < \alpha < 1$.

Definition 1.4.1, continued. With A as above, define A^α = inverse of $A^{-\alpha}$ $(\alpha > 0)$, $D(A^\alpha) = R(A^{-\alpha})$; A^0 = identity on X.

Example. A^1 = inverse of $A^{-1} = A$.

Exercises.

(1)* If $\alpha > 0$, A^α is closed and densely defined.

(2)* If $\alpha \ge \beta$ then $D(A^\alpha) \subset D(A^\beta)$.

(3)* $A^\alpha A^\beta = A^\beta A^\alpha = A^{\alpha+\beta}$ on $D(A^\gamma)$ where $\gamma = \max(\alpha,\beta,\alpha+\beta)$.
(4)* $A^\alpha e^{-At} = e^{-At}A^\alpha$ on $D(A^\alpha)$, $t > 0$.

Theorem 1.4.3. Suppose A is sectorial and $\mathrm{Re}\,\sigma(A) > \delta > 0$. For $\alpha \geq 0$ there exists $C_\alpha < \infty$ such that

$$\|A^\alpha e^{-At}\| \leq C_\alpha t^{-\alpha} e^{-\delta t} \quad \text{for} \quad t > 0,$$

and if $0 < \alpha \leq 1$, $x \in D(A^\alpha)$,

$$\|(e^{-At}-1)x\| \leq \frac{1}{\alpha} C_{1-\alpha} t^\alpha \|A^\alpha x\| .$$

Also, C_α is bounded for α in any compact interval of $(0,\infty)$. (We see below that C_α is also bounded as $\alpha \to 0+$).

Proof. $\|e^{-At}\| \leq Ce^{-\delta t}$, $\|Ae^{-At}\| \leq C\, t^{-1}e^{-\delta t}$ for $t > 0$, by Th. 1.3.4, so for $m = 1,2,3,\ldots$

$$\|A^m e^{-At}\| = \|(Ae^{-At/m})^m\| \leq (Cm)^m t^{-m} e^{-\delta t}.$$

If $0 < \alpha < 1$, $t > 0$,

$$\begin{aligned}\|A^\alpha e^{-At}\| &= \|Ae^{-At} \cdot A^{-(1-\alpha)}\| \\ &\leq \frac{1}{\Gamma(1-\alpha)} \int_0^\infty s^{-\alpha} \|Ae^{-A(t+s)}\|\, ds \\ &\leq C\, t^{-\alpha} e^{-\delta t} \Gamma(\alpha).\end{aligned}$$

Finally, $\|A^{\alpha+\beta}e^{-At}\| \leq \|A^\alpha e^{-At/2}\|\,\|A^\beta e^{-At/2}\| \leq C_\alpha C_\beta 2^{\alpha+\beta}\, t^{-(\alpha+\beta)} e^{-\delta t}$, and putting these cases together gives the general result. The other estimate follows from

$$(e^{-At}-I)x = -\int_0^t A^{1-\alpha} e^{-As} A^\alpha x\, ds.$$

Theorem 1.4.4. If $0 \leq \alpha \leq 1$, $x \in D(A)$, then $\|A^\alpha x\| \leq C\,\|Ax\|^\alpha\, \|x\|^{1-\alpha}$, i.e. $\|A^\alpha x\| \leq \varepsilon\,\|Ax\| + C'\varepsilon^{-\alpha/(1-\alpha)}\,\|x\|$ for all $\varepsilon > 0$. (Here C, C' are constants independent of α.)

Proof. Let $0 < \beta < 1$, $\varepsilon > 0$, so (if $\|e^{-At}\| \leq C$ for $t > 0$)

$$\|\Gamma(\beta)A^{-\beta}x\| = \|(\int_0^\varepsilon + \int_\varepsilon^\infty) t^{\beta-1}e^{-At}x\, dt\|$$

$$\leq C\|x\| \frac{\varepsilon^\beta}{\beta} + \|\varepsilon^{\beta-1}e^{-A\varepsilon}A^{-1}x + (\beta-1)\int_\varepsilon^\infty t^{\beta-2}e^{-At}A^{-1}x\|$$

$$\leq C\|x\| \frac{\varepsilon^\beta}{\beta} + 2C\|A^{-1}x\|\, \varepsilon^{\beta-1}.$$

Minimize the right-hand side over $\varepsilon > 0$ and conclude

$$\|A^{-\beta}x\| \leq \frac{2(2(1-\beta))^{\beta-1}}{\Gamma(1+\beta)} C\|x\|^{1-\beta}\|A^{-1}x\|^\beta.$$

The coefficient is bounded uniformly on $0 < \beta < 1$, so we can replace x by Ax and set $\alpha = 1-\beta$ to get the result claimed.

Remark. If $\|e^{-At}\| \leq C_0 e^{-\delta t}$, $\|Ae^{-At}\| \leq C_1 t^{-1}e^{-\delta t}$ for $t > 0$ then (if C is the constant from Th. 1.4.4)

$$\|A^\alpha e^{-At}\| \leq CC_0^{1-\alpha}C_1^\alpha\, t^{-\alpha}e^{-\delta t},$$

proving C_α in Th. 1.4.3 is bounded as $\alpha \to 0+$.

Exercise 5*. When $\alpha = \beta\theta + (1-\theta)\gamma$, $0 \leq \theta \leq 1$, $\beta \geq 0$, $\gamma \geq 0$, show there is a constant C so

$$\|A^\alpha x\| \leq C\|A^\beta x\|^\theta \|A^\gamma x\|^{1-\theta}.$$

Combining Theorems 1.3.2 and 1.4.4 proves the following.

Corollary 1.4.5. If A is a sectorial operator with $\text{Re}\,\sigma(A) > 0$ and if B is a linear operator such that $BA^{-\alpha}$ is bounded on X for some α in $0 \leq \alpha < 1$, then $A+B$ is sectorial.

Exercises.

(6) If A is positive definite and self adjoint, then so is A^α for all $\alpha > 0$. Compare with exer. 4, sec. 1.3.

(7)* Suppose A is sectorial with $\text{Re}\,\sigma(A) > 0$; then A^{-1} is compact if and only if $A^{-\alpha}$ is compact for all $\alpha > 0$, if and only if e^{-At} is compact for $t > 0$.

(8) For each $x \in X$, $t \to tAe^{-At}x$ is continuous from $0 \leq t < \infty$ to X and

$$\|tAe^{-At}x\| \to 0 \quad \text{as} \quad t \to 0+.$$

(This result is used in Th. 3.4.2.)

(9) Suppose A is self-adjoint and positive definite in a Hilbert space; prove

$$\|A^{\alpha}x\| \leq \|Ax\|^{\alpha}\|x\|^{1-\alpha} \quad \text{for} \quad x \in D(A), \quad 0 \leq \alpha \leq 1.$$

Hint: For $0 < \alpha < 1$, let $g(t) = t^{1/\alpha}$ (a convex function) and $A = \int_0^\infty \lambda\, dE_\lambda$; $d\mu = d\|E_\lambda x\|^2$ $(x \in D(A))$, and apply Jensen's inequality to $g(\int_0^\infty \lambda^{2\alpha}d\mu(\lambda)/\int_0^\infty d\mu(\lambda))$.

(10) If $x \in X$, A is sectorial in X with $\operatorname{Re}\sigma(A) > 0$, then for any α in $0 < \alpha \leq 1$,

$$t^{\alpha}\|A^{\alpha}e^{-At}x\| \to 0 \quad \text{as} \quad t \to 0+.$$

Theorem 1.4.6. Suppose A, B are sectorial operators in X with $D(A) = D(B)$, with $\operatorname{Re}\sigma(A) > 0$, $\operatorname{Re}\sigma(B) > 0$, and for some α in $[0,1)$, $(A-B)A^{-\alpha}$ is bounded on X. Then for any β in $[0,1]$, $A^{\beta}B^{-\beta}$ and $B^{\beta}A^{-\beta}$ are bounded in X.

Proof. By Th. 1.4.4, $\|A^{\beta}(\lambda+A)^{-1}\| \leq C|\lambda|^{\beta-1}$ for $0 \leq \beta \leq 1$, $|\pi-\arg\lambda| \geq \phi$, for some positive constants C and $\phi < \pi/2$. Also for $0 < \beta < 1$

$$B^{-\beta}-A^{-\beta} = \frac{1}{\pi}\sin\pi\beta\int_0^\infty \lambda^{-\beta}(\lambda+B)^{-1}(A-B)(\lambda+A)^{-1}d\lambda$$

- easy estimates then show $B^{\beta}A^{-\beta}$ is bounded. Also $\|A^{\alpha}(\lambda+B)^{-1}\| = O(|\lambda|^{\alpha-1})$ as $\lambda \to +\infty$, since $\{I+A^{\alpha}(\lambda+A)^{-1}(B-A)A^{-\alpha}\}A^{\alpha}(\lambda+B)^{-1} = A^{\alpha}(\lambda+A)^{-1}$, so interchanging A and B in the integral identity above proves $A^{\beta}B^{-\beta}$ is also bounded. The cases $\beta = 0$, $\beta = 1$ follow immediately.

Exercise 11*. [30, p. 177]. Suppose A is sectorial in a Banach space X, $\operatorname{Re}\sigma(A) > 0$, and B is linear from $D(B) \subset X$ to a Banach space Y. Assume $D(B) \supset D(A)$ and, for some α in $0 \leq \alpha < 1$ and constant C (or K) that, for all $x \in D(A)$

$$\|Bx\|_Y \leq C\,\|Ax\|^{\alpha}\;\|x\|^{1-\alpha}$$

or equivalently

$$\|Bx\|_Y \le \varepsilon\|Ax\| + K\varepsilon^{-\alpha/(1-\alpha)}\|x\|$$

for all $\varepsilon > 0$. Prove for any β in $\alpha < \beta \le 1$ that B has a unique extension to a continuous linear operator from X^β to Y (see def. 1.4.7), i.e. that $BA^{-\beta}$ is continuous.

Hint: note BA^{-1} is bounded, and for $x \in D(A^{1+\beta})$ $(\alpha < \beta < 1)$ estimate

$$Bx = \frac{1}{\Gamma(\beta)} \int_0^\infty t^{\beta-1} Be^{-At} A^\beta x \, dt.$$

Definition 1.4.7. If A is a sectorial operator in a Banach space X, define for each $\alpha \ge 0$

$$X^\alpha = D(A_1^\alpha) \text{ with the graph norm}$$

$$\|x\|_\alpha = \|A_1^\alpha x\|, \quad x \in X^\alpha,$$

where $A_1 = A + aI$ with a chosen so $\mathrm{Re}\,\sigma(A_1) > 0$. Different choices of a give equivalent norms on X^α, by Th. 1.4.6, so we supress the dependence on the choice of a.

These spaces X^α will provide the basic topology in all the pages below, so we collect certain of their properties here, merely reformulating results on fractional powers proved above.

Theorem 1.4.8. If A is sectorial in a Banach space X, then X^α is a Banach space in the norm $\|\cdot\|_\alpha$ for $\alpha \ge 0$, $X^0 = X$, and for $\alpha \ge \beta \ge 0$, X^α is a dense subspace of X^β with continuous inclusion. If A has compact resolvent, the inclusion $X^\alpha \subset X^\beta$ is compact when $\alpha > \beta \ge 0$.

If A_1, A_2 are sectorial operators in X with the same domain and $\mathrm{Re}\,\sigma(A_j) > 0$ for $j = 1,2$, and if $(A_1 - A_2)A_1^{-\alpha}$ is a bounded operator for some $\alpha < 1$, then with $X_j^\beta = D(A_j^\beta)$ $(j = 1,2)$, $X_1^\beta = X_2^\beta$ with equivalent norms for $0 \le \beta \le 1$.

Example. For a bounded smooth domain Ω in $\mathbb{R}^n$, take $Au = (-\Delta)^m u$ for $u \in C_0^{2m}(\Omega)$, and extend A to be a self adjoint positive definite operator in $X = L^2(\Omega)$ with domain $H_0^m(\Omega) \cap H^{2m}(\Omega)$. Then for $0 \le \alpha \le 1$, if $m\alpha$ is an integer,

$$X^\alpha = D((-\Delta)^{m\alpha}) = H_0^{m\alpha}(\Omega) \cap H^{2m\alpha}(\Omega),$$

with equivalent norms; this is also true for nonintegral $m\alpha$, when the fractional order Sobolev spaces are defined by interpolation [71].

Other examples are in sec. 1.6.

1.5. Invariant subspaces and exponential bounds

Definition 1.5.1. If A is a linear operator with domain and range in a Banach space X, and $\sigma(A)$ denotes the spectrum, a set $\sigma \subset \sigma(A) \cup \{\infty\} \equiv \hat{\sigma}(A)$ is a *spectral set* if both σ and $\hat{\sigma}(A)\setminus\sigma$ are closed in the extended plane $C \cup \{\infty\}$.

Example. An isolated point of $\sigma(A)$, or a finite collection of isolated points of $\sigma(A)$, is a spectral set; the complement of a spectral set is a spectral set.

Theorem 1.5.2. Suppose A is a closed linear operator in X and suppose σ_1 is a bounded spectral set, and $\sigma_2 = \sigma(A)\setminus\sigma_1$ so $\sigma_2 \cup \{\infty\}$ is another spectral set. Let E_1, E_2 be the projections associated with these spectral sets, and $X_j = E_j(X)$, $j = 1,2$. Then $X = X_1 \oplus X_2$, the X_j are invariant under A, and if A_j is the restriction of A to X_j, then

$$A_1 : X_1 \to X_1 \text{ is bounded, } \sigma(A_1) = \sigma_1;$$

$$D(A_2) = D(A) \cap X_2 \text{ and } \sigma(A_2) = \sigma_2.$$

For the proof, see [23, v.1, Ch. 7 or 99, Th. 5.7, A,B].

Remark. If A has compact resolvent, as frequently happens, then $\sigma(A)$ consists of isolated eigenvalues of finite multiplicity; in this case, X_1 would be finite dimensional.

Theorem 1.5.3. Suppose A is a sectorial operator and σ_1 is a bounded spectral set; forming the operators A_1, A_2 as above, A_1 is bounded and A_2 is a sectorial operator.

If $\operatorname{Re}\sigma_1 = \operatorname{Re}\sigma(A_1) < \alpha$, then $\|e^{-A_1 t}\| \le Ce^{-\alpha t}$ for $t \le 0$;

If $\operatorname{Re}\sigma_2 = \operatorname{Re}\sigma(A_2) > \beta$, then for $t > 0$, $\|e^{-A_2 t}\| \le Ce^{-\beta t}$, $\|A_2 e^{-A_2 t}\| \le Ct^{-1}e^{-\beta t}$.

Proof. If $x \in X_2$, $\|x\| \leq 1$, $\lambda \notin \sigma(A)$, then $\|(\lambda-A_2)^{-1}x\| = \|(\lambda-A)^{-1}x\| \leq C/|\lambda|$ for $|\arg \lambda| > \phi$, $|\lambda| > R$, (some ϕ in $(0,\pi/2)$). Since $\operatorname{Re} \sigma(A_2) > \beta$, it follows that $\|(\lambda-A_2)^{-1}\| \leq C'/|\lambda-\beta|$ for $|\arg(\lambda-\beta)| > \phi'$, some ϕ' in $(0,\pi/2)$ (see ex. 7 following Th. 1.3.4), and so the estimates for $e^{-A_2 t}$ follow from Th. 1.3.4.

Since A_1 is a bounded operator, the estimate for $e^{-A_1 t}$ is elementary.

Exercise 1*. For A_1 as above, the spectral radius $r(e^{A_1}) < e^{\alpha}$ and $r(e^{A_1}) = \lim_{n\to\infty} \|e^{A_1 n}\|^{1/n}$. Show that this implies there is a constant C,

$$\|e^{A_1 s}\| \leq Ce^{\alpha s} \quad \text{for all} \quad s \geq 0.$$

In fact, $\|e^{A_1 s}\| e^{-\alpha s} \to 0$ as $s \to +\infty$. [98, 39].

Example. $Au = -\dfrac{d^2u}{dx^2}$ on $(0,\pi)$ with $u(0) = u(\pi) = 0$; $X = L^2(0,\pi)$, $\sigma(A) = \{1,2^2,3^2,\ldots\}$, $\sigma_1 = \{1,2^2,\ldots,N^2\}$, $\sigma_2 = \{(N+1)^2\ldots.\}$. Then $X_1 = \operatorname{span}\{\sin kx\}_{k=1}^{N}$, X_2 is the orthogonal complement of X_1, and

$$A_2 f = \sum_{N+1}^{\infty} n^2 \phi_n(\phi_n,f), \quad \phi_n(x) = \frac{1}{\sqrt{2}} \sin nx$$
$$e^{-A_2 t} = \sum_{N+1}^{\infty} e^{-n^2 t} \phi_n(\phi_n,f)$$

for $f \in C^2(0,\pi)$ such that $(\phi_j,f) = 0$, $1 \leq j \leq N$, and $f(0) = f(\pi)=0$. Combining Th. 1.5.3 with Th. 1.4.8 proves the following.

Theorem 1.5.4. Suppose A is a sectorial operator, σ_1 a bounded spectral set for A, $\sigma_2 = \sigma(A)\setminus\sigma_1$, $\operatorname{Re} \sigma_2 > \gamma$, and $X = X_1 \oplus X_2$ is the corresponding decomposition.

Assume also B is a sectorial operator with $D(B) = D(A)$, $\operatorname{Re} \sigma(B) > 0$, $(B-A)B^{-\alpha}$ is bounded for some $\alpha < 1$. Then using the norm $\|x\|_{\beta} \equiv \|B^{\beta}x\|$, $0 \leq \beta \leq 1$: for $x \in X_2 \cap D(B^{\beta})$ and $t > 0$,

$$\|e^{-A_2 t}x\|_{\beta} \leq C_1 \|x\| t^{-\beta} e^{-\gamma t}$$

$$\|e^{-A_2 t}x\|_{\beta} \leq C_1 \|x\|_{\beta} e^{-\gamma t}$$

for some constant C_1.

Exercise 2. If A is sectorial in X and $a = \inf \operatorname{Re} \sigma(A)$ then the spectral radius $r(e^{-At}) = e^{-at}$ for any $t \geq 0$.

1.6. An example of fractional powers and an embedding theorem

Suppose Ω is an open set in $\mathbb{R}^n$ (possibly all of $\mathbb{R}^n$) and let u be a complex-valued C^2 function on Ω with support compact in Ω. If $1 < p < \infty$ and λ is a fixed complex number, then

$$\int_\Omega \bar{u}|u|^{p-2}(\lambda u - \Delta u)\,dx = \lambda \int_\Omega |u|^p dx + \int_\Omega J,$$

$$J = |u|^{p-2}\, \nabla u \cdot \nabla \bar{u} + \bar{u}\nabla u \cdot \nabla |u|^{p-2}.$$

Now

$$|u|^2 \nabla u \cdot \nabla \bar{u} = |u|^2 \nabla |u| \cdot \nabla |u| + (\operatorname{Im}(\bar{u}\nabla u))^2$$

$$(\operatorname{Im}(\bar{u}\nabla u) = u_1 \nabla u_2 - u_2 \nabla u_1 \quad \text{when} \quad u = u_1 + iu_2)$$

and so

$$|\operatorname{Im} J| \leq |p-2|\,|u|^{p-2}|\nabla |u||\ \ |\operatorname{Im}(\tfrac{\bar{u}\nabla u}{|u|})|$$

$$\operatorname{Re} J = (p-1)|u|^{p-2}(\nabla |u|)^2 + |u|^{p-2}|\operatorname{Im}(\tfrac{\bar{u}}{|u|}\nabla u)|^2,$$

so

$$|\operatorname{Im} J|/\operatorname{Re} J \leq |p-2|/2\sqrt{p-1}.$$

If $0 < \eta \leq 2\sqrt{p-1}/|p-2|$ then by Hölder's inequality,

$$(1+\eta)\,\|\lambda u - \Delta u\|_{L_p(\Omega)} \geq (\operatorname{Re} \lambda + \eta |\operatorname{Im} \lambda|)\,\|u\|_{L_p(\Omega)}.$$

It follows that the Laplacian on $C_c^2(\Omega)$ is closeable in $L_p(\Omega)$ and we write the closure Δ_D. If the spectrum $\sigma(\Delta_D)$ is not the entire complex plane, then $\sigma(\Delta_D)$ is in the sector $\{\lambda: \operatorname{Re}\lambda + \eta|\operatorname{Im}\lambda| \leq 0\}$ and $-\Delta_D$ is sectorial. This is true when $\Omega = \mathbb{R}^n$ (as we prove below) and when Ω is bounded and $\partial\Omega$ is a C^2 hypersurface separating Ω from $\mathbb{R}^n \setminus \bar{\Omega}$ (see, for example, [30]).

Now we concentrate on the case $\Omega = \mathbb{R}^n$. Suppose $\lambda \in \mathbb{C} \setminus (-\infty, 0]$, $\operatorname{Re}\sqrt{\lambda} > 0$, $u \in C_c^2(\mathbb{R}^n)$ and $(\lambda - \Delta)u = f$; then the Fourier transform

$$\hat{u}(y) = \int_{\mathbb{R}^n} e^{-ix\cdot y}\, u(x)\,dx$$

satisfies

$$\hat{u}(y) = (\lambda+y^2)^{-1} \hat{f}(y)$$

so

$$u = \Gamma_\lambda * f \quad \text{where} \quad \hat{\Gamma}_\lambda(y) = (\lambda+y^2)^{-1}.$$

The function G_α with $\hat{G}_\alpha(y) = (1+y^2)^{-\alpha/2}$ is

$$G_\alpha(x) = \frac{(4\pi)^{-n/2}}{\Gamma(\alpha/2)} \int_0^\infty t^{(\alpha-n)/2} e^{-(t+ x\cdot x/4t)} \frac{dt}{t}$$

$$= \frac{(4\pi)^{-n/2}\xi^{(\alpha-n)/2}}{\Gamma(\alpha/2)} \int_0^\infty s^{(\alpha-n)/2} e^{-\xi(s+1/4s)} \frac{ds}{s}$$

where $\xi = \sqrt{x\cdot x}$.

Examining the limits as $\xi \to 0$ or $\text{Re}\ \xi \to +\infty$, we find

$$|G_\alpha(x)| \leq C_\alpha |\xi|^{\alpha-n/2} (\text{Re}\ \xi)^{\alpha-n/2} e^{-\frac{1}{2}\text{Re}\ \xi} \quad \text{if} \quad \alpha < n$$

$$|G_n(x)| \leq C_n \max\{\ln \frac{1}{\text{Re}\ \xi}, 1\} e^{-\frac{1}{2}\text{Re}\ \xi}$$

for $\text{Re}\ \xi > 0$ and constants C_α, C_n. (See ex. 1.) Note $G_\alpha(x)$ is bounded on $\mathbb{R}^n$ when $\alpha > n$. Now $\hat{\Gamma}_\lambda(y) = \lambda^{-1}\hat{G}_2(y/\sqrt{\lambda})$ so

$$\Gamma_\lambda(x) = (\sqrt{\lambda})^{n-2} G_2(x\sqrt{\lambda}) \qquad (\text{Re}\ \sqrt{\lambda} > 0)$$

and so there is a constant C with

$$\int_{\mathbb{R}^n} |\Gamma_\lambda(x)|\, dx \leq \frac{C}{|\lambda|} \left(\frac{|\sqrt{\lambda}|}{\text{Re}\sqrt{\lambda}}\right)^{\frac{n}{2}+1}.$$

Thus $(\lambda-\Delta)u = f$ implies (see ex. 2 with $q = 1$, $r = p$)

$$\|u\|_{L_p(\mathbb{R}^n)} \leq \frac{C}{(\cos\ \theta/2)^{n/2+1}} \frac{1}{|\lambda|} \|f\|_{L_p(\mathbb{R}^n)}$$

for $1 \leq p \leq \infty$ and $|\arg \lambda| \leq \theta < \pi$. If f is bounded and differentiable with bounded derivative, it is easily shown that $u \equiv \Gamma_\lambda * f$ is C^2 with $(\lambda-\Delta)u = f$, when $\text{Re}\ \sqrt{\lambda} > 0$, and if f has compact support $u(x) \to 0$ exponentially as $|x| \to +\infty$. It follows that $-\Delta_D$ is sectorial in $L_p(\mathbb{R}^n)$, $1 \leq p < \infty$, and in the space of uniformly continuous bounded functions $C_{unif}(\mathbb{R}^n)$, and in the space of continuous functions which vanish at ∞, with $\sigma(\Delta_D) = (-\infty, 0]$ in each case.

<u>Exercise 1</u>. Let $J_p(m) = \int_0^\infty s^{-p-1}e^{-m(s+1/4s)}ds$, $m > 0$; then show

$$J_p(m) \simeq \begin{cases} 4^p\Gamma(p)m^{-p} & \text{as } m \to 0+, \text{ if } p > 0 \\ 2\,\ln \frac{1}{m} & \text{as } m \to 0+ \text{ if } p = 0 \end{cases}$$

$$J_p(m) \simeq 2^p\sqrt{2\pi/m}\,e^{-m} \quad \text{as } m \to +\infty$$

so

$$J_p(m) \le \begin{cases} B_p m^{-p}e^{-m/2} & \text{for } m > 0 \quad (\text{if } p > 0) \\ B_0 \max(\ln \frac{1}{m}, 1)\, e^{-m/2} & (\text{if } p = 0), \end{cases}$$

for some constants B_p.

Let $1 \le p < \infty$, $X = L_p(\mathbb{R}^n)$, $A = 1-\Delta_D$; then A is sectorial in X and $u = e^{-\bar{A}t}\phi$ satisfies $du/dt + Au = 0$, $t > 0$, if $\phi \in C_c^2(\mathbb{R}^n)$ and so

$$(u(\cdot,t))^\wedge(y) = e^{-t(1+|y|^2)}\,\hat{\phi}(y),$$

hence

$$(A^{-\alpha/2}\phi)^\wedge(y) = (1 + |y|^2)^{-\alpha/2}\,\hat{\phi}(y)$$

for $\alpha > 0$, i.e.

$$A^{-\alpha/2}\phi = G_\alpha * \phi.$$

<u>Exercise 2</u>. If $1 \le p,q,r \le \infty$ and $1/p + 1 = 1/q + 1/r$, show

$$\|f*g\|_{L_p(\mathbb{R}^n)} \le \|f\|_{L_q(\mathbb{R}^n)}\,\|g\|_{L_r(\mathbb{R}^n)}.$$

<u>Hint</u>: apply the Hölder inequality for three terms to

$$\int |f(y)|^{1-\alpha}\cdot|g(x-y)|^{1-\beta}\cdot(|f(y)|^\alpha|g(x-y)|^\beta)\,dy,$$

$$\alpha = q/p,\ \beta = r/p.$$

Note the special cases $q = 1$ or $1/q + 1/r = 1$.

<u>Exercise 3</u>. Show $\|G_\alpha\|_{L_q(\mathbb{R}^n)} < \infty$ if $\alpha > n(1 - 1/q)$, $1 \le q \le \infty$ and

$$\|G_\alpha(x+h) - G_\alpha(x)\|_{L_q(\mathbb{R}^n)} \le C|h|^\nu$$

if $\nu = \alpha-n(1 - 1/q)$, $0 < \nu < 1$ and $\alpha < n$, for a constant C.

<u>Hint</u>: $|G_\alpha(x+h)-G_\alpha(x)| \le C|h|\,|x|^{\alpha-n-1}$ for $|x| \ge 2|h|$.

Now for $\alpha > 0$, $1 \leq p < \infty$, define

$$\mathscr{L}^p_\alpha(\mathbb{R}^n) = \{G_\alpha * f \mid f \in L_p(\mathbb{R}^n)\},$$

which is of course $D(A^{\alpha/2})$; this space is provided with the usual norm

$$\|G_\alpha * f\|_{\mathscr{L}^p_\alpha(\mathbb{R}^n)} = \|f\|_{L_p(\mathbb{R}^n)} = \|A^{\alpha/2} G_\alpha * f\|_{L_p}.$$

We also identify $\mathscr{L}^p_0(\mathbb{R}^n)$ with $L_p(\mathbb{R}^n)$. These are the so called Bessel potential spaces [97] and we will derive some of their properties. Our approach uses fairly crude estimates and so, it not entirely self contained; we refer to Calderon [109] or Stein [97] for some of the results. We will, however, prove all the results for the case of strict inequality in the conditions below.

According to exercises 2 and 3, we have

$$\|G_\alpha * f\|_{L_q} \leq \|G_\alpha\|_{L_r} \|f\|_{L_p} \leq \text{Const.} \|G_\alpha * f\|_{\mathscr{L}^p_\alpha}$$

when $\alpha > n(1 - 1/r)$ and $1/q = 1/p + 1/r - 1$, so we have the continuous inclusion

$$\mathscr{L}^p_\alpha(\mathbb{R}^n) \subset L_q(\mathbb{R}^n)$$

when $1/p \geq 1/q > 1/p - \alpha/n$. If $p > 1$ and $q < \infty$ this holds also for $1/q = 1/p - \alpha/n$ (see [109]).

By the definition, it follows that

$$\mathscr{L}^p_\alpha(\mathbb{R}^n) \subset \mathscr{L}^q_\beta(\mathbb{R}^n)$$

when $p \leq q$ and $\alpha - n/p \geq \beta - n/q$, (with strict inequality for $p = 1$). Note that $\alpha - \beta \geq n(1/p - 1/q) \geq 0$. Also, by exercise 3,

$$|G_\alpha * f(x+h) - G_\alpha * f(x)| \leq \text{Constant } |h|^\nu \|G_\alpha * f\|_{\mathscr{L}^p_\alpha}$$

when $\nu = \alpha - n/p$, $0 < \nu < 1$ and $0 < \alpha < n$, so

$$\mathscr{L}^p_\alpha(\mathbb{R}^n) \subset C^\nu(\mathbb{R}^n).$$

Suppose $\nu \leq \alpha - n/p$, $0 < \nu < 1$, but $\alpha \geq n$; we can choose $0 < \beta < n$ and $q \geq p$ so $\nu = \beta - n/q \leq \alpha - n/p$ and so

$$\mathscr{L}^p_\alpha(\mathbb{R}^n) \subset \mathscr{L}^q_\beta(\mathbb{R}^n) \subset C^\nu(\mathbb{R}^n)$$

when $0 < \nu < 1$, $\nu \leq \alpha - n/p$ (strict inequality if $p = 1$). A similar argument (ex. 4) shows $\mathscr{L}^p_\alpha \subset C^\nu$ when

$$\nu \leq \alpha - n/p \text{ (strict inequality if } p = 1 \text{ or } \nu = \text{integer).}$$

Alternatively, we may appeal to the following [97]: when $p > 1$, $\alpha \geq 1$, $f \in \mathscr{L}^p_\alpha(\mathbb{R}^n)$ if and only if f and $\partial f/\partial x_i \in \mathscr{L}^p_{\alpha-1}(\mathbb{R}^n)$ $(i = 1,\ldots,n)$ and

$$\|f\|_{\mathscr{L}^p_{\alpha-1}} + \sum_{i=1}^n \|\partial f/\partial x_i\|_{\mathscr{L}^p_{\alpha-1}}$$

is an equivalent norm. The proof of this is rather difficult, and the result is false when $p = 1$. With this, we easily extend the inclusion $\mathscr{L}^p_\alpha \subset C^\nu$ from the case $0 < \nu < 1$ to any positive nonintegral ν, when $p > 1$.

Since $\mathscr{L}^p_0(\mathbb{R}^n) = L_p(\mathbb{R}^n)$, the result quoted above proves $\mathscr{L}^p_k(\mathbb{R}^n) = W^{k,p}(\mathbb{R}^n)$ for $k = 0,1,2,\ldots$ and $p > 1$; these spaces are not the same when $p = 1$. However for all $p \geq 1$, $\mathscr{L}^p_\alpha(\mathbb{R}^n)$ contains (or is contained in) $W^{k,p}(\mathbb{R}^n)$ when $\alpha < k$ (or $\alpha > k$, respectively); see ex. 5.

<u>Exercise 4</u>. Show $|(\partial/\partial x)^k G_\alpha(x)| \leq C_{\alpha,k} e^{-|x|/2} |x|^{\alpha-n-k}$ when $0 < \alpha < n+k$. (Note $\partial/\partial|x|\ G_\alpha(x) = C_\alpha |x|\ G_{\alpha-2}(x)$ for a constant C_α.) Conclude from this that $\mathscr{L}^p_\alpha(\mathbb{R}^n) \subset W^{k,p}(\mathbb{R}^n)$ if $\alpha > k$, $p \geq 1$. Also conclude $\mathscr{L}^p_\alpha(\mathbb{R}^n) \subset C^\nu(\mathbb{R}^n)$ if $\nu \leq \alpha - n/p$, with strict inequality if $p = 1$ or ν is an integer.

<u>Exercise 5</u>. Prove $\mathscr{L}^p_\alpha(\mathbb{R}^n) \supset W^{k,p}(\mathbb{R}^n)$ when $p \geq 1$ and $\alpha < k$ $(k = 1,2,\ldots)$.
<u>Hint</u>: choose integer $m \geq k/2$ and $\beta \geq 0$ so $\alpha+\beta = 2m$; then

$$\|f\|_{\mathscr{L}^p_\alpha} = \|(1-\Delta)^m G_\beta * f\|_{L^p} \leq C\,\|G_\beta\|_{W^{2m-k,1}}\,\|f\|_{W^{k,p}}.$$

If we define $\mathscr{L}^\infty_\alpha(\mathbb{R}^n) = C^\alpha(\mathbb{R}^n)$ for positive non-integral α (the left side is undefined here when α is an integer), our results can be combined as follows:

$$\mathscr{L}^p_\alpha(\mathbb{R}^n) \subset \mathscr{L}^q_\beta(\mathbb{R}^n)$$

whenever $p \le q$ and $\alpha - n/p \ge \beta - n/q$ (strict inequality if $p = 1$), provided both sides are defined. The connection between the spaces C^α and $\mathscr{L}^p_\alpha$ with $p = \infty$ is, of course, no accident: see [97].

Now we prove a form of the Nirenberg-Gagliardo inequality [30], which includes the above statement as a special case.

By the moment inequality for fractional powers (exercise 5, sec. 1.4), for any non-negative α, β_1, γ_1 and $0 \le \theta \le 1$ with $\alpha = \theta\beta_1 + (1-\theta)\gamma_1$, and $1 \le p < \infty$, there is a constant C such that for all u in $C_c^\infty(\mathbb{R}^n)$,

$$\|u\|_{\mathscr{L}^p_\alpha} \le C\,\|u\|^{\theta}_{\mathscr{L}^p_{\beta_1}}\,\|u\|^{1-\theta}_{\mathscr{L}^p_{\gamma_1}}.$$

In case $p = \infty$, this is an interpolation theorem for the spaces $C^\nu(\mathbb{R}^n)$: see, for example, [8]. If $p \ge q$, $p \ge r$ and $\beta - n/q \ge \beta_1 - n/p$, $\gamma - n/r \ge \gamma_1 - n/p$ (with strict inequality if q or $r = 1$), we have:

$$\|u\|_{\mathscr{L}^p_\alpha} \le C\,\|u\|^{\theta}_{\mathscr{L}^q_\beta}\,\|u\|^{1-\theta}_{\mathscr{L}^r_\gamma}$$

and $\alpha - n/p \le \theta(\beta-n/q) + (1-\theta)(\gamma-n/r)$, with strict inequality if q or $r = 1$, and $p \ge q$, $p \ge r$. Conversely if p,q,r, α,β,γ and θ, $0 \le \theta \le 1$, satisfy these conditions we can choose appropriate β_1, γ_1 to prove the result -- at least if β and γ are large (so $\beta_1 \ge 0$, $\gamma_1 \ge 0$). But we can replace α,β,γ by $\alpha+N$, $\beta+N$, $\gamma+N$ (integer N) without changing anything else, and the original case follows from this.

In particular, we have a version of the *Nirenberg-Gagliardo inequalities:*

(a) $\|u\|_{W^{k,p}} \le C\,\|u\|^{\theta}_{W^{m,q}}\,\|u\|^{1-\theta}_{L_r}$

if $p \ge q$, $p \ge r$, $0 \le \theta \le 1$, and

$$k - n/p \le \theta(m - n/q) - n(1-\theta)/r$$

with strict inequality if q or $r = 1$;

(b) $\|u\|_{C^\nu} \le C\,\|u\|^{\theta}_{W^{m,q}}\,\|u\|^{1-\theta}_{L_r}$

if $0 \le \theta \le 1$ and $\nu \le \theta(m - n/q) - n(1-\theta)/r$, (strict inequality if q or $r = 1$, or if ν is an integer).

<u>Remark</u>. The inequality in Friedman [30] has different conditions: (a) holds when $k - n/p = \theta(m - n/q) - n(1-\theta)/r$, $1 \le p,q,r < \infty$, and

$k/m \le \theta \le 1$, so $1/p \le \theta/q + (1-\theta)/r$. Thus p is less restricted than in our version. We did not mention the domain $\mathbb{R}^n$ in these inequalities because they hold on any fairly nice set $\Omega \subset \mathbb{R}^n$. Specifically, suppose there is an extension map $E: C_c^m(\overline{\Omega}) \to C_c^m(\mathbb{R}^n)$, so $E(\phi)$ restricted to $\overline{\Omega}$ is ϕ, such that for the norms of any of the spaces C^ν or $W^{k,q}$ ($0 \le \nu, k \le m$ and $1 \le q < \infty$) there is a constant $B > 0$ with

$$B^{-1} \|\phi\|_\Omega \le \|E(\phi)\|_{\mathbb{R}^n} \le B \|\phi\|_\Omega .$$

It follows easily that the inequalities (a), (b) also hold when restricted to Ω.

Such an extension map is easily constructed if Ω is bounded and $\partial\Omega$ is a C^m hypersurface separating Ω from $\mathbb{R}^n \setminus \overline{\Omega}$ (see [30]), and a more complicated construction shows $\partial\Omega$ needs only to be Lipschitzian [97] ("minimally smooth"). When such an extension map exists, we say Ω has the *C^m-extension property.*

Exercise 6. Consider $\mathbb{R}^{n-1}$ as a subspace $\mathbb{R}^{n-1}\times\{0\}$ of $\mathbb{R}^n$ and let $\mathscr{R}$ be the restriction map

$$\mathscr{R}u(x') = u(x'0), \quad x' \in \mathbb{R}^{n-1}, \quad u \in C_c^\infty(\mathbb{R}^n).$$

Prove $\mathscr{R}$ extends to a continuous map of $\mathscr{L}_\alpha^p(\mathbb{R}^n)$ to $\mathscr{L}_\beta^q(\mathbb{R}^{n-1})$ when $p \le q$ and $\alpha - n/p > \beta - (n-1)/q$. In fact, restriction to a k-dimensional subspace $\mathbb{R}^k \subset \mathbb{R}^n$ is continuous from $\mathscr{L}_\alpha^p(\mathbb{R}^n)$ to $\mathscr{L}_\beta^q(\mathbb{R}^k)$ when $p \le q$ and $\alpha - n/p > \beta - k/q$.

Hint: if $v(\cdot,t) = e^{-t(1-\Delta)}g$ for $t \ge 0$, then (by Fourier transformation in the x_n-variable)

$$v(x',0,t) = (4\pi t)^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-y^2/4t} e^{-t(1-\Delta_{x'})} g(x',y)\,dy$$

where $\Delta_{x'}$ is the Laplacian in the first $(n-1)$ variables. Thus, if $u = (1-\Delta)^{-\alpha/2}g$,

$$\sqrt{4\pi}\ \Gamma(\alpha/2)u(x',0) = \int_{-\infty}^{\infty} dy \int_0^\infty t^{\frac{\alpha-1}{2}-1} e^{-y^2/4t} e^{-t(1-\Delta_{x'})} g(x',y)\,dt$$

and we estimate the $\mathscr{L}_\gamma^p(\mathbb{R}^{n-1})$ norm when $\alpha - \gamma > 1/p$.

Remark. We can allow $\alpha - n/p = \beta - k/q$ provided $1 < p = q \le 2$ and $\beta > 0$, or $1 < p < q < \infty$. If $p = q = 2$, $\beta > 0$, the restriction map is then surjective. (See [97] and [109].)

Exercise 7. If $1 \le p < \infty$, $\alpha > \beta \ge 0$ and $B, R < \infty$ then show $K = \{u \mid \|u\|_{\mathscr{L}^p_\alpha(\mathbb{R}^n)} \le B,\ u = 0 \text{ a.c. in } |x| > R\}$ is in a compact set in $\mathscr{L}^p_\beta(\mathbb{R}^n)$.
Hint: first show $\|u(\cdot+h) - u\|_{L_p(\mathbb{R}^n)} \le C|h|^\alpha$ when $u \in K$, $h \in \mathbb{R}^n$ and $0 < \alpha < 1$ thus show $\|u(\cdot+h)-u\|_{L_p} \to 0$ as $h \to 0$ uniformly on K for each $\alpha > 0$ and apply the compactness criterion of Fréchet and Kolomogorov [103] to prove the result for $\beta = 0$; and complete the proof with the Nirenberg-Gagliardo inequality

$$\|u\|_{\mathscr{L}^p_\beta} \le C\,\|u\|^\theta_{\mathscr{L}^p_\alpha}\,\|u\|^{1-\theta}_{L_p} \quad \text{when} \quad \beta/\alpha < \theta < 1.$$

Remark. A similar result for $p = \infty$ follows immediately from the Arzela-Ascoli theorem.

Exercise 8. If $1 \le p < \infty$, $p \le q$ and $\alpha - n/p > \beta - n/q$, and if S_R = functions on $\mathbb{R}^n$ supported in the ball of radius R, then the inclusion

$$S_R \cap \mathscr{L}^p_\alpha(\mathbb{R}^n) \subset \mathscr{L}^q_\beta(\mathbb{R}^n)$$

is compact.

Theorem 1.6.1. Suppose $\Omega \subset \mathbb{R}^n$ is an open set having the C^m extension property, $1 \le p < \infty$, and A is a sectorial operator in $X = L_p(\Omega)$ with $D(A) = X^1 \subset W^{m,p}(\Omega)$ for some $m \ge 1$. Then for $0 \le \alpha \le 1$,

$$X^\alpha \subset W^{k,q}(\Omega) \quad \text{when} \quad k - n/q < m\alpha - n/p,\ \ q \ge p,$$

$$X^\alpha \subset C^\nu(\Omega) \quad \text{when} \quad 0 \le \nu < m\alpha - n/p.$$

Proof. By the Nirenberg-Gagliardo inequality (a),

$$\|u\|_{W^{k,q}(\Omega)} \le C\,\|u\|^\theta_{W^{m,p}(\Omega)}\,\|u\|^{1-\theta}_{L_p(\Omega)}$$

provided $k - n/q < \theta(m - n/p) - n(1-\theta)/p = m\theta - n/p$ and $q \ge p$. Thus for $u \in D(A)$,

$$\|u\|_{W^{k,q}(\Omega)} \le C_1\,\|Au\|^\theta_{L_p}\,\|u\|^{1-\theta}_{L_p}$$

so by exercise 11, sec. 1.4, the inclusion map $D(A) \subset W^{m,p}(\Omega) \to W^{k,q}(\Omega)$ extends to a continuous inclusion of $X^\alpha \to W^{k,q}(\Omega)$, provided $\alpha > \theta$. The other case is proved similarly.

Exercise 9. Suppose $\Omega \subset \mathbb{R}^n$ has the C^m extension property, X is a closed subspace of $C_{unif}(\Omega)$ and A is a sectorial operator in X with domain in $C^m(\Omega)$. For $0 < \alpha \leq 1$ prove

$$X^\alpha \subset C^\mu(\Omega) \quad \text{when} \quad \mu < m\alpha.$$

Remark. The special case $A = -\Delta_D$ in $L^2(\Omega)$, $\Omega \subset \mathbb{R}^3$, has $D(A) = W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$, so $X^\alpha \subset C^\nu(\Omega)$ when $\alpha > 3/4$, $\nu < (4\alpha-3)/2$; $X^\alpha \subset W^{1,q}(\Omega)$ for $1/q > (5-4\alpha)/6$ and $\alpha > 1/2$; $X^\alpha \subset L^q(\Omega)$ for $1/q > (3-4\alpha)/6$.

Exercise 10. For A as in the above remark, let $Bu(x) = \sum_1^3 b_j(x)\frac{\partial u}{\partial x_j} + c(x)u$ where $b_j \in L^6(\Omega)$, $c \in L^2(\Omega)$; shows $B: X^\alpha \to X$ is continuous if $\alpha > 3/4$.

Applying Th. 1.6.1 to the operator $A = -\Delta_D$ in $L^p(\Omega)$, $\Omega \subset \mathbb{R}^n$, we have $D(A) \subset W^{2,p}(\Omega)$ [2, 7, 91] so:

$$X^\alpha \subset C^\nu(\Omega) \quad \text{if} \quad 0 \leq \nu < 2\alpha - n/p;$$

$$X^\alpha \subset W^{1,q}(\Omega) \quad \text{if} \quad \alpha > 1/2 \ \text{and} \ 1/q > 1/p - (2\alpha-1)/n;$$

$$X^\alpha \subset L^q(\Omega) \quad \text{if} \quad 1/q > 1/p - 2\alpha/n, \quad q \geq p.$$

Exercise 11. Suppose A, B are sectorial operators in Banach space X,Y and for some $\alpha_1,\alpha_2,\beta_1,\beta_2,\theta$ in $[0,1]$, T is a linear continuous map from X^{α_j} to Y^{β_j} for $j = 1,2$; then T is also continuous from $X^{\alpha+\varepsilon}$ to Y^β (any $\varepsilon > 0$) if $\alpha = \theta\alpha_1 + (1-\theta)\alpha_2$, $\beta = \theta\beta_1 + (1-\theta)\beta_2$.

Chapter 2

Examples of Nonlinear Parabolic Equations in Physical, Biological and Engineering Problems

In this section we display some of the variety of parabolic problems encountered in applications, and hopefully justify the study of the abstract formulation of parabolic equations given later. In particular, this abstract formulation covers ordinary differential equation, semilinear parabolic partial differential equations, and systems of coupled ordinary and partial differential equations.

2.1 Nonlinear heat equation

(For more specific applications, see [10].) Conduction of heat in a stationary medium is described by

$$\rho\, c_p \frac{\partial T}{\partial t} = \operatorname{div}(K \operatorname{grad} T) + \rho q$$

where T is the temperature, ρ = density, c_p = specific heat, K = conductivity, and q = rate of production of heat per unit mass. If the heat source is, say, radioactive decay, it will be essentially independent of T, but if heat is produced by chemical reaction, q will be strongly (and ordinarily nonlinearly) dependent on T. For example, $q = Qe^{-H/RT}$, the Arrhenius factor, is frequently encountered.

If the medium is moving with given velocity $v(x,t)$, we must add a convection term:

$$\rho\, c_p\left(\frac{\partial T}{\partial t} + \vec{v} \cdot \operatorname{grad} T\right) = \operatorname{div}(K \operatorname{grad} T) + \rho q.$$

Similar equations describe the diffusion of a fluid (liquid or gas) in a porous medium, but in these applications the diffusivity (K, in the equations above) frequently depends strongly on the concentration of the fluid (T, in the equations above), leading to quasilinear rather than semilinear equations. Pending extension of the theory to quasilinear systems (existence and uniqueness questions have been studied [93], but little beyond that), a simple trick may reduce the problem to a semilinear one. Suppose $K = K(T)$ is a smooth positive scalar function, and change the time variable to s:

$$\rho c_p \frac{\partial T}{\partial s} = \Delta T + \frac{K'(T)}{K(T)} |\nabla T|^2 + \frac{\rho q}{K(T)}$$

$$\frac{\partial t}{\partial s} = \frac{1}{K(T)} > 0,$$

a semilinear system of coupled ordinary and partial differential equations.

2.2 Flow of electrons and holes in a semiconductor [83]

If p,n are the concentrations of holes and electrons, respectively, and V is the electrical potential,

$$\Delta V = -q(p - n+D)$$

$$\frac{\partial n}{\partial t} = \operatorname{div} q\mu_n(\alpha_n \operatorname{grad} n - n\nabla V) - R_n(n,p)$$

$$\frac{\partial p}{\partial t} = \operatorname{div} q\mu_p(\alpha_p \operatorname{grad} p + p\nabla V) + R_p(n,p)$$

where D, q, μ_p, μ_n, α_p, α_n are positive constants and R_n, R_p are the recombination rates. The first equation (assuming appropriate boundary conditions are given) gives V as a function of p and n: $V = qG(p - n+D)$, where G is the appropriate Green's function. Inserting this in the other equations gives a pair of coupled semilinear parabolic partial differential equations, where the lower order terms involve the integral operator G.

Similar equations appear in the theory of ions in solution [17, p. 191].

2.3 Hodgekin-Huxley equations for the nerve axon [16, 17, 25]

A parabolic equation for the electrical potential V is coupled with a system of ordinary differential equations for the quantities m, h, n which vary between zero and one, and describe the changes in the conductance of the axon membrane for potassium (K) and sodium (Na):

$$C \frac{\partial V}{\partial t} = \frac{1}{r_e+r_i} \frac{\partial^2 V}{\partial x^2} - \bar{g}_k n^4 (V-E_k) - \bar{g}_{Na} m^3 h (V-E_{Na})$$

$$\frac{\partial n}{\partial t} = \alpha_n(v)(n-\bar{n}) - \beta_n(v)n$$

$$\frac{\partial m}{\partial t} = \alpha_m(v)(m-\overline{m}) - \beta_m(v)m$$

$$\frac{\partial h}{\partial t} = \alpha_h(v)(h-\overline{h}) - \beta_h(v)h.$$

Here α_n, β_n, etc. are functions of V alone, for example

$$\alpha_n = .01(V + 10)/(\exp(\frac{V+10}{10}) - 1),$$

$$\beta_n = .125 \exp(V/80)$$

(where V is measured in millivolts). (See Cole [17] for more details.) An analogous simpler system, the Nagumo equation has also been studied as a preliminary to this system; see [16, 74].

2.4 Chemical reactions in a catalyst pellet [33, 4].

Suppose $M_1,\ldots,M_N$ are the N chemical species involved in R independent reactions $\sum_{i=1}^{N} \nu_{ij} M_i = 0$ $(j = 1,\ldots,R)$. If c_i is the concentration of M_i and T is the temperature, then

$$\varepsilon_p \frac{\partial c_i}{\partial t} = \operatorname{div}(D_i \operatorname{grad} c_i) + \sum_{j=1}^{R} \nu_{ij} f_j \quad (i = 1,\ldots,N)$$

$$\rho c_p \frac{\partial T}{\partial t} = \operatorname{div}(k \operatorname{grad} T) - \sum_{j=1}^{R} \sum_{i=1}^{N} \nu_{ij} H_i f_j$$

where $f_j = f_j(c_1,\ldots,c_N,T)$ is the rate of the j^{th} reaction and H_i is the partial molar enthalpy of the i^{th} species (assumed constant).

Similar systems were studied by D. B. Spalding [94] in the theory of flames, and by Amundson and Varma [100] for tubular chemical reactors.

2.5 Population genetics [20, 73]

Suppose A_1, A_2 are the two alleles in a population at a single gene locus for $0 \le p,x \le 1$ and $t > 0$, $\phi(x,t;p)dx$ is the probability that the frequency at A_1 at time t lies in $(x,x+dx)$, given that the A_1-frequency is p at time $t = 0$. Then ϕ satisfies

$$\frac{\partial \phi}{\partial t} = \frac{1}{2} \frac{\partial^2}{\partial x^2} (V(x,t)\phi) - \frac{\partial}{\partial x} (M(x,t)\phi)$$

where M, V are the mean and variance of the change in gene frequency per generation (i.e., per unit time). For example, assuming random mating among a population of size N, $V(x,t) = x(1-x)/2N$, and M describes the action of selection, mutation, etc. on the gene frequencies. The elliptic operator on the right-side of this equation is then singular at $x = 0$ and $x = 1$, and this makes the discussion of these equations difficult [26]. The singularity is not surprising, however, since $x = 0$ and $x = 1$ are absorbing states in the absence of mutation.

The effect of "sampling error" in finite populations is important, in general, but it is difficult to include it together with the effects of migration when geographic variations in gene frequency are allowed. If the population density is large enough that the "law of large numbers" may be invoked in local populations, we obtain equations of the form

$$\partial N/\partial t = m^2/2\Delta N + g(N,p)N$$
$$\partial M/\partial t = m^2/2\Delta M + b(N,p)M/(1-s(1-p)) - d(N,p)M$$

where $p = M/2N$ is the local A_1 gene frequency, N and M are the population and A_1-gene densities, $g = b-d$ is the growth rate (difference of birth and death rates), m is the migration rate, and s the selective advantage of A_1, assuming A_1 is dominant. Since s is small, these equations have almost the same form and we may replace the M-equation by an equation for $p = M/2N$:

$$\partial p/\partial t = (m^2/2N^2)\mathrm{div}(N^2 \mathrm{grad}\ p) + b(N,p)sp(1-p)/[1-s(1-p)].$$

If the population density N is constant, this is the form of equation investigated (in the one dimensional case) by Kolmogoroff, Petrovsky and Piscounoff in 1937 [60]; see sec. 5.4.

Similar equations have been used to study dispersal of plants and animals [90] and the geographic spread of epidemics [6].

2.6 Nuclear reactor dynamics (multigroup neutron diffusion equations)

If ψ_j = neutron flux in the j^{th} energy group and $\Psi = \mathrm{col}(\psi_1,\ldots,\psi_M)$, $V = \mathrm{diag}(v_1,\ldots,v_M)$ where v_j is the speed of energy group j, C_i = precursor density for the i^{th} delayed neutron group $(i = 1,\ldots,N)$ and the matrices A, F, χ, D measure respectively absorption minus scattering, fission, emission spectra, and diffusion for the various classes of particles, then

$$V^{-1}\frac{\partial\Psi}{\partial t} = \operatorname{div}(D \operatorname{grad} \Psi) + ((1-\beta)\chi F^T - A)\Psi + \sum_{i=1}^{N} \lambda_i \chi_i C_i$$

$$\frac{\partial C_i}{\partial t} = \beta_i F^T \Psi - \lambda_i C_i \qquad (i = 1,\dots,N).$$

(See [46, p. 449] for further explanation.) Control of the reactor is exerted by changes in the matrix A. A vastly simplified one-group equation with negative feedback control is discussed in exercise 4, sec. 4.3, and non-negativity of the solutions for the general case is in exercise 5, sec. 3.3.

2.7 Navier-Stokes and related equations

An important system of equations that does not obviously fall in our class of semilinear parabolic equations is the Navier-Stokes system, describing the flow of a viscous incompressible fluid:

$$\frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\operatorname{grad})\vec{v} = \nu\Delta\vec{v} - \frac{1}{\rho}\operatorname{grad} p$$

$$\operatorname{div}\vec{v} = 0,$$

where $\vec{v}$ is the velocity, p the pressure, and ρ,ν are given positive constants (density and kinematic viscosity). One of these equations ($\operatorname{div}\vec{v} = 0$) involves no time derivative, and one of the unknown quantities (p) never appears with a time derivative. Both these problems are resolved, by working in a Banach space of divergence -- free vector fields $\vec{v}$: the second equation is automatically satisfied, and the term $\operatorname{grad} p$ in the first equation disappears. See [32] or sec. 3.8 for the details.

Related systems are studied in the same way. Coupling the Navier-Stokes equations with the heat equation, by including a convection term in the heat equation and a bouyancy term in the Navier-Stokes equations gives the Boussinesq equations:

$$\frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\operatorname{grad})\vec{v} = \nu\Delta\vec{v} - \frac{1}{\rho}\operatorname{grad} p + \mathrm{Gr}\,\theta\vec{g}$$

$$\operatorname{div}\vec{v} = 0$$

$$\frac{\partial\theta}{\partial t} + (\vec{v} \cdot \text{grad})\theta = \frac{1}{Pr} \Delta\theta + \vec{v}\cdot\vec{g},$$

($\vec{g}$ = direction of gravity). Bifurcation of equilibrium solutions for this system (Bénard convection) has been extensively studied [59], along with some work on stability questions.

Gerd Lassner [67] investigates one formulation of the equations of magneto-hydrodynamics in this spirit, using a slight modification of Ohm's law.

R. Hide [47] gives the following system for the flow and the magnetic field within the earth:

$$\frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\text{grad})\vec{v} = \nu\Delta\vec{v} - \frac{1}{\rho}\nabla p - 2\vec{\Omega} \times \vec{v} + \frac{1}{\rho\mu}(\nabla \times \vec{B}) \times \vec{B}$$

$$\frac{\partial\vec{B}}{\partial t} = \lambda\Delta\vec{B} + \nabla \times (\vec{V} \times \vec{B})$$

$$\text{div}\ \vec{v} = 0, \quad \text{div}\ \vec{B} = 0.$$

These equations probably admit a similar treatment.

Chapter 3

Existence, Uniqueness and Continuous Dependence

3.1 Examples and counterexamples

In this chapter we study the initial-value problem for a class of abstract differential equations which includes ordinary differential equations and many semilinear parabolic partial differential equations. Here we shall give a few examples to display some of the ways things can go wrong, before studying ways to ensure they go right.

Ordinary differential equations are included, problems of the form: find differentiable $u(\cdot)$ from $[0,\tau)$ to $\mathbb{R}^n$, for some $\tau > 0$, such that $du/dt = f(t,u)$, $0 < t < \tau$, and $u(0) = u_0$.

Example of nonexistence. If $f(u) = -1$ when $u \geq 0$, $f(u) = 1$ when $u < 0$, there is no absolutely continuous function on $[0,\tau]$, $\tau > 0$, such that $u(0) = 0$ and $du/dt = f(u)$ for a.e. t in $(0,\tau)$.

Second example of nonexistence. If $p \geq 1$, there is no absolutely continuous solution of

$$\frac{du}{dt} = t^{-p} \quad (0 < t < \tau), \quad u(0) = 1.$$

In both of these examples, the difficulty arises from the fact that the right side of the equation is not continuous near the initial point. Here is an example which is certainly continuous:

Example of non-global existence

$$\frac{dx}{dt} = x^2, \quad x(0) = a > 0;$$

in this case $x(t) = a/(1-at)$ for $0 \leq t < a^{-1}$ is the only solution, and it does not exist for all $t > 0$, but only for t sufficiently close to the initial time.
The difficulty here, rapid growth at infinity, cannot be exorcised by smoothness assumptions, and will remain to plague us throughout the following pages.

<u>Example of nonuniqueness</u>. $(0 < \alpha < 1)$. The problem

$$\frac{dx}{dt} = |x|^{\alpha}$$
$$x(0) = 0$$

has the solution $x(t) \equiv 0$ and also infinitely many other solutions; for any $\tau > 0$,

$$x(t) = \begin{cases} 0 & \text{for } 0 \leq t \leq \tau, \\ p^{-p}(t-\tau)^p & \text{for } t \geq \tau, \end{cases}$$

where $p = 1/(1-\alpha)$.

Here the problem is againone of insufficient smoothness: the right side of the equation is not Lipschitz continuous in u.

It may be thought that "real" parabolic equations would not suffer from such infirmities -- but that hope is vain.

Consider the semilinear parabolic problem given $f: \mathbb{R}^2 \to \mathbb{R}$,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(t,u) \qquad (0 < x < 1, \quad t > 0)$$
$$\frac{\partial u}{\partial x} = 0 \quad \text{at} \quad x = 0,1$$
$$u(x,0) = a \quad \text{for} \quad 0 < \alpha < 1;$$

this has as a solution $u(x,t) = v(t)$ provided

$$\frac{dv}{dt} = f(t,v), \qquad v(0) = a.$$

Taking various choices for f and a, we can obtain the phenomena of the examples above including nonexistence, nonuniqueness, and nonglobal existence. Some less artifical examples were given by Fujita [31] and Levine [70].

<u>Exercise</u>. $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(x,u(\cdot,t))$ on $0 < x < \pi$, $u = 0$ at $x = 0,\pi$ where

$$f(x,\phi) = \sin x \; g(\phi_1,\phi_2) + \sin 2x \; h(\phi_1,\phi_2)$$
$$\phi_j = (2/\pi) \int_0^{\pi} \phi(x) \sin jx \, dx \qquad (j = 1,2)$$

has solutions of the form $u(x,t) = a(t) \sin x + b(t) \sin 2x$, provided $da/dt = -a + g(a,b)$, $db/dt = -4b + h(a,b)$. For appropriate choices g,h we can duplicate all the phenomena of O.D.E.s in the plane, including centers and limit cycles. Also, any solution of the partial differential equation converges exponentially to a corresponding solution of the O.D.E., as $t \to +\infty$.

We close with a fairly natural example having nonglobal existence:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u^3 \qquad (0 < x < \pi, \quad t > 0)$$

$$u(0,t) = 0, \quad u(\pi,t) = 0,$$

$$u(x,0) = \phi(x), \text{ a given smooth function.}$$

If $\int_0^\pi(\phi^2 + \phi_x^2)dx$ is sufficiently small, the solution $u(\cdot,t)$ exists for all $t \geq 0$ and tends to zero as $t \to +\infty$ (Th. 5.1.1). However, suppose ϕ is not small: assume $\phi(x) \geq 0$ for $0 \leq x \leq \pi$ and $\int_0^\pi \phi(x)\sin x\, dx > 2$. Then the solution with this initial value must blow up in finite time. Indeed, by maximum principle arguments (or sec. 3.3, ex. 8), $u(x,0) = \phi(x) \geq 0$ on $(0,\pi)$ implies $u(x,t) \geq 0$ for $t \geq 0$, $0 \leq x \leq \pi$, as long as the solution exists. And if we define

$$s(t) = \int_0^\pi \sin x\, u(t,x)dx,$$

then

$$\frac{ds}{dt} = -s + \int_0^\pi \sin x\, u^3(t,x)dx.$$

But, by Hölder's inequality, $s(t) \leq 2^{2/3}(\int_0^\pi u^3 \sin x\, dx)^{1/3}$ so

$$\frac{ds}{dt} \geq -s + \frac{1}{4}s^3 \quad \text{for} \quad t > 0,$$

$$s(0) = \int_0^\pi \phi(x)\sin x\, dx > 2.$$

It is then easily proved that $s(t) \to +\infty$ in finite time -- certainly not later than $t = \frac{1}{2}\log(s(0)+2)/(s(0)-2)$.

3.2 The linear Cauchy problem

First consider the homogeneous problem

$$\frac{dx}{dt} + Ax = 0, \quad t > 0$$
$$x(0) = x_0, \qquad (*)$$

where A is a sectorial operator in a Banach space X and $x_0 \in X$ is given. A *solution* of (*) on $0 < t < T$ is a continuous function $x: [0,T) \to X$ which is continuously differentiable on the open interval $(0,T)$, has $x(t) \in D(A)$ for $0 < t < T$, and satisfies (*) on $(0,T)$ with $x(t) \to x_0$ in X as $t \to 0+$. From Ch. 1, Th. 1.3.4, it is clear $x(t) = e^{-At}x_0$ is a solution of (*); we now prove this is the only solution.

Let $0 \leq s \leq t < T$ and

$$y(t,s) = e^{-A(t-s)}x(s),$$

where $x(\cdot)$ is any solution of (*) on $(0,T)$. Then $s \to y(t,s)$ is continuous on $0 \leq s \leq t$, and continuously differentiable on $0 < s < t$, with

$$\frac{\partial y(t,s)}{\partial s} = e^{-A(t-s)}\frac{dx(s)}{ds} + Ae^{-A(t-s)}x(s) = 0$$

for $0 < s < t$, so $y(t,0) = y(t,t)$, i.e.

$$e^{-At}x_0 = x(t).$$

Now we consider the nonhomogeneous equation

$$\frac{dx}{dt} + Ax = f(t), \quad 0 < t < T$$
$$x(0) = x_0.$$

Lemma 3.2.1. Let $f: (0,T) \to X$ be locally Hölder continuous with $\int_0^\rho \|f(s)\|ds < \infty$ for some $\rho > 0$. For $0 \leq t < T$, define

$$F(t) = \int_0^t e^{-A(t-s)}f(s)ds.$$

Then $F(\cdot)$ is continuous on $[0,T)$, continuously differentiable on $(0,T)$, with $F(t) \in D(A)$ for $0 < t < T$, and $dF(t)/dt + AF(t) = f(t)$ on $0 < t < T$, $F(t) \to 0$ in X as $t \to 0+$.

Proof. For small $\rho > 0$, define

$$F_\rho(t) = \int_0^{t-\rho} e^{-A(t-s)}f(s)ds, \quad \rho \leq t < T,$$

with $F_\rho(t) = 0$ for $0 \leq t \leq \rho$.

Then (setting $f(s) = 0$ for $s < 0$)

$$\|F(t) - F_\rho(t)\| \le \int_{t-\rho}^{t} \|e^{-A(t-s)}\| \|f(s)\| ds$$

which tends to 0 as $\rho \to 0+$, uniformly in $0 \le t \le t_0$ for any $t_0 < T$. Also, F_ρ is continuous, since

$$F_\rho(t+h)-F_\rho(t) = (e^{-Ah}-I)\int_0^{t-\rho} e^{-A(t-s)}f(s)ds + \int_{t-\rho}^{t+h-\rho} e^{-A(t+h-s)}f(s)ds$$

$(0 \le t \le t+h \le t_0)$, which tends to zero as $h \to 0$. Therefore F is continuous on $[0,T)$ into X, and $\|F(t)\| \le \int_0^t \|e^{-A(t-s)}\| \|f(s)\| ds \to 0$ as $t \to 0+$.

Also, if $0 \le s < t$, then $e^{-A(t-s)}f(s)$ is in $D(A)$, so the Riemann sums for $F_\rho(t)$,

$$\sum_{t-s_j \ge \rho} e^{-A(t-s_j)} f(s_j)\Delta s_j, \text{ are in } D(A), \text{ and } \lim_{\Delta s \to 0} A \sum_{s \le t-\rho} e^{-A(t-s)}f(s)\Delta s$$

$$= \int_0^{t-\rho} Ae^{-A(t-s)}f(s)ds.$$

Thus, by closedness of A, $F_\rho(t) \in D(A)$ and

$$AF_\rho(t) = \int_0^{t-\rho} Ae^{-A(t-s)}f(s)ds = \int_0^{t-\rho} Ae^{-A(t-s)}\{f(s)-f(t)\}ds$$
$$+ \{e^{-A\rho}-e^{-At}\}f(t).$$

Now $\|Ae^{-A(t-s)}\| = O((t-s)^{-1})$, $\|f(s)-f(t)\| = O(|t-s|^\theta)$ for some $\theta > 0$ as $s \to t-$, hence as $\rho \to 0+$,

$$AF_\rho(t) \to \int_0^t Ae^{-A(t-s)}\{f(s)-f(t)\}ds + \{I-e^{-At}\}f(t).$$

Thus, again by closedness of A, $F(t) \in D(A)$ for $0 < t < T$.

Consider any strictly interior interval $[t_0,t_1]$, $0 < t_0 < t_1 < T$; then $AF_\rho(t) \to AF(t)$ uniformly on $t_0 \le t \le t_1$, since $\|f(t)-f(s)\| \le K|t-s|^\theta$ for t, s in $[t_0,t_1]$ and some $\theta > 0$, so

$$\|AF_\rho(t)-AF(t)\| = \|\{-I+e^{-A\rho}\}f(t) + \int_{t-\rho}^{t} Ae^{-A(t-s)}\{f(s)-f(t)\}ds\|$$
$$\le \|\{e^{-A\rho}-I\}f(t)\| + C\int_{t-\rho}^{t}(t-s)^{-1+\theta}ds \to 0 \quad \text{as} \quad \rho \to 0+,$$

uniformly in $t_0 \le t \le t_1$.

Finally, $F_\rho(t)$ is differentiable when $t > \rho$, with $\frac{dF_\rho(t)}{dt} = -AF_\rho(t)+e^{-A\rho}f(t-\rho)$, $\rho < t < T$. The right side converges uniformly to $-AF(t) + f(t)$ on $t_0 \le t \le t_1$ $(0 < t_0 < t_1 < T)$ as $\rho \to 0+$, so F is continuously differentiable on the open interval $(0,T)$, with $\frac{dF}{dt} + AF = f(t)$.

Theorem 3.2.2. Suppose A is a sectorial operator in X, $x_0 \in X$, $f\colon (0,T) \to X$ is locally Hölder continuous and $\int_0^\rho \|f(t)\|\,dt < \infty$ for some $\rho > 0$; then there exists a unique (strong) solution $x(\cdot)$ of

$$\frac{dx}{dt} + Ax = f(t), \quad 0 < t < T; \quad x(0) = x_0,$$

namely

$$x(t) = e^{-At}x_0 + \int_0^t e^{-A(t-s)}f(s)ds.$$

Exercise 1 [20]. The problem of "genetic drift" in population genetics is described by the singular diffusion equation $\frac{\partial u}{\partial t} = \frac{1}{4N}\frac{\partial^2}{\partial x^2}(x(1-x)u)$ for $0 < x < 1$, $t > 0$. Examine the operator on the right side of this equation, considered as a self-adjoint operator in $X = \{\phi\colon (0,1) \to \mathbb{R}, \int_0^1 x(1-x)|\phi(x)|^2dx < \infty\}$, and show that it generates an analytic semigroup in X. Show that any u in the domain of the generator has $\sqrt{x(1-x)}u(x)$ continuous in $0 \le x \le 1$, and $\sqrt{x(1-x)}u(x) \to 0$ as $x \to 0$ or 1. If u is in the domain of the square of the generator, it is continuous on $0 \le x \le 1$. Show that $\|u(\cdot,t)\| \to 0$ as $t \to +\infty$. What is the meaning of this last result, in view of the interpretation of $u(\cdot,t)$ as a probability density? (See Sec. 2.5 and [20, 26].)

3.3 Local existence and uniqueness

Now consider the nonlinear equation

$$\begin{aligned} &\frac{dx}{dt} + Ax = f(t,x), \qquad t > t_0 \\ &x(t_0) = x_0, \end{aligned} \tag{*}$$

where we assume A is a sectorial operator so that the fractional powers of $A_1 \equiv A+aI$ are well defined, and the spaces $X^\alpha = D(A_1^\alpha)$

with the graph norm $\|x\|_\alpha = \|A_1^\alpha x\|$ are defined for $\alpha \geq 0$. We assume f maps some open set U in $\mathbb{R} \times X^\alpha$ into X, for some α in $0 \leq \alpha < 1$, and f is locally Hölder continuous in t and locally Lipschitz in x on U. More precisely, if $(t_1,x_1) \in U$, there exists a neighborhood $V \subset U$ of (t_1,x_1) such that for $(t,x) \in V$, $(s,y) \in V$,

$$\|f(t,x) - f(s,y)\| \leq L(|t-s|^\theta + \|x-y\|_\alpha),$$

for some constants $L > 0$, $\theta > 0$. In sections 3.6-3.8 we give some examples where these hypotheses hold, in addition to an example at the end of this section.

<u>Definition 3.3.1</u>. A solution of the Cauchy problem (initial-value problem) on (t_0,t_1) is a continuous function $x\colon [t_0,t_1) \to X$ such that $x(t_0) = x_0$ and on (t_0,t_1) we have $(t,x(t)) \in U$, $x(t) \in D(A)$, $\frac{dx}{dt}(t)$ exists, $t \to f(t,x(t))$ is locally Hölder continuous, and $\int_{t_0}^{t_0+\rho} \|f(t,x(t)\|\,dt < \infty$ for some $\rho > 0$, and the differential equation (*) is satisfied on (t_0,t_1).

<u>Lemma 3.3.2</u>. If x is a solution of (*) on (t_0,t_1), then

$$x(t) = e^{-A(t-t_0)}x_0 + \int_{t_0}^{t} e^{-A(t-s)}f(s,x(s))ds. \qquad (**)$$

Conversely, if x is a continuous function from (t_0,t_1) into X^α, and $\int_{t_0}^{t_0+\rho} \|f(s,x(s)\|\,ds < \infty$ for some $\rho > 0$, and if the integral equation (**) holds for $t_0 < t < t_1$, then $x(\cdot)$ is a solution of the differential equation (*) on (t_0,t_1).

<u>Proof</u>. The first claim is immediate from the definition of the solution and Th. 3.2.2. Suppose x is a solution of the integral equation (**) and $x \in C((t_0,t_1);X^\alpha)$. We first prove x is locally Hölder continuous from (t_0,t_1) to X^α. If $t_0 < t < t+h < t_1$, then

$$x(t+h)-x(t) = (e^{-Ah}-I)e^{-A(t-t_0)}x_0 + \int_{t_0}^{t} (e^{-Ah}-I)e^{-A(t-s)}f(s,x(s))ds + \int_{t}^{t+h} e^{-A(t+h-s)}f(s,x(s))ds.$$

Now if $0 < \delta < 1-\alpha$, then for any $z \in X$,

$$\|(e^{-Ah}-I)e^{-A(t-s)}z\|_\alpha \le C(t-s)^{-(\alpha+\delta)}h^\delta\, e^{a(t-s)}\,\|z\|\,.$$

(Th. 1.4.3) hence for $t \in [t_0^*,t_1^*] \subset (t_0,t_1)$,

$$\|x(t+h) - x(t)\|_\alpha \le \text{Constant } h^\delta.$$

It follows that $t \to f(t,x(t))$ is locally Hölder continuous on (t_0,t_1), so by Th. 3.2.2 x solves the linear equation

$$\frac{dy}{dt} + Ay = f(t,x(t)) \quad \text{on} \quad t_0 < t < t_1,\ y(t_0) = x_0$$

hence x is also a solution of (*) on (t_0,t_1).

Theorem 3.3.3. Assume A is a sectorial operator, $0 \le \alpha < 1$, and $f: U \to X$, U an open subset of $\mathbb{R} \times X^\alpha$, $f(t,x)$ is locally Hölder continuous in t, locally Lipschitzian in x; then for any $(t_0,x_0) \in U$ there exists $T = T(t_0,x_0) > 0$ such that (*) has a unique solution x on (t_0,t_0+T) with initial value $x(t_0) = x_0$.

Proof. By the lemma, it suffices to prove the corresponding result for the integral equation (**), the "variation-of-constants formula". Choose $\delta > 0$, $\tau > 0$, such that the set

$$V = \{(t,x) \;\big|\; t_0 \le t \le t_0+\tau,\ \|x-x_0\|_\alpha \le \delta\}$$

is contained in U, and

$$\|f(t,x_1) - f(t,x_2)\| \le L\,\|x_1-x_2\|_\alpha$$

for (t,x_1), $(t,x_2) \in V$. Also let $B = \max\limits_{[t_0,t_0+\tau]} \|f(t,x_0)\|$ and choose T so that $0 < T \le \tau$ and

$$\|(e^{-Ah}-I)x_0\|_\alpha \le \delta/2 \quad \text{for} \quad 0 \le h \le T,$$

$$M(B + L\delta)\int_0^T u^{-\alpha}e^{au}du \le \delta/2$$

where $\|A_1^\alpha e^{-At}\| \le Mt^{-\alpha}e^{at}$ for $t > 0$.

If S denotes the set of continuous functions $y: [t_0,t_0+T] \to X^\alpha$ such that $\|y(t)-x_0\|_\alpha \leq \delta$ on $t_0 \leq t \leq t_0+T$, provided with the usual "sup"-norm

$$\|y\|^T = \sup\{\|y(t)\|_\alpha ,\ t_0 \leq t \leq t_0+T\},$$

then S is a complete metric space.

For $y \in S$ define $G(y): [t_0,t_0+T] \to X$ by

$$G(y)(t) = e^{-A(t-t_0)}x_0 + \int_{t_0}^{t} e^{-A(t-s)}f(s,y(s))ds.$$

We show that G maps S into itself, and G is a strict contraction.

First note

$$\begin{aligned}\|G(y)(t)-x_0\|_\alpha &\leq \|(e^{-A(t-t_0)}-I)x_0\|_\alpha + \int_{t_0}^{t} \|A_1^\alpha e^{-A(t-s)}\| \,(B+L\delta)ds \\ &\leq \delta/2 + M(B+L\delta)\int_{t_0}^{t_0+T} (t-s)^{-\alpha}e^{a(t-s)}ds \leq \delta, \text{ for } t_0 \leq t \leq t_0+T.\end{aligned}$$

Also $G(y)$ is continuous from $[t_0,t_0+T]$ to X^α, as is easily proved, so G maps S into itself.

If $y,z \in S$ then for $t_0 \leq t \leq t_0+T$,

$$\begin{aligned}\|G(y)(t)-G(z)(t)\|_\alpha &\leq \int_{t_0}^{t} \|A_1^\alpha e^{-A(t-s)}\| \, \|f(s,y(s))-f(s,z(s))\| \,ds \\ &\leq ML\int_{t_0}^{t} (t-s)^{-\alpha}e^{a(t-s)}ds \cdot \|y-z\|^T\end{aligned}$$

so $\|G(y)-G(z)\|^T \leq \frac{1}{2}\|y-z\|^T$ for all $y,z \in S$.

By the contraction mapping theorem, G has a unique fixed point x in S, which is a continuous solution of the integral equation (**) and has $f(t,x(t))$ bounded as $t \to t_0+$. By Lemma 3.3.2, this is the unique solution of (*) on (t_0,t_0+T) with initial value $x(t_0) = x_0$.

Remark. Following F. Browder's terminology, a continuous solution of the integral equation (**) is a *mild solution* of the differential equation (*). One may study mild solutions when $t \to f(t,x)$ is not continuous. This generalization will not be needed here, although the smoothness results below involve mild solutions.

Theorem 3.3.4. Assume A, f are as in Th. 3.3.3 above, and also assume that for every closed bounded set $B \subset U$, the image $f(B)$ is bounded in X. If x is a solution of (*) on (t_0,t_1) and t_1 is maximal, so there is no solution of (*) on (t_0,t_2) if $t_2 > t_1$, then either $t_1 = +\infty$ or else there exists a sequence $t_n \to t_1-$ as $n \to +\infty$ such that $(t_n,x(t_n)) \to \partial U$. (If U is unbounded, the point at infinity is included in ∂U.)

Proof. Suppose $t_1 < +\infty$ but $(t,x(t))$ doesn't enter a neighborhood N of ∂U for $t_2 \le t < t_1$; we may take N of the form $U \setminus B$ where B is a closed bounded subset of U, and $(t,x(t)) \in B$ for $t_2 \le t < t_1$. We prove there exists $x_1 \in B$ such that $x(t) \to x_1$ in X^α as $t \to t_1-$, which implies the solution may be extended beyond time t_1 (with $x(t_1) = x_1$), by Thm. 3.3.3, contradicting maximality of t_1.

Now let $C = \sup\{\|f(t,x)\|, (t,x) \in B\}$; we show first that $\|x(t)\|_\beta$ remains bounded as $t \to t_1-$, for any $\beta < 1$.

Observe that if $\alpha \le \beta < 1$, $t_2 \le t < t_1$,

$$\|x(t)\|_\beta \le \|A_1^{\beta-\alpha} e^{-A(t-t_0)}\| \, \|x(t_0)\|_\alpha + \int_{t_0}^{t} \|A_1^{\beta} e^{-A(t-s)}\| \, \|f(s,x(s))\| \, ds$$
$$\le \text{Const.}\{(t-t_0)^{-(\beta-\alpha)}\|x(t_0)\|_\alpha + \int_{t_0}^{t} (t-s)^{-\beta} ds\}$$

which is bounded as $t \to t_1-$.

Now suppose $t_2 \le \tau < t < t_1$, so

$$x(t)-x(\tau) = \{e^{-A(t-\tau)}-I\}x(\tau) + \int_\tau^t e^{-A(t-s)} f(s,x(s))ds$$

$$\|x(t)-x(\tau)\|_\alpha \le C_1(t-\tau)^{\beta-\alpha}\|x(\tau)\|_\beta + C_2 \int_\tau^t (t-s)^{-\alpha} ds$$
$$\le C_3(t-\tau)^{\beta-\alpha}, \quad (\alpha < \beta < 1).$$

Thus $\lim_{t \to t_1} x(t)$ exists in X^α, and the proof is complete.

Corollary 3.3.5. Suppose A is sectorial, $U = (\tau,\infty) \times X^\alpha$, f is locally Hölder continuous in t, locally Lipschitz in x for $(t,x) \in U$, and also

$$\|f(t,x)\| \le K(t)(1 + \|x\|_\alpha)$$

for all $(t,x) \in U$, where $K(\cdot)$ is continuous on (τ,∞). If $t_0 > \tau$, $x_0 \in X^\alpha$, the unique solution of (*) through (t_0,x_0) exists for all $t \ge t_0$.

Proof. Theorem 3.3.4 applies, and the Corollary can fail only if there exist $t_n \to t_1 < \infty$ such that $\|x(t_n)\|_\alpha \to +\infty$. However

$$\|x(t)\|_\alpha \le \|e^{-A(t-t_0)}x_0\|_\alpha + \int_{t_0}^{t} \|A_1^\alpha e^{-A(t-s)}\| \cdot K(s)(1+\|x(s)\|_\alpha)$$

which implies $\|x(t)\|_\alpha$ remains bounded as $t \to t_1$, by Gronwall's inequality.

Theorem 3.3.6. Assume A, f as in Th. 3.3.3 above, and also assume A has compact resolvent and f maps all sets $\mathbb{R}^+ \times B \subset U \subset \mathbb{R} \times X^\alpha$ with B closed and bounded, into bounded sets in X. If $x(t;t_0,x_0)$ is a solution of (*) on (t_0,∞) with $\|x(t;t_0,x_0)\|_\alpha$ bounded as $t \to +\infty$, then $\{x(t;t_0,x_0)\}_{t>t_0}$ is in a compact set in X^α.

Proof. If $\alpha < \beta < 1$, then $X^\beta \subset X^\alpha$ has compact inclusion (Th. 1.4.8) and it suffices to show $\|x(t;t_0,x_0)\|_\beta$ is bounded for $t \ge t_0+1$. But we may suppose without loss of generality $\text{Re}\,\sigma(A) > \delta > 0$, and $\|f(t,x(t;t_0,x_0)\| \le C$ for all $t \ge t_0$, hence

$$\|x(t;t_0,x_0)\|_\beta \le M(t-t_0)^{-(\beta-\alpha)}e^{-\delta(t-t_0)}\|x_0\|_\alpha + MC\int_{t_0}^{t}(t-s)^{-\beta}e^{-\delta(t-s)}ds,$$

which is bounded for $t \ge t_0+1$.

Remark. The above argument demonstrates a degree of "smoothing", even without assuming compactness of the resolvent: if the solution is bounded in X^α, then it is bounded in X^β with $\alpha < \beta < 1$.

Example.

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2} + f(t,x,u(x,t)) \qquad (0 < x < \pi,\ t > 0)$$
$$u(0,t) = 0,\ u(\pi,t) = 0$$

where $f: \mathbb{R}^+ \times [0,\pi] \times \mathbb{R} \to \mathbb{R}$ is measurable in x, locally Hölder continuous in t and locally Lipschitz continuous in u, uniformly in x, with $|f(t,x,u)| \le h(x)\, g(t,|u|)$, $h \in L^2(0,\pi)$, g continuous and increasing in its second argument.

Take $X = L^2(0,\pi)$, $A = -d^2/dx^2$ with domain $H^2(0,\pi) \cap H_0^1(0,\pi)$, and $D(A^{\frac{1}{2}}) = X^{\frac{1}{2}} = H_0^1(0,\pi)$. We prove that $F: \mathbb{R}^+ \times H_0^1(0,\pi) \to L^2(0,\pi)$,

$$F(t,\phi)(x) = -\phi(x)\phi'(x) + f(t,x,\phi(x)), \quad 0 < x < \pi,$$

satisfies the hypotheses of Th. 3.3.3 and 3.3.4.

First observe that, for $\phi \in H_0^1(0,\pi)$, $\phi(x) = \int_0^x \phi'(\xi)d\xi$, so ϕ is absolutely continuous with $\sup_x|\phi(x)| \leq \sqrt{\pi}\,\|\phi\|_{\frac{1}{2}}$, so

$$\|F(t,\phi)\|_{L^2(0,\pi)} \leq \sqrt{\pi}\,\|\phi\|_{\frac{1}{2}}^2 + \|h\|_{L^2(0,\pi)}\, g(t,\sqrt{\pi}\|\phi\|_{\frac{1}{2}}),$$

and F maps bounded sets in $\mathbb{R}^+ \times X^{\frac{1}{2}}$ into bounded sets in X. Also, if $(t_0,\phi_0) \in \mathbb{R}^+ \times X^{\frac{1}{2}}$, ϕ_0 continuous, then there exists a neighborhood V of the compact set $\{(t_0,x,\phi_0(x)): 0 \leq x \leq \pi\}$ in $\mathbb{R}^+ \times [0,\pi] \times \mathbb{R}$ and positive constants L,θ so that for $(t_1,x,u_1) \in V$, $(t_2,x,u_2) \in V$,

$$|f(t_1,x,u_1)-f(t_2,x,u_2)| \leq L(|t_1-t_2|^\theta + |u_1-u_2|).$$

Hence there is a neighborhood U of (t_0,ϕ_0) in $\mathbb{R}^+ \times X^{\frac{1}{2}}$ so that

$$(t,\phi) \in U \text{ implies } (t,x,\phi(x)) \in V \text{ for a.e. } 0 \leq x \leq \pi;$$

and if $(t_1,\phi_1) \in U$, $(t_2,\phi_2) \in U$ we have

$$\|f(t_1,\cdot,\phi_1(\cdot))-f(t_2,\cdot,\phi_2(\cdot))\|_{L^2(0,\pi)} \leq \sqrt{\pi}\, L(|t_1-t_2|^\theta + \sqrt{\pi}\|\phi_1-\phi_2\|_{\frac{1}{2}}).$$

Also, for any ϕ_1,ϕ_2 in $X^{\frac{1}{2}}$,

$$\begin{aligned}\|\phi_1\phi_1'-\phi_2\phi_2'\|_{L^2(0,\pi)} &\leq \|\phi_1(\phi_1'-\phi_2')\|_{L^2} + \|(\phi_1-\phi_2)\phi_2'\|_{L^2}\\ &\leq \sqrt{\pi}\|\phi_1\|_{\frac{1}{2}}\,\|\phi_1-\phi_2\|_{\frac{1}{2}} + \|\phi_2\|_{\frac{1}{2}}\cdot\sqrt{\pi}\|\phi_1-\phi_2\|\\ &\leq \sqrt{\pi}(\|\phi_1\|_{\frac{1}{2}} + \|\phi_2\|_{\frac{1}{2}})\,\|\phi_1-\phi_2\|_{\frac{1}{2}}.\end{aligned}$$

(Alternatively, $\phi \to \phi\phi'$ is a continuous polynomial from $X^{\frac{1}{2}}$ to X.) Thus all the hypotheses of Th. 3.3.3 and 3.3.4 are verified for this problem.

<u>Exercise 1</u>. Suppose A sectorial, $f: \mathbb{R}^+ \times X^\alpha \to X$ is locally Lipschitzian, and a solution $x(t)$ of $\frac{dx}{dt} + Ax = f(t,x)$ for $t > t_0 \geq 0$ has $\frac{\|f(t,x(t))\|}{1+\|x(t)\|_\alpha}$ bounded on its domain of existence; then the solution exists for all $t \geq t_0$.

Exercise 2. Prove local existence and uniqueness for $\frac{dx}{dt} + \rho(t,x)Ax = f(t,x)$, $t > t_0$

$$x(t_0) = x_0$$

where $\rho(t,x) > 0$ and $\rho: U \to \mathbb{R}$, $f: U \to X$ are locally Lipschitzian on an open set $U \subset \mathbb{R} \times X^\alpha$.

Hint: Examine the system with a different time variable θ,

$$\frac{dx}{d\theta} + Ax = \frac{1}{\rho(t,x)} f(t,x)$$

$$\frac{dt}{d\theta} = \frac{1}{\rho(t,x)} .$$

Exercise 3. Suppose A is sectorial in X, U is an open set in $\mathbb{R} \times X^\alpha \times Y$, and $f: U \to X$, $g: U \to Y$ are locally Lipschitzian. Prove local existence and uniqueness results for the coupled "parabolic" equation and "ordinary" equation:

$$\frac{dx}{dt} + Ax = f(t,x,y), \quad \frac{dy}{dt} = g(t,x,y).$$

Apply this to the Hodgekin-Huxley equation (sec. 2.3) with $X = L^2(\mathbb{R},\mathbb{R})$, $Y = L^\infty(\mathbb{R},\mathbb{R}^3)$.

Exercise 4. (An equation with delay) Suppose A is sectorial in X, U is an open set in $\mathbb{R} \times X^\alpha \times X^\alpha$ and $f: U \to X$ is locally Lipschitzian. Consider the problem:

$$\frac{dx}{dt}(t) + Ax(t) = f(t,x(t),x(t-1)), \quad t > 0$$

$$x(t) = \phi(t) \quad \text{for} \quad -1 \le t \le 0.$$

Prove local existence and uniqueness when ϕ is Hölder continuous from $[-1,0]$ to X^α.

Examine also the more general problem when f is locally Lipschitz from an open set in $\mathbb{R} \times C([-1,0],X^\alpha)$ into X. (cf. [102]).

Maximum principle arguments are used in several of the examples to establish, for example, that a solution which is nonnegative at the initial time remains nonnegative for all later times, as long as the solution exists. The following exercises isolate what seems to be the essential feature of such arguments, starting with the relatively well-known fact that the Green's function for many linear second order elliptic operators is nonnegative. Most PDE books mention the maximum principle, but Protter and Weinberger [81] is especially recommended, along with Friedman [29]. See also [87] for some nonlinear problems.

Exercise 5. Suppose Ω is a bounded domain in $\mathbb{R}^n$ with $\partial\Omega$ smooth, that $a_{ij}(x)$, $b_i(x)$, $c(x)$, $h(x)$ are continuous functions $(i,j = 1,2,\dots,n)$ with $\alpha > 0$ such that for $x \in \Omega$, $a_{ij}(x) = a_{ji}(x)$ and $\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \alpha|\xi|^2$ for all $\xi \in \mathbb{R}^n$, and $h(x) > 0$ on $\partial\Omega$. For $u \in C^2(\overline{\Omega})$ with $\frac{\partial u}{\partial \nu} + h(x)u = 0$ on $\partial\Omega$ (ν = outward normal), define

$$Au(x) = -\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u$$

for $x \in \Omega$. If $u(P) = \min_{\overline{\Omega}} u$ for some $P \in \Omega$, show $Au(P) \leq c(P)u(P)$, but if $P \in \partial\Omega$, show $u(P) \geq 0$.

If λ is so large that $\lambda + c(x) > 0$ in Ω, show that, for $u \in D(A)$ with $Au + \lambda u \geq 0$ in Ω, we have $u \geq 0$ in Ω. For such λ, suppose there is a Green's function G_λ so that $Au + \lambda u = f$ whenever

$$u(x) = (\lambda + A)^{-1}f(x) = \int_\Omega G_\lambda(x,\xi)f(\xi)d\xi$$

(for smooth f), and show $G_\lambda(x,\xi) \geq 0$ in $\Omega \times \Omega$. Conversely, nonnegativity of the Green's function implies: when $f \geq 0$, then $(\lambda + A)^{-1}f \geq 0$.

Exercise 6. Suppose X is a real Banach space with an order relation $\geq$ such that for any $x,y,z \in X$, (a) $x \geq x$; (b) $x \geq y$ and $y \geq z$ implies $x \geq z$; (c) $x \geq y$ implies $x+z \geq y+z$ and $\lambda x \geq \lambda y$ for any real $\lambda \geq 0$; (d) the set $\{x \in X \mid x \geq 0\}$ is closed.

A function $f: X \to X$ is *increasing* if $x \geq y$ implies $f(x) \geq f(y)$.

If Y is a Banach subspace of X, the order induced on Y from X also satisfies (a)-(d).

If $X = L^p(\Omega)$ for some $1 \leq p \leq \infty$, then say $x \geq y$ when $x(t) \geq y(t)$ for almost every $t \in \Omega$; show (a)-(d) hold.

Suppose A is sectorial in a space X with an order (satisfying (a)-(d) above) and suppose $(\lambda + A)^{-1}$ is increasing (we write $(\lambda + A)^{-1} \geq 0$) for every $\lambda > 0$; prove $e^{-At} \geq 0$ for every $t \geq 0$.

Hint: $e^{-At}x = \lim_{n\to\infty} (1 + \frac{t}{n} A)^{-n}_x$.

Remark. It is sometimes useful to study such operators with an affine domain - the translate of a subspace. The arguments here are only slightly changed.

Exercise 7. Suppose X is a Banach space with an order, as in ex. 6, A is sectorial with $(\lambda+A)^{-1} \geq 0$ for every $\lambda > 0$ and $f: [t_0,t_1) \times X^\alpha \to X$ is locally Lipschitzian and satisfies

$$x \in X^\alpha,\ x \geq 0,\ t_0 \leq t < t_1 \ \text{ implies } \ f(t,x) \geq 0.$$

Then prove that, if $x_0 \in X^\alpha$, $x_0 \geq 0$, the solution $x(t) = x(t;t_0,x_0)$ of $dx/dt + Ax = f(t,x)$ with $x(t_0) = x_0$ has $x(t) \geq 0$ for all t in $[t_0,t_1)$ on its interval of existence.

Hint: establish this first for small $t-t_0 > 0$, by a successive approximations construction of the solution.

Exercise 8. Suppose X is a Banach space with an order, A is sectorial with $(\lambda+A)^{-1} \geq 0$ for all $\lambda > \lambda_0$ and $f: [t_0,t_1) \times X^\alpha \to X$ is locally Lipschitzian and on bounded sets $B = \{x \in X^\alpha \mid x \geq 0,$ $\|x\|_\alpha \leq b\}$ there exists a real constant $\beta = \beta(B)$ such that $f(t,x) + \beta x \geq 0$ whenever $x \in B$ and $t_0 \leq t < t_1$. Prove the results of ex. 7 under these more general assumptions. If also $x \to f(t,x)+\beta x$ is increasing on B, then $x_0 \to x(t;t_0,x_0)$ is increasing for $x_0 \geq 0$ and $t \geq t_0$.

Exercise 9. Suppose a, b, k are positive constants and consider

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + au - bu^2, \quad 0 < x < 1, \quad t > 0$$

with $u(0,t) = 0$, $u(1,t) = 0$, and $u(x,0) = u_0(x) \geq 0$ on $0 \leq x \leq 1$, $u_0 \in H_0^1(0,1)$. Prove $u(x,t) \geq 0$ on $0 \leq x \leq 1$, $t > 0$, on its interval of existence; then prove this solution exists for all $t \geq 0$.

If $u(x,0) \leq 0$ on $0 \leq x \leq 1$, then $u(x,t) \leq 0$ for $t > 0$, $0 \leq x \leq 1$, on its interval of existence, but the solution may blow up in finite time. (Compare with the final example 3.1.)

Exercise 10. Suppose $f: \mathbb{R}^+ \times [0,1] \times \mathbb{R} \to \mathbb{R}$ is locally Lipschitz, $f(t,x,0) \geq 0$, and α, β, γ, δ, ε are constants with $\beta \geq 0$, $\gamma \geq 0$, $\varepsilon \geq 0$. If $u(x,0) \geq 0$ on $0 \leq x \leq 1$, $v(0) \geq 0$, $w(0) \geq 0$ and for $t > 0$,

$$u_t = u_{xx} + f(t,x,u), \quad (0 < x < 1, \quad t > 0)$$
$$u_x(0,t) = -v(t), \quad u_x(1,t) + u(1,t) = w(t)$$
$$v_t = \alpha v + \beta w$$
$$w_t = \gamma v + \delta w + \varepsilon \int_0^1 u(x,t)dx$$

Prove $u(x,t) \geq 0$, $v(t) \geq 0$, $w(t) \geq 0$ on the domain of existence. (See ex. 14, sec. 1.3.)

Exercise 11. Consider the equations of nuclear reactor dynamics (sec. 2.6 above) when V, D are constant positive diagonal matrices, F is constant and nonnegative, $A = A(\Psi,c,x,t)$ is a smooth function with off-diagonal terms ≤ 0, $0 \leq \beta \leq 1$, the β_i and λ_i are nonnegative constants and the χ_i are constant column vectors with nonnegative entries. Subject to the boundary condition $\Psi|_{\partial\Omega} = 0$ (Ω a bounded domain in $\mathbb{R}^3$), if $\Psi(x,t) \geq 0$, all $c_i(x,t) \geq 0$ $(x \in \Omega)$ at $t = 0$, then these remain nonnegative for all $t > 0$ on the domain of existence. Also if the off-diagonal terms in A are non-increasing functions of Ψ and c, then if $\{\Psi,c\}$, $\{\hat{\Psi},\hat{c}\}$ are two solutions with $\Psi_j(x,t) \geq \hat{\Psi}_j(x,t) \geq 0$ $(1 \leq j \leq M)$, $c_i(x,t) \geq \hat{c}_i(x,t) \geq 0$ $(1 \leq i \leq N)$ for all $x \in \Omega$ at $t = 0$, then these hold also when $t > 0$.

3.4. Continuous and differentiable dependence of solutions

Theorem 3.4.1. Suppose A is a sectorial operator. Let $\{f_n(t,x), n = 0,1,2,\dots\}$ be a sequence of functions defined on an open set $U \subset \mathbb{R} \times X^\alpha$ (some α in $[0,1)$) into X, each $f_n(t,x)$ locally Lipschitz in x, locally Hölder continuous in t, and such that

$$f_0(t,x) = \lim_{n\to\infty} f_n(t,x)$$

uniformly for (t,x) in a neighborhood of any point of U. Also assume the real numbers $\mu_n \to \mu_0 > 0$ and $\|x_n - x_0\|_\alpha \to 0$ as $n \to \infty$ with $(t_0,x_n) \in U$.

Let $\phi_n(t)$ be the maximally defined solution of

$$\frac{d\phi_n}{dt} + \mu_n A\phi_n = f_n(t,\phi_n), \quad t > t_0$$
$$\phi_n(t_0) = x_n,$$

which exists on (t_0, t_0+T_n). Then $T_0 \geq \limsup_{n\to\infty} T_n$ and $\|\phi_n(t) - \phi_0(t)\|_\alpha \to 0$ uniformly on compact subintervals of $[t_0, t_0+T_0)$.

Proof. Without loss of generality, suppose $t_0 = 0$, $x_0 = 0$, $\mu_0 = 1$, $\frac{1}{2} < \mu_n < 2$, $f_0(t,0) = 0$ and $\phi_0(t) = 0$ on $[0,T_0)$. For otherwise consider the functions $\psi_n(t) = \phi_n(t) - \phi_0(\mu_n t)$ satisfying

$$\frac{d\psi_n}{dt} + \mu_n A\psi_n = f_n(t,\psi_n(t) + \phi_0(\mu_n t)) - \mu_n f_0(\mu_n t, \phi_0(\mu_n(t))).$$

Choose any t_1 in $(0,T_0)$; then $[0,t_1] \times \{0\}$ is a compact set in U, hence there exist $\delta > 0$ and $L > 0$ such that

$$\|f_0(t,x_1) - f_0(t,x_2)\| \leq L\|x_1 - x_2\|_\alpha \quad \text{for } \|x_1\|_\alpha, \|x_2\|_\alpha \leq \delta$$
$$\text{and } 0 \leq t \leq t_1,$$

and $\|f_n(t,x) - f_0(t,x)\| \to 0$ uniformly for $0 \leq t \leq t_1$, $\|x\|_\alpha \leq \delta$.

For sufficiently large n, the ϕ_n will be defined on $[0,t_1]$. Indeed, as long as $\|\phi_n(s)\|_\alpha \leq \delta$ on $0 \leq s \leq t$, we have

$$\|\phi_n(t)\|_\alpha \leq Me^{\mu_n at}\|x_n\|_\alpha + M\mu_n^{-\alpha}\int_0^t (t-s)^{-\alpha} e^{\mu_n a(t-s)} \|f_n(s,\phi_n(s)) - f_0(s,\phi_n(s))\|\, ds$$
$$+ LM\mu_n^{-\alpha}\int_0^t (t-s)^{-\alpha} e^{\mu_n a(t-s)} \|\phi_n(s)\|_\alpha\, ds$$

so $\|\phi_n(t)\|_\alpha \leq C(\|x_n\|_\alpha + \Delta_n)e^{Bt}$, $\Delta_n = \sup\{\|f_n(s,x) - f_0(s,x)\| : 0 \leq s \leq t_1, \|x\|_\alpha \leq \delta\}$, where B, C are independent of n.

If n is sufficiently large that

$$Ce^{Bt_1}(\|x_n\|_\alpha + \Delta_n) < \delta,$$

it follows that ϕ_n exists on $[0,t_1]$, and $\|\phi_n(t)\|_\alpha \to 0$ uniformly in $0 \leq t \leq t_1$ as $n \to \infty$.

Lemma 3.4.2. Suppose A is sectorial in X, $0 < T < \infty$ and $0 \leq \beta \leq \alpha < 1+\beta$. Then the maps

$$(\mu,\xi) \to \{e^{-\mu At}\xi,\ 0 \leq t \leq T\}: \mathbb{R}^+ \times X^\alpha \to C([0,T], X^\alpha),$$

and

$$(\mu,g) \to \{\int_0^t e^{-\mu A(t-s)} g(s)ds,\ 0 \le t \le T\}: \mathbb{R}^+ \times C([0,T],X^\beta) \to C([0,T],X^\alpha)$$

are both analytic.

Proof. For fixed $t > 0$, $(\mu,\xi) \to e^{-\mu At}\xi$ is analytic with Taylor series $e^{-(\mu+\delta)At}\xi = \sum_{n=0}^{\infty} \frac{1}{n!} (-At\delta)^n e^{-\mu At}\xi$, convergent for small $|\delta|$. In fact, if $\|e^{-At}\| \le M$, $\|Ae^{-At}\| \le Mt^{-1}$ on $0 < t \le T$, note $t \to (At)^n e^{-\mu At}\xi$ is continuous from X^α to X^α, $0 \le t \le T$ (see ex. 8, sec. 1.4) and $\frac{1}{n!} \|(-At\delta)^n e^{-\mu At}\xi\|_\alpha \le \frac{n^n}{n!} (\frac{M|\delta|}{\mu})^n \|\xi\|_\alpha$, so the series converges uniformly in $0 \le t \le T$ when $|\delta| < \mu/Me$, proving convergence in $C([0,T],X^\alpha)$. The other case is proved in the same way.

Lemma 3.4.3. Suppose X,Y are Banach spaces, U is open in X, and J is a compact interval in $\mathbb{R}$. If $F: J \times U \to Y$ is continuous, the composition map $x \to F(\cdot,x(\cdot)): C(J,U) \to C(J,Y)$ is continuous. If $(t,x) \to (\frac{\partial}{\partial x})^k F(t,x)$ is continuous on $J \times U$ for $k = 0,1,\ldots,r$, the composition map is C^r. If this holds for all r and the Taylor series for $x \to F(t,x)$ converges uniformly in $t \in J$ near each $x \in U$, the composition map is analytic.

Proof. If x_n, $x \in C(J,U)$ and $x_n(t) \to x(t)$ uniformly in $t \in J$ as $n \to \infty$ but $\|F(\cdot,x_n(\cdot)) - F(\cdot,x(\cdot))\|_{C(J,Y)} \ge \varepsilon > 0$, there exist $t_n \in J$ with $\|F(t_n,x_n(t_n)) - F(t_n,x(t_n))\| \ge \varepsilon/2$ for large n. There is a subsequence $t_n' \to t^* \in J$, and we contradict continuity of F at $(t^*,x(t^*))$. For each $1 \le k \le r$, $\partial^k F/\partial x^k$ satisfies the hypotheses of the case $r = 0$, and is uniformly continuous on $\{(t,x(t)), t \in J\}$ if $x \in C(J,U)$. By the converse Taylor theorem (see sec. 1.2.5) it follows that the composition map is C^r. For the analytic case, direct estimate of the Taylor remainder proves the result.

Theorem 3.4.4. Suppose A is a sectorial operator in a Banach space X, $0 \le \alpha < 1$, U is open in $\mathbb{R} \times X^\alpha$, Λ is open in a Banach space L, and $f: U \times \Lambda \to X$ has f, $D_x f$, $D_\lambda f$ continuous on $U \times \Lambda$, while $t \to f(t,x,\lambda)$ is locally Hölder continuous.

For $\mu > 0$, $\lambda \in \Lambda$, $(\tau,\xi) \in U$, let $x(t) = x(t;\tau,\xi,\lambda,\mu)$ be the maximally defined solution of

$$\frac{dx}{dt} + \mu Ax = f(t,x,\lambda), \quad t > \tau$$
$$x(\tau) = \xi.$$

Then $(\xi,\lambda,\mu) \to x(t;\ \tau,\xi,\lambda,\mu)$ is continuously differentiable from $X^\alpha \times \Lambda \times \mathbb{R}^+$ into X^α, on the domain of existence of the solution. These derivatives $u(t) = D_\xi x(t)$, $v(t) = D_\lambda x(t)$, $w(t) = D_\mu x(t)$ are mild solutions of

$$du/dt + \mu Au = D_x f(t,x(t),\lambda)u, \quad u(\tau) = I;$$
$$dv/dt + \mu Av = D_x f(t,x(t),\lambda)v + D_\lambda f(t,x(t),\lambda), \quad v(\tau) = 0;$$
$$dw/dt + \mu Aw = D_x f(t,x(t),\lambda)w - Ax(t), \quad w(\tau) = 0.$$

(They are ordinary solutions if also $(t,x) \to D_x f$, $D_\lambda f$ are locally Hölder continuous on U.) For the w-equation, note $\|Ax(t)\| = O(t-\tau)^{\alpha-1})$ as $t \to \tau+$.

Proof. It is sufficient to consider a short time interval; say $0 \le t \le T$, $\tau = 0$, without loss. The solution $x(t;\ \tau,\xi,\lambda,\mu)$ is obtained as the fixed point of a map G on the space of continuous functions from $[0,T]$ into X^α,

$$G(x;\ \xi,\lambda,\mu)(t) = e^{-\mu At}\xi + \int_0^t e^{-\mu A(t-s)} f(s,x(s),\lambda)ds, \quad 0 \le t \le T.$$

For (ξ,λ,μ) in a small neighborhood of (ξ_0,λ_0,μ_0), $(0,\xi_0,\lambda_0,\mu_0) \in U \times \Lambda \times \mathbb{R}^+$, G is a uniform contraction of a ball $B \subset C([0,T],X^\alpha)$. Now $(x,\lambda) \to f(\cdot,x(\cdot),\lambda) \in C([0,T],X)$ is continuously differentiable on $B \times \Lambda$, and G is the composition of this map with an analytic map (see lemma), so G is C^1 and thus the fixed point is also C^1 as claimed.

By the same argument, we prove the C^r case.

Corollary 3.4.5. In addition to the hypotheses of Theorem 3.4.4, assume $(x,\lambda) \to f(t,x,\lambda)$ is C^r $(1 \le r \le \infty)$ with derivatives continuous on $U \times \Lambda$, or analytic uniformly in t for (t,x,λ) in a neighborhood of each point in $U \times \Lambda$. Then the map

$$(\xi,\lambda,\mu) \to x(t;\ \tau,\xi,\lambda,\mu)$$

is C^r or analytic, on the interval of existence.

Corollary 3.4.6. In addition to the hypotheses of Theorem 3.4.4, assume $(t,x,\lambda) \to f(t,x,\lambda): U \times \Lambda \to X$ is C^r or analytic. Then

$$(t,\tau,\xi,\lambda,\mu) \to x(t;\ \tau,\xi,\lambda,\mu)$$

is C^r or analytic respectively for $t > \tau$ on the domain of existence. (Smoothness in time may break down as $t \to \tau+$.)

Proof. For any $m > 0$, define $y(s) = y(s;\ 0,\xi,\hat{\lambda},\mu/m)$, $\hat{\lambda} = (\lambda,m,\tau)$, as the solution of

$$\frac{dy}{ds} + (\mu/m)Ay = g(s,y,\hat{\lambda}) \equiv \frac{1}{m} f(\tau + \frac{s}{m},y,\lambda)$$

$$y(0) = \xi.$$

By uniqueness, on the domain of existence

$$x(t;\ \tau,\xi,\lambda,\mu) = y(m(t-\tau);\ 0,\xi,\hat{\lambda},\ \mu/m)$$

for every $m > 0$. In particular, if $t > \tau$ and $m = 1/(t-\tau)$

$$x(t;\ \tau,\xi,\lambda,\mu) = y(1;\ 0,\xi,(\lambda,(t-\tau)^{-1},\tau),\mu(t-\tau)).$$

But for each $s > 0$,

$$(\xi,(\lambda,m,\tau),\mu/m) \to y(s;\ 0,\xi,(\lambda,m,\tau),\mu/m)$$

is C^r or analytic, by Corollary 3.4.5, and the result is proved.

Now we study another notion of continuous dependence, which leads to the "method of averaging". (For ordinary differential equations, this approach originated with Gikhman.) Another approach for linear equations is in sec. 7.5.

Lemma 3.4.7. Suppose A is sectorial in X, $f: [0,T) \times U \times \Lambda \to X$, U is open in X^α, $\alpha < 1$, and Λ is a metric space, and assume

(i) $\|f(s,x,\lambda)\| \le N$ for $0 \le s < T$, $x \in U$, $\lambda \in \Lambda$;

(ii) $\|f(s,x,\lambda)-f(s,y,\lambda)\| \le L\,\|x-y\|_\alpha$ for $0 \le s < T$, $x,y \in U$, $\lambda \in \Lambda$;

(iii) $\lambda \to \int_0^t f(s,x,\lambda)ds$ is continuous for each $x \in U$, uniformly in compact sets of $0 \le t < T$. Here we understand also that the integral exists $(\int_0^t \|f(s,x,\lambda)\|\,ds < \infty)$ when $0 \le t < T$.

Then for any continuous $x: [0,T) \to U$,

$$\lambda \to \int_0^t e^{-A(t-s)} f(s,x(s),\lambda)ds \in X^\alpha$$

is continuous, uniformly in $0 \le t \le t_1$ for any $t_1 < T$.

Proof. First we examine the case $x(s) = x$, a constant. Let $\rho > 0$ and

$$J_\rho(\lambda,t) = \int_0^{t-\rho} e^{-A(t-s)} f(s,x,\lambda)ds, \quad \rho \le t < T,$$

$$J_\rho(\lambda,t) = 0 \quad \text{if} \quad 0 \le t \le \rho.$$

Then for $\rho \le t < T$,

$$J_\rho(\lambda,t) = e^{-At} \int_0^t f(s,x,\lambda)ds - e^{-A\rho} \int_{t-\rho}^t f(s,x,\lambda)ds$$
$$+ \int_0^{t-\rho} Ae^{-A(t-s)} \int_s^t f(\sigma,x,\lambda)d\sigma\, ds,$$

and $\lambda \to J_\rho(\lambda,t) \in X^\alpha$ is continuous by Lebesgue's dominated convergence theorem and the fact

$$\left\|Ae^{-A(t-s)} \int_s^t f(\sigma,x,\lambda)d\sigma\right\|_\alpha \le NM(t-s)^{-\alpha}e^{a(t-s)}$$

for $0 < s < t < T$. Further, $J_\rho(\lambda,t)$ converges uniformly in λ,t as $\rho \to 0$; for

$$\left\|J_\rho(\lambda,t) - \int_0^t e^{-A(t-s)} f(s,x,\lambda)ds\right\|_\alpha \le \left\|e^{-A\rho} \int_{t-\rho}^t f(s,x,\lambda)ds\right\|_\alpha$$
$$+ \int_{t-\rho}^t \left\|Ae^{-A(t-s)} \int_s^t f(\sigma,x,\lambda)d\sigma\right\|_\alpha ds$$
$$\le (MN\,\rho^{1-\alpha} + MN\,\rho^{1-\alpha}/(1-\alpha))e^{a\rho} \to 0 \text{ as } \rho \to 0+.$$

Thus $J_0(\lambda,t) = \int_0^t e^{-A(t-s)} f(s,x,\lambda)$ is continuous in λ, for each $x \in U$, hence also

$$\lambda \to \int_0^t e^{-A(t-s)} f(s,\hat{x}(s),\lambda)ds \quad \text{is continuous}$$

when $\hat{x}(\cdot)$ is any step function on $[0,T) \to U$. Finally, for any continuous $x: [0,T) \to U$ there exist step functions $x_n: [0,T) \to U$ so that $\|x(s)-x_n(s)\|_\alpha \to 0$ uniformly on $0 \le s \le t_1$ (any $t_1 < T$),

$$\left\| \int_0^t e^{-A(t-s)} [f(s,x(s),\lambda) - f(s,x_n(s),\lambda)] ds \right\|_\alpha$$
$$\leq \int_0^t LM(t-s)^{-\alpha} e^{a(t-s)} \|x(s) - x_n(s)\|_\alpha ds \to 0$$

uniformly in $\lambda \in \Lambda$ and $0 \leq t \leq t_1$, and the lemma is proved.

Theorem 3.4.8. Assume A, f satisfy the hypotheses of lemma 3.4.7. Suppose $x(\cdot,\lambda)$ is a solution of $dx/dt + Ax = f(t,x,\lambda)$, $t > 0$, and suppose $x(t,\lambda_0) \in U$ exists on $0 \leq t \leq T$ and $\|x(0,\lambda) - x(0,\lambda_0)\|_\alpha \to 0$ as $\lambda \to \lambda_0$. Then $\|x(t,\lambda) - x(t,\lambda_0)\|_\alpha \to 0$ as $\lambda \to \lambda_0$, uniformly in $0 \leq t \leq T$.

Proof. Let $y(t) = x(t,\lambda) - x(t,\lambda_0)$; then

$$y(t) = e^{-At} y(0) + \int_0^t e^{-A(t-s)} \{f_0(s,\lambda) - f_0(s,\lambda_0)\}$$
$$+ \int_0^t e^{-A(t-s)} \{f(s,x(s,\lambda_0)+y(s),\lambda) - f(s,x(s,\lambda_0),\lambda)\}$$

where we set $f_0(s,\lambda) = f(s,x(s,\lambda_0),\lambda)$. Since $s \to x(s,\lambda_0)$ is continuous on $0 \leq s \leq T$, by the lemma we have $\|y(t)\|_\alpha \leq \delta(\lambda) + LM \int_0^t (t-s)^{-\alpha} \|y(s)\|_\alpha ds$, as long as $x(s,\lambda_0) + y(s) = x(s,\lambda) \in U$ on $0 \leq s \leq t$ and $0 \leq t \leq T$, where $\delta(\lambda) \to 0$ as $\lambda \to \lambda_0$. But it then follows, for λ sufficiently close to λ_0, that on $0 \leq s \leq T$,

$$\|y(s)\|_\alpha < \text{dist}\{x(\cdot,\lambda_0), \partial U\}$$

and so the solution $x(\cdot,\lambda)$ exists and $\sup_{0 \leq t \leq T} \|x(t,\lambda) - x(t,\lambda_0)\|_\alpha = O(\delta(\lambda)) \to 0$ as $\lambda \to \lambda_0$.

We now state the result in a form closer to the method of averaging: we compare solutions of $dx/dt + Ax = g(t/\varepsilon, x)$, $0 < \varepsilon < \varepsilon_0$, with solutions of the averaged equation

$$dy/dt + Ay = g_0(y)$$
$$g_0(y) = \lim_{T \to +\infty} \frac{1}{T} \int_0^T g(s,y) ds.$$

Theorem 3.4.9. Assume A is sectorial in X, $g: \mathbb{R}^+ \times U \to X$ is continuous with U an open set in X^α, and

(i) $t \to g(t,x)$ is locally Hölder continuous, $\|g(t,x)\| \leq N$ and $\|g(t,x) - g(t,y)\| \leq L\|x-y\|_\alpha$ for $t \geq 0$ and $x,y \in U$;

(ii) $\lim_{T\to\infty} \|\frac{1}{T}\int_0^T g(s,x)ds - g_0(x)\| = 0$ for each $x \in U$.

Suppose there exists a solution $\xi(t)$ of

$$d\xi/dt + A\xi = g_0(\xi), \quad t > 0,$$

with $\xi(t) \in U$ on $0 \leq t \leq T < \infty$. Then given $\eta > 0$, there exists $\delta > 0$ and $\varepsilon_1 > 0$ so that if

$$\|x(0) - \xi(0)\|_\alpha \leq \delta \quad \text{and} \quad 0 < \varepsilon \leq \varepsilon_1,$$

and $dx/dt + Ax = g(t/\varepsilon, x)$, $t > 0$ then $\|x(t)-\xi(t)\|_\alpha \leq \eta$ on $0\leq t\leq T$.

Proof. We apply Th. 3.4.8 with $\lambda = \varepsilon$,

$$f(t,x,\lambda) = g(t/\lambda,x) \quad \text{for} \quad 0 < \lambda \leq \varepsilon_0$$

$$f(t,x,0) = g_0(x)$$

for $0 \leq t \leq T$ and $x \in U$.

If $t > 0$, $\lambda > 0$, then for any $x \in U$

$$\int_0^t f(s,x,\lambda)ds = \int_0^t g(s/\lambda,x)ds = \lambda \int_0^{t/\lambda} g(\sigma,x)d\sigma \to tg_0(x) \quad \text{as} \quad \lambda \to 0+.$$

Exercise 1. Suppose A_λ is sectorial in X for each λ in an open set $\Lambda \subset \mathbb{R}^m$, $\lambda \to (\zeta_0 - A_\lambda)^{-1} \in \mathscr{L}(X)$ is analytic for some fixed ζ_0 and either $D(A_\lambda)$ is independent of λ or $\|(\zeta-A_\lambda)^{-1}\| \leq C/|\zeta-a|$ for all ζ on $\arg(\zeta-a) = \pm\theta$, $0 < \theta < \pi/2$, and λ in a complex neighborhood of Λ. Prove $(\lambda,t) \to e^{-A_\lambda t}: \Lambda \times \mathbb{R}^+ \to \mathscr{L}(X)$ is analytic and $(\lambda,x) \to \{e^{-A\lambda t}x, 0 \leq t \leq 1\}: \Lambda \times X \to C([0,1],X)$ is analytic. (Compare Kato [56, p. 498].)

Exercise 2. Suppose A_λ as above has $X^\alpha = D(A_\lambda^\alpha)$ independent of λ for some $\alpha < 1$ and f is a C^k map $(k \geq 1)$ from an open set of $\mathbb{R} \times X^\alpha \times \Lambda$ into X. Prove the solution $x(t) = x(t;\tau,\xi,\lambda)$ of

$$\frac{dx}{dt} + A_\lambda x = f(t,x,\lambda), \quad t > \tau, \quad x(\tau) = \xi \in X^\alpha$$

is a C^k function of all its arguments for $t > \tau$ on its domain of existence. An example of this type is in ex. 11, section 6.4, with $\alpha = 1/2$.

3.5 Smoothing action of the differential equation

In Sec. 3.3, we showed that a certain degree of "smoothing" occurs: with initial value in $X^\alpha = D(A_1^\alpha)$, $0 < \alpha < 1$, the solution is actually in $D(A)$ at any later time. We will need some more precise expressions of this smoothing action.

Lemma 3.5.1. Suppose A is sectorial, $g: (0,T) \to X$ has $\|g(t)-g(s)\| \leq K(s)(t-s)^\gamma$ for $0 < s < t < T < \infty$, where $K(\cdot)$ is continuous on $(0,T)$ and $\int_0^T K(s)ds < \infty$. Then $G(t) \equiv \int_0^t e^{-A(t-s)}g(s)ds$, $0 < t < T$, is continuously differentiable on the open interval $(0,T)$ into X^β, provided $0 \leq \beta < \gamma$, and

$$\|dG(t)/dt\|_\beta \leq Mt^{-\beta}\|g(t)\| + M\int_0^t (t-s)^{\gamma-\beta-1}K(s)ds$$

for $0 < t < T$, where M is a constant independent of γ, β and $g(\cdot)$. Further $t \to \frac{d}{dt}G(t)$ is locally Hölder continuous from $(0,T)$ into X^β, if $\int_0^h K(s)ds = O(h^\delta)$ as $h \to 0+$, for some $\delta > 0$.

Proof. From the proof of Lemma 3.2.1,

$$dG/dt = e^{-At}g(t) + H(t) = -AG(t) + g(t),$$

$$H(t) = \int_0^t Ae^{-A(t-s)}(g(t)-g(s))ds,$$

and the required estimates follow. Note

$$H(t+h)-H(t) = \int_0^h Ae^{-A(t+h-s)}(g(t+h)-g(s))$$
$$+ \int_0^t Ae^{-A(t-s)}[g(t+h)-g(s+h)-g(t)+g(s)]ds.$$

Remark. $t \to AG(t) \in X$ is also locally Holder continuous, but will not have values in X^β unless $g(t) \in X^\beta$.

Theorem 3.5.2. Assume A is sectorial, $f\colon U \to X$ is locally Lipschitzian on an open set $U \subset \mathbb{R} \times X^\alpha$, for some $0 \le \alpha < 1$. Suppose $x(\cdot)$ is a solution on $(t_0,t_1]$ of

$$\frac{dx}{dt} + Ax = f(t,x), \quad x(t_0) = x_0$$

and $(t_0,x_0) \in U$.

Then if $\gamma < 1$, $t \to \frac{dx}{dt}(t) \in X^\gamma$ is locally Hölder continuous for $t_0 < t \le t_1$, with

$$\|dx/dt\|_\gamma \le C(t-t_0)^{\alpha-\gamma-1}$$

for some constant C.

Remark. This result (together with the embedding theorems) is used to prove the solutions of the abstract equation yield classical solutions of the original PDE: see the examples in sec. 3.6-3.8.

Proof. Let $\alpha < \beta < 1$, $t_0 < \tau < t_1$; then $\|x(\tau)\|_\beta \le C_1(\tau-t_0)^{\alpha-\beta}$ and $g(t) \equiv f(t,x(t))$ satisfies $\|g(t)-g(s)\| \le L(|t-s| + \|x(t)-x(s)\|_\alpha)$ for $t_0 \le s \le t \le t_1$ for some constants L, C_1.

Also if $\tau < t < t+h \le t_1$,

$$x(t+h) - x(t) = (e^{-Ah}-1)e^{-A(t-\tau)}x(\tau) + \int_\tau^t e^{-A(t-s)}\{g(s+h)-g(s)\}ds + \int_\tau^{\tau+h} e^{-A(t+h-s)}g(s)ds$$

then

$$\|g(t+h) - g(t)\| \le Lh + L\|x(t+h) - x(t)\|_\alpha$$

$$\le Lh + C_2h((t-\tau)^{-1+\beta-\alpha}\|x(\tau)\|_\beta + (t-\tau)^{-\alpha})$$

$$+ \int_\tau^t M(t-s)^{-\alpha}\|g(s+h) - g(s)\|\,ds.$$

Thus $\|g(t+h)-g(t)\| \le C_3h((t-\tau)^{-1+\beta-\alpha}\|x(\tau)\|_\beta + (t-\tau)^{-\alpha})$ and, by lemma 3.5.1, $t \to dx/dt \in X^\gamma$ $(\gamma < 1)$ is Hölder continuous on $(\tau,t_1]$, with

$$\left\|\frac{dx}{dt}\right\|_\gamma \le C_4[(t-\tau)^{\beta-\gamma-1}\|x(\tau)\|_\beta + (t-\tau)^{-\gamma}]$$

$$\le C_5[(t-\tau)^{\beta-\gamma-1}(\tau-t_0)^{\alpha-\beta} + (t-\tau)^{-\gamma}].$$

Take $t-\tau = \tau-t_0 = \frac{1}{2}(t-t_0)$ so

$$\|dx/dt\|_\gamma \le M(t-t_0)^{\alpha-\gamma-1}.$$

Theorem 3.5.3. Suppose A is sectorial, $f\colon U \to X$ is locally Lipschitzian on an open set $U \subset \mathbb{R} \times X^\alpha$ for some $\alpha < 1$, and suppose $\bar{x}(\cdot)$ is a solution on $[t_0,t_1]$ of

$$dx/dt + Ax = f(t,x)$$

with $(t,\bar{x}(t)) \in U$ and $\|d\bar{x}(t)/dt\|_\alpha$ bounded on $t_0 \le t \le t_1$.

If $\|x_0-\bar{x}(t_0)\|_\alpha$ is sufficiently small, the solution $x(\cdot)$ with $x(t_0) = x_0$ exists on $t_0 \le t \le t_1$ and for any $0 \le \beta < \gamma < 1$, $t \to dx(t)/dt$ is locally Hölder continuous into X^β on $(t_0,t_1]$ and

$$\left\|\frac{dx}{dt}(t) - \frac{d\bar{x}}{dt}(t)\right\|_\beta \le K(t-t_0)^{\alpha-\beta-1}\|x_0-\bar{x}(t_0)\|_\alpha^{1-\gamma},$$

for some constant K independent of t, x_0.

Proof. The curve $\{(t,\bar{x}(t)),\ t_0 \le t \le t_1\}$ is a compact subset of U, so there is a neighborhood of V of this curve with $V \subset U$ and $\|f(t,x)-f(s,y)\| \le L(|t-s| + \|x-y\|_\alpha)$ for $(t,x) \in V$, $(s,y) \in V$.

Define $g(t,z) = f(t,\bar{x}(t)+z) - f(t,\bar{x}(t))$, $t_0 \le t \le t_1$, $\|z\|_\alpha \le \delta$, for some $\delta > 0$ so small that $(t,\bar{x}(t)+z) \in V$ for $t_0 \le t \le t_1$, $\|z\|_\alpha \le \delta$. Then $g(t,0) = 0$, and

$$\|g(t,z_1) - g(t,z_2)\| \le L\,\|z_1-z_2\|_\alpha,$$

$$\|g(t,z) - g(s,z)\| \le 2L\min(\|z\|_\alpha, |t-s|(1+M))$$

$$M = \sup\{\|d\bar{x}/dt\|_\alpha,\ t_0 \le t \le t_1\}.$$

Using the fact $\min\{a,b\} \le a^\gamma b^{1-\gamma}$ whenever $a,b \ge 0$ and $0 \le \gamma \le 1$, it follows that

$$\|g(t,z_1) - g(s,z_2)\| \le L_1(\|z_1-z_2\|_\alpha + \|z_1\|_\alpha^{1-\gamma}|t-s|^\gamma).$$

If $\|z(t_0)\|_\alpha$ is sufficiently small, then the solution $z(t)$ exists on $[t_0,t_1]$ with $\|z(t)\|_\alpha \le \delta$. We estimate the Hölder continuity of $z(\cdot)$: if $t_0 < t < t+h < t_1$, then

$$z(t+h)-z(t) = (e^{-Ah}-I)e^{-A(t-t_0)}z_0 + \int_{t_0}^{t_0+h} e^{-A(t+h-s)}g(s,z(s))ds$$
$$+ \int_{t_0}^{t} e^{-A(t-s)}\{g(s+h,z(s+h))-g(s,z(s))\}ds.$$

If $\|z_0\|_\alpha$ is sufficiently small, $\|z(t)\|_\alpha \leq K_1 \|z_0\|_\alpha$ for $t_0 \leq t \leq t_1$, and so we find

$$\|z(t+h)-z(t)\|_\alpha \leq K_2\, h^\gamma (t-t_0)^{-\theta}\|z_0\|_\alpha^{1-\gamma}$$

where $\theta = \max(\alpha,\gamma)$.

Now if $g(t) \equiv g(t,z(t))$ then on (t_0,t_1)

$$\|g(t)-g(s)\| \leq K_3(s-t_0)^{-\theta}|t-s|^\gamma \|z_0\|_\alpha^{1-\gamma}$$

and the lemma applies: $z(t) = e^{-A(t-t_0)}z(t_0) + G(t)$ so

$$\|\frac{dz}{dt}(t)\|_\beta \leq \text{Const.}\ (t-t_0)^{\alpha-\beta-1}\|z_0\|_\alpha^{1-\gamma}.$$

Exercise 1. Suppose A is sectorial and $f: \mathbb{R} \times X^{s+\alpha} \to X^s$ is Lipschitzian in a neighborhood (t_0,x_0) for some $s \geq 0$ and $0 \leq \alpha < 1$. If $x_0 \in X^{s+\alpha}$, show there exists a unique solution $x(t) = x(t;t_0,x_0)$ of $dx/dt + Ax = f(t,x)$ $(t > t_0)$, $x(t_0) = x_0$, on some interval $t_0 \leq t \leq t_1$ and $t \to x(t) \in X^{s+\alpha}$ is continuous.

Exercise 2. Suppose $f: \mathbb{R} \times X^{m+\alpha} \to X^m$ is locally Lipschitzian for $m = 0,1,2,\ldots,n$. If $x_0 \in X^\alpha$, show the solution $x(t;t_0,x_0)$ through (t_0,x_0) has $x(t;t_0,x_0) \in X^{n+\alpha}$ when $t > t_0$.

Apply this result to the example 3.7 below and show, if $u(\cdot,0) \in W^{1,p}(\mathbb{R}^n)$ for some $p > n/2$, then $u(\cdot,t) \in C^k(\mathbb{R}^n)$ for any $k \geq 1$ and for every $t > 0$. What can you say about differentiability in t? Hint: $W^{k,p}(\mathbb{R}^n)$ is an algebra under pointwise multiplication if $kp > n$.

Exercise 3. Consider a bounded smooth domain $\Omega \subset \mathbb{R}^3$, and let $A = -\Delta_D$ in $L^2(\Omega)$. Show the problem

$$\partial u/\partial t = \Delta u + u^3 \text{ in } \Omega,\ u = 0 \text{ on } \partial\Omega$$
$$u(\cdot,0) = u_0$$

has a unique solution for small $t > 0$, provided $u_0 \in X^\beta$ with $\beta > 1/4$. (Note: $u_0 \in X^\beta$ does not imply that $u_0^3 \in L^2(\Omega)$.)

Exercise 4. (Based on work of N. Alikakos [107].) Suppose $\Omega \subset \mathbb{R}^n$ is such that the Nirenberg-Gagliardo inequalities (sec. 1.6) and the divergence theorem hold. Suppose $a_j(x,t)$, $b(x,t)$ are continuous real functions with

$$|a_j(x,t)| \leq A, \quad b(x,t) \leq B \quad \text{on} \quad \Omega \times \mathbb{R}^+$$

and u is a solution of

$$\partial u/\partial t = \Delta u + \sum_{j=1}^{n} a_j(x,t)\, \partial u/\partial x_j + b(x,t)u$$

on $\Omega \times \mathbb{R}^+$ with $u \frac{\partial u}{\partial N} \leq 0$ on $\partial\Omega \times \mathbb{R}^+$. Assume, for some p $(1 \leq p < \infty)$ $u|_{t=0} \in L_p(\Omega) \cap L_\infty(\Omega)$ and $\|u(\cdot,t)\|_{L_p(\Omega)}$ is uniformly bounded for all $t \geq 0$. Then prove $\|u(\cdot,t)\|_{L_q(\Omega)}$ is bounded uniformly in $t \geq 0$ and in $q = p\cdot 2^m$ $(m = 0,1,2,\ldots)$; thus, for example, an apriori $L_1(\Omega)$ bound gives an $L_\infty(\Omega)$ bound.

Hint: If $s \geq 2$

$$\frac{d}{dt}\int_\Omega |u|^s dx \leq \frac{-4(s-1)}{s}\int_\Omega |\nabla|u|^{s/2}|^2 dx + sB\int_\Omega |u|^s dx$$
$$+ 2A\left(\int_\Omega |\nabla|u|^{s/2}|^2\right)^{\frac{1}{2}}\left(\int_\Omega |u|^s dx\right)^{\frac{1}{2}}$$

and for $0 < \varepsilon \leq 1$ (Nirenberg-Gagliardo)

$$\int_\Omega |u|^s dx \leq \varepsilon\int_\Omega |\nabla|u|^{s/2}|^2 dx + K\varepsilon^{-m}\left(\int_\Omega |u|^{s/2} dx\right)^2$$

where $m > n/2$ and $K = K(\Omega,m) < \infty$. Use these to estimate $\|u(\cdot,t)\|_{L_s(\Omega)}$ in terms of $\sup_{t\geq 0}\|u(\cdot,t)\|_{L_{s/2}(\Omega)}$ and $\|u(\cdot,0)\|_{L_s(\Omega)}$.

Exercise 5. Suppose Ω is a bounded smooth domain in $\mathbb{R}^n$, $f(x,u)$ is a smooth function on $\Omega \times \mathbb{R}$ with $|f(x,u)| \leq C(1+|u|^m)$. Suppose $u(x,t)$ is a solution of $\partial u/\partial t = \Delta u + f(x,u)$, $u = 0$ on $\partial\Omega$, and this solution exists only for $0 \leq t < t_1 < \infty$. Prove $\|u(\cdot,t)\|_{L_q(\Omega)}$ is unbounded as $t \to t_1-$ if $q \geq 1$ and either (i) $q > m$ or (ii) $q > n/2\ (m-2)$. (Hint: for (ii) use the Nirenberg-Gagliardo inequality as above.)

Example 3.6. Ω is a bounded domain in $\mathbb{R}^3$ with $\partial\Omega$ smooth.

$$\frac{\partial u}{\partial t} = \Delta u + f(t,x,u,\text{grad } u) \quad \text{for} \quad t > 0,\ x \in \Omega$$

$$\frac{\partial u}{\partial n} + a(x)u = 0 \quad \text{on} \quad \partial\Omega, \quad t > 0$$

$$u(x,0) = u_0(x) \quad \text{for} \quad x \in \Omega.$$

We assume $f(t,x,u,p)$ is locally Lipschitzian in all its arguments, and for some real k in $1 \le k < 3$, and some continuous $B(t,r)$,

$$|f(t,x,u,p)| \le B(t,|u|)(1 + |p|^k)$$

$$|f(t,x,u,p)-f(t,x,u,q)| \le B(t,|u|)(1+|p|^{k-1}+|q|^{k-1})|p-q|$$

$$|f(t,x,u,p)-f(t,x,v,p)| \le B(t,|u|+|v|)(1+|p|^k)\ |u-v|.$$

Also, $a(x)$ is continuously differentiable on $\partial\Omega$ and $a(x) > 0$.

Take $X = L^2(\Omega)$, and $Au = g$ when $u \in H^1(\Omega)$ and for all $v \in H^1(\Omega)$,

$$\int_\Omega g(x)v(x)dx = \int_\Omega \nabla u(x) \cdot \nabla v(x)dx + \int_{\partial\Omega} a(x)u(x)v(x)ds.$$

Then A is self adjoint and positive definite, and $D(A)$ contains all $u \in C^2(\Omega)$ such that $\partial u/\partial n + au = 0$ on $\partial\Omega$, in which case $Au = -\Delta u$ in Ω. In fact,

$$D(A) = \{u \in W^{2,2}(\Omega) \,\Big|\, \frac{\partial u}{\partial n} + au = 0 \quad \text{on} \quad \partial\Omega\},$$

since this is a regular boundary value problem for a strongly elliptic operator ([2, 30, 71]). It follows that $X^\alpha \subset W^{1,q}(\Omega)$ if $\alpha > 1/2$ and $1/q > (5-4\alpha)/6$, and $X^\alpha \subset L^\infty(\Omega)$ if $\alpha > 3/4$, (Th. 1.6.1). Thus if $1 > \alpha > \max(3/4,\ (5k-3)/4k)$, we have

$$X^\alpha \subset W^{1,2k}(\Omega) \cap L^\infty(\Omega)$$

with continuous inclusion.

We then have, for example, when $u \in X^\alpha$, and

$$F(t,u)(x) = f(t,x,u(x),\text{grad } u(x)), \quad x \in \Omega,$$

that

$$\|F(t,u)\|_{L^2(\Omega)} \le B(t, \|u\|_{L^\infty(\Omega)})((\text{meas } \Omega)^{\frac{1}{2}} + \|u\|^k_{W^{1,2k}(\Omega)}).$$

The other estimates are similar and we see that the hypotheses of Th. 3.3.3 and 3.3.4 hold, so when $u_0 \in X^\alpha$, there is a unique solution $u(t;u_0)$ on some maximal interval $0 \le t < t_1$, and either $t_1 = +\infty$ or $\|u(t;u_0)\|_\alpha \to \infty$ as $t \to t_1-$.

Thus we have a solution of the abstract form of the original equation. But in fact, when $t > 0$, $u(t;u_0) \in D(A)$ and $t \to du/dt \in X^\alpha$ is locally Hölder continuous when $t > 0$, by Th. 3.5.3, so

$$(t,x) \to u(t,x;u_0), \frac{\partial u}{\partial t}(t,x;u_0)$$

are also continuous on $t_0 < t < t_1$ and $x \in \bar{\Omega}$. And $u \in D(A)$ implies $\nabla u \in W^{1,2}(\Omega) \subset L^6(\Omega)$, so $Au = F(t,u) - du/dt \in L^{6/k}(\Omega)$. This implies $u \in W^{2,6/k}(\Omega)$, so $\nabla u \in W^{1,6/k}(\Omega) \subset L^q(\Omega)$ if $1/q > (k-2)/6$. If $k < 2$, $\nabla u(t,\cdot)$ is Hölder continuous; if $k \ge 2$ we continue the argument. If $u(t,\cdot) \in W^{2,p_n}(\Omega)$ then (as above) $u(t,\cdot) \in W^{2,p_{n+1}}(\Omega)$ if $1/p_{n+1} > k(1/p_n - 1/3)$ and if $1 \le k < 3$ we eventually have $p_n > 3$ and $\nabla u(t,\cdot)$ Hölder continuous so $F(t,u) \in C^\delta(\Omega)$ for some $\delta > 0$ and $u(t,\cdot) \in C^{2+\delta}(\Omega)$. Thus for $t > 0$, $(t,x) \to u(t,x;u_0)$ is continuously differentiable in t, twice continuously differentiable in x, and we have a classical solution.

Exercise 1. Consider the special case

$$u_t = \Delta u + \lambda u^3 \quad \text{in } \Omega \quad (\text{constant } \lambda)$$
$$\frac{\partial u}{\partial n} + au = 0 \quad \text{on } \partial\Omega.$$

Prove that, if $\lambda \le 0$, then as long as the solution exists, $t \to \int_\Omega |u(t,x)|^6 dx$ is nonincreasing. Prove that when $\lambda \le 0$, the solution exists for all $t > 0$ with $\|u(t)\|_\alpha$ bounded $(1 > \alpha > 3/4)$, and in fact the orbit $\{u(t), t \ge 0\}$ lies in a compact set in X^α. Show the only limit point of this orbit as $t \to +\infty$ is the origin, i.e. that $\|u(t)\|_\alpha \to 0$ as $t \to \infty$. (Note convergence in X^α implies convergence in L^∞ and in $W^{1,2}$.)

Remark. The polynomial bounds on $p \to f(x,u,p)$ were imposed to allow us to work in $L_2(\Omega)$. If we work instead in $L_q(\Omega)$ with $q > 3$, no such restrictions are needed.

Example 3.7.

$$\frac{\partial u}{\partial t} = \Delta u - \lambda u^3 \quad \text{for} \quad t > 0,\ x \in \mathbb{R}^n$$

$$u(x,0) = u_0), \qquad x \in \mathbb{R}^n,$$

where λ is a fixed nonnegative constant and u_0 is a given smooth function in $L^p(\mathbb{R}^n)$ for some $2 \le p < \infty$ with $p > n/2$.

We take $X = L^p(\mathbb{R}^n)$ and A is the closure in X of the differential operator $-\Delta = -\sum_{j=1}^{n} \partial^2/\partial x_j^2$ in $C_0^\infty(\mathbb{R}^n)$. Let $A_1 = A+I$ and $X^\alpha = D(A_1^\alpha)$, $\alpha \ge 0$. Then $X^\alpha = \mathscr{L}^p_{2\alpha}(\mathbb{R}^n)$, the Bessel potential space [97], which has the following properties:

when $2\alpha = k$ is an integer ≥ 0, $X^\alpha = W^{k,p}(\mathbb{R}^n)$;

when $2p\alpha < n$, $\frac{1}{q} \ge \frac{1}{p} - \frac{2\alpha}{n}$, $\infty > q \ge p$, $X^\alpha \subset L^q(\mathbb{R}^n)$;

when $2p\alpha > n$, $0 \le \nu < 2\alpha - n/p$, $X^\alpha \subset C^\nu(\mathbb{R}^n)$;

when $0 \le k \le 2\alpha$ and $\frac{1}{q} \ge \frac{1}{p} - \frac{2\alpha - k}{n}$, $X^\alpha \subset W^{k,q}(\mathbb{R}^n)$.

Now if $1 > \alpha > n/2p$, we have

$$\begin{aligned}\|u^3 - v^3\|_{L^p(\mathbb{R}^n)} &\le C(\|u\|^2_{L^\infty(\mathbb{R}^n)} + \|v\|^2_{L^\infty(\mathbb{R}^n)})\ \|u-v\|_{L^p(\mathbb{R}^n)} \\ &\le C_1(\|u\|^2_\alpha + \|v\|^2_\alpha)\ \|u-v\|_\alpha\end{aligned}$$

$$\|u^3\|_{L^p(\mathbb{R}^n)} \le C_1\ \|u\|^3_\alpha$$

so the hypotheses of section 3.3 are verified, and local existence and uniqueness follows. Note however the resolvent of A is not compact. (See exercise 1.)

We will prove global existence when $\lambda \ge 0$. As long as the solution exists, we have

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^n} |u(x,t)|^{3p} dt &= 3p \int_{\mathbb{R}^n} |u|^{3p-2} u(\Delta u - \lambda u^3)\, dx \\ &= -\int_{\mathbb{R}^n} (3p\lambda\ |u|^{3p+2} + 3p(3p-1)|u|^{3p-2}|\nabla u|^2)\, dx \le 0.\end{aligned}$$

Thus $\int_{\mathbb{R}^n} |u(x,t)|^{3p} dt \le \int_{\mathbb{R}^n} |u(x,0)|^{3p} dx \le C\,\|u(\cdot,0)\|_\alpha^{3p}$ for all $t > 0$, as long as the solution exists, so the solution exists for all $t > 0$ (ex. 1, sec. 3.3). In fact, the solution has $\|u(\cdot,t)\|_\beta$ bounded for all $t \ge 1$ for any $\beta < 1$; for

$$u(t+1) - e^{-A}u(t) = -\lambda \int_t^{t+1} e^{-A(t+1-s)} u^3(s)\,ds,$$

so $\|u(t+1) - e^{-A}u(t)\|_\beta$ is bounded for $t \ge 0$. But $\|u(t)\| = \|u(t)\|_{L^p(\mathbb{R}^n)}$ is also bounded for all $t \ge 0$, as above, hence for $t \ge 0$,

$$\|u(t+1)\|_\beta \le \|e^{-A}u(t)\|_\beta + C \le C_1.$$

Now the nonlinearity, $-\lambda u^3$, is a bounded polynomial map from X^α to X, so the solution depends analytically on $t > 0$, and $u_0 \in X^\alpha$, with values in X^α. We can also establish smoothness in the x-variable. For example, $u \in X^{1+\alpha}$ implies

$$\begin{aligned}\|Au^3\|_{L^p} &= \|-3u^2\Delta u - 6u|\nabla u|^2\|_{L^p(\mathbb{R}^n)}\\ &\le 3\,\|u\|_{L^\infty}^2 \|Au\| + 6\,\|u\|_{L^\infty}\|u\|_{W^{1,2p}(\mathbb{R}^n)}^2 \le C\,\|u\|_{1+\alpha}^3\end{aligned}$$

for some constant C, since $Au \in X^\alpha \subset L^\infty(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ implies $u \in W^{1,q}(\mathbb{R}^n)$ for any $p \le q < \infty$. It follows that $u(t) \in X^{1+\alpha}$ for any $t > 0$, and by similar arguments, for any $\beta \ge \alpha$,

$$t \to u(t) \in X^\beta \text{ is continuous when } t > 0.$$

(See ex. 1 and 2, sec. 3.5.)

Thus we have a solution u existing for all $t > 0$, provided $u(0) \in X^\alpha$, and such that $(t,x) \to u(t,x)$ is analytic in $t > 0$ and infinitely often differentiable in $x \in \Omega$; in particular, we have a classical solution, which is bounded for all $t > 0$.

Exercise 1. The resolvent of A is not compact; that is, there exists a bounded sequence $\{\phi_n\}_{n=1}^\infty$ in $D(A) = W^{2,p}(\mathbb{R}^n)$ such that there is no subsequences which converge in $L^p(\mathbb{R}^n)$. (Hint: if $\phi \in W^{2,p}(\mathbb{R}^n)$, compare its norm with that of the "translated" function ϕ_h, $\phi_h(x) = \phi(x+h)$ for all $x \in \mathbb{R}^n$.)

Since the orbit $\{u(t), t \geq 0\}$ is not necessarily compact in $L^p(\mathbb{R}^n)$, we cannot examine its limit points, as we did for the corresponding problem in a bounded domain (ex. 1, sec. 3.6). However, let $\phi_q(t) = \int_{\mathbb{R}^n}|u(t,x)|^2dx$; then if $q \geq 2$, $\phi_q(\cdot)$ is nonincreasing, nonnegative and $\frac{d\phi_q}{dt}(t) \leq -q\lambda\ \phi_{q+2}(t)$ for $t > 0$, hence $\int_0^\infty \phi_{q+2}(t)dt < \infty$. Since $\phi_{q+2}(\cdot)$ is also nonincreasing, it follows that

$$\phi_{q+2}(t) \to 0 \quad \text{as} \quad t \to +\infty,$$

i.e. when $q \geq 4$, $\int_{\mathbb{R}^n}|u(t,x)|^q dx \to 0$ as $t \to +\infty$. If $2 \leq p < 4$, $\|u(t,\cdot)\|_{L^{p_1}} \leq \|u(0,\cdot)\|_{L^p}^{\theta}\ \|u(t,\cdot)\|_{L^4}^{1-\theta}$ for any p_1 in $p < p_1 < 4$ and some $\theta = \theta(p,p_1)$ in $0 < \theta < 1$ so $\|u(t,\cdot)\|_{L^{p_1}} \to 0$ as $t \to +\infty$ for any $p_1 > p$.

Exercise 2. Examine this problem in the space $X = H^m(\mathbb{R}^n) = W^{m,2}(\mathbb{R}^n)$ with $m > n/2$.

Example 3.8: The Navier-Stokes equation [32, 65]

Let Ω be a bounded domain in $\mathbb{R}^3$ with $\partial\Omega$ smooth, and consider for $t > t_0$, $x \in \Omega$

$$\partial u_j/\partial t - 1/Re\ \Delta u_j = -\partial p/\partial x_j - \sum_{k=1}^{3} u_k\ \partial u_j/\partial x_k + f_j(x,t)$$

for $j = 1,2,3$,

$$\operatorname{div} u = \sum_{j=1}^{3} \partial u_j/\partial x_j = 0$$

with boundary value $u = 0$ on $\partial\Omega \times [t_0,\infty)$ and initial value $u(x,t_0) = u_0(x)$ for $x \in \Omega$.

At first glance, the theory of section 3.3 does not apply: one of the unknown quantities p (the pressure) does not appear with a time derivative, and one of the equations ($\operatorname{div} u = 0$) also has no time derivative. However, we will choose the space X so that $\operatorname{div} u = 0$ automatically, and then the pressure term drops out of the equations.

If $u: \Omega \to \mathbb{R}^3$ is continuously differentiable, $\operatorname{div} u = 0$, and the normal component u_n vanishes on $\partial\Omega$, then for any scalar $\phi \in C^1(\Omega)$,

$$\int_\Omega u \cdot \text{grad}\ \phi\ dx = 0.$$

Conversely, a smooth vector field u which is orthogonal to all gradients must satisfy $\text{div}\ u = 0$ in Ω, $u_n = 0$ on $\partial\Omega$.

Let H_π be the $L^2(\Omega,\mathbb{R}^3)$ closure of $\{\text{grad}\ \phi \mid \phi \in C^1(\Omega)\}$, and let H_σ be the $L^2(\Omega,\mathbb{R}^3)$ closure of $\{u \in C^1(\Omega,\mathbb{R}^3) \mid \text{div}\ u = 0$ in Ω, $u_n = 0$ on $\partial\Omega\}$. Then H_π, H_σ are closed orthogonal subspaces in $L^2(\Omega,\mathbb{R}^3)$; in fact

$$L^2(\Omega,\mathbb{R}^3) = H_\pi \oplus H_\sigma.$$

To prove this, it is sufficient to show every smooth $u: \Omega \to \mathbb{R}^3$, $u = 0$ near $\partial\Omega$, has the form $u = v + \nabla\phi$ with $v \in H_\sigma$ and $\nabla\phi \in H_\pi$. But we may choose ϕ to be a solution of

$$\Delta\phi = \text{div}\ u \quad \text{in} \quad \Omega,\ \partial\phi/\partial n = u_n = 0 \quad \text{on} \quad \partial\Omega;$$

then ϕ is smooth and $v \equiv u - \nabla\phi$ is also smooth, $\text{div}\ v = 0$ in Ω, $v_n = 0$ on $\partial\Omega$.

Let P denote the orthogonal projection of $L^2(\Omega,\mathbb{R}^3)$ onto H_σ; projecting the Navier-Stokes equation into H_σ, we obtain formally

$$dv/dt + \mu A v = N(v) + f_\sigma(t), \quad v(t) \in H_\sigma,$$

where $\mu = 1/\text{Re}$ (Re - Reynolds number), $A = -P\Delta$ with zero boundary data, $N(v) = -P(v \cdot \text{grad})v$, and $f_\sigma(t) = Pf(\cdot,t)$.

Now if $u,v \in C_0^3(\Omega) \cap H_\sigma$, a dense subspace of H_σ, then $Au \in H_\sigma$,

$$(Au,v) = (u,Av)$$
$$(Au,u) = (-\Delta u,u) = \int_\Omega |\text{grad}\ u|^2 dx \geq 0.$$

Thus we may suppose A is a self adjoint densely defined operator in H_σ. Odqvist [77] investigated the boundary value problem $Au = f$, i.e.

$$-\Delta u + \text{grad}\ p = f,\ \text{div}\ u = 0 \quad \text{in} \quad \Omega$$
$$u = 0 \quad \text{on} \quad \partial\Omega,$$

by the methods of potential theory, and obtained estimates for the Green's function which prove $H_\sigma^1 = D(A) \subset W^{2,2}(\Omega) \cap H_\sigma$ with continuous inclusion. (See [65].) It follows from Th. 1.6.1 that, if

$1/2 < \alpha < 1$, $H_\sigma^\alpha \subset W^{1,q}(\Omega,\mathbb{R}^3)$ provided $1/q > (5-4\alpha)/6$, and $3/4 < \alpha < 1$, $H_\sigma^\alpha \subset L^\infty(\Omega,\mathbb{R}^3)$.

Now consider the nonlinear term: $\|N(v)\| \leq \|v\|_{L^\infty} \|\nabla v\|_{L^2} \leq C\|v\|_\alpha^2$ if $\alpha > 3/4$, so N is a bounded polynomial from H_σ^α to H_σ, if $\alpha > 3/4$.

Now it follows from Th. 3.3.3 that, if $v(t_0) \in H_\sigma^\alpha$ and $t \to f_\sigma(\cdot,t)$ is Hölder continuous into H_σ for $t \geq t_0$, there is a unique solution $v(t)$ on some maximal interval $t_0 \leq t < t_1$, and the solution depends analytically on $v(t_0)$ and the Reynolds number $Re = 1/\mu > 0$.

Suppose in fact that, in the original equation, $(x,t) \to f(x,t)$ is $C^{1+\delta}$, some $\delta > 0$, for $x \in \Omega$, $t_0 \leq t < t_1$; then $(x,t) \to f_\sigma(x,t)$ ($f_\sigma = Pf$) is Lipschitzian in (x,t). We show we then have a classical solution of the original system. Arguing as in the examples above, if $t > t_0$, $(x,t) \to v(x,t)$ is $C^{2+\delta}$ in x, $C^{1+\delta}$ in t, for some $\delta > 0$, so

$$P\{\frac{\partial v}{\partial t}(\cdot,t) - \mu\Delta v(\cdot,t) - (v\cdot\text{grad})v + f(\cdot,t)\} = 0.$$

But the quantity in braces is Hölder continuous on Ω, so it is the gradient of a $C^{1+\delta}$ function $p(\cdot,t)$ for each t, which is also continuous in t.

More general and precise results are in [32].

Exercise 1. If $\|f(\cdot,t)\|_{L^2(\Omega)}$ is bounded on $0 \leq t < \infty$, prove $\|u(\cdot,t)\|_{L^2(\Omega)}$ is bounded on the domain of existence. If it is also assumed that $\|u(\cdot,t)\|_{L^p(\Omega)}$ is bounded on the domain of existence of some solution, for some $p > 3$, prove the solution exists for all $t > 0$.

Hint: for the $L^2(\Omega)$ bound, multiply the original equation by u and integrate over Ω.

Chapter 4

Dynamical Systems and Liapunov Stability

4.1 Dynamical systems

Definition 4.1.1. A *dynamical system* (nonlinear semigroup) on a complete metric space C is a family of maps $\{S(t): C \to C,\ t \geq 0\}$ such that

(i) for each $t \geq 0$, $S(t)$ is continuous from C to C;

(ii) for each $x \in C$, $t \to S(t)x$ is continuous;

(iii) $S(0)$ = identity on C;

(iv) $S(t)(S(\tau)x) = S(t+\tau)x$ for all $x \in C$ and $t,\tau \geq 0$.

Example 1. Suppose A is sectorial in a Banach space X, V is an open set in X^α for some $0 \leq \alpha < 1$, and $f: V \to X$ is locally Lipschitzian. Assume C is a closed subset of V and that for any solution $x(\cdot)$ of $dx/dt + Ax = f(x)$, $x(0) \in C$ implies the solution exists and $x(t) \in C$ for all $t \geq 0$ (C is positively invariant). If $x(t;x_0)$ is the solution of time t with initial value $x(0;x_0) = x_0$, then

$$S(t)x_0 = x(t;x_0), \quad (x_0 \in C,\ t \geq 0)$$

defines a dynamical system on C, in the induced topology of X^α. Verification of conditions (i)-(iv) above is left as an easy exercise.

More specific examples are in sec. 3.7 with $C = X^\alpha = \mathscr{L}^p_{2\alpha}(\mathbb{R}^n)$, and sec. 3.3, ex. 9, with $C = \{\phi \in H^1_0(0,\pi) \mid \phi(x) \geq 0 \text{ for } 0 \leq x \leq \pi\}$, and the example below.

Example 2. (Varma and Amundson [100]). The equations describing a tubular chemical reactor with a single first-order irreversible reaction are

$$\partial y/\partial t = \partial^2 y/\partial x^2 - Pe\ \partial y/\partial x + \beta z r(y) + \gamma(y_a - y)$$

$$Le\ \partial z/\partial t = \partial^2 z/\partial x^2 - Pe\ \partial z/\partial x - z r(y)$$

for $0 < x < 1$, $t > 0$, with boundary conditions $\partial y/\partial x = Pe(y-1)$, $\partial z/\partial x = Pe(z-1)$ at $x = 0$ and $\partial y/\partial x = 0$, $\partial z/\partial x = 0$ at $x = 1$. Here β,γ,y_a,Pe, Le (Peclet and Lewis numbers) are positive constants, and

$r(y) = \alpha \exp(-\delta/y)$ with positive α, δ.

It is physically clear that the (nondimensional) temperature y and concentration z must always be nonnegative. In fact, this system of equations defines a dynamical system on

$$C = \{(y,z) \in H^1(0,1;\mathbb{R}^2) \mid y(x) \geq 0,\ z(x) \geq 0 \ \text{ on } \ 0 \leq x \leq 1\}.$$

To see this, note first that $r(y)$ may be set equal to zero for $y \leq 0$, and we would then have a dynamical system on $H^1(0,1;\mathbb{R}^2)$, since the right-side grows at most linearly (Cor. 3.3.5). Next consider the operator $u \to -u_{xx} + \text{Pe}\, u_x + \lambda u$ for $\lambda > 0$ and C^2 functions u satisfying the boundary conditions: $u_x = \text{Pe}(u-1)$ at $x = 0$; $u_x = 0$ at $x = 1$. If $-u_{xx} + \text{Pe}\, u_x + \lambda u \geq 0$ on $0 < x < 1$, it follows easily (max. principle [81]) that $u \geq 0$ on $0 \leq x \leq 1$. (Otherwise examine the point where a negative minimum occurs, and obtain a contradiction.) We say $(y,z) \in L^2(0,1;\mathbb{R}^2)$ has $(y,z) \geq 0$ if $y(x) \geq 0$ and $z(x) \geq 0$ a.e. in $0 \leq x \leq 1$; then we may apply the results of exercises 7 and 8, sec. 3.3, since $(y,z) \geq 0$ implies $(\beta z r(y) + \gamma(y_a - y), -z r(y)) + K(y,z) \geq 0$, if $K > 0$ is sufficiently large. Thus C is a positively invariant set, and we have a dynamical system in C.

Example 3. One can define dynamical systems generated by nonautonomous differential equations, following Sell [89]. This approach will not be developed here but may lead to interesting results.

Definition 4.1.2. Let $\{S(t),\ t \geq 0\}$ be a dynamical system on C and for any $x \in C$, let $\gamma(x) = \{S(t)x,\ t \geq 0\}$ = the *orbit* (or positive semi-orbit) through x. We say x is an *equilibrium point* if $\gamma(x) = \{x\}$; $\gamma(x)$ is a *periodic orbit* if there exists $p > 0$ such that $\gamma(x) = \{S(t)x,\ 0 \leq t \leq p\} \neq \{x\}$.

An orbit $\gamma(x)$ (or sometimes, the point x) is *stable* if $S(t)y \to S(t)x$ as $y \to x$, $y \in C$, uniformly in $t \geq 0$; i.e. if for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that for all $t \geq 0$, $\text{dist}(S(t)x, S(t)y) < \varepsilon$ whenever $\text{dist}(x,y) < \delta(\varepsilon)$, $y \in C$. An orbit $\gamma(x)$ is *unstable* if it is not stable. An orbit $\gamma(x)$ is *uniformly asymptotically stable* if it is stable and also there is a neighborhood $V = \{y \in C: \text{dist}(x,y) < r\}$ such that

$$\text{dist}(S(t)y, S(t)x) \to 0 \quad \text{as} \quad t \to +\infty,$$

uniformly for $y \in V$.

Exercises

(1) If $\gamma(x)$ is unstable, then so is $\gamma(y)$ for any $y \in \gamma(x)$. Does the corresponding result hold when $\gamma(x)$ is stable?

(2) If $y \in \gamma(x)$ and $\gamma(y)$ is stable, then $\gamma(x)$ is stable.

(3) If $\gamma(x)$ is stable, then it is also *orbitally stable*, i.e. whenever $y \to x$, $y \in C$, we have $\text{dist}\{S(t)y, \gamma(x)\} \to 0$ uniformly in $t \geq 0$. The converse implication fails. Orbital stability is the notion required for studying periodic orbits (Ch. 8).

Definition 4.1.3. Let $\{S(t), t \geq 0\}$ be a dynamical system on C. A *Liapunov function* is a continuous real-valued function V on C such that

$$\dot{V}(x) \equiv \overline{\lim_{t\to 0+}} \frac{1}{t} \{V(S(t)x) - V(x)\} \leq 0$$

for all $x \in C$. We do not exclude the possibility $\dot{V}(x) = -\infty$. Henceforth we use the notation $\|x-y\| = \text{dist}\{x,y\}$, since in most examples we use the induced topology of some Banach space.

Theorem 4.1.4. Let $\{S(t), t \geq 0\}$ be a dynamical system on C, and let 0 be an equilibrium point in C. Suppose V is a Liapunov function on C which satisfies $V(0) = 0$, $V(x) \geq c(\|x\|)$ for $x \in C$, $\|x\| = \text{dist}\{x,0\}$, where $c(\cdot)$ is a continuous strictly increasing function, $c(0) = 0$, and $c(r) > 0$ for $r > 0$. Then 0 is stable.

Suppose in addition that $\dot{V}(x) \leq -c_1(\|x\|)$, where $c_1(\cdot)$ is also continuous, increasing and positive, with $c_1(0) = 0$. Then 0 is uniformly asymptotically stable.

Proof. For each $k > 0$ define $U_k = \{x \in C: V(x) < k\}$; each U_k is a neighborhood of 0. Also, each U_k is positively invariant: $x \in U_k$ implies $V(S(t)x) \leq V(x) < k$ for all $t \geq 0$.

If $V(x) \geq c(\|x\|)$, then for any $\varepsilon > 0$, there exists $k = c(\varepsilon)>0$ such that $V(x) \leq k$ implies $\|x\| \leq \varepsilon$. And by continuity of V, there exists $\delta > 0$ such that $\|x\| < \delta$ implies $x \in U_k$, so $S(t)x \in U_k$ for all $t \geq 0$, so $\|S(t)x\| \leq \varepsilon$ for all $t \geq 0$.

Now $V(S(t)x)$ is a nonincreasing function of t, bounded below by zero. Let $\ell = \lim_{t\to+\infty} V(S(t)x)$; if $\ell > 0$ then $\inf_{t\geq 0} \|S(t)x\|$ is positive so $\sup_{t\geq 0} \dot{V}(S(t)x) \leq -m$ for some $m > 0$. But this contradicts the fact $V(S(t)x) \geq 0$ for all $t \geq 0$. Thus $V(S(t)x)$ and $\|S(t)x\|$ tend to zero as $t \to +\infty$. In fact, this is true uniformly in $\|x\| \leq \varepsilon_0$, $x \in C$ (for some $\varepsilon_0 > 0$) since $V(S(t)x) \leq v(t)$ where $v(\cdot)$ is the maximal solution [18, p. 28] of

$$\frac{dv}{dt} + c_1(b^{-1}(v)) = 0, \quad t > 0, \; v(0) = v_0$$

where b is continuous and increasing with $V(x) \leq b(\|x\|)$, $b(0) = 0$, and $v_0 = \sup\{V(x): \|x\| \leq \varepsilon_0\}$. If ε_0 is sufficiently small, $c_1(b^{-1}(s))$ is continuous on $0 \leq s \leq v_0$, and $v(t) \to 0$ as $t \to +\infty$.

Example 4. $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - u^3$ $(0 < x < \pi,\ t > 0)$, $u(0,t) = 0$, $u(\pi,t) = 0$, defines a dynamical system in $H_0^1(0,\rho)$. (Compare with example 3.6 above.) Define $V_p(u) = \int_0^\pi |u(x)|^p dx$ for any $p \geq 2$; then V_p is continuous in $H_0^1(0,\pi)$ and $\dot{V}_p(u) = -p \int_0^\rho |u|^{p+2}dx - p(p-1)\int_0^\rho |u|^{p-2}u_x^2 dx$ so $\dot{V}_p(u) \leq -p \int_0^\pi |u|^{p+2}dx \,\_\, -p\pi^{-2/p}(V_p(u))^{p+2/p}$ (by Hölder's inequality). Taking $p = 6$ shows that we have a dynamical system on $H_0^1(0,\pi)$, and that any solution $u(\cdot,t)$ satisfies $\int_0^\pi |u(x,t)|^p dx \to 0$ as $t \to +\infty$, for any $p \geq 2$. To show convergence in $H_0^1(0,\pi)$, we use a different Liapunov function:

$$W(u) = \int_0^\pi \{\tfrac{1}{2} u_x^2 + \tfrac{1}{4} u^4\}dx.$$

Then $W(u) \geq \frac{1}{2}\|u\|^2_{H_0^1}$, $W(\cdot)$ is continuous on $H_0^1(0,\pi)$ and $u \in H_0^1(0,\pi) \cap H^2(0,\pi)$

$$\dot{W}(u) = -\int_0^\pi \{u_{xx} - u^3\}^2 dx.$$

It follows that $\dot{W}(u) \leq 0$ for all $u \in H_0^1(0,\pi)$. (Note: if $u(\cdot,t)$ is a solution for $t > 0$, then $t \to W(u(\cdot,t))$ is continuously differentiable for $t > 0$, and for some t^* in $(0,t)$

$$\frac{1}{t}\{W(u(\cdot,t)) - W(u(\cdot,0)) = \dot{W}(u(\cdot,t^*)) \leq 0).$$

In fact $\int_0^\pi u_{xx}^2 dx \geq \int_0^\pi u_x^2 dx$ for $u \in H_0^1(0,\pi)$ so

$$-\dot{W}(u) = \int_0^\pi u_{xx}^2 dx + 6\int_0^\pi u^2 u_x^2 dx + \int_0^\pi u^6 dx \geq \int_0^\pi u_x^2 dx$$

$\dot{W}(u) \leq -\|u\|^2_{H_0^1}$, and uniform asymptotic stability in $H_0^1(0,\pi)$ follows.

Exercise 4. Let $\{S(t), t \geq 0\}$ be a dynamical system on C, and let B be open in C. Let $\Sigma(t)$ be the restriction of $S(t)$ to B; then for each x in B, there exists maximal $T(x)$, $0 < T(x) \leq \infty$, such that $\{\Sigma(t)x, 0 \leq t < T(x)\}$ is in B. This family of maps satisfies:

(i) if $x_n \to x_0$ in B and $0 \leq t < T(x_0)$, then $T(x_n) > t$ eventually and $\Sigma(t)x_n \to \Sigma(t)x_0$ in B;

(ii) if $x \in B$, $t \to \Sigma(t)x$ is continuous for $0 \leq t < T(x)$ into B;

(iii) $\Sigma(0) =$ identity on B;

(iv) if $x \in B$, $t,\tau \geq 0$ and $t+\tau < T(x)$ then $\Sigma(t)(\Sigma(\tau)x) = \Sigma(t+\tau)x$.

Such a family $\{\Sigma;T\}$ we call a *local dynamical system* on B. Examine stability notions for local dynamical systems.

Exercise 5. If A is sectorial in X, $f: X^\alpha \to X$ is locally Lipschitzian for some $\alpha < 1$, then $dx/dt + Ax = f(x)$ defines a local dynamical system $\{\Sigma,T\}$ on any open set $B \subset X^\alpha$.

If $\|f(x)\| \leq M < \infty$ for all $x \in B$, define $g(x) = f(x)\phi(\|f(x)\|)$ for $x \in X^\alpha$, where $\phi(y) = 1$ if $0 \leq y \leq M$, $\phi(y) = M/y$ if $y \geq M$. (Note ϕ is Lipschitz continuous on $\mathbb{R}^+$.) Then $\frac{dx}{dt} + Ax = g(x)$ defines a dynamical system on X^α whose restriction to B is $\{\Sigma,T\}$.

4.2 Converse theorem on asymptotic stability

Theorem 4.2.1. Consider the equation

$$dx/dt + Ax = f(x)$$

where A is sectorial and f is Lipschitz continuous from a neighborhood of 0 in X^α (some $0 \leq \alpha < 1$) to X, with $f(0) = 0$.

Suppose $x = 0$ is uniformly asymptotically stable (in X^α). Then there exist positive constants β, K and r, a continuous strictly increasing function $a(s)$, $0 \leq s \leq r$, $a(0) = 0$, and a real-valued function $V(x)$ defined for $\|x\|_\alpha \leq r$ such that

(i) $a(\|x\|_\alpha) \leq V(x) \leq K\|x\|_\alpha$

(ii) $|V(x) - V(y)| \leq K\|x-y\|_\alpha$

(iii) $\dot{V}(x) \leq -\beta V(x)$

for $\|x\|_\alpha \leq r$, $\|y\|_\alpha \leq r$.

Remark. The proof below follows T. Yoshizawa, *Stability Theory by Liapunov's Second Method*, Univ. Tokyo Press (1966).

Proof of the theorem. Assume $r_0 > 0$ so small that $\|x_0\|_\alpha \leq r_0$ implies $x(t;x_0)$ exists for all $t \geq 0$ and $\|x(t;x_0)\|_\alpha \leq \theta(t) \to 0$ as $t \to +\infty$. We assume that $\theta(t)$ has a continuous negative derivative, and $T(\varepsilon)$ is the inverse function: $T(\theta(t)) = t$, so $T(\varepsilon)$ is continuous on $0 < \varepsilon \leq \theta(0)$ and $T(\varepsilon) \to +\infty$ as $\varepsilon \to 0+$.

If r_0 is small enough, f is uniformly Lipschitzian on the ball of radius $\theta(0)$ and $x_0 \to x(t;x_0)$ has Lipschitz constant $\leq Le^{Mt}$ (for some constants L,M) when $\|x_0\|_\alpha \leq r_0$. Now define

$$g(\varepsilon) = \exp\{-(\beta+M)T(\varepsilon)\}, \quad g(0) = 0,$$

for any $\beta > 0$. Also

$$G_k(z) = \max(0, z - \tfrac{1}{k}), \quad k > 1$$

and for $k = 1,2,3,\ldots$, and $\|x_0\|_\alpha \leq r_0$,

$$V_k(x_0) = g(\tfrac{1}{k+1}) \sup_{t \geq 0} \{e^{\beta t} G_k(\|x(t;x_0)\|_\alpha)\}.$$

Observe that this supremum may be taken only on the interval $0 \leq t \leq T_k \equiv T(\frac{1}{k+1})$, and so

$$0 \leq V_k(x_0) \leq g(\tfrac{1}{k+1}) e^{\beta T_k} \theta(0) \leq \theta(0)$$

$$\begin{aligned} |V_k(x_0) - V_k(y_0)| &\leq g(\tfrac{1}{k+1}) \sup_{0 \leq t \leq T_k} \{e^{(\beta+M)t} L \|x_0 - y_0\|_\alpha\} \\ &\leq g(\tfrac{1}{k+1}) L e^{(\beta+M)T_k} \|x_0 - y_0\|_\alpha \\ &= L\|x_0 - y_0\|_\alpha . \end{aligned}$$

Finally, for small $h > 0$,

$$\begin{aligned} V_k(x(h;x_0)) &= e^{-\beta h} g(\tfrac{1}{k+1}) \sup_{t \geq h} \{e^{\beta t} G_k(\|x(t;x_0)\|_\alpha)\} \\ &\leq e^{-\beta h} V_k(x_0). \end{aligned}$$

so $\dot V_k(x_0) \le -\beta V_k(x_0)$.

If we let $V(x) = \sum_{k=1}^{\infty} 2^{-k} V_k(x)$, $\|x\|_\alpha \le r$, then

$$|V(x) - V(y)| \le L\|x-y\|_\alpha, \quad \dot V(x) \le -\beta V(x), \; V(0) = 0$$

and if $0 < \|x\|_\alpha \le r$,

$$V(x) \ge \sum_{k=1}^{\infty} 2^{-k} g(\tfrac{1}{k+1}) G_k(\|x\|_\alpha) = a(\|x\|_\alpha) > 0$$

and $a(\cdot)$ is Lipschitz continuous and strictly increasing.

Remark. If we choose smooth functions behaving like the G_k, we can have $a(\cdot)$ also smooth.

Exercise 1*. Let T be a Lipschitz continuous map of a neighborhood of 0 in a Banach space X into X, with $T(0) = 0$. We will say that 0 is asymptotically stable if there is a neighborhood V of the origin such that $T^n(x)$ is defined for all $n \ge 0$ and all $x \in V$ and

$$\operatorname{diam} T^n(V) \to 0 \quad \text{as} \quad n \to \infty.$$

Prove (in analogy with Theorem 4.1.4 and 4.2.1 above) the necessary and sufficient condition that 0 is asymptotically stable is the existence of positive constants K, r and $\theta < 1$, a continuous strictly increasing function $a(s)$, $0 \le s \le r$, $a(0) = 0$, and a real-valued function $V(x)$ for $\|x\| \le r$ such that

$$a(\|x\|) \le V(x) \le K\|x\|$$

$$|V(x) - V(y)| \le K\|x-y\| \quad \text{for } \|x\| \le r, \|y\| \le r$$

and $V(T(x)) \le \theta V(x)$ near 0.

Find such a Liapunov function proving stability of $x = 0$ for $x \to x - x^3 : \mathbb{R} \to \mathbb{R}$.

Now consider (E_ε): $dx/dt + Ax = f(x,\varepsilon)$, where $f(0,0) = 0$. We prove a weak version of stability with respect to constantly acting disturbances [62, 41]; a stronger result is outlined in exercise 3.

Corollary 4.2.3. Suppose $x = 0$ is asymptotically stable for the equation (E_0) with $\varepsilon = 0$. Also assume $\|f(x,\varepsilon) - f(y,\varepsilon)\|_\alpha \le L\|x-y\|_\alpha$

for $\|x\|_\alpha$, $\|y\|_\alpha \le r$ and $|\varepsilon| \le \varepsilon_0$, and that

$$\|f(x,\varepsilon)-f(x,0)\|_\alpha \to 0 \quad \text{as} \quad \varepsilon \to 0, \text{ uniformly in } \|x\|_\alpha \le r.$$

Then there exists a neighborhood U of the origin in X^α, $U \subset \{\|x\|_\alpha < r\}$, which is positively invariant for (E_ε) whenever $|\varepsilon|$ is sufficiently small.

Proof. Let $V(x)$ be the Liapunov function provided by Theorem 4.2.1 for (E_0). We will estimate the derivative $\dot{V}_\varepsilon$ of V along solutions of (E_ε).

$$\begin{aligned}\dot{V}_\varepsilon(x_0) &= \overline{\lim_{h\to 0+}} \frac{1}{h}(V(x(h,x_0,\varepsilon)) - V(x_0)) \\ &\le \dot{V}(x_0) + \overline{\lim_{h\to 0}} \frac{1}{h}(V(x(h;x_0,\varepsilon)) - V(x(h;x_0,0)) \\ &\le \dot{V}(x_0) + K(r)\overline{\lim_{h\to 0+}} \frac{1}{h}\|x(h;x_0,\varepsilon) - x(h;x_0,0)\|_\alpha .\end{aligned}$$

Now assume $\|x_0\|_\alpha < r$ and $t > 0$ is small;

$$\begin{aligned}x_\varepsilon(t)-x_0(t) \equiv x(t;x_0,\varepsilon)-x(t;x_0,0) &= \int_0^t e^{-A(t-s)}[f(x_\varepsilon(s),\varepsilon)-f(x_0(s),\varepsilon)]ds \\ &+ \int_0^t e^{-A(t-s)}[f(x_0(s),\varepsilon)-f(x_0(s),0)]ds\end{aligned}$$

so $\|x_\varepsilon(t)-x_0(t)\|_\alpha \le M\int_0^t L\|x_\varepsilon(s)-x_0(s)\|_\alpha ds + Mt\,\Delta(\varepsilon)$,

$$\Delta(\varepsilon) = \sup_{\|x\|_\alpha \le r} \|f(x,\varepsilon)-f(x,0)\|_\alpha .$$

Thus $\|x(h;x_0,\varepsilon)-x(h;x_0,0)\|_\alpha \le M_2 h\,\Delta(\varepsilon)$ for sufficiently small $h > 0$, so

$$\dot{V}_\varepsilon(x_0) \le \dot{V}(x_0) + K(r)M_2\,\Delta(\varepsilon).$$

Choose $\ell > 0$ so that

$$U = \{x;\ V(x) < \ell\} \subset \{\|x\|_\alpha < r\}$$

When $V(x_0) = \ell$, then $\dot{V}(x_0) \le -\beta V(x_0) \le -\beta\ell$, and so $\dot{V}_\varepsilon(x_0) \le -\ell\beta + K(r)M_2\Delta(\varepsilon) < 0$ provided $|\varepsilon|$ is sufficiently small. But this implies that, if $V(x_0) < \ell$, then $V(x(t;x_0,\varepsilon)) < \ell$ for all $t > 0$, i.e. U is positively invariant.

Exercise 2. It is proved below (4.3, example) that the zero solution is globally asymptotically stable (in $H_0^1(0,\pi)$) for $u_t = u_{xx} + \alpha u - \beta u^3$, $0 < x < \pi$,

$$u(0,t) = 0, \quad u(\pi,t) = 0,$$

when $\alpha = 1$ and $\beta > 0$. Assuming this, prove there exists, inside any given neighborhood of the origin a neighborhood U of the origin in $H_0^1(0,\pi)$ which is positively invariant whenever $|\alpha-1|$ is sufficiently small (fixed $\beta > 0$). Can you generalize to $u_t = u_{xx} + f_\varepsilon(x,u(x,t))$ where f_ε is smooth and $f_0(x,u) = u - \beta u^3$? What about $u_t = u_{xx} + f_\varepsilon(x,u(x,t),u_x(x,t))$ with $f_0 = u - \beta u^3$?

Exercise 3*. Referring to the exercise 1 on asymptotic stability of a fixed point under iteration of a map, prove the generalization of Cor. 3.9.6, when it is assumed merely that $\|f(x,\varepsilon)-f(y,\varepsilon)\| \le L\|x-y\|_\alpha$ for $\|x\|_\alpha$, $\|y\|_\alpha \le r$ and $\|f(x,\varepsilon)-f(x,0)\| \to 0$ as $\varepsilon \to 0$, uniformly in $\|x\|_\alpha \le r$. Take T_ε as the map $x_0 \to x(1;x_0,\varepsilon)$, translation along the trajectory for one time unit and use the Banach space X^α; $x = 0$ is asymptotically stable under iteration of T_0 in this space.

Exercise 4. Suppose A is sectorial in X, $0 \le \alpha < 1$, U is a neighborhood of 0 in X^α and $f: \mathbb{R} \times U \to X$ is locally Hölder continuous in t and uniformly Lipschitz in x:

$$\|f(t,x_1) - f(t,x_2)\| \le L\,\|x_1-x_2\|_\alpha$$

for x_1,x_2 in U. Suppose $f(t,0) \equiv 0$ and $x = 0$ is uniformly asymptotically stable. (See sec. 5.1 for definition.) Show there exists $V(t,x)$ with

$$|V(t,x) - V(t,y)| \le B\,\|x-y\|_\alpha$$

$$B\,\|x\|_\alpha \ge V(t,x) \ge a(\|x\|_\alpha)$$

and if $dx/dt + Ax = f(t,x)$ on $(t,t+h]$

$$\dot V(t,x(t)) = \overline{\lim_{h\to 0+}}\ \frac{1}{h}\{V(t+h,x(t+h)) - V(t,x(t))\} \le -V(t,x)$$

(for all t and for $\|x\|_\alpha \le r$). Here r,B are positive constants and $a(\cdot)$ is a continuous strictly increasing function.

(Hint: in analogy with the proof of Th. 4.2.1 take

$$V_k(t_0;x_0) = g(\frac{1}{k+1}) \sup_{t\geq 0} e^{\beta t} G_k(\|x(t_0+t;t_0,x_0)\|_\alpha).$$

4.3 Invariance principle

Despite the converse theorem above, finding a Liapunov function in any particular case is usually difficult, and finding one satisfying all requirements of Th. 4.1.4 is even more difficult. J. P. LaSalle developed an "invariance principle" for ordinary differential equations, extended to other dynamical systems by Hale [38], which enables us to exploit Liapunov functions which do not satisfy all the conditions of Th. 4.1.4.

Definition 4.3.1. Let $\{S(t), t \geq 0\}$ be a dynamical system on a complete metric space C. A set $K \subset C$ is *invariant* if, for any $x_0 \in K$, there exists a continuous curve $x\colon \mathbb{R} \to K$ with $x(0) = x_0$ and

$$S(t)x(\tau) = x(t+\tau) \quad \text{for} \quad -\infty < \tau < \infty, \quad t \geq 0.$$

Definition 4.3.2. If $x_0 \in C$, $\gamma(x_0) = \{S(t)x_0;\ t \geq 0\}$ is the orbit through x_0, then the *ω-limit set* (omega limit set) for x_0 or for the orbit $\gamma(x_0)$ is

$$\omega(x_0) = \omega(\gamma(x_0)) = \{x \in C \mid \text{there exist } t_n \to \infty \text{ such that } S(t_n)x_0 \to x\}.$$

Exercise 1. If $x_1 \in \gamma(x_0)$, then $\omega(x_0) = \omega(x_1)$. Also

$$\omega(x_0) = \bigcap_{\tau\geq 0} \text{Closure}\{S(t)x_0\colon t \geq \tau\}$$
$$= \bigcap_{\tau\geq 0} \overline{\gamma(S(\tau)x_0)}.$$

Theorem 4.3.3. Suppose $x_0 \in C$ and $\{S(t)x_0, t \geq 0\}$ lies in a compact set in C; then $\omega(x_0)$ is nonempty, compact, invariant, and connected, and $\text{dist}(S(t)x_0, \omega(x_0)) \to 0$ as $t \to +\infty$.

Proof. By the exercise above, $\omega(x_0)$ is the intersection of a decreasing collection of nonempty compact sets, so $\omega(x_0)$ is compact and nonempty.

To prove invariance, note that if $y_0 \in \omega(x_0)$ then there exist $t_n \to +\infty$ such that $S(t_n)x_0 \to y_0$, hence for any $t \geq 0$, $S(t+t_n)x_0 \to S(t)y_0$: $S(t)(\omega(x_0)) \subset \omega(x_0)$, so $\omega(x_0)$ is positively invariant. Now by compactness, there exists a subsequence $t_{n_1} \to +\infty$ such that $\lim_{n\to\infty} S(t_{n_1}-1)x_0 \equiv y_1$ exists. Taking further subsequences, and then the diagonal subsequence in the usual way (cf. [38]) we find $t_{n'} \to +\infty$ such that for $j = 0,1,2,3,...$

$$S(t_{n'}-j)x_0 \to y_j \quad \text{as} \quad n' \to \infty.$$

Define $y(t) = S(t+j)y_j$ if $j > -t$, $j \geq 0$, $-\infty < t < \infty$; this is consistent because

$$S(t+k)y_k = S(t+j)y_j \quad \text{when} \quad k \geq j \geq -t.$$

This curve satisfies the requirements of definition 4.3.1.

If $\omega(x_0)$ is not connected, there exist nonempty closed disjoint sets A,B such that $\omega(x_0) = A \cup B$. But then A,B are compact and $\text{dist}(A,B) = 3\delta > 0$. There exist $t_n, t_n' \to +\infty$, $t_n < t_n' < t_{n+1}$, such that $\text{dist}(S(t_n)x_0,A) < \delta$, $\text{dist}(S(t_n')x_0,B) < \delta$ so there exists t_n'' in $t_n < t_n'' < t_n'$ with $\text{dist}(S(t_n'')x_0,A\cup B) > \delta$. But $\{S(t_n'')x_0\}$ lies in a compact set, and any accumulation point y of this set lies in $\omega(x_0)$ but also has $\text{dist}(y, A\cup B) \geq \delta > 0$, a contradiction. Similarily, by compactness, there cannot exist $t_n \to +\infty$ with $S(t_n)x_0$ bounded away from $\omega(x_0)$, so $S(t)x_0 \to \omega(x_0)$ as $t \to +\infty$.

<u>Theorem 4.3.4</u>. Let V by a Liapunov function on C (so $\dot V(x) \leq 0$) and define $E = \{x \in C: \dot V(x) = 0\}$, M = maximal invariant subset of E. If $\{S(t)x_0, t \geq 0\}$ lies in a compact set in C, then $S(t)x_0 \to M$ as $t \to +\infty$.

<u>Proof</u>. By hypothesis, $V(S(t)x_0)$ is nonincreasing for $t \geq 0$ and is bounded below so $\ell = \lim_{t\to+\infty} V(S(t)x_0)$ exists. If $y \in \omega(x_0)$, then $V(y) = \ell$, so also $V(S(t)y) = \ell$, $t \geq 0$, and so $\dot V(y) = 0$. Thus $\omega(x_0) \subset E$, so $\omega(x_0) \subset M$ and the result is proved.

<u>Remark</u>. For dynamical systems defined by $dx/dt + Ax = f(x)$ with A sectorial and having compact resolvent, bounded orbits are generally precompact (see Th. 3.3.6), and boundedness of orbits frequently follows

from the existence of a Liapunov function such that $\{x \in C: V(x) < k\}$ is a bounded set for appropriate $k > 0$.

Theorem 4.3.5. Assume x_0 is an equilibrium point in C, N is a neighborhood (in C) of x_0 and U is an open set in C with x_0 in the closure of U. Assume (i) V is a Liapunov function on $\overline{G}$, $G = N \cap U$; (ii) the only possible invariant set in $\overline{G} \cap \{x: \dot{V}(x) = 0\}$ is $\{x_0\}$; (iii) $V(x_0) = \eta$, $V(x) < \eta$ for $x \in G \setminus \{x_0\}$; (iv) $V(x) = \eta$ on $N \cap \partial G$.

If N_0 is any bounded neighborhood of x_0 properly contained in N, if $x_1 \in G \cap N_0 \setminus \{x_0\}$, then either $\overline{\gamma(x_1)}$, the closure of the orbit, is a noncompact subset of $\overline{G \cap N_0}$, or $S(t)x_1 \in \partial N_0$ for some $t > 0$.

Proof. If $\gamma(x_1) \subset N_0$ then $\gamma(x_1) \subset G \cap N_0$; for $V(S(t)x_1) \le V(x_1) < \eta$ for all $t \ge 0$ so $V(S(t)x_1)$ cannot reach $\partial G \cap N$, by (iv). If $\gamma(x_1)$ is in a compact subset of $G \cap N_0$, then $\omega(x_1)$ is a nonempty invariant set in $\overline{G \cap N_0} \cap \{\dot{V} = 0\}$ so $\omega(x_1) = \{x_0\}$. But $V(\omega(x_1)) \le V(x_1) < V(x_0)$, a contradiction.

Example. $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + u - au^3$ for $0 < x < \pi$, $t > 0$

$$u(0,t) = 0, \quad u(\pi,t) = 0$$

where a is a nonzero constant. We prove that if $a > 0$, the origin is globally asymptotically stable in $H_0^1(0,\pi)$, and if $a < 0$, the origin is unstable in $H_0^1(0,\pi)$. (Note that, for $a = 0$, the origin is stable but not asymptotically stable: $u(t,x) = c \sin x$ is a solution for any constant c.)

Taking $A = -d^2/dx^2$ in $L^2(0,\pi)$, it is easily seen that the initial-value problem is well-posed in $D(A^{1/2}) = H_0^1(0,\pi)$. For $\phi \in H_0^1(0,\pi)$, define

$$V(\phi) = \int_0^\pi \{(\phi'(x))^2 - \phi^2(x) + a/2\ \phi^4(x)\}dx,$$

a continuous polynomial functional on $H_0^1(0,\pi)$. Using the (local) dynamical system defined by this differential equation on bounded sets in $H_0^1(0,\pi)$,

$$\dot{V}(\phi) = -2\int_0^\pi \{\phi''(x) + \phi(x) - a\phi^3(x)\}^2 dx$$

for $\phi \in D(A)$, so V is a Liapunov function.

If $a > 0$ then $V(\phi) \geq \frac{a}{2} \int_0^\pi \phi^4 dx$ and for any solution $u(x,t)$, $\frac{a}{2} \int_0^\pi u^4(x,t)dx \leq V(u(\cdot,t)) \leq V(u(\cdot,))$, so $\int_0^\pi u^4 dx$ and $\int_0^\pi u^2 dx$ are bounded, so also $||u(\cdot,t)||_{H_0^1}$ is bounded, and we have a dynamical system on $H_0^1(0,\pi)$ with all orbits bounded. But A has compact resolvent (as an operator in $L^2(0,\pi)$) and it follows by Th. 3.3.6 that every orbit is precompact. Thus $u(\cdot,t) \to \omega(u(\cdot,0)) \subset \{\phi \mid \dot{V}(\phi) = 0\}$ in $H_0^1(0,\pi)$ as $t \to +\infty$. But $\dot{V}(\phi) = 0$ implies

$$\phi'' + \phi - a\phi^3 = 0 \quad \text{on} \quad (0,\pi),\ \phi(0) = 0,\ \phi(\pi) = 0,$$

so (multiplying by ϕ and integrating)

$$a \int_0^\pi \phi^4 dx = \int_0^\pi (\phi''+\phi)\phi\ dx = \int_0^\pi \{-(\phi')^2+\phi^2\}dx \leq 0$$

which implies $\phi = 0$. Thus $u(\cdot,t) \to 0$ in $H_0^1(0,\pi)$ as $t \to +\infty$, provided $a > 0$.

Now suppose $a < 0$, and consider again the functional V but restrict attention to the set

$$G = \{\phi \in H_0^1(0,\pi) : V(\phi) < 0\}$$

Observe that G is a nonempty open set in $H_0^1(0,\pi)$, since V is continuous and $\phi \in G$ whenever $\phi(x) = c \sin x$, $c \neq 0$.

<u>Exercise 2</u>. If $a < 0$, $\phi'' + \phi - a\phi^3 = 0$ on $(0,\pi)$, $\phi(0) = 0$, $\phi(\pi) = 0$, and $\phi \not\equiv 0$, then

$$\frac{\pi}{2n} = \int_0^c \frac{d\phi}{\sqrt{c^2 - \frac{a}{2}c^4 - \phi^2 + \frac{a}{2}\phi^4}} \quad \text{for some } c > 0$$

and some integer $n \geq 1$. This follows from $(\phi')^2 + \phi^2 - \frac{a}{2}\phi^4 =$ constant for $0 \leq x \leq \pi$.

Changing variables, the requirement is $\frac{\pi}{2n} = J(c)$ $J(c) = \int_0^{\pi/2} [1 - \frac{a}{2} c^2(1+\sin^2\theta)]^{-\frac{1}{2}} d\theta$. But $J(0+) = \pi/2$, $\pi/2 > J(c) > \pi/4$ if $0 < c \leq \sqrt{3/|a|}$. Thus the only solution ϕ having $\max |\phi(x)| \leq \sqrt{3/|a|}$ is $\phi \equiv 0$.

Example, continued. Choosing N as a small neighborhood of the origin, it follows that $\dot{V}(\phi) = 0$, $\phi \in N$, only if $\phi = 0$. Then Th. 4.3.5 applies (with $\eta = 0$) and if $\phi \in N$, $V(\phi) < 0$, then the solution $u(\cdot,t;\phi)$ with initial-value ϕ eventually reaches ∂N. In particular, the origin is unstable. Observe that Th. 5.1.1 and Th. 5.1.3 below do not apply, since the linearization has a zero eigenvalue. This example is also discussed using the critical manifold in section 6.3.

Exercise 3. If $a < 0$, examine $V_1(\phi) = -\int_0^\pi \phi(x) \sin x\, dx$ in the cone $C = \{\phi \in H_0^1(0,\pi) \mid \phi(x) \geq 0$ on $0 \leq x \leq 1\}$. Observe that $\dot{V}_1(\phi) = a \int_0^\pi \phi^3(x) \sin x\, dx \leq 0$ for $\phi \in C$, but Th. 4.3.5 does not apply directly, since C has no interior in $H_0^1(0,\pi)$.

Show C is positively invariant; then apply Th. 4.3.5 to the dynamical system on C. (With the induced topology of $H_0^1(0,\pi)$, C is a complete metric space.) Prove that, if $u_0 \in C\setminus\{0\}$, then $u(t;u_0)$ eventually leaves every bounded set in C, and in fact $V_1(u(t;u_0)) \to \infty$ in finite time. (Alternatively, use an argument like that of the final example in sec. 3.1.)

Exercise 4. (Kastenberg, [55]). A simple model of feedback control of a nuclear reactor leads to $\partial\phi/\partial t = \partial^2\phi/\partial x^2 + \lambda\phi - \rho\phi^2$ $(0 < x < \pi)$, $\phi(0,t) = 0$, $\phi(\pi,t) = 0$, where λ, ρ are positive constants. Here ϕ is the neutron flux, which must be nonnegative; but a maximum principle argument shows $C = \{\phi \in H_0^1(0,\pi): \phi(x) \geq 0$ on $0 \leq x \leq \pi\}$ is a positively invariant set (see 3.3, ex. 9). Using the Liapunov function $V(\phi) = \int_0^\pi\{(\phi')^2 - \lambda\phi^2 + 2\rho/3\ \phi^3\}dx$ for $\phi \in C$, $V(\phi) \geq (1-\lambda)\|\phi\|^2 + \frac{2\rho}{3\sqrt{\pi}}\|\phi\|^3$ where $\|\phi\| = (\int_0^\pi \phi^2 dx)^{1/2}$. It follows that we have a dynamical system on C.

If $\phi \in C$, $\dot{V}(\phi) = 0$, then either (i) $0 < \lambda \leq 1$ and $\phi = 0$, or (ii) $\lambda > 1$ and $\phi = 0$ or ϕ is the unique solution ϕ^+ of $\phi'' + \lambda\phi - \rho\phi^2 = 0$ on $(0,\pi)$, $\phi(0) = 0$, $\phi(\pi) = 0$, $\phi(x) > 0$ on $0 < x < \pi$. It follows that when $\lambda \leq 1$ $\|\phi(\cdot,t)\|_{H_0^1} \to 0$ as $t \to +\infty$, and if $\lambda > 1$, $\|\phi(\cdot,t)-\phi^+\|_{H_0^1} \to 0$ as $t \to +\infty$ unless $\phi(x,0) \equiv 0$. (For this last result, examine also $\frac{d}{dt}\int_0^\pi\phi(x,t)\sin x\, dx$.)

We end this section with some simple results on asymptotically autonomous equations, following Sell [89].

Theorem 4.3.6. Suppose A is sectorial in X and has compact resolvent, and suppose U is an open set in X^α, $\alpha < 1$, and $f: \mathbb{R}^+ \times U \to X$ is locally Lipschitzian and $f(\mathbb{R}^+ \times B)$ is bounded in X for any closed bounded $B \subset U$. Finally assume $\|f(t,x)-g(x)\| \to 0$ as $t \to \infty$, uniformly in a neighborhood of each $x \in U$, and $g(\cdot)$ is locally Lipschitzian in U.

Then if $dx/dt + Ax = f(t,x)$ for $t \geq t_0 \geq 0$ has a solution $x(\cdot)$ which is in a closed bounded set $B \subset U$ on $t_0 \leq t < \infty$, then

$$\operatorname{dist}_{X^\alpha}\{x(t),M\} \to 0 \quad \text{as} \quad t \to \infty,$$

where M is the maximal invariant subset of B for $dy/dt + Ay = g(y)$.

Proof. First note $\{x(t)\}_{t \geq t_0}$ is in a compact set $K \subset B$ (Th. 3.3.6), so $\sup_{x \in K} \|f(t,x)-g(x)\| \to 0$ as $t \to \infty$. Suppose y_0 is any limit point of the solution so $y_0 = \lim_{n\to\infty} x(t_n)$ for some $t_n \to +\infty$. If $y(t)$ is a solution of $dy/dt + Ay = g(y)$, $t > 0$, and $y(0) = y_0$, then $y(t)$ exists for all $t > 0$. For otherwise there exists $0 < T < \infty$ so that the solution exists on $[0,T]$ and $y(T) \notin B$. But, by continuous dependence (Th. 3.4.1), $\sup_{0 \leq t \leq T} \|y(t)-x(t+t_n)\|_\alpha \to 0$ as $n \to \infty$, a contradiction.

Now (cf. proof of Th. 4.3.3) there exists a subsequence $\{t_{n'}\}$ of $\{t_n\}$ such that $\lim_{n'\to\infty} x(t_{n'}-k) = y_k$ exists for $k = 0,1,2,\ldots$, and it follows that the solution $y(\cdot)$ can be extended on $-\infty < t < \infty$ with $y(-k) = y_k$, $k = 0,1,\ldots$. This proves $y_0 \in M$, for every limit point y_0 of the trajectory, hence $x(t) \to M$ as $t \to \infty$.

Theorem 4.3.7. Suppose A, f, g as above but assume also $f(t,0) \equiv 0$ and $g(0) = 0$, and assume that $y = 0$ is uniformly asymptotically stable for the limiting equation $dy/dt + Ay = g(y)$. Then $x = 0$ is uniformly asymptotically stable for $dx/dt + Ax = f(t,x)$, $t > t_0 \geq 0$.

Proof. Let $T(x_0) = y(1;x_0)$, the "time one" map for $dy/dt + Ay = g(y)$, $y(0) = x_0$. Similarly let $T_{t_0}(x_0) = x(t_0+1;t_0,x_0)$ for $dx/dt + Ax = f(t,x)$. Observe that $\|T_{t_0}(x)-T(x)\|_\alpha \to 0$ uniformly in a neighborhood of 0 in X^{α}, as $t_0 \to +\infty$.

Choose any $\varepsilon > 0$, sufficiently small that on $B_{\varepsilon} = \{x \in X^{\alpha} \mid \|x\|_{\alpha} < \varepsilon\}$, we have $\|T_{t_0}(x) - T(x)\|_{\alpha} \to 0$ uniformly; by exercises 1 and 3 of sec. 4.3, there is a neighborhood U of the origin, contained in B_{ε}, and such that U is positively invariant under any T_{t_0} with $t_0 \geq C = C(\varepsilon)$. We may also assume, by continuous dependence, that $x_0 \in U$ implies $\|x(t;t_0,x_0)\|_{\alpha} < \varepsilon$ on $0 \leq t - t_0 \leq 1$, and also (by asymptotic stability of the limiting equation) that $\|x_0\|_{\alpha} \leq \varepsilon$ implies $\|T^n(x_0)\|_{\alpha} \to 0$ as $n \to \infty$.

Now if $\|x_0\|_{\alpha} \leq \delta = \delta(\varepsilon)$, some $\delta > 0$, then $x(t;t_0,x_0) \in U$ on $0 \leq t_0 \leq t \leq C(\varepsilon)$, so $\|x(t;t_0,x_0)\|_{\alpha} \leq \varepsilon$ for all $t \geq t_0$. Therefore by the previous theorem

$$\|x(t;t_0,x_0)\|_{\alpha} \to 0 \quad \text{as} \quad t \to +\infty,$$

since $M = \{0\}$ is the only invariant set in B_{ε}.

<u>Remark</u>. Exercise 5, sec. 5.2, involves an asymptotically autonomous system, but the limiting equations do not have an asymptotically stable equilibrium.

Chapter 5

Neighborhood of an Equilibrium Point

5.1 Stability and instability by the linear approximation [79, 59]

Let A be a sectorial linear operator in a Banach space X, and let $f: U \to X$ where U is a cylindrical neighborhood in $\mathbb{R} \times X^\alpha$ (for some $\alpha < 1$) of $(\tau,\infty) \times \{x_0\}$. We say x_0 is an *equilibrium point* if $x(t) \equiv x_0$ is a solution of

$$\frac{dx}{dt} + Ax = f(t,x), \quad t > t_0,$$

i.e. if $x_0 \in D(A)$ and $Ax_0 = f(t,x_0)$ for all $t > t_0$.

A solution $\bar{x}(\cdot)$ on $[t_0,\infty)$ is *stable* (in X^α) if, for any $\varepsilon > 0$, there exists $\delta > 0$ such that any solution x with $\|x(t_0)-\bar{x}(t_0)\|_\alpha < \delta$ exists on $[t_0,\infty)$ and satisfies $\|x(t)-\bar{x}(t)\|_\alpha < \varepsilon$ for all $t \geq t_0$; that is, if $x_0 \mapsto x(t;t_0,x_0)$ is continuous (in X^α) at $x_0 = \bar{x}(t_0)$, uniformly in $t \geq t_0$. The solution $\bar{x}$ is *uniformly stable* if $x_1 \mapsto x(t;t_1,x_1)$ is continuous as $x_1 \to \bar{x}(t_1)$, uniformly in $t \geq t_1$ and $t_1 \geq t_0$.

The solution $\bar{x}(\cdot)$ is uniformly *asymptotically* stable if it is uniformly stable and $x(t;t_1,x_1) - \bar{x}(t) \to 0$ as $t-t_1 \to +\infty$, uniformly in $t_1 \geq t_0$ and $\|x_1-\bar{x}(t_1)\|_\alpha < \delta$, for some constant $\delta > 0$.

Simple examples of these and other related stability notions are in [11, 37, 62, 68, 89].

<u>Theorem 5.1.1</u>. Let A,f be as above and let x_0 be an equilibrium point. Suppose

$$f(t,x_0+z) = f(t,x_0) + Bz + g(t,z)$$

where B is a bounded linear map from X^α to X and $\|g(t,z)\| = o(\|z\|_\alpha)$ as $\|z\|_\alpha \to 0$, uniformly in $t > \tau$, and $f(t,x)$ is locally Hölder continuous in t, locally Lipschitzian in x, on U.

If the spectrum of $A-B$ lies in $\{\mathrm{Re}\,\lambda > \beta\}$ for some $\beta > 0$, or equivalently if the linearization

$$\frac{dz}{dt} + Az = Bz$$

is uniformly asymptotically stable, then the original equation has the solution x_0 uniformly asymptotically stable in X^α. More precisely, there exist $\rho > 0$, $M \geq 1$ such that if $t_0 > \tau$ and $\|x_1 - x_0\|_\alpha \leq \rho/2M$ then there exists a unique solution of

$$dx/dt + Ax = f(t,x), \quad t > t_0, \quad x(t_0) = x_1$$

existing on $t_0 \leq t < \infty$ and satisfying for $t \geq t_0$

$$\|x(t;t_0,x_1) - x_0\|_\alpha \leq 2Me^{-\beta(t-t_0)}\|x_1 - x_0\|_\alpha.$$

<u>Proof</u>. Under the stated hypotheses, $L = A-B$ is sectorial and if $0 < \beta < \beta' < \operatorname{Re}\sigma(L)$, there exists $M \geq 1$ such that for $t > 0$, $z \in X^\alpha$,

$$\|e^{-Lt}z\|_\alpha \leq Me^{-\beta' t}\|z\|_\alpha$$

$$\|e^{-Lt}z\|_\alpha \leq Mt^{-\alpha}e^{-\beta' t}\|z\|.$$

Choose $\sigma > 0$ so small that

$$M\sigma \int_0^\infty s^{-\alpha}e^{-(\beta'-\beta)s}\,ds < 1/2,$$

and choose $\rho > 0$ so small that

$$\|g(t,z)\| \leq \sigma\|z\|_\alpha \quad \text{for} \quad \|z\|_\alpha \leq \rho, \quad t > \tau.$$

Let $z(t) = x(t;t_0,x_1) - x_0$; if $\|x_1 - x_0\|_\alpha \leq \rho/2M$, the solution will exist and have $\|z(t)\|_\alpha \leq \rho$ on some time interval. As long as $\|z(t)\|_\alpha$ remains less than ρ, we have

$$\begin{aligned}
\|z(t)\|_\alpha &= \|e^{-L(t-t_0)}z(t_0) + \int_{t_0}^t e^{-L(t-s)}g(s,z(s))ds\|_\alpha \\
&\leq Me^{-\beta(t-t_0)}\|z(t_0)\|_\alpha + \sigma M\int_{t_0}^t (t-s)^{-\alpha}e^{-\beta'(t-s)}\|z(s)\|_\alpha \\
&\leq \rho/2 + \rho\sigma M\int_{-\infty}^t (t-s)^{-\alpha}e^{-\beta'(t-s)}ds < \rho.
\end{aligned}$$

If $\|z(t)\|_\alpha < \rho$ on $t_0 \leq t < t_1$ with t_1 chosen as large as possible, then either $t_1 = +\infty$ or $\|z(t_1)\|_\alpha = \rho$. But the second case contradicts this calculation, so the solution exists and has $\|z(t)\|_\alpha < \rho$ for all $t \geq t_0$.

If $u(t) = \sup\{\|z(s)\|_\alpha e^{\beta(s-t_0)}, t_0 \le s \le t\}$ then

$$\|z(t)\|_\alpha e^{\beta(t-t_0)} \le M\|z(t_0)\|_\alpha + M\sigma\int_{t_0}^{t}(t-s)^{-\alpha}e^{-(\beta'-\beta)(t-s)}ds\cdot u(t)$$
$$\le M\|z(t_0)\|_\alpha + \frac{1}{2}u(t),$$

and so $u(t) \le 2M\|z(t_0)\|_\alpha$, as claimed.

Exercise 1. Suppose $\rho: X^\alpha \to \mathbb{R}^+, f: X^\alpha \to X$ are continuously differentiable near the origin, A is sectorial, $f(0) = 0$, $\rho(0) = 1$, and $\mathrm{Re}\ \sigma(A-Df(0)) > 0$. Prove the zero solution of $dx/dt + \rho(x)Ax = f(x)$ is uniformly asymptotically stable in X^α.

(Hint: first change the time variable as in Ex. 2, Sec. 3.3.)

Exercise 2*. Suppose $\{T_n\}_{n=1}^\infty$ is a family of nonlinear operators from a Banach space X to itself with $T_n(x) = Lx + N_n(x)$, L a continuous linear operator and $\|N_n(x)\| = o(\|x\|)$ as $x \to 0$ uniformly in $n \ge 1$. Assume the spectral radius $r(L) < 1$. Prove there exists $\rho > 0$, $M > 0$, $\theta < 1$, such that if $\|x_0\| < \rho/M$ and $x_n = T_n(x_{n-1})$ for $n = 1,2,\ldots$, then $\|x_n\| \le M\theta^n\|x_0\|$.

(Hint: if $r(L) < \nu < 1$, there exists an equivalent norm on X, $\|x\|^* = \sum_{n=0}^\infty \nu^{-n}\|L^n x\|$, in which L has norm $\le \nu$, $\|Lx\|^* \le \nu\|x\|^*$.) If we define $T_n(x_0) = x(t_0+n;t_0+n-1,x_0)$ on X^α, this gives another proof of Th. 5.1.1.

Theorem 5.1.2 (Asymptotic behavior). Let A,f be as in Th. 5.1.1, but assume $\|g(t,z)\| = O(\|z\|_\alpha^{1+\delta})$ as $\|z\|_\alpha \to 0$, uniformly in $t \ge \tau$, for some $0 < \delta$.

Assume also that with $L = A-B$, $\sigma(L) \subset \{\beta\} \cup \{\mathrm{Re}\ \lambda > \beta'\}$, where β is a simple positive eigenvalue of L and $\beta' > \beta > 0$. Then for any γ in $\beta < \gamma < \min(\beta', \beta(1+\delta))$, there exists $\rho > 0$ and M so that $\|x_1 - x_0\|_\alpha \le \rho/2M$ implies

$$x(t;t_0,x_1) = x_0 + K(x_1,t_0)e^{-\beta(t-t_0)} + \varepsilon(t,t_0)$$

where $\|\varepsilon(t,t_0)\|_\alpha \le C\|x_1-x_0\|_\alpha e^{-\gamma(t-t_0)}$. Here $K(x_0,t_0) = 0$ and $K(\cdot,t_0)$ is a continuous map from a neighborhood of x_0 in X^α into the one-dimensional space $N(L-\beta I)$, and if E_1 is the associated projection onto $N(L-\beta I)$,

$$K(x_1,t_0) = E_1(x_1-x_0) + O(\|x_1-x_0\|_\alpha^{1+\delta}) \quad \text{as} \quad x_1 \to x_0.$$

Proof. Let $X = X_1 \oplus X_2$, $X_1 = N(L-\beta I)$, $X_2 = R(L-\beta I)$; then $L_2 = L|X_2$ satisfies for $t > 0$

$$\|e^{-L_2 t} E_2 x\|_\alpha \le Me^{-\beta' t}\|x\|_\alpha, \quad Mt^{-\alpha}e^{-\beta' t}\|x\| .$$

Also, by Th. 5.1.1, if $0 < \beta\theta < \beta$, there exists $\rho > 0$ such that $\|z(t;t_0,z_0)\|_\alpha \le 2M\|z_0\|_\alpha e^{-\beta\theta(t-t_0)}$, provided $\|z_0\|_\alpha \le \rho/2M$. We may assume $\beta\theta(1+\delta) > \gamma$. Writing $z(t;t_0,z_0) = z(t) = z_1(t) + z_2(t) \in X_1 \oplus X_2$, it follows $z_2(t) = e^{-L_2(t-t_0)}E_2 z_0 + \int_{t_0}^t e^{-L_2(t-s)}E_2 g(s,z(s))ds$ so $\|z_2(t)\|_\alpha e^{\gamma(t-t_0)} \le M\|E_2 z_0\|_\alpha + C\|z_0\|_\alpha^{1+\delta} \le C_1\|z_0\|_\alpha$ for some constants C, C_1.

Also $\int_{t_0}^\infty e^{\beta(s-t_0)}E_1 g(s,z(s))ds$ converges, so

$$K_1(z_0,t_0) \equiv \lim_{t\to\infty} z_1(t)e^{\beta(t-t_0)} = E_1 z_0 + \lim_{t\to\infty}\int_{t_0}^t e^{\beta(s-t_0)}E_1 g(s,z(s))ds$$

is well defined. Finally, $E_2 K_1(z_0,t_0) = 0$ and

$$\|K_1(z_0,t_0) - E_1 z_0\|_\alpha \le C_2\|z_0\|_\alpha^{1+\delta}\int_{t_0}^\infty e^{\beta(s-t_0)}e^{-\beta\theta(1+\delta)(s-t_0)}$$

$$= O(\|z_0\|_\alpha^{1+\delta}) \quad \text{as} \quad \|z_0\|_\alpha \to 0.$$

Remark. Another approach to study of asymptotic behavior is outlined in ex. 7 and 8 below. This approach allows us to use the detailed results available for finite dimensional ODE's.

Example. (In the theory of combustion: see [34]). If Ω is a bounded smooth domain in $\mathbb{R}^3$, and n,T are the (nondimensional) concentration and temperature of a substance consumed in a first-order exothermic reaction, we might expect

$$\partial n/\partial t = D\Delta n - \varepsilon n f(T) \quad \text{in} \quad \Omega, \quad \frac{\partial n}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega,$$

$$\partial T/\partial t = \Delta T + qnf(T) \quad \text{in} \quad \Omega, \quad T = 1 \quad \text{on} \quad \partial\Omega$$

where D,q,ε are positive constants, ε small, and $f(T) = \exp(-H/T)$,

H is a positive constant. It is not difficult to show that, if $T \geq 0$, $n \geq 0$ at the initial time, then these remain true for all later time (cf. sec. 4.1, example 2). Further $(n,T) \to (0,1)$ in $H^1(\Omega,\mathbb{R}^2)$ as $t \to \infty$ (for $d/dt \int_\Omega \frac{1}{2} n^2 dx \leq 0$ and it follows easily that $nf(T)$ is bounded in L^2 so (n,T) remains bounded in $H^2(\Omega)$.)

But the linearization about $(0,1)$ has $\lambda_0 = \varepsilon f(1)$ as a simple eigenvalue with eigenfunction (n_0,T_0), $n_0(x) = 1$, $T_0 = -(\Delta_D + \lambda_0)^{-1} qf(1)$, and all other eigenvalues in $\text{Re}\,\lambda \geq c > 0$ for some c independent of (small) $\varepsilon > 0$. Applying Th. 5.1.2, we see as $t \to +\infty$, $(n,T) = (0,1) + \bar{n}(1,T_0(x))e^{-\lambda_0 t} + O(e^{-2\lambda_0 t})$ for some constant $\bar{n} \geq 0$. Information on the behavior of this system when εt is not necessarily large can be found in sec. 6.1.

Theorem 5.1.3. (Instability). Assume A is sectorial and $f(t,x)$ is locally Lipschitz in x, Hölder continuous in t on a cylindrical neighborhood of $\mathbb{R} \times \{x_0\}$ in $\mathbb{R} \times X^\alpha$. Assume also $Ax_0 = f(t,x_0)$ for $t \geq t_0$,

$$f(t,x_0+z) = f(t,x_0) + Bz + g(t,z),\ g(t,0) = 0,$$

$$\|g(t,z_1)-g(t,z_2)\| \leq k(\rho)\|z_1-z_2\|_\alpha \quad \text{for} \quad \|z_1\|_\alpha \leq \rho,\ \|z_2\|_\alpha \leq \rho$$

and $k(\rho) \to 0$ as $\rho \to 0+$.

If $L = A-B$, assume $\sigma(L) \cap \{\text{Re}\,\lambda < 0\}$ is a nonempty spectral set. Then the equilibrium solution x_0 is unstable. Specifically, there exist $\varepsilon_0 > 0$ and $\{x_n, n \geq 1\}$ with $\|x_n - x_0\|_\alpha \to 0$ as $n \to \infty$, but for all n,

$$\sup_{t \geq t_0} \|x(t;t_0,x_n)-x_0\|_\alpha \geq \varepsilon_0 > 0.$$

Here the supremum is taken over the maximal interval of existence of $x(\cdot;t_0,x_n)$.

Remark. The corollary to Th. 5.1.5 below proves instability with weaker assumptions on $\sigma(A-B)$.

Proof. Let $\sigma_1 = \sigma(L) \cap \{\text{Re}\,\lambda < 0\}$, $\sigma_2 = \sigma(L)\setminus\sigma_1$, and let $X = X_1 \oplus X_2$ be the corresponding decomposition into L-invariant subspaces, and $(j = 1,2)$

$$\sigma(L_j) = \sigma_j \quad \text{where} \quad L_j = \text{restriction to } L \text{ to } X_j,$$

For some $\beta > 0$, $M \geq 1$ we have the following estimates: for $t > 0$,

$$\|e^{-L_2 t} E_2 x\|_\alpha \leq Me^{-\beta t}\|x\|_\alpha, \quad Mt^{-\alpha}e^{-\beta t}\|x\| \;;$$

and for $t \leq 0$,

$$\|e^{-L_1 t} E_1 x\|_\alpha \leq Me^{3\beta t}\|x\|_\alpha, \quad Me^{3\beta t}\|x\| \;.$$

Now for small $a \in X_1^\alpha$, consider the integral equation

$$y(t) = e^{-L_1(t-\tau)}a + \int_\tau^t e^{-L_1(t-s)}E_1 g(s,y(s))ds + \int_{-\infty}^t e^{-L_2(t-s)}E_2 g(s,y(s))ds, \quad \text{for } t \leq \tau.$$

Choose $\rho > 0$ so small that

$$Mk(\rho)(\|E_1\|\beta^{-1} + \|E_2\|\int_0^\infty u^{-\alpha}e^{-\beta u}du) \leq \frac{1}{4M} < \frac{1}{2},$$

Then if $a \in X_1^\alpha$, $\|a\|_\alpha \leq \rho/2M$, we shall prove the integral equation has a unique solution $y(t)$ on $-\infty < t \leq \tau$ with $\|y(t)\|_\alpha \leq \rho e^{2\beta(t-\tau)}$. In fact, denoting this solution $y(t) = y^*(t;\tau,a)$, we show

$$\|y^*(t;\tau,a)\|_\alpha \leq 2M\|a\|_\alpha e^{2\beta(t-\tau)}$$

and $\|y^*(\tau;\tau,a)\|_\alpha \geq \frac{1}{2}\|a\|_\alpha$. Also $y^*(\cdot;\tau,a)$ is a solution of

$$\frac{dz}{dt} + Lz = g(t,z) \quad \text{for } t < \tau.$$

The conclusion follows from this, for if $z_n = y^*(t_0;t_0+n,a)$, then the solution $z(t;t_0,z_n)$ has $z(t;t_0,z_n) = y^*(t;t_0+n,a)$ for $t_0 \leq t \leq t_0+n$, $\sup_{t\geq t_0}\|z(t;t_0,z_n)\|_\alpha \geq \|z(t_0+n;t_0,z_n)\|_\alpha \geq \frac{1}{2}\|a\|_\alpha > 0$, while $\|z_n\|_\alpha \leq \rho e^{-2\beta n} \to 0$ as $n \to \infty$.

Now the integral equation above defines a strict contraction on the space of all continuous $y\colon (-\infty,\tau] \to X^\alpha$ with $E_1 y(\tau) = a$ and $\|y(t)\|_\alpha \leq \rho e^{2\beta(t-\tau)}$ for $t \leq \tau$. If $y(t)$ solves the integral equation, set $\gamma(s) = g(s,y(s))$; we show $y(\cdot)$ solves the differential equation. Choose any $t_0 \leq \tau$; then for $t_0 \leq t \leq \tau$

$$E_2y(t) \equiv \int_{-\infty}^{t} e^{-L_2(t-s)} E_2\gamma(s)ds$$
$$= e^{-L_2(t-t_0)}\left(\int_{\infty}^{t_0} e^{-L_2(t_0-s)} E_2\gamma(s)ds\right) + \int_{t_0}^{t} e^{-L_2(t-s)} E_2\gamma(s)ds,$$

so by lemma 3.3.2, the solution $y(t) = E_2y(t) + E_1y(t)$ of the integral equation also solves the differential equation $dx/dt + Ly = \gamma(t)$ on $t_0 < t < \tau$. Finally, $\|y^*(t;\tau,a)\|_\alpha \le 2M\|a\|_\alpha e^{2\beta(t-\tau)}$ for $t \le \tau$, so

$$\|y^*(\tau;\tau,a)-a\|_\alpha = \left\|\int_{-\infty}^{\tau} e^{-L_2(\tau-s)} E_2 g(s,y^*(s))ds\right\|_\alpha$$
$$\le \int_{-\infty}^{\tau} M(\tau-s)^{-\alpha}e^{\beta(\tau-s)} \|E_2\| \, k(\rho) \cdot 2M\|a\|_\alpha e^{2\beta(s-\tau)}ds$$
$$\le \frac{1}{2}\|a\|_\alpha, \text{ by choice of } \rho.$$

Example. $u_t = u_{xx} + au - bu^3$ $(0 < x < \pi,\ t > 0)$ with $u = 0$ at $x = 0,\pi$; the zero solution is asymptotically stable if $a < 1$, unstable if $a > 1$. If $a = 1$, there is a zero eigenvalue and the zero solution may be stable or unstable: see sec. 4.3, example.

Exercise 3. Under the hypotheses of Th. 5.1.3, prove the following stronger result: there exist $\{x_n,\ n \ge 1\}$ with $\|x_n-x_0\|_\alpha \to 0$ but for each $n \ge 1$,

$$\sup_{t\ge t_0} \|x(t;t_0,x_n)-x_0\| \ge \varepsilon_0 > 0.$$

(Notice that the X norm is used, not the X^α norm.)

Lemma 5.1.4. Suppose X is a real Banach space, M is a continuous linear operator on X with spectral radius $r > 0$. Given any $\delta > 0$ and $N_0 \ge 0$, there exists an integer $N \ge N_0$ and $u \in X$, $\|u\| = 1$, such that

$$\|M^n u\| \le (\sqrt{2} + \delta)r^n \quad \text{for} \quad 0 \le n \le N$$

and $\|M^N u\| \ge (1-\delta)r^N$.

Proof. There exists $\lambda = re^{i\theta}$ in $\sigma(M)$; choose integer $N \ge N_0$ so $\cos N\theta - |\sin N\theta| \ge 1-\delta/2$. Since λ is an approximate eigenvalue, we may choose u,v in X, $\|u\| = 1 \ge \|v\|$, so

$$\|\mathrm{Re}\ \{M^n(u+iv) - \lambda^n(u+iv)\}\| < \frac{\delta}{2} r^n$$

for $0 \le n \le N$. It follows that

$$\|M^n u\| \le \frac{\delta}{2} r^n + r^n\|\cos n\theta u - \sin n\theta v\| \le (\sqrt{2} + \delta) r^n \quad \text{for } 0 \le n \le N$$

and

$$\|M^N u\| \ge r^N(\cos N\theta - |\sin N\theta|) - \frac{\delta}{2} r^N \ge (1-\delta) r^N.$$

Theorem 5.1.5. If X is a real Banach space and T_n is a continuous map from a neighborhood of the origin of X into X with $T_n(0) = 0$ $(n = 1,2,3,\ldots)$ and M is a continuous linear operator on X with spectral radius greater than one, and

$$T_n(x) = Mx + O(\|x\|^p) \quad \text{as} \quad x \to 0$$

uniformly in $n \ge 1$, for some constant $p > 1$, then the origin is unstable. Specifically, there exists a constant $c > 0$ and there exist x_0 arbitrarily close to 0 such that, if $x_n = T_n(x_{n-1})$ for $n \ge 1$, then for some N (depending on x_0), the sequence $x_0, x_1, \ldots, x_N$ is well-defined and $\|x_N\| \ge C$.

Before proving this we note the

Corollary 5.1.6. Assume A,B,f,X as in Theorem 5.1.3 except that the requirements on $g(t,z)$, $\sigma(A-B)$ are replaced by: for a constant $p > 1$

$$\|f(t,x_0+z) - f(t,x_0) - Bz\| = O(\|z\|_\alpha^p)$$

as $z \to 0$ in X^α, uniformly in $t \ge t_0$, and $\sigma(A-B) \cap \{\mathrm{Re}\ \lambda < 0\}$ is nonempty. Then the equilibrium x_0 is unstable.

The corollary is proved by examining the maps

$$T_n\colon z \to x(t_0+n; x_0+z, t_0+n-1) - x_0$$

for $n = 1,2,\ldots$. This corollary is a valuable extension for problems in unbounded domains when the linearization has continuous spectrum: see the example below.

Proof of Thm. 5.1.5. If r = spectral radius of M, then $r > 1$ and we may choose $\eta > 0$ so $r^p > r+\eta$. Also there exists $K < \infty$ so that

$\|M^n\| \le K(r+\eta)^n$ for all $n \ge 0$; and there exist, $a > 0$, $b > 0$ so the $T_n(x)$ are defined for $\|x\| \le a$ and satisfy

$$\|T_n(x) - Mx\| \le b\,\|x\|^p \quad \text{for } \|x\| \le a.$$

Choose δ in $0 < \delta < 1/2$ and $\sigma > 0$ with

$$\sigma \le a/R, \quad \frac{bkR^p}{r^p - r - \eta}\,\sigma^{p-1} \le \frac{1}{2}$$

where $R = 2(\sqrt{2} + \delta)$. We show there exist x_0 arbitrarily small so $x_n = T_n(x_{n-1})$ $(n \ge 1)$ are in $\|x_n\| \le a$ for $1 \le n \le N$ while $\|x_N\| \ge (\frac{1}{2} - \delta)\sigma$ which proves instability of the origin.

Choose any $N_0 \ge 0$; choose $\|u\| = 1$ and $N \ge N_0$ according to the lemma; we take $x_0 = \varepsilon u$, $\varepsilon = \sigma/r^N$. Since $r > 1$, we see $\|x_0\| = \varepsilon$ can be arbitrarily small. An easy induction proves

$$x_n = M^n x_0 + \sum_{k=0}^{n-1} M^{n-k-1}(T_{k+1}(x_k) - Mx_k)$$

for $n \ge 0$ (as long as these are defined). Certainly $\|x_k\| \le \varepsilon R r^k$ holds for $k = 0$, and if it holds for $0 \le k < n \le N$ then

$$\|x_n\| \le (\sqrt{2} + \delta)r^n\varepsilon + \sum_{k=0}^{n-1} K(r+\eta)^{n-k-1} b(\varepsilon R r^k)^p$$

and the sum is bounded by

$$bK(\varepsilon R)^p r^{p(n-1)} \sum_0^{n-1} \left(\frac{r+\eta}{r^p}\right)^{n-k-1}$$
$$\le \frac{bK(\varepsilon R)^p r^{np}}{r^p - r - \eta} \le \frac{1}{2}\,\varepsilon r^n \le \frac{R\varepsilon r^n}{2}$$

so $\|x_n\| \le \varepsilon R r^n \le R\sigma \le a$ for all $n \le N$. Also

$$\|x_N\| \ge (1-\delta)r^N\varepsilon - \frac{1}{2}\,\varepsilon r^N = (\tfrac{1}{2} - \delta)\sigma$$

and the result is proved.

Example. Suppose $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is C^p in a neighborhood of 0 $(1 < p \le 2)$ and $f(0) = 0$. Let $u = \mathrm{col}(u_1, \ldots, u_n)$ and consider

$$u_t = \Delta u + f(u), \quad x \in \mathbb{R}^m, \quad t > 0.$$

This has the solution $u = 0$ and defines a C^p local dynamical system near 0 in $W^{1,q}(\mathbb{R}^m,\mathbb{R}^n)$, if $q > m$. The linearization about 0 has spectrum consisting of all λ such that

$$\det\{(\lambda-s)I + f'(0)\} = 0$$

for some $s \geq 0$, so the spectrum is

$$\{-\alpha+s \mid s \geq 0, \alpha \text{ an eigenvalue of } f'(0)\}.$$

Thus if all eigenvalues of the matrix $f'(0)$ have negative real part, the origin is asymptotically stable. And if some eigenvalue of $f'(0)$ has positive real part, the origin is unstable.

If $n = 1$ (or $f'(0)$ has special form) the instability result may be proved by the maximum principle (ex. 8 below); but in general, there seems to be no alternative to Thm. 5.1.5 and its corollary.

<u>Exercise 4</u>. Suppose $L_n \in \mathscr{L}(X)$, $\|T_n(x)-L_nx\| = O(\|x\|^p)$ as $x \to 0$ uniformly in $n \geq 0$ for some $p > 1$, and with $L_{n,m} = L_n \cdot L_{n-1} \cdot \ldots \cdot L_{m+1}$ ($L_{m,m} = I$) we have

$$\overline{\lim_{n\to\infty}} \|L_{n,0}\|^{1/n} = r_0 \quad \text{and} \quad \|L_{n,m}\| \leq Kr_1^{n-m} \quad (n \geq m \geq 0)$$

and $1 < r_0 \leq r_1 < r_0^p$. Prove the zero solution of $x_n = T_n(x_{n-1})$, $n \geq 1$, is unstable.

(<u>Hint</u>: choose $r < r_0$ so $r_1 < r^p$.)

<u>Exercise 5*</u>. Suppose X is a real Banach space and T is a C^1 map on a neighborhood of the origin in X into X, with $T(0) = 0$ and $T'(0) = L$, and there exists $\theta > 1$ such that $\sigma(L) \cap \{|\mu| > \theta\}$ is a nonempty spectral set. Then there exists a nontrivial sequence $\{x_n: n = 0,-1,-2,-3,\ldots\}$ such that $x_{n+1} = T(x_n)$ for all $n \leq 0$ and $\|x_n\| \to 0$ as $n \to -\infty$. Thus $x = 0$ is unstable under iteration of T. (Note: if we suppose f is independent of t in Th. 5.1.3, and take $T(\xi) = x(1;0,\xi+x_0) - x_0$, for ξ near 0 in X^α, we have a generalization of this theorem where we assume merely that $\sigma(A-B) \cap \{\text{Re } \lambda < \gamma\}$ is a nonempty spectral set for some $\gamma \leq 0$, but Cor. 5.1.6 is even more general.)

<u>Hint</u>: Let $T(x) = Lx + N(x)$, $N(0) = 0$, $N'(0) = 0$, $X = X_1 \oplus X_2$

is the L-invariant decomposition with $L_j = L|_{X_j}$ $(j = 1,2)$ and suppose $\|L_2\| \le \theta_1 < \theta$, $\|L_1^{-1}\| \le \theta_2^{-1} < \theta^{-1}$ (see ex. 2 above). Modify $N(x)$ for $\|x\| \ge r$ so as to have small Lipschitz constant on the whole space. Look for solutions $x_n(a)$, $n \le 0$ of

$$x_n = L_1^n a - \sum_{k=n}^{-1} L_1^{n-1-k} E_1 N(x_k) + \sum_{-\infty}^{n-1} L_2^{n-1-k} E_2 N(x_k) \quad (n = 0,-1,-2,\dots)$$

with $\|x_n(a)\| \le \rho\theta^n$ for $n = 0,-1,-2,\dots$, $a \in X_1$, $\|a\| \le \rho/2$.

<u>Exercise 6*</u>. (Asymptotic stability of a family of equilibria). Assume A is sectorial, $f: X^\alpha \to X$ is C^1 and $f(0) = 0$, $f'(0) = 0$. Assume also that there is a C^2 curve $\hat{x}(\lambda) \in X^\alpha$ for λ near 0, $A\hat{x}(\lambda) = f(\hat{x}(\lambda))$, $\hat{x}(0) = 0$, $\frac{d\hat{x}}{d\lambda}(0) \ne 0$. Finally, assume $\sigma(A)$ contains 0 as a simple eigenvalue, while the remainder of the spectrum has real part greater than $\beta > 0$. Observe that $N(A) = \text{span}\,\{\frac{d\hat{x}}{d\lambda}(0)\}$, and let $X_1 = N(A)$, $X_2 = R(A)$.

Introduce new coordinates: $(y,\lambda) \in X_2 \times \mathbb{R}$

$$x = \hat{x}(\lambda) + y.$$

Let $A^*v = 0$, $\langle v,\hat{x}'(0)\rangle = 1$; then $dx/dt + Ax = f(x)$ becomes

$$d\lambda/dt = \phi(\lambda,y), \quad dy/dt + A_2 y = g(\lambda,y)$$

where $\phi(\lambda,y) = \langle v, f(\hat{x}(\lambda)+y) - f(\hat{x}(\lambda))\rangle / \langle v,\hat{x}'(\lambda)\rangle$

$$g(\lambda,y) = E_2\{f(\hat{x}(\lambda)+y) - f(\hat{x}(\lambda)) - \hat{x}'(\lambda)\phi(\lambda,y)\},$$

so ϕ, g are C^1 functions with

$$|\phi(\lambda,y)| + \|g(\lambda,y)\| \le \gamma(\rho)\,\|y\|_\alpha \quad \text{when} \quad |\lambda| + \|y\|_\alpha \le \rho,$$

and $\gamma(\rho) \to 0$ as $\rho \to 0+$.

Suppose $|\lambda(0)| + \|y(0)\|_\alpha$ is small; as long as $|\lambda(t)|$ remains less than some $\delta > 0$, we have $\|y(t)\|_\alpha \le Ke^{-\beta t}\|y(0)\|_\alpha$, and so $|\frac{d\lambda}{dt}(t)| = O(e^{-\beta t}\|y(0)\|_\alpha)$. Thus $|\lambda(t)| < \delta$ for all $t > 0$, and there exists λ_∞ so $|\lambda(t)-\lambda_\infty| + \|y(t)\|_\alpha = O(e^{-\beta t})$.

This result may easily be generalized to prove asymptotic stability of an n-dimensional family of equilibria $\{\hat{x}(\lambda)$, λ near 0

in $\mathbb{R}^n\}$ when $\hat{x}(0) = 0$, rank $\{\frac{\partial\hat{x}}{\partial\lambda}(0)\} = n$ and $\sigma(A) \subset \{0\} \cup \{\text{Re } z > \beta > 0\}$, with $\{0\}$ of multiplicity n. If the multiplicity of the eigenvalue 0 exceeds the dimension of λ, the problem is more complicated and is discussed briefly in section 6.2.

Exercise 7. Assume A is sectorial, $f\colon X^\alpha \times \mathbb{R} \to X$ $(\alpha < 1)$ and its derivative $f_x\colon X^\alpha \times \mathbb{R} \to \mathcal{L}(X^\alpha, X)$ are continuous near the origin, and $f(0,0) = 0$. Assume $\sigma(A - f_x(0,0)) \cap \{\text{Re } \lambda < 0\}$ is nonempty and prove there exists $\rho > 0$ such that if $\|x_0\|_\alpha + |\varepsilon| < \rho$ and x_0 is an equilibrium point of

$$dx/dt + Ax = f(x,\varepsilon),$$

then x_0 is unstable.

This result seems to answer the question raised in [59, p. 297] as to the instability of equilibrium solutions of the Taylor and Bénard problems which bifurcate at the higher eigenvalues of the linearized problem.

Exercise 8. Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is C^1, $f(0) = 0$, $f'(0) < 0$; then the zero solution of $u_t = \Delta u + f(u)$ (x in $\mathbb{R}^n$, $t > 0$) is asymptotically stable in $X = C_{unif}(\mathbb{R}^n)$. In fact, suppose $f(u) > 0$ on $(\alpha,0)$ and $f(u) < 0$ on $(0,\beta)$ where $-\infty \le \alpha < 0 < \beta \le \infty$. If $dw/dt = f(w)$, $\alpha < w(0) < \beta$, then $w(t) \to 0$ as $t \to +\infty$. If u is a solution of the PDE with $\inf u(\cdot,0) = w_-(0) > \alpha$ and $\sup u(\cdot,0) = w_+(0) < \beta$ and $w_\pm(t)$ are solutions of $\dot{w} = f(w)$, prove for $t \ge 0$ and all x,

$$w_-(t) \le u(x,t) \le w_+(t)$$

and $w_\pm(t) \to 0$ as $t \to +\infty$. Thus the region of attraction of $u = 0$ includes all $\phi \in X$ with

$$\inf \phi > \alpha \quad \text{and} \quad \sup \phi < \beta.$$

Hint: $w(x,t) = w(t)$ solves $w_t = \Delta w + f(w)$; use the maximum principle (ex. 8, sec. 3.3). Suppose $f'(0) > 0$ and prove instability of 0 in $C_{unif}(\mathbb{R})$.

Exercise 9. Suppose $f\colon \mathbb{R}^m \to \mathbb{R}^m$ is C^1, $f(0) = 0$, and define $a \le b$ (for a,b in $\mathbb{R}^m$) when $a_k \le b_k$ for $k = 1,\dots,m$. Suppose there exist $a \le 0 \le b$ in $\mathbb{R}^m$ such that

(i) $\frac{\partial f_i}{\partial u_j}(u) \geq 0$ when $i \neq j$, $a \leq u \leq b$

(ii) if $dw/dt = f(w)$, $w(0) = a$ or b, then for $t > 0$, $a \leq w(t) \leq b$ and $w(t) \to 0$ as $t \to +\infty$.

If $u_t = \Delta u + f(u)$ on $\mathbb{R}^n \times \mathbb{R}_+$ and

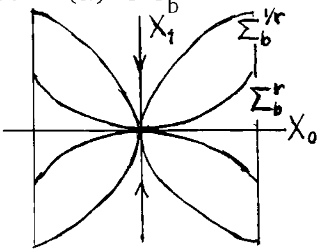
$$a \leq u(x,0) \leq b \quad \text{for all} \quad x$$

prove $u(x,t) \to 0$ as $t \to +\infty$ uniformly in x.

Exercise 10. Suppose X is a Banach space, $L \in \mathscr{L}(X)$ and the spectrum of L is disjoint from the circle of radius $a > 0$. Decompose the space $X = X_0 \oplus X_1$ and $L = L_0 \oplus L_1$ as usual, with $r_\sigma(L_1) < a$, $r_\sigma(L_0^{-1}) < a^{-1}$. We may choose norms so that the corresponding inequalities hold: $\|L_1\| < a$, $\|L_0^{-1}\| < a^{-1}$. Assume $r \geq 1$ is such that $\|L_1\| \, \|L_0^{-1}\|^s < 1$ for $1/r \leq s \leq r$, and suppose a map T is defined near 0 with

$$\|T(x) - Lx\| = o(\|x\|^r) \quad \text{as} \quad x \to 0.$$

(i) If $\Sigma_b^s = \{x = x_0 + x_1 \mid \|x_1\| \leq b\|x_0\|^s\}$ for some $b > 0$ and $\frac{1}{r} \leq s \leq r$, show there exists $\delta > 0$ so that $\|x\| \leq \delta$, $x \in \Sigma_b^s$ implies $T(x) \in \Sigma_b^s$



(ii) If $x^n = T^n(x) \to 0$ as $n \to \infty$, $x^n \neq 0$, and there exists $b > 0$ so $x^n \in \Sigma_b^{1/r}$ for arbitrarily large n, prove

$$\|x_1^n\| = o(\|x_0^n\|^r)$$

and

$$\underline{\lim}_{n\to\infty} \|x_0^n\|^{1/n} \geq \|L_0^{-1}\|^{-1}$$

(iii) If $x^n = T^n(x) \to 0$ as $n \to \infty$ and $\underline{\lim}_{n\to\infty} \|x^n\|^{1/n} < \|L_0^{-1}\|^{-1}$ then $\|x_0^n\| = o(\|x_1^n\|^r)$ as $n \to \infty$ and $\overline{\lim}_{n\to\infty} \|x^n\|^{1/n} \leq \|L_1\|$.

(Hint for (ii): if $x^m \in \Sigma_b^{1/r}$, $\|x^m\|$ small, then eventually $x^n \in \Sigma_1^{1/r}$, so eventually $x^n \in \Sigma_1^1$, so eventually $x^n \in \Sigma_\beta^r$ (for any $\beta > 0$).)

Exercise 11. Suppose A is sectorial in X, f is locally Lipschitz near the origin from X^α to X, $\|f(x)\| = o(\|x\|_\alpha^r)$ as $x \to 0$ for some $r \geq 1$, and suppose $\sigma(A)$ is disjoint from $\{\lambda \mid \beta \leq \operatorname{Re} \lambda \leq \gamma\}$, $\beta < \gamma$, and $\gamma - \beta s > 0$ for $1/r \leq s \leq r$.

(i) Show the time-one map in X^α for $\dot{x} + Ax = f(x)$ satisfies the requirements of ex. 10 above;

(ii) Conclude that a solution $x(t) \to 0$ as $t \to +\infty$ either satisfies

(a) $\|x(t)\|_\alpha = o(e^{-\beta t})$ as $t \to +\infty$ (or this holds for a sequence $t_n \to +\infty$), in which case $\|x(t)\|_\alpha = o(e^{-\gamma t})$ and

$$\|x_0(t)\|_\alpha = o(\|x_1(t)\|_\alpha^r) \quad \text{as} \quad t \to +\infty;$$

or else

(b) $\|x(t)\|_\alpha > e^{-\gamma t}$ for some arbitrarily large t, in which case $\|x_1(t)\|_\alpha = o(\|x_0(t)\|_\alpha^r)$ and $\varliminf_{t\to\infty} \|x(t)\|_\alpha^{1/t} \geq e^{-\beta}$.

Remark. If $\dim X_0 < \infty$ in the above -- true if A has compact resolvent -- solutions of type (b) may be analyzed in more detail, for $y = x_0(t)$ solves the finite dimensional ODE

$$dy/dt + A_0 y = E_0 f(x_1(t) + y)$$

(E_0 = projection from $X = X_0 \oplus X_1 \to X_0$) and $\|x_1(t)\|_\alpha = o(\|y(t)\|^r)$. For example, from (Coppel [18]), $\|x_0(t)\|^{1/t}$ tends to a finite limit $e^{-\mu}$ as $t \to +\infty$ where $0 \leq \mu < \beta$, $\mu = \operatorname{Re} \lambda$ for some $\lambda \in \sigma(A)$. Further, if $\mu > 0$, $r > 1$ and f is C^1 near 0 with $f'(x) = O(\|x\|_\alpha^{r-1})$, there exists a unique nontrivial solution $y_0(t)$ of $\dot{y}_0 + A_0 y_0 = 0$, $\|y_0(t)\|^{1/t} \to e^{-\mu}$, with $\|x(t) - y_0(t)\|_\alpha = o(\|y_0(t)\|_\alpha)$ as $t \to +\infty$. (See [18].)

Another remark. For many second-order parabolic equations, lower bounds of the form $\varliminf_{t\to+\infty} \|x(t)\|^{1/t} > 0$ hold for each nontrival solution: see [29].

5.2 The saddle-point property [37, 43]

Theorem 5.2.1. Suppose A, f, x_0 as in Th. 5.1.2, with

$$f(t,x_0+z) = Ax_0 + Bz + g(t,z),$$

$B \in \mathcal{L}(X^\alpha, X)$, $g(t,0) = 0$ and $\|g(t,z_1)-g(t,z_2)\| \leq k(\rho)\|z_1-z_2\|_\alpha$ for $\|z_1\|_\alpha \leq \rho$, $\|z_2\|_\alpha \leq \rho$, where $k(\rho) \to 0$ as $\rho \to 0+$; we may assume $k(\cdot)$ is nondecreasing.

Suppose $L = A-B$ and $\sigma(L)$ is disjoint from the imaginary axis, and decompose the space $X = X_1 \oplus X_2$ corresponding to the spectral sets $\sigma_1 = \sigma(L) \cap \{\mathrm{Re}\,\lambda < 0\}$ and $\sigma(L) \cap \{\mathrm{Re}\,\lambda > 0\}$, and let E_1, E_2 be the projections onto X_1, X_2. Then there exists $\rho > 0$, $M \geq 1$ such that the following hold:

(i) The stable manifold $S = S(t_0,\rho)$

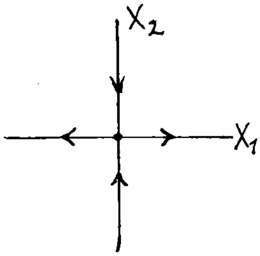
$$S = \{z_0 : \|E_2 z_0\|_\alpha \leq \rho/2M, \|z(t;t_0,z_0)\|_\alpha \leq \rho \quad \text{for} \quad t \geq t_0\}$$

is homeomorphic under $E_2\big|_S$ to the closed ball of radius $\rho/2M$ in X_2^α. Further S is tangent to X_2^α at the origin and when $z_0 \in S$,

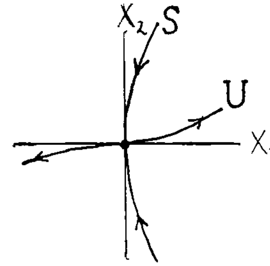
$$\|z(t;t_0,z_0)\|_\alpha \to 0 \quad \text{as} \quad t \to +\infty.$$

(ii) The unstable manifold $U = U(t_0,\rho)$, $U = \{z_0 : \|E_1 z_0\|_\alpha \leq \rho/2M$, $z(t;t_0,z_0)$ is a solution on $(-\infty,t_0)$, $\|z(t;t_0,z_0)\|_\alpha \leq \rho$ for $t \leq t_0\}$ is homeomorphic under $E_1\big|_U$ to the closed ball of radius $\rho/2M$ in X_1. Further U is tangent to X_1 at the origin and when $z_0 \in U$, $z(t;t_0,z_0) \to 0$ as $t \to -\infty$.

(iii) If $\|E_{1,2} z_0\|_\alpha \leq \rho/2M$ and $\|z(t;t_0,z_0)\|_\alpha \leq \rho$ for all $t \geq t_0$ or all $t \leq t_0$, then $z_0 \in S \cup U$.



linearized equation



nonlinear equation

Proof. Assume without loss of generality $\operatorname{Re}\sigma(A) > 0$. Suppose $M > 0$, $\beta > 0$ such that

$$\|A^{\alpha}e^{-L_1 t}\| \leq Me^{\beta t},\ \|e^{-L_1 t}\| \leq Me^{\beta t} \quad \text{for} \quad t \leq 0,$$

$$\|A^{\alpha}e^{-L_2 t}E_2A^{-\alpha}\| \leq Me^{-\beta t},\ \|A^{\alpha}e^{-L_2 t}\| \leq Mt^{-\alpha}e^{-\beta t} \quad \text{for} \quad t > 0.$$

Assume $z_0 \in S$; then $z(t) = z_1(t) + z_2(t) \in X_1 \oplus X_2$

$$z_1(t) = e^{-L_1(t-t_0)}E_1z_0 + \int_{t_0}^{t} e^{-L_1(t-s)}E_1g(s,z(s))ds$$

so

$$e^{L_1(t-t_0)}z_1(t) = e^{L_1t_0}E_1z_0 + \int_{t_0}^{t} e^{L_1s}E_1g(s,z(s))ds \to 0 \text{ as } t \to +\infty.$$

Thus $E_1z_0 = -\int_{t_0}^{\infty} e^{L_1(s-t_0)}g(s,z(s))ds$, so for $t \geq t_0$

$$z(t) = e^{-L_2(t-t_0)}a + \int_{t_0}^{t} e^{-L_2(t-s)}E_2g(s,z(s))ds - \int_{t}^{\infty} e^{-L_1(t-s)}E_1g(s,z(s))ds,$$

where $a = E_2z(t_0)$.

Conversely suppose $a \in X_2$, $\|a\|_{\alpha} \leq \rho/2M$; we show (for $\rho > 0$ sufficiently small) there is a unique solution $z(t) = z(t;t_0,a)$ of the integral equation with $E_2z_0 = E_2z(t_0;t_0,a) = a$ and $\|z(t;t_0,a)\|_{\alpha} \leq \rho$ for all $t \geq t_0$.

Specifically, if $\rho > 0$ is chosen so small that

$$Mk(\rho)\{\|E_2\| \int_0^{\infty} u^{-\alpha}e^{-\beta u}du + \|E_1\| \int_0^{\infty} e^{-\beta u}du\} < 1/2$$

then the right side of this integral equation defines a contraction map of the space of continuous $z\colon [t_0,\infty) \to X$ with $\sup \|z(t)\|_{\alpha} \leq \rho$ and

and $E_2 z(t_0) = a$, provided $\|a\|_\alpha \le \rho/2M$, and therefore there exists a unique fixed point $z(t;t_0,a)$. This solution of the integral equation is a Lipschitz continuous function of $a \in X_2^\alpha$, $\|a\|_\alpha \le \rho/2M$, in the norm $\|\cdot\|_\alpha$. It may then be shown that $t \to z(t;t_0,a)$ is locally Hölder continuous, and so $z(\cdot;t_0,a)$ is the solution of $\frac{dz}{dt} + Lz = g(t,z)$, $t > t_0$, with initial value

$$h(a) \equiv z(t_0;t_0,a) = a - \int_{t_0}^{\infty} e^{-L_1(t_0-s)} E_1 g(s,z(s;t_0,a))ds.$$

Then $E_2 h(a) = a$, and $h(\cdot)$ is Lipschitz continuous, so

$$S = \{h(a) \mid a \in X_2^\alpha, \|a\|_\alpha \le \rho/2M\}$$

is the representation claimed above.

Also $\|h(a)-a\|_\alpha \le \int_{t_0}^{\infty} Me^{\beta(t_0-s)} \|E_1\| \, \|g(s,z(s;t_0,a))\|$ and $\sup_{s\ge t_0} \|z(s;t_0,a)\| = O(\|a\|_\alpha)$ as $\|a\|_\alpha \to 0$, $a \in X_2$, so $\|h(a)-a\|_\alpha = o(\|a\|_\alpha)$, which proves S is tangent to X_2^α at the origin. The proof that, when $z(t_0) \in S$, then $\|z(t;t_0,z_0)\|_\alpha \to 0$ exponentially as $t \to +\infty$ is fairly straightforward using the integral equation above.

The corresponding argument for the unstable manifold uses an integral equation like that in the proof of Th. 5.1.3 above, but we can work in the uniform norm rather than the exponentially-weighted norm used there.

Exercise 1. A singular initial value problem. Suppose $c(0) = 0$, $c(t) > 0$ for $t > 0$, and consider

$$c(t)\frac{dx}{dt} + Ax = f(x), \quad x(0+) = x_0$$

where A is sectorial, $f: X^\alpha \to X$ is C^1 near the origin and $f(0) = 0$. Change time variable to $s = \int_{t_0}^{t} \frac{dt}{c(t)}$, $t_0 > 0$, and consider the cases (i) $\int_{t_0}^{0+} \frac{dt}{c(t)}$ finite; (ii) $\int_{t_0}^{0+} \frac{dt}{c(t)}$ infinite, but $\sigma(A-f_x(0))$ is disjoint from the imaginary axis.

Compare with the problem in $X = \mathbb{R}^1$: $c(t)\frac{dx}{dt} + \lambda x = 0$, $t > 0$; $x(0+) = x_0$.

Theorem 5.2.2. Suppose A is sectorial in X, $\alpha < 1$, U is an open set in X^α, $f: U \to X$ is continuously differentiable, and x_0 is an unstable equilibrium point in U of

$$dx/dt + Ax = f(x).$$

Assume $\sigma(A-f'(0))$ does not intersect the imaginary axis, and let S be the stable manifold given by Th. 5.2.1. Then S is a C^1 manifold in X^α, and if $TS(x_1)$ is the tangent space to S at $x_1 \in S$, $\|x_1-x_0\|_\alpha$ small, then $TS(x_1)$ is a closed proper subspace in X^α and, in fact, is not dense in X.

Remark. This last fact has little claim on the intuition, but turns out to be useful in studying the global behavior of the stable manifolds for the Chafee-Infante problem (see 5.3 and 7.3).

Proof. Adopting notation from the proof of Th. 5.2.1, the stable manifold is

$$S = \{x_0+h(a) \mid a \in X_2^\alpha, \|a\|_\alpha \le \rho/2M\}$$

where $h(a) = z(0;a)$,

$$z(t;a) = e^{-L_2 t}a + \int_0^t e^{-L_2(t-s)}E_2 g(z(s;a))ds - \int_t^\infty e^{-L_1(t-s)}E_1 g(z(s;a))ds$$

for $t \ge 0$. We know $a \to z(t;a)$ is uniformly Lipschitz continuous near the origin in X_2^α, $t \ge 0$, and it is not difficult to show it is actually differentiable with $y(t) = (\partial z(t;a)/\partial a)b$ satisfying for any $b \in X_2$,

$$\begin{aligned} y(t) = e^{-L_2 t}b &+ \int_0^t e^{-L_2(t-s)}E_2 Dg(z(s;a))y(s)ds \\ &- \int_t^\infty e^{-L_1(t-s)}E_1 Dg(z(s;a))y(s)ds, \quad t \ge 0. \end{aligned}$$

Trivially, from the estimates of Th. 5.2.1, $\|y(t)\|_\alpha \le \text{Constant } \|b\|_\alpha$ for $t \ge 0$, $b \in X_2^\alpha$.

Let $|||y||| = \sup_{t>0}\{t^\alpha e^{\beta t/2}\|y(t)\|_\alpha\}$; then

$$|||y||| \le M\|b\| + Mk(\rho)\{\|E_2\|C_2 + \|E_1\| C_1\}\ |||y|||$$

where

$$C_1 = \sup_{t \ge 0}\int_0^t t^\alpha (t-s)^{-\alpha}e^{-\beta(t-s)/2}s^{-\alpha}ds$$

$$C_2 = \sup_{t\geq 0} \int_t^\infty e^{-3\beta(s-t)/2} t^\alpha s^{-\alpha} ds.$$

It follows that for $a \in X_2^\alpha$, $\|a\|_\alpha \leq \rho$ sufficiently small, $\|y(t)\|_\alpha \leq 2Mt^{-\alpha}e^{-\beta t/2}\|b\|$ for $t > 0$ and any $b \in X_2^\alpha$. Substituting this in the integral equation, we find $\|y(t)\| = \|\frac{\partial z}{\partial a}(t;a)b\| \leq \text{Const.} \|b\|$ for all $t \geq 0$, $b \in X_2^\alpha$. This implies that $Dh(a)$ extends to a bounded linear operator from X_2 to X, provided a is small enough, and proves the theorem.

<u>Exercise 2*</u>. (The center-stable manifold). Suppose A is sectorial, $f: X^\alpha \to X$ is continuously differentiable with $f(0) = 0$, $\sigma(A-f'(0)) \cap \{\mathrm{Re}\,\lambda > 0\}$ is a nonempty spectral set and $X = X_1 \oplus X_2$ is the corresponding decomposition. With $L = A-f'(0)$, $L_j = L|_{X_j}$ $(j = 1,2)$, L_1 is bounded and for some $\beta > 0$, $\|e^{-L_1 t}\| \leq Me^{3\beta t}$ for $t \leq 0$ while $\|e^{-L_2 t}\|_\alpha \leq Me^{\beta t}\|x\|_\alpha$, $Mt^{-\alpha}e^{\beta t}\|x\|$ for $t > 0$, $x \in X_2^\alpha$. There exists $r > 0$ and a local invariant manifold S, tangent to X_2^α at 0, such that any solution $x(\cdot)$ with $\|x(t)\|_\alpha \leq r$ for all $t \geq 0$ must have $x(0) \in S$ (hence, $x(t) \in S$ for $t \geq 0$). If f is C^1 and X has a C^1 norm, and if $x(0) \in S$ with $\|x(0)\|_\alpha$ small and $x(t) \to 0$ as $t \to \infty$, then S is differentiable at $x(0)$ and the tangent space at $x(0)$ is not dense in X. (It is sufficient that there exists $\phi: X \to [0,1]$ which is C^1, $\phi(x) = 1$ near $x = 0$, $\phi(x) = 0$ outside a bounded set.)

<u>Hint</u>: modify $g(x) = f(x)-f'(0)x$ outside $\|x\|_\alpha \leq r$ so it will have a small Lipschitz constant $\gamma(r)$ (or small derivative bound) on the whole space $(\gamma(r) \to 0$ as $r \to 0)$, and examine the integral equation

$$x(t) = e^{-L_2 t}a + \int_0^t e^{-L_2(t-s)}E_2 g(x(s))ds - \int_t^\infty e^{-L_1(t-s)}E_1 g(x(s))ds, \quad t \geq 0$$

for $a \in X_2^\alpha$, $\|a\|_\alpha \leq \rho/2M$, in the class of continuous $x: \mathbb{R}^+ \to X^\alpha$, $\|x(t)\|_\alpha \leq \rho e^{2\beta t}$ for $t \geq 0$.

<u>Exercise 3</u>. Under the assumptions of ex. 2 above, if $\sigma(A) \cap \{\mathrm{Re}\,\lambda < 0\}$ is a nonempty spectral set, Th. 5.1.3 tells us the origin is unstable in X^α. In fact, there exists $r > 0$ and an open dense set U in $\{\|x\|_\alpha < r\}$ such that, if $x(\cdot)$ is a solution with $\|x(0)\|_\alpha < r$ and $x(0) \in U$, then $x(t)$ leaves the r-ball at some finite $t > 0$.

Exercise 4. If $S \subset X$ is homeomorphic under projection to a set in Y, where Y is a proper closed subspace of X, then S has no interior in X.

Exercise 5. A model of D. C. Leigh [69] for swelling of channel flow of a viscoelastic fluid reduces eventually to the form

$$\text{(ODE)}: \qquad da/dt = f(a,b,t), \quad db/dt = g(a,b,t)$$

$$\text{(PDE)}: \qquad \begin{cases} a_1^2(t)\,\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial y^2} + h(a,b,u,v,y,t) & (0 < y < 1) \\ \qquad\qquad \dfrac{\partial u}{\partial y} = 0 \quad \text{at} \quad y = 0,\ y = 1 \\ \dfrac{\partial v}{\partial t} = k(a,b,u,v,y,t) & (0 < y < 1) \end{cases}$$

where $a(t) \in \mathbb{R}^2$, $b(t) \in \mathbb{R}^5$, $u(y,t) \in \mathbb{R}^1$, $v(y,t) \in \mathbb{R}^3$ and f, g, h, k are smooth functions of their arguments provided $a_1 > 0$. (Here a_1 is proportional to the width of the fluid stream, and several of the functions involved blow up if $a_1 \to 0+$.)

The ODE system for a,b is uncoupled from the rest of the system, and $f(a,b,t) = f_0(a,b) + f_1(a,b,t)$ where $f_1(a,b,t) = O(e^{-\beta t})$ as $t \to +\infty$, some $\beta > 0$, and similarly for g. The limiting system

$$da/dt = f_0(a,b), \quad db/dt = g_0(a,b),$$

has $f_0(a,b) = 0$ and $g_0(a,b) = 0$ if and only if $b = 0$. Suppose $a^* \in \mathbb{R}^2$ has $a_1^* > 0$, and assume for the 5×5 matrix $\frac{\partial g_0}{\partial b}(a^*,0)$ that all eigenvalues have real part negative. Prove that if a,b is a solution of (ODE) with $|a(T)-a^*| + |b(T)|$ sufficiently small for some sufficiently large T, then $a(t) = a_\infty + O(e^{-\delta t})$, $b(t) = O(e^{-\delta t})$ as $t \to +\infty$ for some $a_\infty \in \mathbb{R}^2$, $|a_\infty - a^*| < \Delta$, and some $\delta > 0$ depending on a^*.

The system (PDE) is also asymptotically autonomous and the limit system (with obvious notation) is

$$(a_1)_\infty^2\,\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial y^2} + h_0(a_\infty,0,u,v,y), \quad 0 < y < 1$$

$$\partial u/\partial y = 0 \quad \text{at} \quad 0,1$$

$$\frac{\partial v}{\partial t} = k_0(a_\infty,0,u,v,y), \quad 0 < y < 1.$$

This is a linear system in u, v, it has zero as a double eigenvalue (with two independent eigenvectors), and this two dimensional collection of equilibrium points is exponentially asymptotically stable (all other e.v. have real part $\geq \delta > 0$). The time dependent part of this system (PDE) is $O(e^{-\beta t})$ for some $\beta > 0$.

Prove that, for the full system (ODE) + (PDE), if the solution $(a,b,u,v)|_{t=T}$ is sufficiently close to such an equilibrium point of the limiting equation for some sufficiently large $T > 0$, the solution converges (exponentially) to an equilibrium point of the limiting system.

Exercise 6. Assume the hypotheses of Th. 5.2.1 and in addition that

$$\|f(t,x+h) - \sum_{k=0}^{m} \frac{1}{k!} D_x^k f(t,x)h^k\| \leq k(\|h\|_\alpha)\ \|h\|_\alpha^m$$

for all t, $\|x-x_0\|_\alpha \leq \rho_0$ and $\|h\|_\alpha \leq \rho_0$, where $k(\rho) \to 0$ as $\rho \to 0+$. Prove that $U(t_0,\rho)$ and $S(t_0,\rho)$ are C^m manifolds -- i.e. for $S(t_0,\rho)$, show the map h (which is the inverse of $E_2|S(t_0,\rho)$) is C^m. Hint: Use the converse Taylor theorem to show the uniform contraction map defining $z(t;t_0,a)$ is a C^m function of its arguments.

5.3 The Chafee-Infante problem and gradient flows

N. Chafee and E. Infante [14] in 1971 studied global stability questions for the nonlinear heat equation

$$\partial u/\partial t = \partial^2 u/\partial x^2 + \lambda f(u), \qquad (0 < x < \pi,\ t > 0)$$

$$u(0,t) = 0, \quad u(\pi,t) = 0$$

$$u(x,0) = u_0(x)$$

where λ is a nonnegative constant and f is a C^2 function satisfying

(i) $f(0) = 0$, $f'(0) = 1$

(ii) $\overline{\lim}_{|u|\to\infty} f(u)/u \leq 0$

(iii) $f''(u) < 0$ when $u > 0$, $f''(u) > 0$ when $u < 0$.

For example, we might consider $f(u) = u-au^3$, $a > 0$, as in the example following Th. 4.3.5.

Let $X = L^2(0,\pi)$, $A\phi(x) = -\phi''(x)$ for smooth ϕ which vanish at $x = 0,\pi$, and let A be extended to a positive definite, self-adjoint densely defined operator in X. Then $D(A) = H_0^1(0,\pi) \cap H^2(0,\pi)$, $D(A^{\frac{1}{2}}) = H_0^1(0,\pi)$, A has compact resolvent, and $\sigma(A)$ consists of simple eigenvalues $\{1,2^2,3^2,\ldots,n^2,\ldots\}$. Also if $\phi,\psi \in H_0^1(0,\pi) = X^{\frac{1}{2}}$, with $\|\phi\|_{\frac{1}{2}} \le \rho$, $\|\psi\|_{\frac{1}{2}} \le \rho$, then $\|f(\phi)-f(\psi)-f'(\phi)(\phi-\psi)\| \le k(\rho)\|\phi-\psi\|_{\frac{1}{2}}$ and $\|f(\phi)-f(\psi)\|_{\frac{1}{2}} \le L(\rho)\|\phi-\psi\|_{\frac{1}{2}}$ where k,L are continuous, $k(0) = 0$. The equation defines a local dynamical system in $X^{\frac{1}{2}} = H_0^1(0,\pi)$.

The linearization about the zero solution, which is an equilibrium point for any λ, is

$$\partial v/\partial t = \partial^2 v/\partial x^2 + \lambda v,\ 0 < x < \pi;\ v = 0 \text{ at } x = 0,\pi.$$

But $\sigma(A-\lambda I) = \{n^2-\lambda;\ n = 1,2,3,\ldots\}$, so by Th. 5.1.1 and 5.1.3,

$u = 0$ is asymptotically stable in $H_0^1(0,\pi)$ if $\lambda < 1$;

$u = 0$ is unstable in $H_0^1(0,\pi)$ if $\lambda > 1$.

Now consider the Liapunov function

$$V(\phi) = \int_0^\pi \{\tfrac{1}{2}(\phi'(x))^2 - \lambda F(\phi(x))\}dx, \qquad \phi \in H_0^1(0,\pi)$$

where $F(s) = \int_0^s f(t)dt$. If u is a solution of the problem for $t > 0$ then

$$\dot V(u(\cdot,t)) = \frac{d}{dt} V(u(\cdot,t)) = -\int_0^\pi u_t^2(x,t)dx$$

so it follows that $\dot V(\phi) \le 0$ for any $\phi \in H_0^1(0,\pi)$. Further, it follows from (ii) that for any $\varepsilon > 0$, there exist a constant C_ε such that $F(s) \le \varepsilon s^2 + C_\varepsilon$, $-\infty < s < \infty$. Choose $0 < \varepsilon \le 1/4\lambda$, and note $(A\phi,\phi) = \int_0^\pi (\phi')^2 dx \ge \int_0^\pi \phi^2 dx$ so

$$V(\phi) \ge (\tfrac{1}{2} - \varepsilon\lambda)\int_0^\pi (\phi'(x))^2 dx - \pi\lambda C_\varepsilon \ge \tfrac{1}{4}\|\phi\|_{\frac{1}{2}}^2 - \pi\lambda C_\varepsilon, \text{ for any } \phi \in H_0^1(0,\pi).$$

If $u(x,t)$ is a solution with $u(\cdot,t) \in H_0^1(0,\pi)$ on $0 \le t < t_1$, then

$$V(u(\cdot,t)) \le V(u(\cdot,0)) < \infty$$

so $\|u(\cdot,t)\|_{\frac{1}{2}}^2 \le 4(\pi\lambda C_\varepsilon + V(u(\cdot,0)))$ for $0 \le t < t_1$. It follows that the solution exists for all $t \ge 0$ and we have a dynamical system on $H_0^1(0,\pi)$.

Also, every orbit is bounded in $H_0^1(0,\pi)$, and (Th. 3.3.6) in fact precompact in $H_0^1(0,\pi)$. Thus for any solution u, $u(\cdot,t) \to \omega(u(\cdot,0))$ as $t \to \infty$, where $\omega(u(\cdot,0))$ is a nonempty compact connected, invariant set in $E = \{\phi: \dot{V}(\phi) = 0\}$. That is, the ω-limit set of an orbit is a connected subset of the collection of equilibrium points:

$$\dot{V}(\phi) = 0 \text{ only if } \phi''(x)+\lambda f(x\phi(x)) = 0,\ 0 < x < \pi, \text{ with } \phi(0) = 0,\ \phi(\pi) = 0.$$

Below we study these equilibrium points in detail and prove that E is a finite set of points, for each $\lambda \geq 0$. This implies $\omega(u(\cdot,0))$ is a single equilibrium point.

Exercise 1. Prove for $\lambda \leq 1$ that the origin is globally asymptotically stable in $H_0^1(0,\pi)$, i.e. that $\phi'' + \lambda f(\phi) = 0$ on $(0,\pi)$, $\phi = 0$ at $x = 0,\pi$, and $\lambda \leq 1$ implies $\phi \equiv 0$. (Hint: compare with the special case in the example following Th. 4.3.5.)

Exercise 2. (Euler's elastica). The problem of minimizing $V(\phi) = \int_0^\pi \{\frac{1}{2}(\phi'(x))^2 - \lambda(1-\cos\phi(x))$ over $\phi \in H_0^1(0,\pi)$ may be formulated as above. First note that a critical point of $V(\phi)$ must solve the Euler equation: $\phi'' + \lambda\sin\phi = 0$ on $(0,\pi)$, $\phi = 0$ at $x = 0,\pi$, and such a function necessarily has $|\phi(x)| \leq \pi-\varepsilon$ for all x, some $\varepsilon > 0$ (depending on λ).

Let $f(u) = \sin u$ for $|u| \leq \pi-\varepsilon$, and extend $f(u)$ smoothly for all u in such a way that it satisfies the conditions (i), (ii), (iii) above. Then critical points of V are equilibrium points of $u_t = u_{xx} + \lambda f(u)$, $u = 0$ at $x = 0,\pi$. Also, a critical point is a minimum or a saddle point, determined by the quadratic terms, when the equilibrium point is stable or unstable in the linear approximation. Anti ipating results below, it follows that for $0 \leq \lambda \leq 1$, the absolute minimum of V is attained at the origin, while for $\lambda > 1$, the absolute minimum is attained at $\pm\phi$ where $\phi'' + \lambda\sin\phi = 0$ on $(0,\pi)$, $\phi(0) = 0$, $\phi(\pi) = 0$, $\phi(x) > 0$ on $(0,\pi)$. Thus for $\lambda > 1$, this "buckled" state minimizes the potential energy of the elastica. (The solution $\phi(x,\lambda)$ is expressible in terms of the Jacobi elliptic functions.)

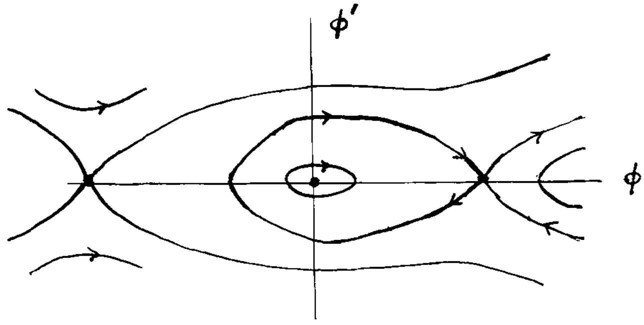
Exercise 3. For the general problem, suppose $\phi_0 \in H_0^1(0,\pi)$ minimizes the Liapunov function V; prove ϕ_0 is a stable equilibrium point for the original equation. (See also Auchmuty [5] for some more general results.)

Now we make a fairly detailed study of the equilibrium points and prove the following. If $n^2 < \lambda \le (n+1)^2$ ($n = 0,1,2,\ldots$) there are precisely $2n+1$ equilibrium points, designated $\phi_0 \equiv 0$ and $\phi_k^{\pm}$ $(k = 1,\ldots,n)$, where $\frac{d}{dx}\phi_k^+ > 0$ at $x = 0$, $\frac{d}{dx}\phi_k^- < 0$ at $x = 0$, and $\phi_k^{\pm}$ vanishes $(k-1)$ times in $0 < x < \pi$. Also, if $\lambda > 1$ then $\phi_1^{\pm}$ are asymptotically stable by the linear approximation, but ϕ_0 and $\phi_k^{\pm}$ $(2 \le k \le n)$ are all unstable by the linear approximation.

If ϕ solves $\phi'' + \lambda f(\phi) = 0$ on $(0,\pi)$, $\phi(0) = \phi(\pi) = 0$, then

$$\frac{1}{2}(\phi'(x))^2 + \lambda F(\phi(x)) = \text{Constant}, \quad 0 \le x \le \pi.$$

Now $sF'(s) = sf(s) > 0$ for small nonzero s, and we let (a_-,a_+) be the maximal interval about 0 where $sF'(s) > 0$ for $s \neq 0$, $-\infty \le a_- < 0 < a_+ \le +\infty$. We have the following phase portrait



in the case when $a_\pm$ are finite and $F(a_-) > F(a_+)$, with obvious modifications for the other cases.

If $\frac{1}{2}(\phi')^2 + \lambda F(\phi) = \lambda E$ with $0 < E < \min\{F(a_-),F(a_+)\}$, there is a solution ϕ with $\phi'(0) = \pm\sqrt{2\lambda E}$ if and only if, for some integer $n \ge 0$,

$$\pi = n(\ell_+(E) + \ell_-(E)) + \{0 \text{ or } \ell_+(E) \text{ or } \ell_-(E)\}$$

where $\ell_+(E)[\ell_-(E)]$ is the x-distance required by the solution ϕ to traverse one "loop" in $\phi > 0$ $[\phi < 0]$. More precisely

$$\ell_+(E) = (2\lambda)^{-\frac{1}{2}} \int_0^{m_+(E)} \{E-F(\xi)\}^{-\frac{1}{2}} d\xi$$

where $0 < m_+(E) < a_+$, $F(m_+(E)) = E$, with a similar expression for $\ell_-(E)$. If $F(a_+) \le E < F(a_-)$, the only possibility is that $\ell_-(E) = \pi$, and that there is a solution with one arch in $\phi < 0$, $\phi'(0) = -\sqrt{2\lambda E}$. If $E \geq \max(F(a_+), F(a_-))$, there are no solutions.

Now observe that $\ell_\pm(E) \to \pi/\sqrt{\lambda}$ as $E \to 0+$, and $\ell_\pm(E) \to +\infty$ as $E \to F(a_\pm)$; we can show that $\ell_+(E)$, $\ell_-(E)$ are strictly increasing functions of E. From this, the claims about the number and disposition of equilibrium solutions follow; for more details, see [14].

<u>Exercise 4</u>. $\sqrt{2\lambda}\ \ell_+(E) = 2\int_0^{\pi/2} \frac{(F(\xi))^{\frac{1}{2}}}{F'(\xi)}\, d\theta$ where $F(\xi) = E \sin^2\theta$, so $\sqrt{2\lambda}\ \frac{d\ell_+(E)}{dE} = 2E^{\frac{1}{2}} \int_0^{\pi/2} \frac{d}{d\xi}\{\frac{F(\xi)}{(F'(\xi))^2}\}d\theta$. But $\frac{d}{d\xi}\{(F'(\xi))^2 - 2F(\xi)F''(\xi)\} = -2F(\xi)f''(\xi) > 0$ for $\xi > 0$ so $\frac{d}{d\xi}\{\frac{F(\xi)}{(F'(\xi))^2}\} = \{(F'(\xi))^2 - 2F(\xi)F''(\xi)\}/(F'(\xi))^3 > 0$ for $\xi > 0$, thus $\ell_+(E)$ is strictly increasing.

We investigate now the stability of these equilibria. Observe that if ϕ is a nontrivial solution of $\phi'' + \lambda f(\phi) = 0$, $\phi(0) = \phi(\pi) = 0$, and

$$\psi''(x) + \lambda f'(\phi(x))\psi(x) = 0, \quad 0 < x < \pi$$

$$\psi(0) = 0, \quad \psi'(0) = 1$$

then ϕ is asymptotically stable if $\psi(x) > 0$ on $0 < x \leq \pi$, and is unstable if $\psi(x) < 0$ somewhere in $0 < x < \pi$. This is proved by a standard comparison theorem (see exercise 5) applied to ψ and the first eigen function $\theta(x)$ (which is positive):

$$\theta'' + (\mu + \lambda f'(\phi(x)))\theta = 0 \quad \text{on} \quad 0 < x < \pi,$$

$$\theta(0) = 0, \quad \theta(\pi) = 0, \ \theta'(0) = 1,$$

and μ is chosen as small as possible. If $\mu < 0$, then $\psi(x) < \theta(x)$ on $(0,\pi]$, as long as ψ is positive, and if $\mu > 0$, $\psi(x) > \theta(x)$ on $(0,\pi]$.

Now consider, for example, $\phi(x) = \phi_1^+(x,\lambda)$ when $\lambda > 1$. We have $\phi'(0) > 0$, $\phi(x) > 0$ and $f(\phi(x)) > 0$ on $0 < x < \pi$, so if

$$\chi(x) = -(\lambda\phi'(0))^{-1}\phi''(x) = (\phi'(0))^{-1}f(\phi(x))$$

then $\chi(x) > 0$ on $0 < x < \pi$, $\chi(0) = 0$, $\chi(\pi) = 0$, $\chi'(0) = 1$ and on $(0,\pi)$,

$$\chi'' + \lambda f'(\phi(x))\chi = f''(\phi(x))(\phi'(x))^2/\phi'(0) < 0.$$

By the comparison theorem, $\psi(x) > \chi(x) \geq 0$ on $0 < x \leq \pi$, which proves asymptotic stability. Similarly ϕ_1^- is asymptotically stable.

Now suppose ϕ is a nontrivial equilibrium with $\phi'(0) > 0$ and ϕ vanishes somewhere in $(0,\pi)$. Then ϕ has a negative minimum at some point x_1 in $(0,\pi)$, so $\phi(x_1) < 0$, $\phi'(x_1) = 0$. But $\psi(x)$ and $\phi'(x)$ both solve the equation $\psi'' + \lambda f'(\phi)\psi = 0$, so their Wronskian is constant:

$$\psi'(x)\phi'(x) - \psi(x)\phi''(x) = \text{Constant} = \phi'(0) > 0.$$

Thus at $x = x_1$, $-\psi(x_1)\phi''(x_1) = \phi'(0)$, so $\psi(x_1) < 0$ and instability of ϕ is proved.

Exercise 5. Suppose ϕ,ψ are C^2 functions with $\phi(0) = \psi(0) = 0$, $\phi'(0) = \psi'(0) = 1$ and $\phi'' + a(x)\phi > \psi'' + a(x)\psi$ on $0 < x < x_1$, and $\psi(x) > 0$ on $0 < x < x_1$; then prove $\phi(x) > \psi(x)$ on $0 < x \leq x_1$. (Hint: examine $\frac{d}{dx}(\psi\phi' - \phi\psi')$, and then $\frac{d}{dx}(\phi/\psi)$).

It is plausible from these results of Chafee and Infante that, if $\lambda > 1$, most solutions will tend to $\phi_1^{\pm}$ as $t \to +\infty$; we shall prove this is the case for an open dense set of initial conditions in $H_0^1(0,\pi)$, by proving the region of attraction of the unstable equilibria is nowhere dense.

We first discuss a generalization, which we term gradient flows. Suppose Ω is a bounded domain in $\mathbb{R}^n$ with boundary of class C^2 and

(i) $a_{ij}(x) = a_{ji}(x)$ is Hölder continuous on $\bar{\Omega}$ $(1 \leq i,j \leq n)$ and $\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq a|\xi|^2$ for $x \in \Omega$, $\xi \in \mathbb{R}^n$, for some constant $a > 0$;

(ii) $f: \Omega \times \mathbb{R} \to \mathbb{R}$ is continuous and locally Lipschitz continuous in its second argument;

(iii) for any $\varepsilon > 0$ there exists a constant C_ε such that

$$uf(x,u) \leq \varepsilon u^2 + C_\varepsilon \quad \text{on} \quad \Omega \times \mathbb{R};$$

(iv) if $n > 1$, there are positive constants b,c,q so

$$|f(x,u)| \leq b + c|u|^q \quad \text{on} \quad \Omega \times \mathbb{R}.$$

Consider the initial-value problem for

$$u_t = \sum_{i,j=1}^{n} (a_{ij}(x)u_{x_i})_{x_j} + f(x,u) \quad \text{on} \quad \Omega \times \mathbb{R}^+$$

$$u = 0 \quad \text{on} \quad \partial\Omega \times \mathbb{R}^+.$$

If $p \geq 2$, $n < p < \infty$, it is easy to show this initial-value problem is locally well-posed in the space $X = W_0^{1,p}(\Omega)$. We first show the solution exists for all $t > 0$. Choose $k \geq 2$, $k \geq pq$; on the domain of existence if $v = |u|^{k/2}$

$$\frac{d}{dt}\int_\Omega v^2 dx = -\frac{4(k-1)}{k}\int_\Omega \Sigma a_{ij} v_{x_i} v_{x_j} dx + k\int uf(x,u)|u|^{k-2}dx$$
$$\leq -\frac{4(k-1)}{k} a \int_\Omega |\nabla v|^2 + k\varepsilon \int_\Omega v^2 + kC_\varepsilon \int_\Omega |v^2|^{1-2/k}$$

Choosing $\varepsilon > 0$ small enough and using Hölder's inequality, $\psi(t) = \int_\Omega v^2 dx = \int_\Omega |u|^k dx$ satisfies (for some positive constants α, β)

$$\frac{d\psi}{dt} \leq -\alpha\psi + \beta\psi^{1-2/k}$$

so $\int_\Omega |u|^{pq} dx$ is uniformly bounded on the domain of existence, so $\|f(\cdot,u(\cdot,t))\|_{L_p(\Omega)}$ is uniformly bounded and the solution must exist for all $t > 0$. (In the case $n = 1$, we obtain uniform bounds in H_0^1 from the Liapunov function V, as in the Chafee-Infante problem.)

Let $F(x,u) = \int_0^u f(x,s)ds$; and define

$$V(u) = \int_\Omega \left(\frac{1}{2}\sum_{i,j=1}^{n} a_{ij} u_{x_i} u_{x_j} - F(x,u)\right)dx.$$

For any solution u of the initial-value problem

$$\dot{V}(u) = -\int_\Omega u_t^2 dx \leq 0$$

so V is a Liapunov function on $W_0^{1,p}(\Omega) = X$. Every solution has $\|f(\cdot,u(\cdot,t))\|_{L_p(\Omega)}$ uniformly bounded so the orbit $\{u(\cdot,t),\ t \geq 0\}$ is in a compact set in X and consequently must tend to a nonempty connected compact invariant subset of E,

$$E = \{\phi \in X \mid \dot{V}(\phi) = 0\}$$
$$= \{\phi \in W_0^{1,p} \cap W^{2,p}(\Omega) \mid \Sigma(a_{ij}\phi_{x_i})_{x_j} + f(\cdot,\phi) = 0\}.$$

If E is a discrete set -- as in the Chafee-Infante problem -- then each solution tends to an equilibrium point as $t \to +\infty$.

Exercise 6. Suppose, instead of requiring $u = 0$ on the boundary, we require $\partial u/\partial\nu + h(x)u = 0$ on $\partial\Omega$ where ν is the conormal direction ($\nu_i = \sum_{j=1}^{n} a_{ij}N_j$, N = unit outward normal) and $h(x)$ is Hölder continuous and $h(x) \geq 0$ on $\partial\Omega$. If $h(x) \equiv 0$, require in (iii) above that the inequality holds for some $\varepsilon < 0$. Prove the initial value problem is well-posed in $W^{1,p}(\Omega)$, and every solution exists for all $t > 0$ and tends to the set of equilibria as $t \to +\infty$. (Hint: there is an extra term in the Liapunov function.)

Suppose E consists entirely of hyperbolic equilibrium points, i.e. the linearization

$$\psi \to \Sigma(a_{ij}\psi_{x_i})_{x_j} + \frac{\partial f}{\partial u}(\cdot,\phi)\psi$$

with the boundary conditions, has no eigenvalues on the imaginary axis (0 is not an eigenvalue) for each ϕ in E. Here we assume f and $\partial f/\partial u$ are continuous. Now E is compact and, by the implicit function theorem, E is discrete hence finite and we have the disjoint union

$$X = \bigcup_{\phi\in E} W^s(\phi)$$

where $W^s(\phi) = \{u_0 \in X \mid \text{solution } u(t;u_0) \to \phi \text{ as } t \to +\infty\}$. We will prove (Th. 6.1.9, 6.1.10) that each stable manifold $W^s(\phi)$ is a C^1 imbedded submanifold of X and if ϕ is unstable, $W^s(\phi)$ has codimension ≥ 1. Thus X is expressed as a finite union of open connected sets ($W^s(\phi)$ for stable $\phi \in E$), together with a closed nowhere-dense remainder. In particular, for the Chafee-Infante problem, if $\lambda > 1$ and $\lambda \neq n^2$ for integer n, $W^s(\phi_1^+) \cup W^s(\phi_1^-)$ is an open dense set in $H_0^1(0,\pi)$. This actually holds even for $\lambda = n^2$, if we use the center-stable manifold at 0.

Finally choose a bounded connected open set $B \subset X$ which contains E (for example, a large ball about the origin) and let

$$K = \{\psi \mid \text{there exist } \phi_n \in B \text{ and } t_n \to +\infty \text{ with } u(t_n;\phi_n) \to \psi\}.$$

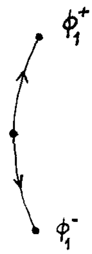
It is easily shown (cf. Th. 4.3.3) that K is a compact connected invariant set containing E; we show $K = \bigcup_{\phi \in E} W^u(\phi)$, so K is finite dimensional and is the maximal bounded invariant set. For any $u_0 \in K$, the solution $u(t;u_0)$ exists and remains in K on $(-\infty,\infty)$, by invariance, so $\phi = \lim_{t\to-\infty} u(t;u_0)$ exists as before with $\phi \in E$. Thus $K \subset \bigcup_{\phi\in E} W^u(\phi)$ and the reverse inclusion is clear so we have equality.

We now study the maximal bounded invariant set K_λ for the Chafee-Infante problem with parameter value λ. In general, $K_\lambda = \bigcup_{\phi\in E_\lambda} W^u(\phi) =$ closure of $W^u(0)$, K_λ is n-dimensional for $n^2 < \lambda < (n+1)^2$. If $0 < \lambda < 1$, $K_\lambda = \{0\}$. If $1 < \lambda < 4$, K_λ is one-dimensional and contains 0, ϕ_1^+ and ϕ_1^-. In fact, the unstable manifold at 0 points into the cone of non-negative functions (and its negative), since the eigenfunction of the linearization is positive. Also ϕ_1^+ (or ϕ_1^-) is the only nontrivial equilibrium in this cone, so $W^u(0)$ consists of 0, a unique orbit from 0 to ϕ_1^+ in the cone of non-negative functions, and a unique orbit from 0 to ϕ_1^- in the cone of non-positive functions.

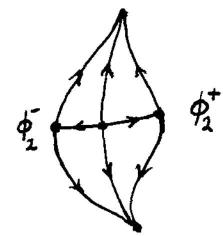
If $4 < \lambda < 9$, K_λ is two-dimensional. As above, there is an orbit joining 0 to ϕ_1^+ and one joining 0 to ϕ_1^-. In fact, $W^u(0) \cap W^s(\phi_1^+)$ and $W^u(0) \cap W^s(\phi_1^-)$ are nonempty sets open in $W^u(0)$. Since $W^u(0)$ is connected, these cannot exhaust $W^u(0)$ and there is a nonconstant orbit $u(t) \to 0$ as $t \to -\infty$ which does not tend to ϕ_1^+ or ϕ_1^- as $t \to +\infty$: suppose for definiteness $u(t) \to \phi_2^+$ as $t \to +\infty$. Then if $\tilde{u}(x,t) = u(\pi-x,t)$, $\tilde{u}$ is also a solution and $\tilde{u}$ goes from 0 to ϕ_2^-. Now $W^u(\phi_2^\pm)$ are one-dimensional, and the eigenfunction of the linearization is positive. Thus there exists a solution $v(t) \to \phi_2^+$ as $t \to -\infty$ with $v(x,t) > \phi_2^+(x)$ for large negative t, $0 < x < \pi$. By the maximum principle, $v(x,t) > \phi_2^+(x)$ for all $t > 0$ so $\psi = \lim_{t\to+\infty} u(\cdot,t)$ is an equilibrium with $\psi(x) > \phi_2^+(x)$ for $0 < x < \pi$, which means $\psi = \phi_1^+$. The reflection $\tilde{v}(x,t) = v(\pi-x,t)$ then joins ϕ_2^- to ϕ_1^+, and a similar argument on the other half of $W^u(\phi_2^+)$ gives orbits joining $\phi_2^\pm \to \phi_1^-$.

Thus we can justify the following pictures of the maximal bounded invariant set K_λ:

$0 < \lambda < 1$



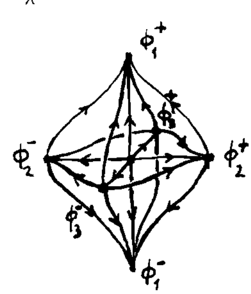
$1 < \lambda < 4$



$4 < \lambda < 9$

In this range of λ, there exist orbits joining any equilibria $\phi \to \psi$ provided $\dim W^u(\phi) > \dim W^u(\psi)$, or equally $V(\phi) > V(\psi)$. If $9 < \lambda < 16$ and $f(u)$ is an odd function, this remains true and we can draw a picture of the flow in K_λ as above. The arguments are as above, but now a solution u yields four solutions: u, $\tilde{u}$, $-u$ and $-\tilde{u}$ ($\tilde{u}$ = reflection in $\pi/2$ of u). Without this extra symmetry, I haven't been able to show there are orbits from 0 to each of $\phi_2^\pm$ and $\phi_3^\pm$. But the picture is worth an extra hypothesis.

For $9 < \lambda < 16$, K_λ is



provided f is odd.

<u>Exercise 7</u>. Suppose Ω is a bounded set in $\mathbb{R}^n$, $\partial\Omega$ is C^2, $f: \mathbb{R} \to \mathbb{R}$ is C^1, $f(0) = 0$, $f(u)/u$ is strictly monotonic for $u > 0$ and $\overline{\lim}_{u\to+\infty} f(u)/u < 0$. Then any solution $u \geq 0$ of $u_t = \Delta u + f(u)$ in $\Omega \times \mathbb{R}^+$ with $u = 0$ on $\partial\Omega$ (or $\partial u/\partial N + hu = 0$ on $\partial\Omega$, given $h \geq 0$) will tend to a nonnegative equilibrium. If the linearization about $u = 0$ has no positive eigenvalue, $u(\cdot,t) \to 0$ as $t \to +\infty$. Otherwise u tends to the unique positive equilibrium solution, or $u \equiv 0$.

The key, of course, is to show there is exactly one positive equilibrium, and for this we follow Stakgold and Payne [96]. There is a maximal and a minimal positive equilibrium $u_\pm$ ($0 < u_- \le u_+$ in Ω and any positive equilibrium u has $u_- \le u \le u_+$), and $u_+ = u_-$ since

$$0 = \int_\Omega (u_+ f(u_-) - u_- f(u_+))dx.$$

5.4 Traveling waves of parabolic equations

Traveling waves are solutions which, viewed in an appropriately moving coordinate system, appear constant. For example, the equations for chemical reactions in a gas at rest have the form

$$u_t = D\Delta u + f(u)$$

where $u = \mathrm{col}(u_1,\ldots,u_n)$ and D is a constant positive diagonal matrix. Suppose these have a plane-wave solution depending only on the scalar variable $s = k\cdot x - Vt$, V and k constant, $|k| = 1$; then $u(x,t) = \phi(s)$ where

$$D\phi''(s) + V\phi'(s) + f(\phi(s)) = 0.$$

Suppose the reaction goes from one equilibrium state $u = a$ far ahead of the wave ($s \to +\infty$) to another equilibrium $u = b$ far behind the wave:

$$\phi(s) \to a \quad \text{as} \quad s \to +\infty, \ \phi(s) \to b \quad \text{as} \quad s \to -\infty$$

$$f(a) = 0, \quad f(b) = 0.$$

Such a solution might describe the propagation of a flame, for example, though generally three-dimensional effects might enter and complicate the situation. In some problems, the hypothesis of one-dimensional behavior is more natural, for example propagation of a pulse along a nerve [25]; in any case it simplifies the mathematics and we need all the help we can get, so we deal only with a single space variable.

Our first example is a simplified form of the Fitzhugh-Nagumo equation -- in turn, a simplified form of the Hodgekin-Huxley equations (see [17]):

$$u_t = u_{xx} + f(u) \qquad (-\infty < x < \infty, \quad t > 0)$$

$$f(u) = u(1-u)(u-a)$$

where a is constant, $0 < a < 1/2$. A traveling wave $u(x,t) = \phi(x+Vt)$ satisfies

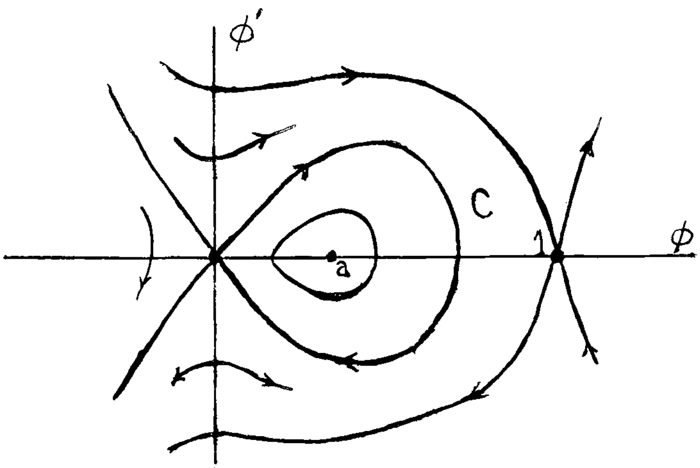
$$\phi''(s) - V\phi'(s) + f(\phi(s)) = 0, \quad -\infty < s < \infty.$$

We show that, for some $V > 0$, there is a solution to this equation with $\phi(s) \to 0$ as $s \to -\infty$, $\phi(s) \to 1$ as $s \to +\infty$.

First suppose $V = 0$; then any solution ϕ satisfies $\frac{1}{2}\phi'(s)^2 + F(\phi(s)) = \text{constant}$

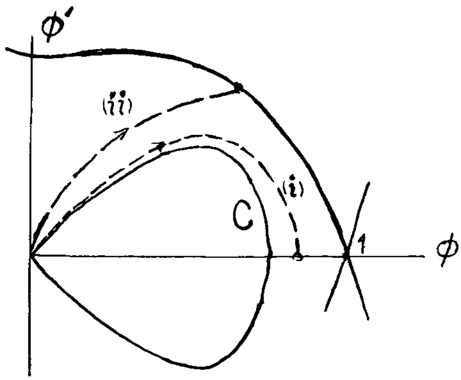
$$F(u) = \int_0^u f(v)dv$$

so the orbit is one of the level curves indicated in the first figure;



the curve C surrounding the point $(a,0)$, going from 0 to 0, is $\frac{1}{2}p^2 + F(\phi) = 0$ and does not reach as far as $\phi = 1$ since $F(\phi) \leq 0$ on C but $F(1) = \frac{1}{6}(\frac{1}{2} - a) > 0$. For any $V \geq 0$, the origin is a saddle point with the unstable manifold pointing into the first (and third) quadrant, with steep slope when V is large.

We take ϕ to be the solution starting from the unstable manifold in the first quadrant, given $V > 0$, and choose V so that it satisfies $\phi(s) \to 1$ as $s \to +\infty$. Now $\frac{d}{ds}(\frac{1}{2}\phi'(s)^2 + F(\phi(s))) = V\phi'(s)^2 > 0$ as long as the solution remains in the first quadrant, so it moves through increasing values of the level curves indicated in the first diagram. If $V > 0$ is small, the solution will nearly follow the curve C and will hit the ϕ-axis at a point with $\phi < 1$ (case (i) in the figure). On the other hand, if V is large, the solution



(i) small $V > 0$
(ii) large $V > 0$

will reach the level curve $\frac{1}{2}\phi'^2 + F(\phi) = F(1)$ at a point with $\phi' > 0$. Since the unstable manifold and the solution ϕ depend continuously on V, there will be a value $V > 0$ for which $\phi(s) \to 1$ as $s \to +\infty$. Observe that this solution has $\phi'(s) > 0$ for all s and $\phi'(s) \to 0$ exponentially as $s \to \pm\infty$, since $(0,0)$ and $(1,0)$ are both saddle points.

The explicit solution $V = \sqrt{2}\,(\frac{1}{2} - a)$, $\phi(s) = 1/(1 + \exp(-s/\sqrt{2}))$ was found by Huxley.

Exercise 1. Show, for any $V > 0$, there is a solution of the above equation which goes from a at $-\infty$ to 0 at $+\infty$. When $V^2 < 4a(1-a)$ it spirals away from a, but when $V^2 > 4a(1-a)$ it is monotonic.

McKean [74] presents a detailed picture of the solutions for $0 < a < 1/2$ and various choices $V \geq 0$.

We now consider stability of this wave. We change to moving coordinates with $\xi = x + Vt$; then

$$u_t = u_{\xi\xi} - Vu_\xi + f(u), \qquad -\infty < \xi < \infty$$

In these variables, $u = \phi$ is an equilibrium point. The linearization about ϕ is

$$u_t = u_{\xi\xi} - Vu_\xi + f'(\phi(\xi))u \equiv -Au$$

and we need information about the spectrum of A. Using a result proved below (see Appendix to this chapter) the essential spectrum of A (i.e., the spectrum aside from isolated eigenvalues of finite multiplicity) lies in $\{\lambda \mid \mathrm{Re}\,\lambda \geq \min(a,1-a) = a\}$. Here the underlying space may be $L_p(\mathbb{R})$, $1 \leq p < \infty$ or $C_{unif}(\mathbb{R})$ or $C_0(\mathbb{R})$, the space of continuous functions which vanish at ∞. The set $\{\phi(\cdot + c) \mid -\infty < c < \infty\}$ is a curve of equilibrium points and $A\phi' = 0$; if we show 0 is a simple eigenvalue of A and the remainder of the spectrum lies in $\mathrm{Re}\,\lambda > 0$, then according to exercise 6, sec. 5.1, a solution which starts sufficiently close to ϕ will approach exponentially a translate of ϕ: here the norm may be $W^{1,p}(\mathbb{R})$ or $C_{unif}(\mathbb{R})$.

With the essential spectrum out of the way we need only determine whether there are any eigenvalues in $\mathrm{Re}\,\lambda \leq 0$, $\lambda \neq 0$. Suppose then

$$v'' - Vv' + f'(\phi(\xi))v + \lambda v = 0$$

with $\mathrm{Re}\,\lambda \leq 0$ and v bounded as $\xi \to \pm\infty$. Examination of the characteristic equation for the limits $\xi \to \pm\infty$ shows $v(\xi)$ must actually tend to zero, at least $O(e^{-V|\xi|})$, when $\xi \to \pm\infty$. Let $w(\xi) = v(\xi)e^{-V\xi/2}$ so $w(\xi) = O(e^{-V|\xi|/2})$ and

$$w'' + (\lambda + V^2/4 + f'(\phi(\xi)))w = 0.$$

We can consider this as a self-adjoint problem in $L_2(\mathbb{R})$, regardless of the original space, so any eigenvalue must be real: suppose $\lambda \leq 0$ and W is real. The smallest eigenvalue is

$$\lambda = \min\{\int_{-\infty}^{\infty} w'^2 - (f'(\phi(\xi)) + V^2/4)w^2 d\xi \mid w \in H^1(\mathbb{R}),\ \int w^2 d\xi = 1\}.$$

If w is the function which achieves the minimum then $|w|$ gives the same value, so we can suppose the eigenfunction w is nonnegative and in fact strictly positive. (If $w = 0$ and $w' = 0$ at some point, then $w \equiv 0$.) Let $\psi = \phi'(\xi)e^{-V\xi/2}$; then $\psi'' + (V^2/4 + f'(\phi))\psi = 0$ and $\psi > 0$, and integration by parts shows $\lambda = \int_{-\infty}^{\infty}\psi^2(d/d\xi(w/\psi))^2 d\xi$. Thus λ cannot be negative and if $\lambda = 0$ then $w/\psi = v/\phi' =$ constant.

Therefore 0 is a simple eigenvalue of A (in $L_p(\mathbb{R})$ or $C_{unif}(\mathbb{R})$) and for any solution u with $\|u(\cdot,0)-\phi\|$ sufficiently small (norm of $W^{1,p}$ or C_{unif}) there exists real c so

$$\|u(\cdot,t) - \phi(\cdot+c)\| = O(e^{-\beta t}), \quad t > 0,$$

for a constant $\beta > 0$.

This problem was treated by Sattinger [131] using a special weighted L_∞-norm, and by Aronson and Weinberger [108] in the uniform norm. Fife and McLeod [116] treat more complicated initial values in the norm.

Now consider the traveling wave corresponding to the solution $\tilde{\phi}(s) \to a$ as $s \to -\infty$, $\tilde{\phi}(s) \to 0$ as $s \to +\infty$ found in exercise 1. For this case, if $V^2 < 4a(1-a)$, $\tilde{\phi}(s)$ spirals in toward a as $s \to -\infty$ so $\tilde{\phi}'(s)$ does not have constant sign; this suggests instability, in light of the variational argument above. When $V^2 > 4a(1-a)$, $\tilde{\phi}'(s) < 0$ for all s. However the essential spectrum of the linearization reaches as far as $-a(1-a)$, so $\tilde{\phi}$ is unstable (cor. 5.1.6) in any of the spaces mentioned above. This is true independent of $V > 0$, so monotonicity of the solution is no guarantee of stability

Finally note the equilibrium solution $u = \hat{\phi}(x) > 0$ $(1/2\ \hat{\phi}'^2 + F(\hat{\phi}) = 0$, $\hat{\phi}(x) \to 0$ as $x \to \pm\infty)$ is unstable in $W^{1,p}(\mathbb{R})$ or $C_{unif}(\mathbb{R})$, since the linearization about $\hat{\phi}$ $(L\colon \psi \to -\psi'' - f'(\hat{\phi}(x))\psi)$ has a negative eigenvalue. Indeed, 0 is an eigenvalue with eigenfunction $\hat{\phi}'$ which changes sign, and the least eigenvalue is

$$\mu = \min\{\int_{-\infty}^{\infty}(\psi'^2 + f'(\hat{\phi})\psi^2)dx \mid \int_{-\infty}^{\infty}\psi^2 = 1\} \le 0$$

which is a simple eigenvalue with a positive eigenfunction, hence $\mu \neq 0$.

We can give some interesting results for a general scalar equation $u_t = u_{xx} + f(u,u_x)$ where $f(u,p)$ is C^1. Suppose there exists $\phi(x)$,

$$\phi''(x) + f(\phi(x),\phi'(x)) = 0 \quad \text{on} \quad -\infty < x < \infty,$$

and $\phi(x) \to \alpha$ as $x \to -\infty$, $\phi(x) \to \beta$ as $x \to +\infty$, where $f(0,\alpha) = 0$, $f(0,\beta) = 0$. The linearization about ϕ is

$$-Lv = v'' + a(x)v' + b(x)v$$

where $a(x) = \frac{\partial f}{\partial p}(\phi(x),\phi'(x))$, $b(x) = \frac{\partial f}{\partial u}(\phi(x),\phi'(x))$. Let $a_\pm$, $b_\pm$ denote the limits as $x \to \pm\infty$. Then by results given in the appendix, $\sigma_e(L)$ is in the right half-plane if and only if $b_+ < 0$ and $b_- < 0$, i.e. if and only if the solution $\phi(x)$ goes from one saddle point to another. If we consider the equation in a space with an appropriate exponential weight w, i.e. in the norm

$$\left(\int_{-\infty}^{\infty} |v(x)/w(x)|^p dx\right)^{1/p},$$

we can achieve $\text{Re}\ \sigma_e(L) > 0$, provided $b_\pm < a_\pm^2/4$ for each sign, i.e. provided the linearization at the equilibria $(\alpha,0)$, $(\beta,0)$ has real distinct eigenvalues. We must choose $w(x)$ so $w(x) \sim e^{\mu_\pm x}$ as $x \to \pm\infty$ where μ_+, μ_- are chosen so that (for each choice of sign)

$$\mu^2 + a\mu + b < 0.$$

But $\phi'(x)$ behaves asymptotically like $e^{\lambda x}$ where $\lambda^2 + a\lambda + b = 0$, so μ lies strictly between the two roots of this equation. Suppose we are dealing with the limit $x \to +\infty$ and both roots are negative (a stable node); then ordinarily $\phi'(x)$ will behave like the weaker (less negative) exponential and so $\phi'(x)/w(x)$ will blow up as $x \to +\infty$. Thus we have stabilized the essential spectrum only at the expense of having ϕ' excluded from the space, and the problem becomes essentially meaningless. In this weighted norm, translation $(h \to \phi(\cdot+h))$ is not continuous.

Summarizing:

(i) if ϕ joins two saddle points, the essential spectrum of the linearization is stable in L_p (and in L_p with certain weights);

(ii) if ϕ joins two nodes or a saddle and a node, the solution is unstable in $L_p(\mathbb{R})$, but with an appropriate weight function we can stabilize the essential spectrum, at the same time (usually) pushing ϕ' out of the space.

(iii) If ϕ approaches a spiral point at $+\infty$ or $-\infty$, it is unstable.

In case (i), we can argue as in the example to show $\{\phi(\cdot+c) \mid -\infty<c<\infty\}$ is asymptotically stable with asymptotic phase when ϕ is monotonic, and it is unstable when ϕ is not monotonic. Ordinarily, in a saddle-saddle orbit, the velocity V of the traveling wave is uniquely determined but this is not true in cases (ii) or (iii). Sattinger [131] treats this general case also but has more confidence in the significance of waves joining nodes.

Perhaps I overstate the case in dismissing traveling waves joining nodes. Arguments using the maximum principle ([108] and especially [116]) have lead to some beautiful results -- including stability results -- for such cases, but the dependence on the initial value is rather delicate. For example, a particular case treated by

Kolomogorov, Petrovsky and Piscounov (in the 1937 paper [60] which really initiated the subject) is

$$u_t = u_{xx} + u(1-u) \qquad (-\infty < x < \infty, \quad t > 0).$$

For any $V \geq 2$, there is a traveling wave with velocity V going from the saddle at $u = 1$ to the stable node at 0. If $u_0(x,t)$ is the solution with initial value $u_0(x,0) = 1$ for $x < 0$, $u_0(x,0) = 0$ for $x > 0$, then u_0 approaches the traveling wave ϕ with speed 2 in the weak sense that

$$u_0(x+\xi(t),t) \to \phi(x) \quad \text{as} \quad t \to +\infty$$

where $\xi(t)$ is a function with derivative $\xi'(t) \to 2$ as $t \to +\infty$. However Larson [124] proves $u_0(x+2t,t) \to 0$ as $t \to +\infty$. Further, if $u(x,t)$ is a solution in $0 < u < 1$ which approaches a traveling wave as $t \to +\infty$ and $e^{\alpha x}u(x,0)$ approaches a finite nonzero limit as $x \to +\infty$ for some $0 < \alpha < 1$, then the traveling wave must have velocity $\alpha + 1/\alpha$. (Larson [124, Th. 51]). Such delicate dependence is bound to raise doubts about the usefulness of the enterprise. Are the topologies too stringent? Should we seek only stability of the entire family of traveling waves (allowing the velocity to vary as well as the phase)? Clearly the subject is far from settled, even if we leave out related problems where the traveling wave is periodic or asymptotically periodic in the space variable.

Exercise 2. Burger's equation $u_t + uu_x = \nu u_{xx}$ $(-\infty < x < \infty,\ t > 0;\ \nu > 0$ constant) has an equilibrium solution $\phi(x) = -a \tanh(ax/2\nu)$ for any $a \geq 0$. If $\bar{x} = x + ct$, $\bar{u} = u + c$ for a constant c then $\bar{u}_t + \bar{u}\,\bar{u}_{\bar{x}} = \nu \bar{u}_{\bar{x}\bar{x}}$ so

$$u = -c - a \tanh(a(x+ct)/2\nu)$$

is a traveling wave solution with $u(x,t) \to -c \pm a$ as $x \to \pm\infty$, and it is sufficient to study the stationary solution ϕ.

Exercise 3. With ϕ as above, $a > 0$, $\xi = ax/2\nu$, $\tau = a^2t/4\nu$, and $b = 2/a$, $v(\xi,\tau) = u-\phi$ satisfies

$$v_\tau = v_{\xi\xi} + 2 \tanh \xi\, v_\xi + 2 \operatorname{sech}^2\xi v - bvv_\xi.$$

The linearization about $v = 0$ is $v_\tau = -Lv$

$$-Lv = v_{\xi\xi} + 2 \tanh \xi\, v_\xi + 2 \operatorname{sech}^2\xi\, v$$

and 0 is in the essential spectrum (considered in $L_p(\mathbb{R})$). Use the weighted norm $\|\psi\| = \|\psi(\xi)\cosh \xi\|_{L_2(\mathbb{R})}$ and then $L_w\psi = -\psi'' + (1 - 2\operatorname{sech}^2\xi)\psi$ where $L_w = \frac{1}{w} Lw$, so $\sigma_e(L_w) = [1,\infty)$. If $v = w \operatorname{sech} \xi$ then

$$w_\tau = w_{\xi\xi} - (1-2 \operatorname{sech}^2\xi)w - b \operatorname{sech} \xi(ww_\xi - w^2\tanh \xi)$$

and

$$\int_{-\infty}^{\infty} w\{-w_{\xi\xi} + (1 - 2 \operatorname{sech}^2\xi)w\}d\xi$$
$$= \int_{-\infty}^{\infty} (w_\xi + \tanh \xi\, w)^2 d\xi \geq 0$$

so 0 is a simple isolated eigenvalue and the remainder of the spectrum is in $[\beta,\infty)$ for some $\beta > 0$. The nonlinear term in the w-equation is a continuous polynomial from $H^1(\mathbb{R})$ to $L_2(\mathbb{R})$ and ex. 6, sec. 5.1 applies. Note $\phi'(\xi)\cosh \xi = -a \operatorname{sech} \xi$ is in $L_2(\mathbb{R})$ and $h \to (\phi(\xi+h) - \phi(\xi))\cosh \xi$ is continuously differentiable from $\mathbb{R}$ to $L_2(\mathbb{R})$. Also from the equation $\frac{d}{d\tau} \int_{-\infty}^{\infty} v\, d\xi = 0$ for $\tau > 0$ so if $\|v(\xi,0)\cosh\xi\|_{H^1(\mathbb{R})}$ is small enough, there exists real c with

$$\|(u(\xi,\tau) - \phi(\xi+c))\cosh \xi\|_{H^1(\mathbb{R})} \to 0 \quad \text{as} \quad \tau \to \infty$$

and

$$\int_{-\infty}^{\infty} u(\xi,0)d\xi = \int_{-\infty}^{\infty} \phi(\xi+c)d\xi.$$

<u>Exercise 4</u>. Suppose $b > 0$. Prove the zero solution of $u_t = u_{xx} - u - bu^2$ $(-\infty < x < \infty,\ t > 0)$ is asymptotically stable in $H^1(\mathbb{R})$ and the equilibrium solution $\phi(x) = -3/2b \operatorname{sech}^2(x/2)$ is unstable. (<u>Hint</u>: $\phi(x) < 0$ and so $\phi(-\phi'' + \phi + 2b\phi^2) = b\phi^3 < 0$.) Note that for any bounded uniformly continuous initial value with $\inf_x u(x,0) > -1/b$, we have $u(x,t) \to 0$ as $t \to +\infty$ uniformly in x, by ex. 8, sec. 5.1.

<u>Exercise 5</u>. $u_t = u_{xx} + u - bu^3$ $(b > 0)$ on $-\infty < x < \infty$, $t > 0$, has an equilibrium $u = \phi(x)$ with $\phi(x) \to b^{-1/2}$ (or $-b^{-1/2}$) as $x \to +\infty$ (or $-\infty$) and $\phi'(x) > 0$ for all x. Prove, when $\|u(\cdot,0)-\phi\|_{H^1}$ is small, there exists real c so $\|u(\cdot,t) - \phi(\cdot+c)\|_{H^1} \leq Ke^{-\beta t}\|u(\cdot,0)-\phi\|_{H^1}$ for all

$t > 0$, for some positive constants K, β. Also examine the stability of $u = 0$.

Exercise 6. For every $V \neq 0$ prove the equation in ex. 5 above admits exactly two nontrivial bounded traveling wave solutions with velocity V, one the negative of the other, going from the unstable node 0 at $x = -\infty$ to $\pm b^{-\frac{1}{2}}$ at $x = +\infty$ provided $V > 0$ and $u(x,t) = \phi(x+vt)$. If $|V| < 2$, ϕ is oscillatory and unstable; if $|V| > 2$, ϕ is monotonic, but unstable in $W^{1,p}(\mathbb{R})$ or $C_{unif}(\mathbb{R})$. When $|V| > 2$, the linearization about ϕ is stable in an appropriate weighted norm (which excludes ϕ' from the space) but the nonlinear terms are not well-behaved in that norm.

Appendix. Essential spectrum of some ordinary differential operators

Definition. If L is a linear operator in a Banach space, a *normal point* for L is any complex number which is in the resolvent set, or is an isolated eigenvalue of L of finite multiplicity. Any other complex number is in the *essential spectrum*.

Remark. Several definitions of essential spectrum are in use. Ours (following Gohberg and Krein [118]) is more restrictive and less stable under perturbation than some others, but gives correspondingly more information once it is computed.

Theorem A.1. Suppose X is a Banach space, $T: D(T) \subset X \to X$ is a closed linear operator, $S: D(S) \subset X \to X$ is linear with $D(S) \supset D(T)$ and $S(\lambda_0 - T)^{-1}$ is compact for some λ_0. Let U be an open connected set in $\mathbb{C}$ consisting entirely of normal points of T; then either U consists entirely of normal points of $T+S$, or entirely of eigenvalues of $T+S$.

The proof is a slight modification of the proof given in [118, p. 22].

Exercise 1. If $X = \{(x_n)_{n=-\infty}^{\infty} \mid x_n \in \mathbb{C}, \|x\| = (\sum_{-\infty}^{\infty} |x_n|^2)^{\frac{1}{2}} < \infty\}$ and $(Tx)_n = x_{n-1}$ for all n, and $(Sx)_n = 0$ for $n \neq 1$, $(Sx)_1 = -x_0$, then prove $\sigma(T) =$ unit circle, $\sigma(T+S) =$ closed unit disc. In fact

$\sigma(T+tS)$ = unit circle for $0 \le t < 1$, but the interior of the disc consists of eigenvalues when $t = 1$.

Exercise 2. Let $Tu(x) = u'' + a(x)u' + b(x)u$, $-\infty < x < \infty$, be considered in $L_p(\mathbb{R})$, $1 \le p < \infty$, or $C_0(\mathbb{R})$ or $C_{unif}(\mathbb{R})$ (here a,b are continuous and bounded). Let $Su(x) = m(x)u' + n(x)u$ where m,n are continuous and $m(x), n(x) \to 0$ as $x \to \pm\infty$. Prove $S(\lambda_0 - T)^{-1}$ is compact for large real λ_0. (Hint: first suppose m,n have compact support.)

We will study first order systems with asymptotically constant coefficients:

$$\frac{du}{dx} + A(x)u = f(x), \quad -\infty < x < \infty$$

$$A(x) \to A_{\pm} \quad \text{as} \quad x \to \pm\infty.$$

In fact we will study the case when

$$A(x) = A_+ \quad \text{for} \quad x > 0, \quad A(x) = A_- \quad \text{for} \quad x < 0$$

and then apply Theorem A.1, since these differ by a relatively compact operator.

Lemma 1. Suppose $A(x) = A_+$ for $x > 0$, $A(x) = A_-$ for $x < 0$ and the complex $n \times n$ matrices $A_{\pm}$ have no eigenvalues on the imaginary axis. Let E_+, E_- be the projections corresponding to the eigenvalues of A_+, A_- in the right half-plane.

Then $u'(x) + A(x)u(x) = f(x)$, $-\infty < x < \infty$, has a unique bounded solution for every bounded measurable f (or equivalently, for every continuous f with compact support) if and only if $\mathbb{C}^n = R(E_+) \oplus R(I-E_-)$, i.e. if and only if $R(E_+)$ and $R(I-E_-)$ together span $\mathbb{C}^n$ and intersect only at 0. In this case, $\|u\|_{L_p(\mathbb{R})} \le C_A \|f\|_{L_p(\mathbb{R})}$, $1 \le p \le \infty$ for a constant C_A depending only on A.

Proof. First consider $x > 0$. There exist positive M, α so

$$|e^{-A_+ x} E_+| \le Me^{-\alpha x} \quad \text{for} \quad x \ge 0$$

$$|e^{-A_+ x}(1-E_+)| \le Me^{-\alpha|x|} \quad \text{for} \quad x \le 0$$

and if u, f are bounded on $[0,\infty)$ then

$$(1-E_+)u(0) + \int_0^\infty (1-E_+)e^{A_+y} f(y)dy = 0.$$

Conversely when this is true,

$$u(x) = e^{-A_+x}E_+u(0) + \int_0^x e^{-A_+(x-y)}E_+f(y)dy - \int_x^\infty e^{-A_+(x-y)}(1-E_+)f(y)dy$$

for $x \geq 0$ and if $1 \leq p \leq \infty$

$$\|u\|_{L_p(\mathbb{R}_+)} \leq M\alpha^{-1/p}|E_+u(0)| + 2M\alpha^{-1}\|f\|_{L_p(\mathbb{R}_+)};$$

in particular $(p = \infty)$, u is bounded. Here we use the familiar consequence of Hölder's inequality

$$\|f*g\|_{L_p(\mathbb{R})} \leq \|f\|_{L_p(\mathbb{R})}\|g\|_{L_1(\mathbb{R})}, \qquad 1 \leq p \leq \infty.$$

Similarly on $(-\infty,0]$, u is bounded if and only if

$$E_-u(0) - \int_{-\infty}^0 e^{A_-y}E_-f(y)dy = 0,$$

and if this holds, $\|u\|_{L_p(\mathbb{R}_-)} \leq C_A(|(I-E_-)u(0)| + \|f\|_{L_p(\mathbb{R}_-)})$.

In order that there exist a bounded solution u for every continuous f supported in $(-1,1)$ it is necessary (and sufficient, for every bounded f) that

$$R(E_-) + R(1-E_+) \subset R(E_+) + R(1-E_-)$$

or equivalently, $\mathbb{C}^n = R(E_+) + R(1-E_-)$. It is easily seen that uniqueness holds if and only if $R(E_+) \cap R(1-E_-) = \{0\}$, and if both hold then $\|u\|_{L_p(\mathbb{R})} \leq C_A\|f\|_{L_p(\mathbb{R})}$.

Lemma 2. Suppose the matrices $A_+(\lambda)$, $A_-(\lambda)$ are analytic functions of $\lambda \in \mathbb{C}$. Let

$$S_\pm = \{\lambda \mid A_\pm(\lambda) \text{ has an imaginary eigenvalue}\}.$$

Let $A(x,\lambda) = A_+(\lambda)$ for $x > 0$, $A_-(\lambda)$ for $x < 0$ and define the differential operator $L(\lambda)u = du/dx + A(\cdot,\lambda)u$ in any of the spaces $L_p(\mathbb{R})$, $1 \le p < \infty$, or $C_0(\mathbb{R})$ or $C_{unif}(\mathbb{R})$; we may consider $L(\lambda)$ as closed and densely defined.

Then if G is any open connected set in $\mathbb{C}\setminus(S_+ \cup S_-)$, either

(i) $0 \in \sigma(L(\lambda))$ for all λ in G, or

(ii) $0 \in \rho(L(\lambda))$ for all λ in G except at isolated points; the exceptional points are poles of $L(\lambda)^{-1}$ of finite order.

Also, $0 \in \sigma(L(\lambda))$ whenever $\lambda \in S_+ \cup S_-$.

<u>Proof</u>. The projections $E_+(\lambda)$, $E_-(\lambda)$ defined as in lemma 1 are analytic functions of λ in G, and the condition

$$\mathbb{C}^n = R(E_+(\lambda)) \oplus R(1-E_-(\lambda))$$

which is necessary and sufficient for $0 \in \rho(L(\lambda))$ is equivalent to requiring

$$\det[p_1(\lambda),\dots,p_m(\lambda),\ q_{m+1}(\lambda),\dots,q_n(\lambda)] \not\equiv 0$$

for some choice of columns $p_j(\lambda)$ from $E_+(\lambda)$ and $q_k(\lambda)$ from $I-E_-(\lambda)$, when $R(E_+(\lambda))$ is m-dimensional and $R(1-E_-(\lambda))$ is (n-m) dimensional. The dimension condition either holds everywhere in G or nowhere in G, and if it holds we have $0 \in \rho(L(\lambda))$ in G aside from isolated points where the above determinants (analytic in λ) all vanish, or they all vanish everywhere in G and $0 \in \sigma(L(\lambda))$ for all λ. In case (ii), zeros of the relevant determinants are of finite order, yielding poles of $L(\lambda)^{-1}$ of finite order.

If $\lambda \in S_+$ (or S_-, similarly) there is a solution $u(x)$ of the homogeneous equation which is bounded and bounded from zero as $x \to +\infty$. Let $\phi(x)$ be smooth, $\phi(x) = 1$ for $|x| < 1$, $\phi(x) = 0$ for $|x| > 2$ and let $u_m(x) = u(x)\phi(\frac{x-3m}{m})$, then if $1 \le p \le \infty$,

$$\frac{\|L(\lambda)u_m\|_{L_p(\mathbb{R})}}{\|u_m\|_{L_p(\mathbb{R})}} = O(\tfrac{1}{m}) \to 0 \quad \text{as} \quad m \to \infty$$

so $0 \in \sigma(L(\lambda))$.

<u>Remark</u>. The result also holds for asymptotically periodic coefficients -- and even more generally. In this case $\lambda \in S_\pm$ if the limiting

periodic equation has a nontrivial solution bounded on $(-\infty,\infty)$. The proof is virtually the same as that above, but computation of $S_\pm$ is much more difficult.

Now we apply the results to second order systems.

Theorem A.2. Suppose $M(x)$, $N(x)$ are bounded real matrix functions and $M(x), N(x) \to M_\pm, N_\pm$ as $x \to \pm\infty$, and suppose D is constant symmetric and positive. In any of the spaces $L_p(\mathbb{R})$, $1 \le p < \infty$, or $C_0(\mathbb{R})$ or $C_{unif}(\mathbb{R})$, (for column-vector functions $u(x)$) define

$$Au(x) = -Du_{xx} + M(x)u_x + N(x)u, \quad -\infty < x < \infty.$$

We consider A as a closed, densely defined linear operator.

Let $S_\pm = \{\lambda \mid \det(\tau^2 D + i\tau M_\pm + N_\pm - \lambda I) = 0$ for some real τ, $-\infty < \tau < \infty\}$. Then $S_\pm$ consists of a finite number of algebraic curves which are symmetric about the real axis and are asymptotically parabolas: $\lambda = \tau^2\delta + O(\tau)$ as $\tau \to \pm\infty$ where δ is an eigenvalue of D. Let P denote the union of the regions inside or on the curves S_+, S_-; thus $\mathbb{C}\setminus P$ is the component of $\mathbb{C}\setminus(S_+ \cup S_-)$ containing a left half-plane. Then the essential spectrum of L is contained in P, and in particular includes $S_+ \cup S_-$.

Proof. Writing the corresponding first-order system, we obtain the result when M, N are replaced by $M_\pm, N_\pm$ on each half-line; note that a pole of $(\lambda - A)^{-1}$ of order m is an eigenvalue of A of multiplicity $\le 2mn$. Applying Thm. A.1 gives the general case. Since large negative numbers are not eigenvalues, $\mathbb{C}\setminus P$ can't consist entirely of eigenvalues, so it contains no essential spectrum.

Example. $-Lu = u_{xx} + a(x)u_x + b(x)u$, $-\infty < x < \infty$, (scalar equation) with $a(x), b(x) \to a_\pm, b_\pm$ as $x \to \pm\infty$; then the essential spectrum

$$\sigma_e(L) \subset \{\lambda \mid \operatorname{Re}\lambda - \frac{(\operatorname{Im}\lambda)^2}{a_\pm^2} \ge -b_\pm\}$$

where we use the corresponding ray $[-b_\pm, \infty)$ when $a_\pm = 0$. In particular

$$\min \operatorname{Re} \sigma_e(L) = \min(-b_+, -b_-)$$

and this is positive if and only if both b_+ and b_- are negative.

Suppose $w(x) > 0$ is a smooth weight-function and

$$\frac{w'(x)}{w(x)} \to \mu_\pm, \quad \frac{w''(x)}{w(x)} \to \mu_\pm^2 \quad \text{as} \quad x \to \pm\infty.$$

This can be achieved with appropriate choice of $w(x) = e^{\lambda x} + e^{\nu x}$ or $1/(e^{\lambda x} + e^{\nu x})$ for arbitrary choice of real μ_+ and μ_-. Applying the similarity transformation

$$-L_w v = -\frac{1}{w} L(wv) = v'' + (a + 2\frac{w'}{w})v' + (b + \frac{w''+aw'}{w})v$$

(which is equivalent to using a weighted norm in $L_p(\mathbb{R})$, $1 \le p \le \infty$) we find

$$\min \operatorname{Re} \sigma_e(L_w) = \min(-b_\pm - \mu_\pm^2 - a_\pm\mu_\pm).$$

If $b_\pm < a_\pm^2/4$ for each sign, we can choose a weight function w so $\operatorname{Re} \sigma_e(L_w) > 0$; this is an improvement over the previous condition $b_\pm < 0$.

Exercise 3. Consider a system of n equations

$$-Lu = Du'' + Vu' + M(x)u$$

where D is symmetric and positive, V is a constant scalar and $M(x) \to M_\pm$ as $x \to \pm\infty$. Show that, for $\operatorname{Re} \sigma_e(L) > 0$, it is necessary that all eigenvalues of $M_\pm$ have negative real part. When D is also a scalar, this is necessary and sufficient.

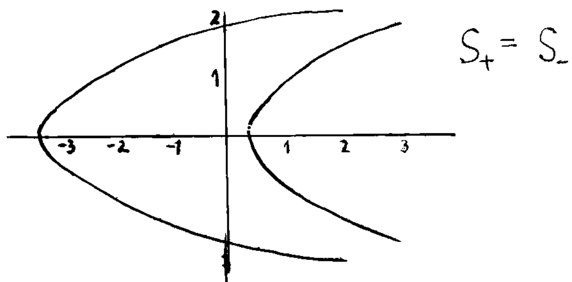
Another example. This is an artificial example, meant only to display the form of the curves in S_+, S_-:

$$-L\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} u_{xx} \\ v_{xx} \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} u_x \\ v_x \end{pmatrix} + \begin{pmatrix} 0 & 2 \\ \frac{1}{2} & 3 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}, \quad -\infty < x < \infty.$$

$S_+ = S_-$ is given by the solutions λ, for $-\infty < \tau < \infty$, of

$$0 = \lambda^2 + \lambda\{3-5\tau^2 + 3i\tau\} + 4\tau^4 - 14\tau^2 - 1 + i(-9\tau^3 + 3\tau).$$

When $\tau = 0$, $\lambda = \frac{1}{2}(-3 \pm \sqrt{13}) \approx -3.3$ or 0.3. These are in fact the left-most points of their corresponding curves; computation of the roots λ for a few values in $0 < \tau < 2$ gives the picture



For the constant coefficient case, the essential spectrum of L is simply the union of these curves. For a system whose coefficients approach those of L as $x \to \pm\infty$, the essential spectrum must be confined within (and on) the outermost curve.

Exercise 4. If $-Lu = u'' + 2 \tanh x \cdot u'$ is considered in $L_p(\mathbb{R})$ or $C_{unif}(\mathbb{R})$, show $S_+ = S_- = \{\lambda: 4 \operatorname{Re} \lambda = (\operatorname{Im} \lambda)^2\}$, $\sigma(L) = \sigma_e(L) = \{\lambda \mid 4 \operatorname{Re} \lambda \geq (\operatorname{Im} \lambda)^2\}$. (Hint: let $v = u \cdot \cosh x$.) This shows the region inside the curves $S_\pm$ can be filled with essential spectrum.

Chapter 6

Invariant Manifolds Near an Equilibrium Point

6.1 Existence and stability of an invariant manifold

The results of this section are stated in rather general form to cover the applications in sections 6.2, 6.3 and 6.4. Examination of these applications might make the assumptions below seem less arbitrary.

Definition 6.1.1. A set $S \subset \mathbb{R} \times X^\alpha$ is a *local invariant manifold* for a differential equation $dx/dt + Ax = f(t,x)$ provided for any $(t_0,x_0) \in S$, there exists a solution $x(\cdot)$ of the differential equation on an open interval (t_1,t_2) containing t_0 with $x(t_0) = x_0$ and $(t,x(t)) \in S$ for $t_1 < t < t_2$. S is an *invariant manifold* if we can always choose $(t_1,t_2) = (-\infty,\infty)$. When the differential equation is autonomous and $S = \mathbb{R} \times S_1$, we may also call S_1 an invariant manifold.

Under certain assumptions below, we shall prove the existence of an invariant manifold for the coupled system

$$\frac{dx}{dt} + Ax = f(t,x,y)$$

$$\frac{dy}{dt} = g(t,x,y)$$

in the form $S = \{(t,x,y) \mid x = \sigma(t,y)\}$. Roughly speaking, the assumptions say that $dx/dt + Ax = 0$ is asymptotically stable, f is small, and the solutions of the y-equation do not converge too rapidly. (Compare with [37, Ch. 7] [58], [41] and [72].) Other cases are treated in exercises 4-8 below and in Chapters 8 and 9.

Theorem 6.1.2. Let X, Y be Banach spaces and assume A is sectorial in X. Let U be a neighborhood of the origin in X^α for some $\alpha < 1$, and suppose $f\colon \mathbb{R} \times U \times Y \to X$ and $g\colon \mathbb{R} \times U \times Y \to Y$ are locally Lipschitzian with $\|f(t,x,y)-f(t,x',y')\| \le \lambda(\|x-x'\|_\alpha + \|y-y'\|)$, $\|f(t,x,y,)\| \le N$.

Assume that $\|e^{-At}\| \le Me^{-\beta t}$ and $\|A^\alpha e^{-At}\| \le Mt^{-\alpha}e^{-\beta t}$ for $t > 0$ and also that the solution $y(t) = y(t;\tau,\eta,x(\cdot))$ of $dy/dt = g(t,x(t),y)$ for $t \le \tau$, $y(\tau) = \eta$, exists on $(-\infty,\tau]$ for any continuous curve $x\colon (-\infty,\tau] \to U$ and $\eta \in Y$. Also we suppose for any choices

η, η', x, x',

$$\|y(t;\tau,\eta,x(\cdot)) - y(t;\tau,\eta',x'(\cdot))\|$$
$$\leq M_1 e^{\mu(\tau-t)}\|\eta-\eta'\| + M_2 \int_t^\tau e^{\mu(s-t)} \|x(s)-x'(s)\|_\alpha \, ds$$

for $t \leq \tau$. Here λ, N, M, β, M_1, M_2, μ are non-negative constants, and we assume for some positive constants Δ, D that

a) $\{x \mid \|x\|_\alpha \leq D\} \subset U$ and $MN \int_0^\infty u^{-\alpha}e^{-\beta u}du < D$;

b) $\theta \equiv \lambda M \int_0^\infty u^{-\alpha}e^{-\beta u}e^{(\mu+\Delta M_2)u}du$ has $M_1\theta \leq \Delta/(1+\Delta)$ and

$\theta \max\{1, \frac{(1+\Delta)M_2}{\mu+M_2\Delta}\} < 1.$

Then there exists an invariant manifold

$$S = \{(t,x,y) \mid x = \sigma(t,y), \; -\infty < t < \infty, \; y \in Y\}$$

with $\|\sigma(t,y)\|_\alpha \leq D$, $\|\sigma(t,y_1)-\sigma(t,y_2)\|_\alpha \leq \Delta\|y_1-y_2\|$. If f,g are periodic in t with period $p > 0$ then $\sigma(t+p,y) = \sigma(t,y)$; if f,g are independent of t, σ is independent of t.

If we assume also $\|f(t,x,y)-f(t',x,y)\| \leq B|t-t'|$ and the solution $y^h(t) = y^h(t;\tau,\eta,x(\cdot))$ of $dy^h/dt = g(t+h,x(t),y^h)$, $y^h(\tau) = \eta$, satisfies

$$\|y^h(t) - y^0(t)\| \leq B_1 e^{\mu(\tau-t)}|h|, \quad t \leq \tau,$$

then there exists a constant K so

$$\|\sigma(t+h,y) - \sigma(t,y)\|_\alpha \leq K|h|.$$

Remark. If $\mu < \beta$ and λ is sufficiently small, we can easily satisfy b) above. If $\|g(t,x,y)-g(t,\bar{x},\bar{y})\| \leq M_2\|x-\bar{x}\|_\alpha + \mu\|y-\bar{y}\|$, the requirements for the y-equation are met.

Proof of Th. 6.1.2. Let $\sigma: \mathbb{R} \times Y \to X^\alpha$ be any continuous function satisfying

(*) $\quad \|\sigma(t,y)\|_\alpha \leq D$, $\|\sigma(t,y)-\sigma(t,y')\|_\alpha \leq \Delta\|y-y'\|$

on $\mathbb{R} \times Y$. Let $y(t) = \psi(t;\tau,\eta,\sigma)$ be the solution of

$$dy/dt = g(t,\sigma(t,y),y) \quad \text{for} \quad t < \tau,\ y(\tau) = \eta;$$

then $\psi(t;\tau,\eta,\sigma)$ is well defined for all $t \leq \tau$. Finally, define $G(\sigma): \mathbb{R} \times Y \to X^{\alpha}$ by

$$G(\sigma)(\tau,\eta) = \int_{-\infty}^{\tau} e^{-A(\tau-s)} f(s,\sigma(s,y(s)),y(s))ds$$

where $y(s) = \psi(s;\tau,\eta,\sigma)$.

We prove that $G(\sigma)$ also satisfies (*), and in fact that G is a contraction on this class of functions in the uniform norm. First observe that, by assumption a),

$$\|G(\sigma)\|_{\alpha} \leq \int_{-\infty}^{\tau} M(\tau-s)^{-\alpha} e^{-\beta(\tau-s)} N\, ds \leq D.$$

Next, suppose σ and σ' are two functions satisfying (*) and $\eta,\eta' \in Y$, and set $y(t) = \psi(t;\tau,\eta,\sigma)$, $y'(t) = \psi(t;\tau,\eta',\sigma')$. Then for $t \leq \tau$,

$$\|y(t)-y'(t)\| \leq e^{\mu(\tau-t)} M_1\|\eta-\eta'\| + M_2\int_t^{\tau} e^{\mu(s-t)}\{\Delta\|y(s)-y'(s)\| + |||\sigma-\sigma'|||\}\, ds$$

where $|||\sigma-\sigma'||| = \sup\{\|\sigma(t,y)-\sigma'(t,y)\|_{\alpha} \mid (t,y) \in \mathbb{R} \times Y\}$. It follows (Gronwall inequality) that

$$\|y(t)-y'(t)\| \leq e^{(\mu+\Delta M_2)(\tau-t)}[M_1\|\eta-\eta'\| + \frac{\nu}{\mu+M_2\Delta}|||\sigma-\sigma'|||\} - M_2|||\sigma-\sigma'|||/(\mu+M_2\Delta),$$

thus

$$\|G(\sigma)(\tau,\eta)-G(\sigma')(\tau,\eta')\|_{\alpha} \leq \int_{-\infty}^{\tau} M(\tau-s)^{-\alpha} e^{-\beta(\tau-s)}\lambda\{|||\sigma-\sigma'||| + (1+\Delta)\|y(s)-y'(s)\|\}$$

$$\leq \lambda M(1 - \frac{M_2(1+\Delta)}{\mu+M_2\Delta})\int_0^{\infty} u^{-\alpha}e^{-\beta u}du\ |||\sigma-\sigma'||| + (1+\Delta)\theta[M_1\|\eta-\eta'\| + \frac{M_2}{\mu+M_2\Delta}|||\sigma-\sigma'|||]$$

$$\leq \Delta\|\eta-\eta'\| + \theta'|||\sigma-\sigma'|||,\quad \theta' = \max(1, \frac{(1+\Delta)M_2}{\mu+M_2\Delta})\theta < 1.$$

Therefore there exists a unique fixed point $\sigma = G(\sigma)$ in the class (*). It remains to show σ is a Lipschitz function of t, and that S is invariant.

For $\sigma = G(\sigma)$ as above, define σ^h, $|h| < h_0$, by $\sigma^h(t,y) = \sigma(t+h,y)$; then σ^h satisfies (*) and $y^h(t)$, the solution of

$$dy/dt = g(t+h,\sigma^h(t,y),y),\ t < \tau,\ y(\tau) = \eta$$

is well-defined. It follows, as above, that

$$\|G(\sigma)(\tau+h,\eta)-G(\sigma)(\tau,\eta)\|_\alpha = \left\|\int_{-\infty}^{\tau} e^{-A(\tau-s)}[f(s+h,\sigma(s+h,y^h(s)),y^h(s)) - f(s,\sigma(s,y(s)),y(s))]ds\right\|_\alpha$$

$$\leq \int_{-\infty}^{\tau} M(\tau-s)^{-\alpha}e^{-\beta(\tau-s)}[B|h| + \lambda|||\sigma^h-\sigma||| + \lambda(1+\Delta)\|y^h(s)-y(s)\|]ds,$$

and

$$\|y^h(t)-y(t)\| \leq e^{(\mu+M_2\Delta)(\tau-t)}[B_1|h| + \frac{M_2}{\mu+M_2\Delta}|||\sigma^h-\sigma|||] - \frac{M_2}{\mu+M_2\Delta}|||\sigma^h-\sigma|||$$

so

$$\|\sigma(\tau+h,\eta)-\sigma(\tau,\eta)\|_\alpha \leq B|h|\int_0^\infty Mu^{-\alpha}e^{-\beta u}du + (1+\Delta)B_1\theta|h| + \theta'|||\sigma^h-\sigma|||\ ,$$

which proves the result.

It remains only to prove invariance of S. Let $(t_0,x_0,y_0) \in S$, so $x_0 = \sigma(t_0,y_0)$, and define $y(t)$, $-\infty < t < \infty$, by

$$dy/dt = g(t,\sigma(t,y),y),\quad y(t_0) = y_0;$$

then set $x(t) = \sigma(t,y(t))$ for all t. This defines a curve $(t,x(t),y(t)) \in S$, $-\infty < t < \infty$, through the point (t_0,x_0,y_0); it suffices to prove $x(\cdot)$ satisfies

$$\frac{dx}{dt} + Ax = f(t,\sigma(t,y(t)),y(t)),\quad -\infty < t < \infty.$$

But this equation has a unique solution $x_b(t)$ which remains bounded as $t \to -\infty$, namely

$$x_b(t) = \int_{-\infty}^{t} e^{-A(t-s)}f(s,\sigma(s,y(s)),y(s))ds,$$

so $x_b(t) = \sigma(t,y(t))$ is indeed a solution.

If f, g are periodic in t with period $p > 0$, we may restrict attention in the above argument to p-periodic functions σ which satisfy (*). In that case, $y(t+p;\tau+p) = y(t;\tau)$ and $G(\sigma)(\tau+p,\eta) = G(\sigma)(\tau,\eta)$, and the argument proceeds as before.

Exercise 1. Assume f, g depend also on a parameter ε, $|\varepsilon| < \varepsilon_0$, and the estimates above are uniform in ε. Also assume (with obvious notation) $\|y(t;\tau,\eta,x(\cdot),\varepsilon) - y(t;\tau,\eta,x(\cdot),\varepsilon')\| \le K_1|\varepsilon-\varepsilon'|e^{\mu(\tau-t)}$ for $t \le \tau$, $|\varepsilon| < \varepsilon_0$, $|\varepsilon'| < \varepsilon_0$. Prove that the invariant manifold $S_\varepsilon = \{(t,x,y) \mid x = \sigma(t,y,\varepsilon)\}$ is a Lipschitz function of ε.

Definition 6.1.3. Let S be a local invariant manifold of a differential equation. A subset Σ of S is *stable* if, for any t_0 and any $\varepsilon > 0$, there exists $\delta > 0$ such that when (t_0,x_0) is in a δ-neighborhood of the time-slice $\Sigma \cap \{t_0\}$, then $(t,x(t;t_0,x_0))$ is in an ε-neighborhood of $\Sigma \cap \{t\}$ for all $t \ge t_0$, where $x(\cdot;t_0,x_0)$ is the solution of the differential equation through (t_0,x_0). We say Σ is *stable with respect to the flow in* S provided the stability claim holds for $(t_0,x_0) \in S$, in a δ-neighborhood of $\Sigma \cap \{t_0\}$. Similarly define instability and uniform asymptotic stability of Σ.

Theorem 6.1.4. Under the hypotheses of Thm. 6.1.2 above, assume also $(x,y) \to g(t,x,y)$: $U \times Y \to Y$ satisfies a uniform Lipschitz condition and, with $\mu' = \mu + M_2\Delta$,

$$r \equiv \theta(1 + M_1M_2(1+\Delta)/(\beta-\mu')) < (\beta/(\beta-\mu'))^{1-\alpha}$$

so $\gamma = \beta-(\beta-\mu')r^{1/(1-\alpha)} > 0$. Then the invariant manifold S is exponentially attracting. Specifically, if $(x(t),y(t))$ is any solution on $t_0 \le t \le t_1$ with $x(t) \in U$ on this interval, then on $[t_0,t_1]$

$$\|x(t)-\sigma(t,y(t))\|_\alpha \le KMe^{-\gamma(t-t_0)}\|x(t_0)-\sigma(t_0,y(t_0))\|_\alpha$$

where K is a constant depending only on α.

Further if $\Sigma \subset S$ is uniformly asymptotically stable with respect to the flow in S, then Σ is uniformly asymptotically stable. If $\Sigma \subset S$ is unstable with respect to the flow in S, then S is unstable.

Remark. In the applications we ordinarily have in mind either

$\Sigma = \{(t,x_0,y_0) \mid -\infty < t < \infty\}$ where (x_0,y_0) is an equilibrium point, or $\Sigma = \{(t,x(t),y(t)) \mid \text{all } t\}$ where $(x(t),y(t))$ is a periodic solution, or a family of periodic solutions. It is easily shown that (in case f, g are independent of t) asymptotic stability of Σ is equivalent to asymptotic stability for the equilibrium point, or orbital asymptotic stability of the periodic solution.

<u>Proof of Thm. 6.1.4</u>. Suppose $x(t)$, $y(t)$ is a solution on $t_0 \le t \le t_1$ with $x(t) \in U$ and let $\xi(t) = x(t) - \sigma(t,y(t))$. Let $y(s;t)$, $s \le t$, be the solution of

$$\frac{dy}{ds} = g(s,\sigma(s,y),y), \quad s \le t$$
$$y = y(t) \quad \text{when} \quad s = t.$$

Then by Gronwall's inequality, with $\mu' = \mu + M_2\Delta$,

$$\|y(s;t)-y(s)\| \le M_2 \int_s^t e^{\mu'(\theta-s)}\|\xi(\theta)\|_\alpha d\theta, \quad t_0 \le s \le t,$$

and

$$\|y(s;t)-y(s;t_0)\| \le M_1M_2 \int_{t_0}^t e^{\mu'(\theta-s)}\|\xi(\theta)\|_\alpha d\theta, \quad s \le t_0 \le t.$$

For $t_0 \le t \le t_1$ we have

$$\xi(t)-e^{-A(t-t_0)}\xi(t_0) = \int_{t_0}^t e^{-A(t-s)}[f(s,x(s),y(s)) - f(s,\sigma(s,y(s;t)),\ y(s;t))]$$
$$+ \int_{-\infty}^{t_0} e^{-A(t-s)}[f(s,\sigma(s,y(s;t_0)),y(s;t_0)) - f(s,\sigma(s,y(s;t)),y(s;t))]ds$$

so

$$\|\xi(t)\|_\alpha \le Me^{-\beta(t-t_0)}\|\xi(t_0)\|_\alpha + \lambda M \int_{t_0}^t \|\xi(s)\|_\alpha J(t,s)ds$$

where

$$J(t,s) = (t-s)^{-\alpha}e^{-\beta(t-s)} + M_1M_2(1+\Delta)e^{\mu'(s-t)} \int_{t-s}^\infty u^{-\alpha}e^{-(\beta-\mu')u}du$$
$$\le (t-s)^{-\alpha}e^{-\beta(t-s)}\{1 + M_2M_1(1+\Delta)/(\beta-\mu')\}.$$

If $b = \lambda M\{1 + M_1M_2(1+\Delta)/(\beta-\mu')\}$, we have

$$e^{\beta(t-t_0)}\|\xi(t)\|_\alpha \le M\|\xi(t_0)\|_\alpha + b\int_{t_0}^{t}(t-s)^{-\alpha}e^{\beta(s-t_0)}\|\xi(s)\|_\alpha ds.$$

Such integral inequalities are discussed in sec. 7.1 and lemma 7.1.1 shows, for a constant K depending only on α,

$$\|\xi(t)\|_\alpha e^{\beta(t-t_0)} \le KM\|\xi(t_0)\|_\alpha\, e^{q(t-t_0)}$$

where $q = (b\Gamma(1-\alpha))^{1/(1-\alpha)} = (\beta-\mu')r^{1/(1-\alpha)} = \beta-\gamma$, so the estimate is proved.

Now the flow in S is simply the restriction to S of the flow in $U\times Y$, so instability of Σ with respect to the flow in S trivially implies instability of Σ. Suppose Σ is uniformly asymptotically stable for the flow in S. By an argument analogous to that in Thm. 4.2.1 (or ex. 4 following it, or see Yoshizawa [135]) there exists $V\colon S\to\mathbb{R}$ such that with $z(t) = (\sigma(t,y),y)\in S$ and $dy/dt = g(t,\sigma(t,y),y)$, we have

$$a(\mathrm{dist}((t,z),\Sigma_t)) \le V(t,z) \le \mathrm{dist}(t,z),\Sigma_t)$$

$$|V(t,z_1)-V(t,z_2)| \le \|z_1-z_2\| \le (1+\Delta)\,\|y_1-y_2\|\ ,$$

and

$$\dot V(t,z) = \overline{\lim_{h\to0+}}\ \frac{1}{h}\{V(t+h,z(t+h))-V(t,z(t))\} \le -V(t,z).$$

Here $a(\cdot)$ is continuous and strictly increasing with $a(0) = 0$, and $\Sigma_t = \Sigma\cap\{t\}$ is the time-slice at t.

If (t_0,x_0,y_0) is near Σ_{t_0}, $x_0-\sigma(t_0,y_0) = \xi_0$ and $z_0 = (\sigma(t_0,y_0),y_0)$, define

$$W(t_0,x_0,y_0) = V(t_0,z_0) + P\|\xi_0\|_\alpha$$

where P is a positive constant chosen below. Choose $T>0$ so large that

$$e^{-T}\le 1/2 \quad\text{and}\quad KMe^{-\gamma T}\le 1/4.$$

If (t_0,x_0,y_0) is sufficiently close to S, i.e. $\|\xi(t_0)\|_\alpha$ is small enough, the solution through this point will exist with $x(t)\in U$ on $t_0\le t\le t_0+T$, and so

$$W(t_0+T,x(t_0+T),y(t_0+T)) = V(t_0+T,z(t_0+T)) + P\|\xi(t_0+T)\|_\alpha$$

$$\leq e^{-T}V(t_0,z(t_0)) + (1+\Delta)e^{-T}\|y(t_0;t_0+T)-y(t_0)\|$$

$$+ PMK\ e^{-\gamma T}\|\xi(t_0)\|_\alpha$$

$$\leq \frac{1}{2} V(t_0,z(t_0)) + \frac{1}{4} P\|\xi(t_0)\|_\alpha$$

$$+ (1+\Delta)M_2KM\ e^{-T}\left(\frac{e^{(\mu'-\gamma)T}-1}{\mu'-\gamma}\right)\|\xi(t_0)\|_\alpha$$

$$\leq \frac{1}{2} W(t_0,x(t_0),y(t_0))$$

provided P is chosen large enough (depending only on the given constants and T). We also choose P so large that $\|\xi(t_0)\|_\alpha \leq 1/P$ implies the solution exists with $x(t) \in U$ on $t_0 \leq t \leq t_0+T$, uniformly in t_0. Then $\hat{V}(t_0,x(t_0),y(t_0)) \leq 1$ implies the solution exists for all $t \geq t_0$ and $\hat{V}(t,x(t),y(t)) \leq \text{Constant} \cdot 2^{-(t-t_0)/T}$, which proves uniform asymptotic stability of Σ.

<u>Corollary 6.1.5</u>. Suppose the hypotheses of Th. 6.1.4 are strengthened so $r < 1$ and $\dim Y < \infty$. Then S is uniformly asymptotically stable with asymptotic phase. Specifically, there exist $\delta > 0$, $c > 0$ so that any solution $(x(t),y(t))$ starting with $\|x(t_0)-\sigma(t_0,y(t_0))\|_\alpha < \delta$ exists for all $t \geq t_0$ and there is a solution $\bar{y}(t)$ of $d\bar{y}/dt = g(t,\sigma(t,\bar{y}),\bar{y})$ such that for $t \geq t_0$,

$$\|y(t)-\bar{y}(t)\| + \|x(t)-\sigma(t,\bar{y}(t))\|_\alpha$$

$$\leq Ce^{-\gamma(t-t_0)}\|x(t_0)-\sigma(t_0,y(t_0))\|_\alpha .$$

<u>Proof</u>. Observe $\|\sigma(t,y)\|_\alpha \leq MN \int_0^\infty u^{-\alpha}e^{-\beta u}du < D$ so $\|x(t)\|_\alpha \leq \|\sigma(t,y(t))\|_\alpha + KMe^{-\gamma(t-t_0)}\delta < D$ on the domain of existence $(t \geq t_0)$, if δ is sufficiently small, so the solution exists for all $t \geq t_0$. Also (with notation from Th. 6.1.4))

$$\|y(s)-y(s;t)\| \leq M_2 \int_s^t e^{\mu'(\theta-s)}MKe^{-\gamma(\theta-s)}\|\xi(s)\|_\alpha d\theta$$

$$\leq M_2MK/(\gamma-\mu')\|\xi(s)\|_\alpha \quad \text{for} \quad t_0 \leq s \leq t.$$

(Note $\gamma > \mu'$ since $r < 1$.) Choose $t_n \to +\infty$ so $y(t_0;t_n)$ converges in Y and let $\bar{y}(s)$ be the solution in S with $\bar{y}(t_0) = \lim_n y(t_0;t_n)$. Then for every $s \geq t_0$, $\bar{y}(s) = \lim_n y(s;t_n)$ and

$$\|y(s)-\bar{y}(s)\| \leq \frac{M_2(MK)^2}{\gamma-\mu'}\|\xi(t_0)\|_\alpha\, e^{-\gamma(s-t_0)}$$

which implies the result.

Remark. If Y is infinite dimensional but $\eta \to g(t,\sigma(t,y(t)+\eta), y(t)+\eta)$ is uniformly C^1 in a neighborhood of $\eta = 0$ for $t \geq t_0$, we again have asymptotic phase. This fact was pointed out by Prof. Jack Carr.

Lemma 6.1.6. Let X,Y be Banach spaces, and U an open set in X; then a closed ball in $C^r(U,Y)$ ($r \neq$ integer) or in $C^k_{\mathrm{Lip}}(U,Y)$ ($k = 0,1,2,\ldots$) is also closed in the uniform norm of $C^0(U,Y)$. (Note: the conclusion fails for $C^k(\mathbb{R},\mathbb{R})$, $k = 1,2,\ldots,$)

Proof. The result is trivial for $0 < r < 1$ or C^0_{Lip}; we prove the result for the first derivative and the general case follows easily.

Suppose $u_n: U \to Y$, $\|u_n(x)\| \leq B$, $\|u_n'(x)\| \leq B$ and $\|u_n'(x)-u_n'(y)\| \leq B\|x-y\|^s$ for x,y in U and $n = 1,2,3,\ldots$; here $B > 0$ and s, $0 < s \leq 1$, are fixed. If $u_n(x) \to u(x)$ as $n \to \infty$ uniformly for $x \in U$, we show u is differentiable and satisfies the same estimates. Let $d(x) = \min\{1, \operatorname{dist}(x,\partial U)\}$; then for $x \in U$ and $h \in X$, $\|h\| < d(x)$, and any $f \in C^{1+s}(U,Y)$,

$$\begin{aligned}\|f'(x)h\| &\leq \|f(x+h)-f(x)\| + \|f(x+h)-f(x)-f'(x)h\| \\ &\leq 2\|f\|_{C^0} + \|f\|_{C^{1+s}}\|h\|^{1+s}\end{aligned}$$

so with $f = u_n-u_m$

$$\|u_n'(x)-u_m'(x)\| \leq \frac{2}{\|h\|}\|u_n-u_m\|_{C_0} + 2B\|h\|^s \quad \text{for } \|h\| < d(x)$$

and if $\|u_n-u_m\|_{C^0} \leq 1$,

$$d(x)\|u_n'(x)-u_m'(x)\| \leq 4(1+B)\cdot\|u_n-u_m\|_{C^0}^{s/(1+s)}.$$

It follows that u is differentiable and $u_n'(x) \to u'(x)$ for each $x \in U$ (uniformly if x is bounded away from ∂U) so $\|u'(x)\| \leq B$

and $\|u'(x)-u'(y)\| \le B\|x-y\|^s$.

Exercise 2. Show the C^0-closure of the unit ball in $C^1(\mathbb{R})$ is the unit ball in $C^0_{Lip}(\mathbb{R})$.

Exercise 3. If $u_n: U \to Y$ satisfy $\|u_n'(x)\| \le B$ and $\|u_n'(x)-u_n'(\bar{x})\| \le \omega(\|x-\bar{x}\|)$ for $x,\bar{x}$ in U and all $n \ge 1$, where $\omega(\cdot)$ is a continuous increasing function and $\omega(0) = 0$, and if $u_n(x) \to u(x)$ uniformly in $x \in U$ as $n \to \infty$, prove u is differentiable, $\|u'(x)\| \le B$ and $\|u'(x)-u'(\bar{x})\| \le \omega(\|x-\bar{x}\|)$.

Theorem 6.1.7. Assume in addition to the hypotheses of Th. 6.1.2 that $m > 1$ and $(f(t,\cdot,\cdot), g(t,\cdot,\cdot))$ is uniformly bounded in $C^m(U \times Y, X \times Y)$ (or $C^{m-1}_{Lip}(U \times Y, X \times Y)$ when m is an integer). Let $\mu' = \mu+M_2\Delta$, $m\mu' < \beta$,

$$\theta_p = \lambda M \int_0^\infty t^{-\alpha} e^{-\beta t} e^{p\mu' t} dt$$

and assume

$$\theta_p M_1^p (1 + M_1(1+\Delta)\sup\|g_x\|/(p-1)\mu') < 1$$

for every integer p in $1 < p < m$ and for $p = m$. Then $\sigma(t,\cdot)$: $Y \to U \subset X^\alpha$ is uniformly bounded in C^m or C^{m-1}_{Lip}, respectively.

Remark. Smoothness proofs in C^m, m = integer, are much more difficult.

Proof. By the lemma, it is sufficient to show the map G in the proof of Thm. 6.1.2 takes some closed ball in $S \cap C^m(Y,X^\alpha)$ into itself; we use notation from that proof, and S is the class of $\sigma: \mathbb{R} \times Y \to X^\alpha$ satisfying condition (*) (p.144). We prove the result for C^m, $1 < m < 2$ or C^1_{Lip} $(m = 2)$, and sketch the required estimates for the general case.

Let $m = 1+\delta$, $0 < \delta \le 1$, $\sigma \in S$ with y-derivative $\sigma'(t,y)$ bounded (in fact $\|\sigma'(t,y)\| \le \Delta$), and $\|\sigma'(t,y)-\sigma'(t,\bar{y})\| \le H_\delta(\sigma'(t))\,\|y-\bar{y}\|^\delta$ with Hölder constant $H_\delta(\sigma'(t))$ bounded. Let $y(t) = y(t;\tau,\eta,\sigma)$ be the solution of $dy/dt = g(t,\sigma(t,y),y)$, $t \le \tau$, $y(\tau) = \eta$, and let $\bar{y}(t) = y(t;\tau,\bar{\eta},\sigma)$. First by Gronwall's inequality if $y'(t) = \frac{\partial}{\partial\eta}y(t;\tau,\eta,\sigma)$, we have

$$\|y'(t)\| \leq M_1 e^{\mu'(\tau-t)} \quad \text{for} \quad t \leq \tau,$$

and then

$$\|y'(t)-\bar{y}'(t)\| \leq \int_t^\tau M_1 e^{\mu(s-t)} \{\|g_x\| H_\delta(\sigma') + (1+\Delta)^m \|g\|_{C^m}\} \cdot$$

$$\cdot \|y(s)-\bar{y}(s)\|^\delta \|y'(s)\| ds$$

so

$$H_\delta(y'(t)) \leq M_1^{m+1} [\|g_x\| H_\delta(\sigma') + (1+\Delta)^m \|g\|_{C^m}] \frac{e^{m\mu'(\tau-t)}}{(m-1)\mu'} \equiv M_m e^{m\mu'(\tau-t)}.$$

Also $\|\frac{\partial}{\partial\eta} G(\sigma)(\tau,\eta)\| \leq \Delta$ as proved before

$$H_\delta[\tfrac{\partial}{\partial\eta} G(\sigma)(\tau,\cdot)] \leq \int_{-\infty}^t M(\tau-s)^{-\alpha} e^{-\beta(\tau-s)} \lambda(1+\Delta) M_m e^{m\mu'(\tau-s)}$$

$$+ (\lambda H_\delta(\sigma') + (1+\Delta)^m \|f\|_{C^m})(M_1 e^{\mu'(\tau-s)})^m$$

$$\leq \theta_m [(1+\Delta)M_m + M_1^m H_\delta(\sigma')] + M M_1^m (1+\Delta)^m \|f\|_{C^m} \int_0^\infty u^{-\alpha} e^{-\beta u} e^{m\mu'\mu} du$$

$$\leq \theta_m M_1^m [1 + M_1(1+\Delta)\|g_x\|/(m-1)\mu'] H_\delta(\sigma')$$

$$+ \{\text{terms independent of } H_\delta(\sigma')\}.$$

Since the coefficient of $H_\delta(\sigma')$ is less than one, there exists $B > 0$ so $H_\delta(\sigma') \leq B$ implies $H_\delta(\frac{\partial}{\partial\eta} G(\sigma)) \leq B$ and the result follows.

For the higher order derivatives, we use the estimates:

$$\|y^{(k)}(t)\| = \|\frac{\partial^k}{\partial\eta^k} y(t;\tau,\eta,\sigma)\| \leq M_k e^{k\mu'(\tau-t)}, \quad t \leq \tau,$$

and

$$H_\delta(y^{(k)}(t)) \leq M_{k+\delta} e^{(k+\delta)\mu'(\tau-t)} \qquad (0 < \delta \leq 1)$$

where

$$M_k = \|g_x\| M_1^{k+1} \|\sigma^{(k)}\|/(k-1)\mu' + \{\text{terms in } \|g\|_{C^k} \text{ and } \|\sigma\|_{C^{k-1}})\}$$

$$(m=k+\delta) \quad M_m = \|g_x\| M_1^{m+1} H_\delta(\sigma^{(k)})/(m-1)\mu' + \{\text{terms in } \|g\|_{C^m} \text{ and } \|\sigma\|_{C^k}\}.$$

Also

$$\|(\tfrac{\partial}{\partial\eta})^k G(\sigma)(\tau,\eta)\| \leq \theta_k [(1+\Delta)M_k + M_1^k \|\sigma^{(k)}\|]$$

$$+ \{\text{terms in } \|f\|_{C^k} \text{ and } \|\sigma\|_{C^{k-1}}\}$$

and if $m = k+\delta$ $(0 < \delta \leq 1)$

$$H_\delta((\frac{\partial}{\partial\eta})^k G(\sigma)(\tau,\eta)) \leq \theta_m[(1+\Delta)M_m + M_1^m H_\delta(\sigma^{(k)})]$$

$$+ \{\text{terms in } \|f\|_{C^m} \text{ and } \|\sigma\|_{C^k}\}.$$

The coefficient of the highest order derivative of σ (or of $H_\delta(\sigma^{(k)})$) is less than one so the result follows as before.

Remark. When $(x,y) \to (f(t,x,y), g(t,x,y))$ is uniformly continuously differentiable, one can prove $y \to \sigma(t,y)$ is uniformly continuously differentiable along the lines of [115], but the argument is much more difficult than that above. I don't know if the result holds without such a uniformity condition. Hirsch, Pugh and Shub [120] develop a method of "Lipschitz jets" which may be useful for these questions, but we shall generally just avoid the C^m case, m = integer.

Definition 6.1.8. Suppose X is a Banach space, U is an open set in X and $T: U \to X$ is continuous. A set $\Sigma \subset U$ is *positively invariant* under T if $T(\Sigma) \cap U \subset \Sigma$; it is *negatively invariant* under T if $\Sigma \cap T(U) \subset T(\Sigma)$; and Σ is *invariant* if it satisfies both conditions. Also, Σ is *locally invariant* (or locally positively or negatively invariant) under T if each point of U has an open neighborhood V such that $\Sigma \cap V$ is invariant (or positively or negatively invariant) under the restriction $T|V$.

Remark. We have *not* assumed that T is injective or that $T(U) \subset U$. Similar definitions can be given for local invariance under a dynamical system (cf. def. 4.3.1) and then we might take T as the time-one map or the Poincaré map on a surface of section (see sec. 8.4).

We have found locally invariant manifolds -- the local stable and unstable manifolds (Thm. 5.2.1) and the local strongly-unstable and center-stable manifolds (ex. 4, sec. 5.1, and ex. 2, sec. 5.2) -- and others will be examined in sec. 6.2 and Chapter 8. We now give conditions which allow such locally invariant manifolds to be extended to invariant manifolds, preserving smoothness.

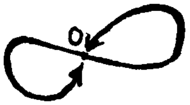
Theorem 6.1.9. Suppose X is a Banach space, U is an open set in X, $T: U \to X$ is a C^r map $(1 \leq r \leq \infty$ or $r = \omega$, when T is analytic) and Σ is a C^r submanifold of U.

(i) If Σ is finite dimensional and locally negatively invariant, and if T and the derivative $T'(x)$ are injective at each point x of $\Sigma^+ = \bigcup_{n=0}^{\infty} T^n(\Sigma)$, then Σ^+ is an injectively immersed C^r manifold in U with the same dimension as Σ, which is positively invariant and locally negatively invariant. If Σ is negatively invariant then Σ^+ is invariant.

(ii) If Σ has finite codimension and is locally positively invariant, and if T is injective and the derivative $T'(x)$ has dense range at each point x of $\Sigma^- = \bigcup_{n=0}^{\infty} T^{-n}(\Sigma)$ then Σ^- is an injectively immersed C^r manifold in U with the same codimension as Σ which is negatively invariant and locally positively invariant under T. If Σ is positively invariant, then Σ^- is invariant.

Remarks. Verification that T or $T'(x)$ is injective requires a backward uniqueness theorem, and we show the range of $T'(x)$ is dense by proving the adjoint map is injective, using backward uniqueness in the adjoint equation. These points are discussed in sec. 7.3.

The extended manifolds $\Sigma^{\pm}$ are merely immersed rather than imbedded: the topology they inherit from Σ may not agree with the relative topology of $\Sigma^{\pm}$ in U. The standard example is the image of $\mathbb{R}$ in $\mathbb{R}^2$ pictured here:



where the two ends of $\mathbb{R}$ approach 0 without reaching it. Oscillatory behavior (like $\sin 1/x$ as $x \to 0$) is also possible and may be

more common. A class of gradient flows was studied in sec. 5.3 and in this case the global stable and unstable manifolds are imbedded (see Thm. 6.1.10).

Proof of Thm. 6.1.8. The invariance properties of $\Sigma^{\pm}$ are immediate.

(i) Σ is represented locally as $\{h(\xi) \mid \xi \in \mathbb{R}^k, |\xi| < 1\}$ where h is a C^r map from $\mathbb{R}^k$ to X whose derivative $h'(\xi)$ has rank k at each ξ, so $T^n(\Sigma)$ is represented locally as the image of the C^r map $T^n \cdot h$. Since $(T^n)'(h(\xi))$ is injective, the derivative of $\xi \to T^n(h(\xi))$ has rank k and the result follows.

(ii) Σ is represented locally as $\{x \in V \mid g(x) = 0\}$ where V is an open set in U, $g: V \to \mathbb{R}^k$ is C^r and $g'(x)$ has rank k at each point where $g = 0$. Thus $T^{-n}(\Sigma)$ is represented locally as $(g \cdot T^n)^{-1}(0)$ and $x \to g(T^n x)$ is a C^r map whose derivative has rank k at each point where $g(T^n x) = 0$.

Theorem 6.1.10. Suppose U is an open set in a Banach space X, $T: U \to X$ is C^r $(r \geq 1)$ and injective. Assume $x_0 = T(x_0) \in U$, $\sigma(T'(x_0))$ is disjoint from the unit circle and there are finitely many eigenvalues (counted with their multiplicity) outside the unit circle. Define the positively (or negatively) invariant C^r manifold $W^s_{loc}(x_0)$ (or $W^u_{loc}(x_0)$) and suppose these may be extended to invariant C^r injectively immersed manifolds $W^s(x_0)$, $W^u(x_0)$ as in Thm. 6.1.9. Finally assume there exists continuous $V: U \to \mathbb{R}$ such that $T(x) \neq x$ implies $V(T(x)) < V(x)$. Then $W^s(x_0)$, $W^u(x_0)$ are imbedded submanifolds of U which intersect only at x_0, and there is an open neighborhood Q of $W^s(x_0)$ such that if $x \in Q \setminus W^s(x_0)$, there exists an integer N such that $T^n(x)$ is defined for $n \leq N$ and $T^n(x) \notin Q$ for all $n \geq N$, as long as $T^n(x)$ is defined.

We prove these manifolds are imbedded by showing, if $x \in W^s(x_0)$ (or $W^u(x_0)$), there is a neighborhood U_1 of x and an integer $N \geq 0$ so

$$U_1 \cap W^s(x_0) \subset T^{-N}(W^s_{loc}(x_0))$$

or

$$U_1 \cap W^u(x_0) \subset T^{N}(W^u_{loc}(x_0)).$$

First we need a result of Chafee [110]:

Lemma 6.1.11. Under the hypotheses of the theorem, there exists a neighborhood U_0 of x_0 such that, if $x \in U_0 \setminus W^s_{loc}(x_0)$, there is an integer $N \geq 0$ (depending on x) so $T^n(x)$ is defined for $n \leq N$

and $T^n(x) \notin U_0$ for all $n \geq N$, as long as $T^n(x)$ is defined. Thus $T^n(x)$ eventually leaves U_0, never to return.

Proof. Choose $\delta_1 > 0$ so $T^n(x) \in B_{\delta_1}(x_0)$ for all $n \geq 0$ (or, all $n \leq 0$) implies $x \in W^s_{loc}(x_0)$ (or $W^u_{loc}(x_0)$). Suppose also $\mathrm{dist}(T^n(x), W^u_{loc}(x_0)) \leq Ka^n$ for $n \geq 0$ as long as $T^n(x)$ remains inside $B_{\delta_1}(x_0)$, where K,a are constants with $0 < a < 1$. Also choose $\eta > 0$ so $\sup V(W) < V(x_0)$ where

$$W = \{x \mid \delta_0 \leq \|x\| \leq \delta_1, \ \mathrm{dist}(x, W^u_{loc}(x_0)) < \eta\}$$

and $0 < \delta_0 < \delta_1$ is chosen so

$$T(B_{\delta_0}(x_0)) \subset B_{\delta_1}(x_0).$$

Choose $n_0 > 0$ so $Ka^{n_0} < \eta$, and then choose U_0 as a neighborhood of x_0 such that $T^n(U_0) \subset B_{\delta_0}(x_0)$ for $n \leq n_0$ and $\sup V(W) < \inf V(U_0)$. If $x \in U_0 \setminus W^s_{loc}(x_0)$ then $T^n(x)$ eventually leaves $B_{\delta_1}(x_0)$ and there exists N such that $T^n(x) \in B_{\delta_1}(x_0)$ for $n \leq N$ and $T^{N+1}(x) \notin B_{\delta_1}(x_0)$. Since $N \geq n_0$, we have $T^N(x) \in W$ and $V(T^n(x)) \leq V(T^N(x)) < \inf V(U_0)$ for $n \geq N$, so $T^n(x) \notin U_0$ for $n \geq N$.

Proof of Thm. 6.1.10. Suppose $x \in W^s(x_0)$; there exists $N \geq 0$ so $T^N(x) \in W^s_{loc}(x_0) \cap U_0$, where U_0 is the neighborhood provided by the lemma above. There is a neighborhood U_1 of x so $T^N(U_1) \subset U_0$ and by the lemma,

$$T^N(U_1 \cap W^s(x_0)) \subset W^s_{loc}(x_0),$$

which proves $W^s(x_0)$ is embedded. For the last statement of the theorem, take $Q = \bigcup_{n \geq 0} T^{-n}(U_0)$.

Now suppose $x \in W^u(x_0) \setminus W^u_{loc}(x_0)$; then (in notation of the lemma) $T^{-n}(x) \to 0$ as $n \to \infty$ but $\|T^{-n}(x)\| > \delta_1$ for some $n \geq 0$ and $T^{-N_0}(x) \in \overline{W} = W \cap W^u_{loc}(x_0)$ for some $N_0 > 0$. Since $\overline{W}$ is compact, there exists $N_1 > 0$ so

$$\sup V(T^{N_1}(\overline{W})) < V(x)$$

and there is a neighborhood U_1 of x such that

$$\sup V(T^{N_1}(\overline{W})) < \inf V(U_1).$$

If $y \in U_1 \cap W^u(x_0)$, there exists $N \geq 0$ with $T^{-N}(y) \in \overline{W}$ and so $N < N_1$ and

$$U_1 \cap W^u(x_0) \subset T^{N_1}(W^u_{loc}(x_0)).$$

An example from the theory of combustion. In section 5.1 we discussed the system

$$\partial n/\partial t = D\Delta n - \varepsilon n f(T), \quad \partial n/\partial \nu = 0 \text{ on } \partial\Omega$$

$$\partial T/\partial t = \Delta T + q n f(T), \quad T = 1 \text{ on } \partial\Omega$$

where $f(T) = e^{-H/T}$, and q, D, H, ε are positive constants (ε small) and Ω is a bounded smooth domain. We showed that non-negative initial data $(n,T)|_{t=0}$ gives a non-negative solution which tends to $n = 0$, $T = 1$ as $t \to +\infty$. For small $\varepsilon > 0$, the convergence is $O(e^{-\varepsilon f(1)t})$, i.e. rather slow. Also it is known that n, T may undergo rather large motions -- commonly called explosions -- before settling down, so it is of interest to follow the behavior for all $t > 0$ -- at least for $t \leq O(1/\varepsilon)$. This problem was treated by Sattinger [132] with "two-time" methods; using invariant manifold theory and simple estimates, we can give a more straightforward treatment without Sattinger's restrictions on the initial data.

First observe that, for small $\varepsilon > 0$, the concentration (n) equation is nearly uncoupled from the temperature (T) equation. Write $n = n_0 + n_1$ where $n_0 = \langle n \rangle = \frac{1}{|\Omega|} \int_\Omega n \, dx$ is the space average, and n_1 has average value zero. Given any bounded, locally Hölder continuous curve $\{T(t), -\infty < t < \infty\}$ of non-negative $L_2(\Omega)$ functions, $T(t) = T(0)$ for $t \leq 0$, we can find an invariant manifold for the concentration equation in the form

$$n_1(t) = \sigma(T_t, \varepsilon) n_0(t),$$

which is linear in n_0 and depends only on the segment $T_t = \{T(s), s \leq t\}$ of the curve. Theorem 6.1.2 does not directly apply to linear equations, but following the proof we find

$$\sigma(T_t,\varepsilon) = -\varepsilon \int_{-\infty}^{0} e^{As} E(f(T(t+s)))\cdot$$
$$\cdot\exp(\varepsilon\int_{t+s}^{t} \langle f(T(\theta))(1+\sigma(T_\theta,\varepsilon))\rangle)ds$$

where A is $-D\Delta$ on functions with mean value zero and normal derivative zero on $\partial\Omega$, $\langle\phi\rangle = \frac{1}{|\Omega|}\int_\Omega \phi\, dx$ and $E\phi = \phi-\langle\phi\rangle$. This equation can be solved by iteration when ε is small (note $f(T)$ is uniformly bounded) and

$$\|\sigma(T_t,\varepsilon)\|_{L_2(\Omega)} \leq 2\varepsilon/D\lambda_1$$

for small $\varepsilon > 0$, where λ_1 is the second eigenvalue (the first positive eigenvalue) of the Neumann problem in Ω.

We prove now that $\|n_1(t)\|_{L_2(\Omega)}$ -- after an initial transient - becomes and remains small, hence it approaches the invariant manifold $n_1 = \sigma(T_t)n_0$ exponentially as $t \to \infty$.

Specifically note first that

$$\frac{d}{dt}\int_\Omega n^2 \leq -2D\int_\Omega |\nabla n|^2 dx \leq -2D\lambda_1 \int_\Omega n_1^2 dx$$

and

$$\frac{d}{dt} n_0 = -\varepsilon\langle nf(T)\rangle$$

so (recalling n,T are ≥ 0)

$$\|n(t)\|_{L_2} \leq \|n(0)\|_{L_2}, \quad n_0(t) \leq n_0(0)$$

and

$$\frac{d}{dt}\int_\Omega n_1^2 dx \leq -2D\lambda_1 \int_\Omega n_1^2 dx + 2\varepsilon n_0\langle nf(T)\rangle$$
$$\leq -2D\lambda_1 \int_\Omega n_1^2 dx + 2\varepsilon\|n(0)\|_{L_2}^2$$

so

$$\int_\Omega n_1^2 dx \leq (\int_\Omega n_1^2 dx)_{t=0} e^{-2D\lambda_1 t} + \frac{\varepsilon}{D\lambda_1}\|n(0)\|_{L_2}^2.$$

Once $\|n_1\|_{L_2(\Omega)}$ becomes small, it enters the region of attraction of the invariant manifold, and so

$$\|n_1(t) - \sigma(T_t,\varepsilon)n_0(t)\|_{L_2} = O(e^{-\frac{1}{2}D\lambda_1 t})$$

as $t \to \infty$.

To study the equations on a time-interval $O(1/\varepsilon)$ it is convenient to change the time variable to $\tau = \varepsilon t$; then

$$\|n_1(\tau)-\sigma(T_\tau,\varepsilon)n_0(\tau)\|_{L_2} = O(e^{-\lambda_1 D\tau/2\varepsilon})$$

$$\frac{dn_0}{d\tau} = -\langle f(T(\tau))(1+\sigma(T_\tau,\varepsilon))\rangle n_0(\tau)$$

$$(k = \lambda_1 D/2) \qquad\qquad + O(e^{-k\tau/\varepsilon})$$

and

$$\varepsilon \frac{\partial T}{\partial \tau} = \Delta T + qn_0(\tau)f(T(\tau))(1+\sigma(T_\tau,\varepsilon)) + O(e^{-k\tau/\varepsilon}).$$

Also

$$\sigma(T_\tau,\varepsilon) = -\varepsilon A^{-1}E(f(T(\tau))) + O(\varepsilon^2)$$

if $\tau \to T(\tau)$ is Lipschitz continuous at this point (or $t \to T(t)$ has Lipschitz constant $O(\varepsilon)$, in the original time variable).

To study the temperature equation more closely, we first estimate $\int_\Omega |\nabla n_1|^2$:

$$\frac{d}{d\tau}\int_\Omega |\nabla n_1|^2 dx = -\frac{2D}{\varepsilon}\int_\Omega (\Delta n_1)^2 + 2\int_\Omega (\Delta n_1) nf(T)$$

$$\le -\frac{D}{\varepsilon}\int_\Omega (\Delta n_1)^2 + \frac{\varepsilon}{D}\int_\Omega n^2 dx$$

$$\le -\frac{D\lambda_0}{\varepsilon}\int_\Omega |\nabla n_1|^2 + \frac{\varepsilon}{D}\|n(0)\|^2_{L_2}$$

where λ_0 is the first eigenvalue of the Dirichlet problem in Ω, so

$$\|\nabla n_1(\tau)\|^2_{L_2} \le \|\nabla n_1(0)\|^2_{L_2}\, e^{-\lambda_0 D\tau/\varepsilon} + \frac{\varepsilon^2}{\lambda_0 D^2}\|n(0)\|^2_{L_2}.$$

Now define $F(T) = \int_1^T f(s)ds$ and

$$Q_n(T) = \int_\Omega (\tfrac{1}{2}|\nabla T|^2 - qnF(T))dx;$$

then

$$\frac{d}{d\tau} Q_n(T(\tau)) + \frac{1}{\varepsilon}\|\Delta T+qnf(T)\|^2_{L_2} = -q\int_\Omega F(T)\frac{\partial n}{\partial \tau}dx$$

$$= q\int_\Omega nf(T)F(T)dx + \frac{qD}{\varepsilon}\int f(T)\nabla n_1\cdot\nabla T\, dx.$$

Following the initial transient period (i.e. for $\tau \ge \varepsilon\, \ell n(\frac{1}{\varepsilon}\|\nabla n_1(0)\|)$) the right-side of this equation is bounded compared to $\|\Delta T + qnf(T)\|_{L_2}$ or even $Q_n(T)$, as $\varepsilon \to 0+$. Further $|F(T)| \leq |T-1|$ so $Q_n(T)$ is bounded below. Therefore, after the initial transients have died down, the temperature T will generally be close to a quasi-steady state (given n -- or rather, n_0), with possible complications arising when T "slips" from one near-equilibrium to another. (Here is a nice problem for a catastrophist!)

If $n(0)$ -- or, more to the point, $n_0(0) = \langle n(0)\rangle$ -- is not too large, there will be only one possible equilibrium and we would have (after initial transients)

$$\Delta T + qn_0 f(T) \approx 0 \text{ in } \Omega, \quad T = 1 \text{ on } \partial\Omega,$$

$$\frac{dn_0}{d\tau} \approx -n_0\langle f(T)\rangle$$

and

$$n_1(\tau) \approx -\varepsilon A^{-1}Ef(T(\tau)).$$

Higher approximations are readily computed, using the invariant manifold. This is the case treated by Sattinger [132], with a different method, and more details may be found in this paper.

We return to this problem in Chapter 10.

Exercise 4. Suppose X, Y, Z are Banach spaces, A is sectorial in X, $0 \leq \alpha < 1$, $B \in \mathscr{L}(Z)$ and consider the system

$$\dot{x} + Ax = f(t,x,y,z)$$
$$\dot{y} = g(t,x,y,z)$$
$$\dot{z} + Bz = h(t,x,y,z)$$

where $(f,g,h): \mathbb{R} \times X^\alpha \times Y \times Z \to X \times Y \times Z$ are locally Hölder continuous in t, locally Lipschitz in (x,y,z) and for $\|(x,z)\| \equiv \max(\|x\|_\alpha, \|z\|) \leq D_0$ satisfy

(i) $\|f(t,0,y,0)\| \leq N$, $\|h(t,0,y,0)\| \leq N$ and $y \to f(t,x,y,z)$, $h(t,x,y,z)$ have Lipschitz constant γ and they have Lipschitz constant δ with respect to (x,z)

(ii) $\|e^{-At}x\|_\alpha \leq Me^{-\beta t}\|x\|_\alpha$, $Mt^{-\alpha}e^{-\beta t}\|x\|$ for $t > 0$

$\|e^{-Bt}y\| \leq Me^{\beta t}\|y\|$ for $t \leq 0$ $\quad(\beta > 0)$

(iii) If $p(t) = (x(t),z(t))$ is any continuous curve in $\|p\| \le D_0$ the solution $y(t) = y(t;\tau,\eta,p)$ of $\dot{y} = g(t,x(t),y,z(t))$, $y(\tau) = \eta \in Y$ exists on $-\infty < t < \infty$ and satisfies

$$\|y(t;\tau,\eta_1,p_1) - y(t;\tau,\eta_2,p_2)\|$$
$$\le M_1\|\eta_1-\eta_2\| e^{\mu(t-\tau)} + M_2\left|\int_\tau^t e^{\mu|t-s|}\|p_1(s)-p_2(s)\|ds\right|$$

(iv) For some positive constants Δ, D with $D \le D_0$,

a) $M(N+\delta D)\max(1/\beta,\ \Gamma(1-\alpha)/\beta^{1-\alpha}) < D$

b) $MM_1(\gamma+\delta\Delta)\max(1/\overline{\beta},\ \Gamma(1-\alpha)/\overline{\beta}^{1-\alpha}) \le \Delta$ where $\overline{\beta} = \beta-\overline{\mu} > 0$, $\overline{\mu} = \mu+M_2\Delta$

c) $M\{(\delta-\overline{\gamma}/\overline{\mu})/\beta^{1-\alpha} + \overline{\gamma}/(\overline{\mu}\ \overline{\beta}^{1-\alpha})\}\Gamma(1-\alpha) < 1$ with $\overline{\gamma} = M_2(\gamma+\delta\Delta)$, and

$$M\{\delta/\beta + \overline{\gamma}/\beta\overline{\beta}\} < 1.$$

Prove there exists an invariant manifold

$$S = \{(t,x,y,z) \mid (x,z) = \sigma(t,y),\quad (t,y) \in \mathbb{R} \times Y\}$$

with $\|\sigma(t,y)\| \le D$, $\|\sigma(t,y_1)-\sigma(t,y_2)\| \le \Delta\|y_1-y_2\|$, and any invariant set in $\{(t,x,y,z) \mid \|x\|_\alpha \le D,\ \|z\| \le D\}$ is contained in S.

<u>Hint</u>: If $y(t) = y(t;\tau,\eta,\sigma\cdot y)$ is the solution of $\dot{y} = g(t,\sigma_x(t,y), y,\sigma_z(t,y))$, $y(\tau) = \eta$ for some $\sigma = (\sigma_x,\sigma_z)$ with $\|\sigma\| \le D$ and Lipschitz bound Δ, and if $\overline{y}(t) = y(t;\tau,\overline{\eta},\overline{\sigma}\cdot\overline{y})$ is defined similarly, then

$$\|\overline{y}(t)-y(t)\| \le M_1\|\overline{\eta}-\eta\|\, e^{\overline{\mu}|t-\tau|} + M_2|||\overline{\sigma}-\sigma|||\,(e^{\overline{\mu}|t-\tau|}-1)/\overline{\mu}.$$

Given such σ, define $\tilde{\sigma}$ by

$$\tilde{\sigma}_x(\tau,\eta) = \int_{-\infty}^{\tau} e^{-A(\tau-s)}f(s,\sigma_x(s,y(s)),y(s),\sigma_z(s,y(s)))$$
$$\tilde{\sigma}_z(\tau,\eta) = -\int_{\tau}^{\infty} e^{-B(\tau-s)}h(s,\sigma_x(s,y(s)),y(s),\sigma_z(s,y(s)))$$

where $y(s) = y(s;\tau,\eta,\sigma\cdot y)$, and show $\sigma \to \tilde{\sigma}$ is a contraction in the uniform norm.

Exercise 5. Use the notation and assumptions of ex. 4. Suppose (x,y,z) is a solution for $t \geq \tau$ satisfying $\|x(t)\|_\alpha \leq D$, $\|z(t)\| \leq D$ for all $t > \tau$, and assume for some $\varepsilon > 0$

$$k = M(\delta + M_2(\gamma+\delta\Delta)/\overline{\beta})\Gamma(1-\alpha) \qquad (\overline{\beta}-\varepsilon)^{1-\alpha} < 1.$$

Then $\xi(t) = x(t) - \sigma_x(t,y(t))$, $\zeta(t) = z(t) - \sigma_z(t,y(t))$ satisfy

$$\max(\|\xi(t)\|_\alpha, \|\zeta(t)\|) \leq \frac{M}{1-k}\|\xi(\tau)\|_\alpha e^{-\varepsilon(t-\tau)}, \quad t > \tau.$$

If, in addition, $\dim Y < \infty$ and we can take $\varepsilon > \overline{\mu}$ then there is a solution $\hat{y}$ of

$$d\hat{y}/dt = g(t,\sigma_x(t,\hat{y}),\hat{y},\sigma_z(t,\hat{y}))$$

such that

$$\|y(t)-\hat{y}(t)\| \leq \frac{MM_2\|\xi(\tau)\|_\alpha}{(1-k)(\varepsilon-\overline{\mu})} e^{-\varepsilon(t-\tau)}.$$

(Hint: $z(t)$ must satisfy

$$z(t) = -\int_t^\infty e^{B(s-t)}h(s,x,y,z)ds.)$$

Exercise 6. With the hypotheses of ex. 4 and $k < 1$ (ex. 5), there exist $0 < D_1 \leq D$ and a manifold

$$W^s(\Sigma) = \{(t,x,y,z) \mid z = s(t,x,y), \ \|x\|_\alpha \leq D_1\}$$

such that for any solution (x,y,z) with

$$\|x(\tau)\|_\alpha \leq D_1 \quad \text{and} \quad \|z(\tau)\| \leq D$$

satisfies either

a) $\|z(t)\| > D$ for some $t > \tau$

or

b) the solution exists for all $t > \tau$ with $\|x(t),z(t)\| \leq D$, which is true if and only if $(\tau,x(\tau),y(\tau),z(\tau)) \in W^s(\Sigma)$.

In case b), the solution is in $W^s(\Sigma)$ for all large t and approaches Σ at an exponential rate as $t \to +\infty$.

Exercise 7. A is sectorial in X, $0 \le \alpha < 1$, $A(t,y) = A + B(t,y)$

$$\|B(t,y_1) - B(t,y_2)\|_{\mathscr{L}(X^\alpha, X)} \le B_2\|y_1 - y_2\|$$

and $(f,g): \mathbb{R} \times U \times Y \to X \times Y$ (U open in X^α) satisfy the conditions of Thm. 6.1.2 and also $\|g(t,x,y)\| \le L$. Assume that, for each curve $y: \mathbb{R} \to Y$ with $\|y(t_1) - y(t_2)\| \le L|t_1 - t_2|$, the evolution operator $T_y(t,s)$ (see Ch. 7) for $\dot{x} + A(t,y(t))x = 0$ satisfies

$$\|T_y(t,s)x\|_\alpha \le Me^{-\beta(t-s)}\|x\|_\alpha$$

$$\|T_y(t,s)x\|_\alpha \le M(t-s)^{-\alpha}e^{-\beta(t-s)}\|x\|$$

for $t > s$, $x \in X^\alpha$. Finally assume, in place of (b) in Thm. 6.1.2, that

$$\text{(b')} \qquad M_1\{NB_2M^2\Gamma(1-\alpha)^2\, \overline{\beta}^{1-2\alpha} + \theta(1+\Delta)\} \le \Delta$$

and

$$NB_2M_2M^2\Gamma(1-\alpha)^2\, \overline{\beta}^{1-2\alpha}/\overline{\mu}$$

$$+ \lambda M\Gamma(1-\alpha)\{(1 - M_2/\overline{\mu})/\beta^{1-\alpha} + M_2/\overline{\mu}\; \overline{\beta}^{1-\alpha}\} < 1$$

where $\overline{\mu} = \mu + M_2\Delta$ and $\overline{\beta} = \beta - \overline{\mu} > 0$. If $\mu < \beta$ and λ and NB_2 are sufficiently small, these can be satisfied.

Then there is an invariant manifold

$$S = \{(t,x,y) \mid x = \sigma(t,y)\}$$

for $\dot{x} + A(t,y)x = f(t,x,y)$, $\dot{y} = g(t,x,y)$ with $\|\sigma(t,y)\|_\alpha \le D$, $\|\sigma(t,y_1) - \sigma(t,y_2)\|_\alpha \le \Delta\|y_1 - y_2\|$.

(Hint: if $y(t)$ solves $\dot{y} = g(t,\sigma(t,y),y)$, $y(\tau) = \eta$ and $\overline{y}(t)$ solves $\dot{\overline{y}} = g(t,\overline{\sigma}(t,\overline{y}),\overline{y})$, $\overline{y}(\tau) = \overline{\eta}$ then for $\tau > s > t$

$$\frac{\partial}{\partial s}\{(T_{\overline{y}}(\tau,s) - T_y(\tau,s))T_y(s,t)\} = T_{\overline{y}}(\tau,s)(A(s,\overline{y}(s)) - A(s,y(s)))T_y(s,t)$$

and

$$\|T_{\bar{y}}(\tau,s)-T_y(\tau,s)\|_{\mathscr{L}(X,X^\alpha)} \le q\cdot B_2 \frac{M^2\Gamma(1-\alpha)^2}{\Gamma(2-2\alpha)} (\tau-s)^{1-2\alpha} e^{-\bar{\beta}(\tau-s)}$$

where $q = M_1\|\eta-\bar{\eta}\| + \frac{M_2}{\mu} |||\bar{\sigma}-\sigma|||$.

Exercise 8. Suppose A is sectorial in X, $A(t,y)-A \in (X^\alpha,X)$ for each $(t,y) \in \mathbb{R}\times Y$, $\|A(t,y_1)-A(t,y_2)\|_{\mathscr{L}(X^\alpha,X)} \le B_2\|y_1-y_2\|$ and the evolution operator $T_y(t,s)$ for $\dot{x} = A(t,y(t))x = 0$ satisfies $\|T_y(t,s)\|_{\mathscr{L}(X^\alpha)} \le Me^{-\beta(t-s)}$, $t \ge s$, whenever $y(t)$ is any solution of $\dot{y} = g_0(t,0,y,0)$ in Y. Assume $B(t,y) \in (Z)$ has $\|B(t,y_1)-B(t,y_2)\| \le B_3\|y_1-y_2\|$ and the evolution operator $U_y(t,s)$ for $\dot{z} + B(t,y(t))z = 0$ satisfies $\|U_y(t,s)\|_{(Z)} \le Me^{-\beta(s-t)}$ for $s \ge t$ when $y(t)$ is as above. Assume

$$(x,y,z) \to (f(t,x,y,z), g_0(t,x,y,z), g_1(t,x,y,z), h(t,x,y,z))$$

satisfy uniform Lipschitz conditions and are bounded on $\{(t,x,y,z) \mid \|x\|_\alpha \le D_0, \|z\| \le D_0\} \subset \mathbb{R}\times X^\alpha\times Y\times Z$ to $X\times Y\times Y\times Z$. Also $\|g_0(t,0,y_1,0)-g_0(t,0,y_2,0)\| \le \mu\|y_1-y_2\|$ with $\mu < \beta$. Prove, for sufficiently small ε,

$$\dot{x} + A(t,y)x = \varepsilon f(t,x,y,z)$$
$$\dot{y} = g_0(t,x,y,z) + \varepsilon g_1(t,x,y,z)$$
$$\dot{z} + B(t,y)z = \varepsilon h(t,x,y,z)$$

has an invariant manifold $S_\varepsilon = \{(t,x,y,z) \mid (x,z) = \sigma_\varepsilon(t,y)\}$ with $\sigma_\varepsilon(t,y) \to 0$ and $\mathrm{Lip}_y\, \sigma_\varepsilon(t,\cdot) \to 0$ as $\varepsilon \to 0$.
(Hint: for sufficiently small δ, $0 < \delta \le \delta_0$, and any curve $\{y(t)\}$ in Y with $\|\dot{y}(t)-g_0(t,0,y(t),0)\| \le \delta$ for all t, we have

$$\|T_y(t,s)\|_{\mathscr{L}(X^\alpha)} \le M_{\delta_0} e^{-\beta'(t-s)} \quad (t \ge s)$$

for some constants M_{δ_0} $\beta'(\delta)$ depending only on the given constants $\beta'(\delta) \to \beta$ as $\delta \to 0$, and a similar estimate holds for $U_y(t,s)$. See sec. 7.4, in particular ex. 1.)

Exercise 9. Suppose $f\colon [0,\pi] \times \mathbb{R} \to \mathbb{R}$ is twice continuously differentiable, $k > 0$, and consider

$$u_t = ku_{xx} + f(x,u(x,t)), \qquad (0 < x < \pi, \quad t > 0)$$
$$u(0,t) = 0, \quad u(\pi,t) = 0.$$

Then $\|f(\cdot,\phi)-f(\cdot,\psi)\|_{H_0^1(0,\pi)} \le L\|\phi-\psi\|_{H_0^1(0,\pi)}$, $\|f(\cdot,\phi)\|_{H_0^1} \le K$ provided $\|\phi\|_{H_0^1} \le B$ and $\|\psi\|_{H_0^1} \le B$, where L, K depend only on B and $\sup_{\substack{0\le x\le\pi \\ |u|\le B\sqrt{\pi/2}}} \{|f_x|,|f_u|,|f_{xu}|,|f_{uu}|\}$. If $P_N\phi(x) = \sum_{n=1}^{N} \phi_n \sin nx$ where $\phi_n = \frac{2}{\pi}\int_0^\pi \phi(\xi) \sin n\xi \, d\xi$ for $1 \le n \le N$, and if $Q_N = I-P_N$, then $\|Q_N\phi\|_{L_2(0,\pi)} \le \frac{1}{(N+1)^2} \|\phi\|_{H_0^1(0,\pi)}$, hence

$$\|Q_N f(\cdot,\phi)\|_{L_2} \le \frac{K}{(N+1)^2}$$
$$\|Q_N f(\cdot,\phi)-Q_N f(\cdot,\psi)\|_{L_2} \le \frac{L}{(N+1)^2} \|\phi-\psi\|_{H_0^1}$$

for $\|\phi\|_{H_0^1} \le B$, $\|\psi\|_{H_0^1} \le B$. Assume $f(x,u)$ is modified outside $|u| \le B\sqrt{\pi/2}$ so these estimates hold everywhere.

Consider $u = u_N + v_N$, $u_N = P_N u$, $v_N = Q_N u$, so

$$\left(\frac{\partial}{\partial t} - k\frac{\partial^2}{\partial x^2}\right)v_N = Q_N f(\cdot,u_N + v_N), \quad t > 0.$$

while u_N satisfies an ordinary differential equation, $u_N = \sum_{n=1}^{N} c_n(t) \sin nx$,

$$\frac{dc_n}{dt} + kn^2 c_n = \frac{2}{\pi}\int_0^\pi \sin nx \, f(x, \sum_1^N c_m(t)\sin mx + v_N(x,t))dx$$
$$(n = 1,\ldots,N).$$

Then Th. 6.1.2 applies with $\alpha = 1/2$, $\beta = K(N+1)^2-K$, $M = N+1$, $\lambda = L/(N+1)^2$, $\mu = KN^2+L$, for sufficiently large N, and there exists an N-dimensional attracting local invariant manifold

$S_N = \{u(x) = \sum_1^N c_n \sin nx + v_N(x;c_1,\dots,c_N)\}$. If u is a solution with $\|u(\cdot,t)\|_{H_0^1} \le B$ for all $t \ge 0$, then its ω-limit set is contained in S_N, and the limiting behavior of all such solutions is determined by the ordinary differential equation describing the flow in S_N.

Compare with the example of Hopf [50] (exercise, sec. 6.4).

Exercise 10. Let D, V, L, β_0, β_1 be real $n \times n$ matrices with D diagonal and positive and define (for real ε)

$$A_\varepsilon u = -Du'' + \varepsilon V u' + \varepsilon L u, \quad 0 < x < 1$$

for $u \in H^2(0,1;\mathbb{R}^n)$ with

$$Du'(0) = \varepsilon\beta_0 u(0), \qquad Du'(1) = -\varepsilon\beta_1 u(1).$$

Show A_ε is sectorial for each ε and the conditions of ex. 2 (section 3.4) hold for $A_\varepsilon + I$ when ε is small, and $D(A_\varepsilon + I)^{1/2} = H^1(0,1;\mathbb{R}^n)$ is independent of ε.

Exercise 11. (cf. A. Poore [130].) With D, V, β_0, β_1 as above, consider the equations for a tubular chemical reactor with large diffusivity:

$$\frac{\partial u}{\partial t} = \frac{1}{\varepsilon} D \frac{\partial^2 u}{\partial x^2} - V \frac{\partial u}{\partial x} + F(x,u), \qquad 0 < x < 1$$

$$\frac{1}{\varepsilon} D \frac{\partial u}{\partial x} = \beta_0 u \quad \text{at} \quad x = 0, \qquad \frac{1}{\varepsilon} D \frac{\partial u}{\partial x} = -\beta_1 u \quad \text{at} \quad x = 1$$

for $0 < \varepsilon << 1$. Here the components of u are the concentrations of various chemical species involved and the temperature, $\frac{1}{\varepsilon} D$ is the diffusivity, V is the convective velocity and $F(x,u)$ gives the rate of reaction. We assume $F(x,u)$ is modified for $|u|$ large so as to be smooth and bounded as a function of u.

Take A_ε as in exercise 10 and let $P_\varepsilon = \operatorname{Res}_{|\lambda|<\delta} (\lambda - A_\varepsilon)^{-1}$ be the projection of $L^2(0,1;\mathbb{R}^n)$ onto an n-dimensional subspace Y_ε of (almost constant) functions:

$$P_\varepsilon u(x) = \int_0^1 u(x)dx + O(\varepsilon\|u\|_{L_2}).$$

Prove there is an n-dimensional attracting invariant manifold represented as a graph over Y_ε, for sufficiently small $\varepsilon > 0$,

$$u = y + \sigma_\varepsilon(y), \quad y = P_\varepsilon u,$$

with $\sigma_\varepsilon(y) \to 0$ as $\varepsilon \to 0+$. The flow in this manifold is given by

$$\frac{dy}{dt} = \overline{F}(y) - (\beta_0+\beta_1)y + O(\varepsilon),$$

uniformly on bounded sets of $y \in Y_\varepsilon$, where $\overline{F}(y) = \int_0^1 F(x,y)dx$. Further, if u is a solution with $\|u(\cdot,t)\|_{H^1} \le B$ on $t_0 \le t \le t_1$, ε sufficiently small, then $z = u - (P_\varepsilon u + \sigma_\varepsilon(P_\varepsilon u))$ satisfies

$$\|z(t)\|_{H^1} \le Ke^{-bt/\varepsilon}\|z(t_0)\|_{H^1}, \quad t_0 \le t \le t_1$$

where K and $b > 0$ are independent of $\varepsilon > 0$.
(<u>Hint</u>: change the time variable to $\tau = t/\varepsilon$.)

<u>6.2 Critical cases of stability [40, 41, 45]</u>

<u>Theorem 6.2.1</u>. Suppose A is sectorial in X, $0 \le \alpha < 1$, U is a neighborhood of the origin in X^α and $h: U \to X$ is continuously differentiable with $h(0) = 0$. Assume $L = A-h'(0)$ has $\mathrm{Re}\,\sigma(L) \ge 0$ and $\sigma(L) \cap \{\mathrm{Re}\,\lambda = 0\}$ is a spectral set. (If A has compact resolvent, this is a finite collection of eigenvalues.) We study the stability of the origin for $dx/dt + Ax = h(x)$.

Let $X = X_1 \oplus X_2$ be the corresponding decomposition into L-invariant subspaces with $\mathrm{Re}\,\sigma(L_1) = 0$, $\mathrm{Re}\,\sigma(L_2) > 0$, $L_j = L|_{X_j}$ $(j = 1,2)$. There exists a Lipschitzian local invariant manifold $S = \{x = x_1 + \sigma(x_1) \mid x_1 \in X_1, \|x_1\| \le r\}$, tangent to X_1 at the origin. The flow in S may be represented by the ordinary differential equation in X_1

$$dx_1/dt + L_1x_1 = E_1g(x_1 + \sigma(x_1))$$

where $g(x) = h(x)-h'(0)x$ and E_1 is the projection of X onto X_1 (along X_2).

If the origin (in X_1) is asymptotically stable for the flow in S, the origin is asymptotically stable in X^α; if the origin is unstable for the flow in S, then it is unstable in X^α.

Remark. If h is smooth near 0, the first few terms in the Taylor series for ϕ may be calculated (see Th. 6.2.3) and X_1 is ordinarily finite dimensional -- usually dimension one or two -- so stability of instability in S can usually be determined by sufficient calculation.

Proof of the theorem. We reduce this to the case dealt with in sec. 6.1: $X = X_1 \oplus X_2$, so $x = x_1 + x_2$ with

$$dx_1/dt + L_1x_1 = E_1g(x_1+x_2), \quad dx_2/dt + L_2x_2 = E_2g(x_1+x_2)$$

E_j = projection of X onto E_j $(j = 1,2)$, with $\|e^{-L_1t}\| \le M_\varepsilon e^{\varepsilon|t|}$ for all $t \le 0$ (for any $\varepsilon > 0$) and $\|e^{-L_2t}\| \le Me^{-\beta t}$, $\|L_2^\alpha e^{-L_2t}\| \le Mt^{-\alpha}e^{-\beta t}$ for $t > 0$ (for some $\beta > 0$). We may choose $\varepsilon > 0$ arbitrarily small, but generally $M_\varepsilon \to +\infty$ as $\varepsilon \to 0+$. We modify the equations outside a neighborhood of the origin in X_1 so the hypotheses of Th. 6.1.2 hold with $Y = X_1$ and U a small neighborhood of the origin in X_2^α; the resulting invariant manifold, restricted to a neighborhood of the origin in $X_1 \times X_2^\alpha$, gives a local invariant manifold for the original system, and the theorem follows.

Choose a Lipschitzian function $\psi: X_1 \to [0,1]$ with $\psi(x_1) = 1$ if $\|x_1\| \le 1$, $\psi(x_1) = 0$ if $\|x_1\| \ge 2$, and for $\rho > 0$ define

$$f_\rho(x_1,x_2) = E_2g(x_1\psi(x_1/\rho) + x_2),$$

$$g_\rho(x_1,x_2) = E_1g(x_1\psi(x_1/\rho) + x_2).$$

There exists continuous increasing $k(\cdot)$ with $k(0) = 0$ such that $\|g(x)-g(x')\| \le k(\rho)\|x-x'\|_\alpha$ when $\|x\|_\alpha \le \rho$, $\|x'\|_\alpha \le \rho$, hence f_ρ, g_ρ have Lipschitz constant on $X_1 \times U_\rho = \{(x_1,x_2) \mid x_1 \in X_1,\ x_2 \in X_2^\alpha,\ \|x_2\|_\alpha \le 2\rho\}$ of order $k(2\rho)$, and also $\|f_\rho(x_1,x_2)\| + \|g_\rho(x_1,x_2)\| \le C\rho k(2\rho)$ on $X_1 \times U$.

Now if $\{\xi(t),\ t \le 0\}$ is any continuous curve with values in U_ρ, define $y(t) = \phi(t;\eta,\xi(\cdot)) = \phi(t+\tau;\tau,\eta,\xi(\cdot))$ by

$$dy/dt + L_1y = g_\rho(y,\xi(t)), \quad t \le 0,$$

$$y(0) = \eta \in X_1.$$

For any other curve $\xi': (-\infty,0] \to U$ and any $\eta' \in X_1$,

$$\begin{aligned}\|y'(t)-y(t)\| &= \|\phi(t;\eta',\xi'(\cdot))-\phi(t;\eta,\xi(\cdot))\| \\ &\le \|e^{-L_1 t}\|\|\eta-\eta'\| + \int_t^0 \|e^{-L_1(t-s)}\| \, \|g_\rho(y'(s),\xi'(s))-g_\rho(y(s),\xi(s))\| ds \\ &\le M_\varepsilon e^{-\varepsilon t}\|\eta-\eta'\| + M_\varepsilon \int_t^0 e^{\varepsilon(s-t)} Ck(2\rho) [\|y'(s)-y(s)\| \\ &\qquad + \|\xi'(s)-\xi(s)\|_\alpha] ds\end{aligned}$$

when $Ck(2\rho)$ is the Lipschitz constant for g_ρ on $X_1 \times U$. It follows that for $t \le 0$

$$\|y'(t)-y(t)\| \le M_\varepsilon \|\Delta\eta\| e^{\mu|t|} + \mu \int_t^0 e^{\mu(s-t)}\|\xi'(s)-\xi(s)\|_\alpha ds$$

where $\mu = CM_\varepsilon k(2\rho) + \varepsilon$. If ε and ρ are chosen sufficiently small, it follows that $\mu < \beta$ and estimates (a), (b), (c) of Th. 6.1.2 and 6.1.4 hold for small ρ, so these theorems apply.

Finally, $\sigma(\xi) = \int_{-\infty}^0 e^{L_2 s} f_\rho(\sigma(x_1(s;\xi)),x_1(s;\xi))ds$ for ξ in X_1, with $\|x_1(s;\xi)\| \le M_\varepsilon \|\xi\| e^{-\mu(1+\Delta)s}$ for $s \le 0$. It follows that, for any $B > 0$,

$$\begin{aligned}\|\sigma(\xi)\|_\alpha/\|\xi\| &\le M\int_{-B}^0 |s|^{-\alpha} e^{\beta s}\|E_2\| \, k((1+\Delta)M_\varepsilon\|\xi\| \, e^{\mu(1+\Delta)|s|}) M_\varepsilon e^{-\mu(1+\Delta)s} ds \\ &\quad + MCk(2\rho)M_\varepsilon(1+\Delta) \int_{-\infty}^{-B} |s|^{-\alpha} e^{\beta s} e^{-\mu(1+\Delta)s} ds.\end{aligned}$$

Take B large and $\|\xi\|$ small to show $\|\sigma(\xi)\|_\alpha = o(\|\xi\|)$ as $\xi \to 0$ in X_1.

Corollary 6.2.2. Assume A, h satisfy the hypotheses of Th. 6.2.1 and also the derivative h' is Lipschitz continuous on a neighborhood of the origin in X^α. Assume also that in the subspace X_1 there exists a function $\psi: X_1 \to [0,1]$ which is continuously differentiable with a Lipschitz continuous derivative, which is equal to one on some neighborhood of the origin and vanishes everywhere outside some bounded set in X_1. (Note: this is trivial if X_1 is finite dimensional, or if X is a Hilbert space or has a C^2 norm.)

Then the local invariant manifold (the "critical" or "center" manifold) constructed above is continuously differentiable with a Lipschitz continuous derivative, and σ satisfies $\|\sigma(x_1)\|_\alpha = O(\|x_1\|^2)$

as $x_1 \to 0$ and

$$\sigma'(x_1)[L_1x_1 - E_1g(x_1+\sigma(x_1))] = L_2\sigma(x_1) - E_2g(x_1+\sigma(x_1))$$

in a neighborhood of the origin in X_1.

Now we show that the critical manifold may be computed (following Hausrath [45]), in the sense that an asymptotic power series for σ may be constructed, directly from the differential equation for σ given in this corollary. Another approach is used in examples below, and its justification is the agreement with the method given here.

Theorem 6.2.3. Assume the hypotheses of Th. 6.2.1 and Cor. 6.2.2, and assume ϕ is a continuously differentiable function with Lipschitzian derivative from a neighborhood of the origin in X_1 to X_2^α, with range in $D(L_2)$, such that (when ϕ' = derivative of ϕ)

$$\phi'(x_1)[L_1x_1-E_1g(x_1+\phi(x_1))]-L_2\phi(x_1)+E_2g(x_1+\phi(x_1)) = \Delta(x_1),$$

with $\|\Delta(x_1)\| \leq K\|x_1\|^p$ near the origin of X_1 for some $p > 1$. Then if $\sigma(\cdot)$ defines the critical manifold,

$$\|\sigma(x_1)-\phi(x_1)\|_\alpha = O(\|x_1\|^p) \quad \text{as} \quad x_1 \to 0 \quad \text{in} \quad X_1.$$

If $g: X^\alpha \to X$ is p-times continuously differentiable near the origin, there exists a unique polynomial function ϕ of order p satisfying the conditions above. Actually, it suffices that there exists a polynomial $\hat{g}$ so that $\|g(x)-\hat{g}(x)\| = O(\|x\|_\alpha^p)$ as $\|x\|_\alpha \to 0$.

Proof. Suppose ϕ is as above, and is extended to all of X_1 subject to all these conditions but with $E_{1,2}g(x_1+\phi(x_1))$ replaced by f_ρ, $g_\rho(x_1,\phi(x_1))$ respectively. Suppose $x_1(t)$, $x_2(t) = \sigma(x_1(t))$ is a solution of the equation, and define $z(t) = \sigma(x_1(t))-\phi(x_1(t))$ so $dz/dt + L_2z = k_\rho(x_1,z)$ where

$$k_\rho(x_1,z) = \Delta(x_1)+g_\rho(x_1,\phi(x_1)+z)-g_\rho(x_1,\phi(x_1))-\phi'(x_1)\cdot$$

$$\cdot[f_\rho(x_1,\phi(x_1)+z)-f_\rho(x_1,\phi(x_1))]$$

$\|k_\rho(x_1,0)\| = \|\Delta(x_1)\| \leq K\|x_1\|^p$. Then we have

$z(0;\xi) = \int_{-\infty}^{0} e^{L_2 s} k_\rho(x_1(s;\xi)z(0;x_1(s;\xi)))ds$ with $dx_1/ds + L_1x_1 = g_\rho(x_1,\phi(x_1) + z(0;x_0))$, $s \le 0$, $x_1(0;\xi) = \xi$. Arguing as above, $\xi \to z(0;\xi)$ is the fixed point of a certain contraction map, and it suffices to prove this maps a class of functions with $\|z(0;\xi)\|_\alpha$ - $K_2\|\xi\|^p$, into itself. But if $z(0;\xi)$ satisfies such an estimate, then $\|x_1(s;\xi)\| \le M_\varepsilon \; e^{-\mu(1+\Delta)s}\|\xi\|$ for $s \le 0$ so

$$\begin{aligned}
&\left\|\int_{-\infty}^{0} e^{L_2 s} k_\rho(x_1(s;\xi),z(0;x_1(s;\xi)))ds\right\|_\alpha \\
&\quad \le \int_{-\infty}^{0} M|s|^{-\alpha}e^{\beta s}(K\|x_1(s;\xi)\|^p + \|k_\rho\|_{Lip}\, K_2\|x_1(s;\xi)\|^p)ds \\
&\quad \le \int_{-\infty}^{0} M|s|^{-\alpha}e^{(\beta-p\mu(1+\Delta))s}\, M_\varepsilon^p(K + \|k_\rho\|_{Lip}K_2)ds\cdot \|\xi\|^p \\
&\quad \le K_2\|\xi\|^p,
\end{aligned}$$

provided $p\mu(1+\Delta) < \beta$, K_2 is chosen sufficiently large and ρ sufficiently small (so $\|k_\rho\|_{Lip}$ is small). For sufficiently small ρ, the condition $p\mu(1+\Delta) < \beta$ will be satisfied, as before. This proves the first claim, and the second follows from the lemma below.

<u>Lemma 6.2.4</u>. Suppose $\gamma: X_1 \to X_2$ is a continuous homogeneous polynomial of order N; then there exists a continuous homogeneous polynomial $\phi: X_1 \to X_2^\alpha$ of order N such that

$$\phi'(x_1)L_1x_1 - L_2\phi(x_1) + \gamma(x_1) = 0.$$

In fact, $\phi(x_1) = \int_{-\infty}^{0} e^{L_2 s}\gamma(e^{-L_1 s}x_1)ds$, and $\|\gamma(x_1)\| \le C\|x_1\|^N$ implies $\|\phi(x_1)\|_\alpha \le (MC\int_{-\infty}^{0} |s|^{-\alpha}e^{\beta s}\|e^{-L_1 s}\|^N ds)\,\|x_1\|^N$.

<u>Proof</u>. Set $\psi(t) = \phi(e^{-L_1 t}x_1)$ for $t \le 0$, where ϕ is defined by the integral above. Then $d\psi/dt + L_2\psi = \gamma(e^{-L_1 t}x_1)$ for $t \le 0$, and the result follows.

<u>Example</u>. $u_t = u_{xx} + u - au^3$ for $t > 0$, $0 < x < \pi$, $u(0,t) = 0$ and $u(\pi,t) = 0$, and a is a nonzero constant. We take $X = L^2(0,\pi)$, $A = -d^2/dx^2$ as in Sec. 4.3.

Linearize about the equilibrium $u = 0$: $v_t = v_{xx} + v$, with $v = 0$ at $x = 0,\pi$. In the notation of Th. 6.2.1, $\sigma(L) = \{n^2-1 \mid n = 1,2,\ldots\}$, a collection of simple eigenvalues, and $X_1 = \mathrm{span}\{\sin x\}$,

$X_2 = \{\phi \mid \int_0^\pi \phi(x) \sin x \, dx = 0\}$.

The critical manifold may be represented in the form

$$S = \{u(x) = s \sin x + \sum_2^\infty c_n(s) \sin nx, \ |s| < s_0\}$$

where $c_n(s) = O(s^2)$ as $s \to 0$. In fact, the $c_n(s)$ must be odd functions of s, since $u(x,t)$ is a solution only if $-u(x,t)$ is a solution, so $c_n(s) = O(s^3)$. The flow in S satisfies

$$\frac{ds}{dt}(\sin x + \sum_2^\infty c_n'(s) \sin nx) + \sum_2^\infty (n^2-1) c_n(s) \sin nx$$

$$= -a \sum_{\ell,m,p} c_\ell c_m c_p \sin \ell \sin mx \sin px$$

(with $c_1(s) = s$). It follows (integrating against $\sin x$) that $\frac{ds}{dt} = -as^3 \cdot \frac{2}{\pi} \int_0^\pi \sin^4 x \, dx + O(s^5)$, i.e. $ds/dt + \frac{3}{4} as^3 + O(s^5) = 0$.

Thus from Th. 6.2.1 $u = 0$ is asymptotically stable if $a > 0$, and unstable if $a < 0$.

Further calculation (integrating against $\sin kx$, $k \geq 2$) reveals

$$S = \{u(x) = s \sin x + \frac{as^3}{32} \sin 3x + O(s^5)\}.$$

We verify this calculation using Th. 6.2.3. When $u_1(x) = s \sin x$, $|s| < s_0$, let

$$\phi(u_1)(x) = \frac{as^3}{32} \sin 3x.$$

Then $g(u) = -au^3$, $g(u_1+\phi(u_1))(x) = -as^3 \sin^3 x + O(s^5)$ so $E_1 g(u_1+\phi(u_1))(x) = -as^3 \sin x \cdot \frac{2}{\pi} \int_0^\pi \sin^4 \xi \, d\xi + O(s^5)$ and

$$\phi(u_1)[L_1 u_1 - E_1 g(u_1+\phi(u_1))] - L_2 \phi(u_1) + E_2 g(u_1+\phi(u_1))$$

$$= \frac{3as^2}{32} \sin 3x \cdot \frac{2}{\pi} \int_0^\pi \sin\xi (0 + \frac{3a}{4} s^3 \sin \xi + O(s^5)) d\xi$$

$$- 8 \cdot \frac{as^3}{32} \sin 3x - as^3(-\frac{1}{4} \sin 3x) + O(s^5) = O(s^5),$$

which proves our calculation is correct to the order specified.

Exercise 1. If S has the representation

$$u_2(x) = \frac{as^3}{32} \sin 3x + s^5 \phi_5(x) + O(s^7)$$

where $s = \frac{2}{\pi} \int_0^\pi u_1(x) \sin x \, dx$ (i.e. $u_1(x) = s \sin x$), substitute in the equation for σ to show $\phi_5''(x) + \phi_5(x) + \frac{3a^2}{128} \sin 3x + \frac{3a^2}{128} \sin 5x = 0$, $\int_0^\pi \phi_5(x) \sin x \, dx = 0$, i.e. $\phi_5(x) = \frac{a^2}{1024}(3 \sin 3x + \sin 5x)$. Thus the flow in S is given by

$$\frac{ds}{dt} + \frac{3}{4} as^3 - \frac{3}{128} a^2 s^5 + \frac{3}{4096} a^3 s^7 + O(s^9) = 0,$$

while S has the representation

$$\{u(x) = s \sin x + \frac{as^3}{32} \sin 3x + \frac{a^2 s^5}{1024}(3 \sin 3x + \sin 5x) + O(s^7), \quad \text{for small } s\}.$$

Using the critical manifold simplifies problems such as that in Exercise 5, Sec. 5.1. In that case we have a C^2 curve of equilibrium points $\hat{x}(\lambda)$, $\hat{x}(0) = 0$, $\hat{x}'(0) \neq 0$ and it is assumed the linearization about 0 has 0 as a simple eigenvalue (with eigenvector $\hat{x}'(0)$) and the remaining eigenvalues are stable. In this case the critical manifold is precisely the curve $\hat{x}(\lambda)$ and the "flow" in this manifold is given by $d\lambda/dt = 0$. In a neighborhood of the curve, introduce coordinates (λ, y), $x = \hat{x}(\lambda) + y$, where y lies in a subspace complementary to $\text{span}\{\hat{x}'(0)\}$; for $\|y\|_\alpha + |\lambda| \leq r$, the flow has $|d\lambda/dt| \leq C\|y(t)\|_\alpha$ and $\|y(t)\|_\alpha + |\lambda(t)| \leq r$ for $t_0 \leq t \leq t_1$ implies $\|y(t)\|_\alpha \leq K e^{-\beta(t-t_0)} \|y(t_0)\|_\alpha$ on $t_0 \leq t \leq t_1$; for some positive constants C, K, β, r. If $|\lambda(t_0)| + K(1 + 1/\beta)\|y(t_0)\|_\alpha \leq r$ then $|\lambda(t)| + \|y(t)\|_\alpha \leq r$ for all $t \geq t_0$ and $\|y(t)\|_\alpha + |\lambda(t) - \lambda_\infty| \to 0$ exponentially as $t \to \infty$, for some constant λ_∞. Precisely the same argument proves the analogous result for an n-parameter family of equilibria when 0 is an n-fold eigenvalue of the linearization. Alternatively, Cor. 6.1.5 shows $\{\hat{x}(\lambda)\}$ has asymptotic phase.

Suppose now that $\hat{x}(\lambda)$ is a curve of equilibria as above, but 0 is a double eigenvalue with $\hat{x}'(0)$ the only eigenvector (other eigenvalues stable). This case is more subtle but the final result is geometrically reasonable. It is sufficient (as above) to examine stability of the curve in the two-dimensional critical manifold. Suppose the flow in the critical manifold is given by $dz/dt = F(z)$ where $z = (x,y) \in \mathbb{R}^2$ near 0, and the curve is represented as

$z = \phi(\lambda)$, $\phi(0) = 0$, $\phi'(0) \neq 0$. Then the curve of equilibria is asymptotically stable with asymptotic phase, independent of terms of order $O(|z|^N)$, if and only if

$$(\text{div } F)_{z=\phi(\lambda)} = a\lambda^q + o(\lambda^q) \quad \text{as} \quad \lambda \to 0$$

for some $a < 0$ and even integer q, $2 \leq q < N$. Since F vanishes along the curve, $F'(\phi(\lambda))$ has zero as an eigenvalue and so $(\text{div } F)_{z=\phi(\lambda)}$ is the nonzero eigenvalue of the derivative which is required to be negative for stability.

To prove this, first observe that we may change variables so the curve is represented as $y = \phi(x)$, $\phi(0) = 0$, $\phi'(0) = 0$ and $F'(0) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. This involves re-parametrization of the curve and a linear change of coordinates in the x-y plane, without changing the assumption on div F. If we put $y = \phi(x) + \eta$, the system becomes $\dot{x} = \eta + q_1(x,\eta)$, $\dot{\eta} = q_2(x,\eta)$ with q_1, q_2 quadratic and $q_1(x,0) = q_2(x,0) = 0$, and $\partial q_2/\partial\eta(x,0) = (\text{div } F)_{(x,\phi(x))}$. If $q_1(x,\eta) = \eta\hat{q}_1(x,\eta)$, $q_2(x,\eta) = \eta\hat{q}_2(x,\eta)$ then on any solution curve with $\eta \neq 0$,

$$d\eta/dx = \hat{q}_2(x,\eta)/(1 + \hat{q}_1(x,\eta))$$

so (recalling $\hat{q}_2(x,0) \sim ax^q$ near 0)

$$\eta = c + ax^{q+1}/(q+1) + O(|cx|) + o(|x|^{q+1})$$

for a small constant c, depending on the solution. Substituting in the x-equation we find $x(t)$ converges to a stable equilibrium and $\eta \to 0$ as $t \to +\infty$ in case q is even and $a < 0$; otherwise the solution $\eta \equiv 0$ is unstable.

6.3. Bifurcation and transfer of stability for equilibrium points [85, 86, 41]

Suppose A is sectorial in X and $f(x,\varepsilon)$ is smooth from a neighborhood of the origin in $X^\alpha \times \mathbb{R}$ to X. Assume the origin is always an equilibrium point, $f(0,\varepsilon) = 0$, but it changes from stability to instability as ε increases through zero. Specifically, suppose $L_\varepsilon = A - f_x(0,\varepsilon)$ has spectrum

$$\sigma(L_\varepsilon) \subset \{\lambda(\varepsilon)\} \cup \{\mathrm{Re}\ \lambda > \beta\}$$

for some $\beta > 0$, where $\lambda(\varepsilon)$ is a simple real eigenvalue with $\lambda(0) = 0$, $\frac{d\lambda}{d\varepsilon}(0) < 0$.

Both $\lambda(\varepsilon)$ and the corresponding projection E_ε are smooth functions of ε [56]. We may suppose $E_\varepsilon = v_\varepsilon \otimes w_\varepsilon$, i.e. $E_\varepsilon x = v_\varepsilon \langle w_\varepsilon, x\rangle$ for any $x \in X$, where $L_\varepsilon v_\varepsilon = \lambda_\varepsilon v_\varepsilon$, v_ε is in X, and w_ε is a continuous linear functional on X satisfying the adjoint equation $L^*_\varepsilon w_\varepsilon = \lambda_\varepsilon w_\varepsilon$, with $\langle w_\varepsilon, v_\varepsilon\rangle = 1$. We may also suppose v_ε, w_ε are smooth functions of ε.

An argument similar to that in sec. 6.2 shows there is a one-dimensional local invariant manifold

$$S_\varepsilon = \{x = sv_\varepsilon + \phi(s,\varepsilon),\ s = \langle w_\varepsilon, x\rangle,\ |s| < \rho_0\}$$

and ϕ is a continuously differentiable function of its arguments, with $\phi(0,\varepsilon) = 0$, $\frac{\partial\phi}{\partial s}(0,0) = 0$. The differential equation in S_ε then takes the form (with $g(x,\varepsilon) = f(x,\varepsilon) - f_x(0,\varepsilon)x$)

$$\frac{ds}{dt} + \frac{\lambda(\varepsilon)s - \langle w_\varepsilon, g(sv_\varepsilon + \phi(s,\varepsilon)\varepsilon)\rangle}{1 + \langle w_\varepsilon, \partial\phi(s,\varepsilon)/\partial s\rangle} = 0$$

as long as $|s| < \rho$, for some positive $\rho \leq \rho_0$, i.e., $ds/dt + h(s,\varepsilon) = 0$ where $h(0,\varepsilon) = 0$, $\frac{\partial h}{\partial s}(0,\varepsilon) = \lambda(\varepsilon)$. The stability properties of any solutions near the origin are determined by the stability properties of this one-dimensional equation, and these are easily determined.

Lemma 6.3.1. Suppose $h(s,\varepsilon)$ is a continuously differentiable real function of (s,ε) near $(0,0)$, and for some $\alpha > 0$, $\beta \neq 0$, and integer $m \geq 2$, $h(s,\varepsilon) = -\alpha s\varepsilon + \beta s^m + o(|s|^m + |s\varepsilon|)$ and $\frac{\partial h}{\partial s}(s,\varepsilon) + \alpha\varepsilon - m\beta s^{m-1} = o(|\varepsilon| + |s|^{m-1})$, as $(\varepsilon,s) \to (0,0)$. Then in a small neighborhood of the origin we have one of the following cases:

(a) m is even (so $h(s,0)$ doesn't change sign for $0 < |s| \leq \rho$), when there is a single non-zero s_ε near 0 (for small $\varepsilon \neq 0$) with $h(s_\varepsilon,\varepsilon) = 0$, and $\partial h(s_\varepsilon,\varepsilon)/\partial s$ is negative for $\varepsilon < 0$, positive for $\varepsilon > 0$.

(b) m is odd and $\beta > 0$ (so $sh(s,0) > 0$ for $0 < |s| \leq \rho$), when there are no small zeros of $h(\cdot,\varepsilon)$ for small $\varepsilon \leq 0$ other than $h(0,\varepsilon) = 0$ -- but for small $\varepsilon > 0$ there is a pair $s^\pm_\varepsilon$, $h(s^\pm_\varepsilon,\varepsilon) = 0$ with $\partial h(s^\pm_\varepsilon,\varepsilon)/\partial s > 0$, and $s^\pm_\varepsilon \to 0$ as $\varepsilon \to 0+$.

(c) m is odd and $\beta < 0$ (so $sh(s,0) < 0$ for $0 < |s| \le \rho$) when there is a pair $s_\varepsilon^\pm$ for small $\varepsilon < 0$, $h(s_\varepsilon^\pm,\varepsilon) = 0$, $\partial h(s_\varepsilon^\pm,\varepsilon)/\partial s < 0$, with $s_\varepsilon^\pm \to 0$ as $\varepsilon \to 0-$, and there are no small zeros of $h(\cdot,\varepsilon)$ for small $\varepsilon > 0$, except $s = 0$.

Note that if $h(s_\varepsilon,\varepsilon) = 0$, then s_ε is an equilibrium point of $ds/dt + h(s,\varepsilon) = 0$, and it is asymptotically stable (or unstable) if $\partial h(s_\varepsilon,\varepsilon)/\partial s > 0$ (or < 0), so the possibilities above may be reformulated in terms of stability and instability of equilibrium points.

Proof of lemma. The proof is a modification of Newton's polygon (see, in particular, [21]). Let $s = \sigma\varepsilon^{1/(m-1)}$, so if $h(s,\varepsilon) = -\alpha s\varepsilon + \beta s^m + \Delta(s,\varepsilon)$, $h(\sigma\varepsilon^{1/(m-1)},\varepsilon) = \varepsilon^{m/(m-1)}(-\alpha\sigma+\beta\sigma^m) + \Delta(\sigma\varepsilon^{1/(m-1)},\varepsilon)$. Now $\sigma \to \varepsilon^{-m/(m-1)}\Delta(\sigma\varepsilon^{1/(m-1)},\varepsilon)$ is continuously differentiable for any small $\varepsilon \neq 0$, and $\partial/\partial\sigma\{\varepsilon^{-m/(m-1)}\Delta(\sigma\varepsilon^{1/(m-1)},\varepsilon)\} = \varepsilon^{-1}\partial\Delta/\partial s(\sigma\varepsilon^{1/(m-1)},\varepsilon) \to 0$ as $\varepsilon \to 0$. Thus, by the implicit function theorem, if $\sigma_0^{m-1} = \alpha/\beta$, there exists a unique $\sigma(\varepsilon) \to \sigma_0$ as $\varepsilon \to 0$ such that $h(\sigma(\varepsilon)\varepsilon^{1/(m-1)},\varepsilon) = 0$, The cases (a), (b), (c) above appear when sign restrictions on ε are made to ensure all the functions here are real. The statements about the sign of $\partial h/\partial s$ follow from

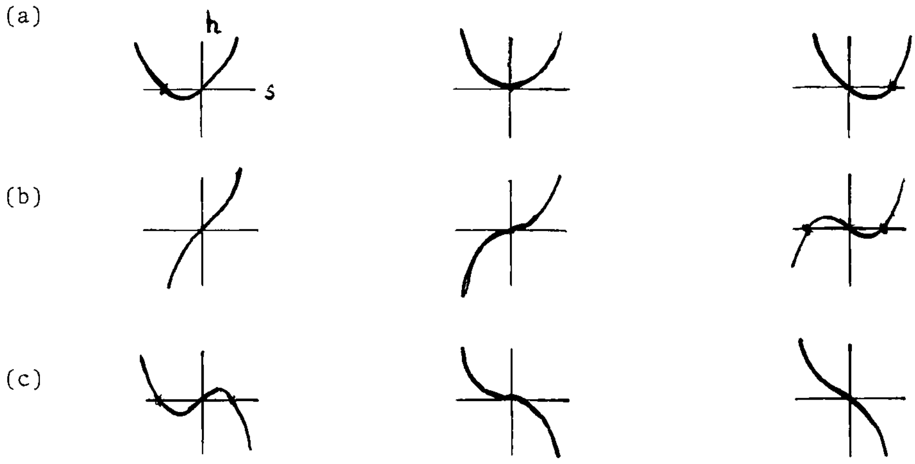
$$\partial h/\partial s(\sigma(\varepsilon)\varepsilon^{1/(m-1)},\varepsilon) = \varepsilon\{(m-1)\alpha + o(1)\}$$

It remains to show that every small solution is obtained in this way. Suppose $s(\varepsilon) \neq 0$ for small $\varepsilon \neq 0$, $h(s(\varepsilon),\varepsilon) = 0$ and $s(\varepsilon) \to 0$ as $\varepsilon \to 0$; we show $\sigma(\varepsilon) \equiv s(\varepsilon)\varepsilon^{-1/(m-1)}$ must tend to some σ_0 with $\alpha\sigma_0 = \beta\sigma_0^m$, as $\varepsilon \to 0$, and then the implicit function theorem proves uniqueness. But $h(s,\varepsilon) = -\alpha s\varepsilon + \beta s^m + \Delta(s,\varepsilon)$ has $|\Delta(s(\varepsilon),\varepsilon)| \le \delta_\varepsilon(|s(\varepsilon)|^m + |\varepsilon s(\varepsilon)|)$, with $\delta_\varepsilon \to 0$ as $\varepsilon \to 0$, hence

$$0 = \varepsilon^{-m/(m-1)}h(\sigma(\varepsilon)\varepsilon^{1/(m-1)},\varepsilon) = -\alpha\sigma + \beta\sigma^m + \varepsilon^{-m/(m-1)}\Delta(\sigma\varepsilon^{1/(m-1)},\varepsilon)$$

so $|\alpha\sigma(\varepsilon) - \beta\sigma(\varepsilon)^m| \le \delta_\varepsilon(|\sigma(\varepsilon)|^m + |\sigma(\varepsilon)|)$ for small ε. This implies $\sigma(\varepsilon)$ remains bounded as $\varepsilon \to 0$, and so $\alpha\sigma(\varepsilon) - \beta\sigma(\varepsilon)^m \to 0$ as $\varepsilon \to 0$.

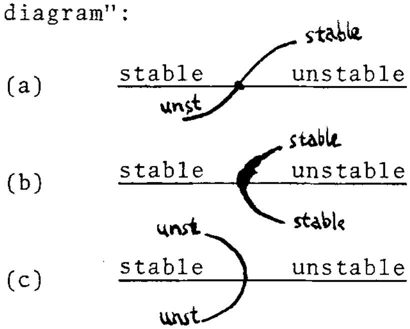
Remark. The results of this lemma are summarized in the following diagrams.



$\varepsilon < 0$ $\quad$ $\varepsilon = 0$ $\quad$ $\varepsilon > 0$

$(\frac{\partial h}{\partial s}(0,\varepsilon) > 0)$ $\quad$ $(\frac{\partial h}{\partial s}(0,0) = 0)$ $\quad$ $(\frac{\partial h}{\partial s}(0,\varepsilon) < 0)$

The stability interpretation may be expressed in a "bifurcation diagram":



In Sattinger's summary [86, 88]: nontrivial solutions which bifurcate subcritically ($\varepsilon < 0$) are unstable but supercritical bifurcating solutions are stable. (Note: this will not generally hold except in the case treated here, of a *simple* eigenvalue $\lambda(\varepsilon)$ with $\lambda(0) = 0$ and $\lambda'(0) < 0$.)

We collect our results in a theorem.

Theorem 6.3.2. Suppose A is sectorial in X, U is a neighborhood of the origin in X^α for some $\alpha < 1$, and for some $\varepsilon_0 > 0$, $f: U \times (-\varepsilon_0, \varepsilon_0) \to X$ is twice continuously differentiable and $f(0,\varepsilon) = 0$ for $|\varepsilon| < \varepsilon_0$.

We consider the equation

$$(E_\varepsilon) \qquad dx/dt + Ax = f(x,\varepsilon)$$

for small ε, comparing with the limiting equation

$$(E_0) \qquad dx/dt + Ax = f(x,0).$$

Assume $L_\varepsilon = A - f_x(0,\varepsilon)$ has a simple real eigenvalue $\lambda(\varepsilon)$ for $|\varepsilon| < \varepsilon_0$, with $\lambda(0) = 0$ and $\frac{d\lambda}{d\varepsilon}(0) < 0$, and suppose the remainder of the spectrum of L_ε lies in $\{\mathrm{Re}\,\lambda > \beta\}$ for some $\beta > 0$, provided ε is sufficiently small.

If the flow in the critical manifold for (E_0) is represented by $ds/dt + h_0(s) = 0$, then $h_0(s) = O(s^2)$ as $s \to 0$, but we assume $h_0(s)$ does not vanish to infinite order at $s = 0$, and in fact $h_0(s) = \beta s^m + o(s^m)$ as $s \to 0$ for some $\beta \neq 0$ and some $m \geq 2$. Suppose f is C^m near the origin.

Then we have one of the following cases: either

(a) there exists a nontrivial equilibrium point $x(\varepsilon)$ for small $\varepsilon \neq 0$ which is unstable for $\varepsilon < 0$ but asymptotically stable for $\varepsilon > 0$;

or (b) there exists a pair of nontrivial equilibria $x^+(\varepsilon), x^-(\varepsilon)$, for small $\varepsilon > 0$, and these are asymptotically stable, but there are no small nontrivial equilibria for small $\varepsilon < 0$;

or (c) there exists a pair of nontrivial equilibria $x^\pm(\varepsilon)$ for small $\varepsilon < 0$ and these are unstable, but there are no small nontrivial equilibria for small $\varepsilon > 0$.

<u>Example</u>.
$$u_t = u_{xx} + \mu f(u), \qquad (0 < x < \pi, \quad t > 0)$$
$$u(0,t) = 0, \quad u(\pi,t) = 0$$

where f is twice continuously differentiable, $f(0) = 0$, $f'(0) = 1$, and μ is a non-negative constant. If $\mu < 1$, the zero solution is asymptotically stable in $H_0^1(0,\pi)$ (see Th. 5.1.1) but if $\mu > 1$, the zero solution is unstable (Th. 5.1.2). Let $\mu = 1 + \varepsilon$; then for smooth u which vanish at $x = 0,\pi$,

$$L_\varepsilon u(x) = -u''(x) - (1+\varepsilon)u(x), \quad 0 < x < \pi,$$

so (in $X = L^2(0,\pi)$)

$$\sigma(L_\varepsilon) = \{n^2 - 1 - \varepsilon \mid n = 1,2,3,\cdots\}$$

and $\lambda(\varepsilon) = -\varepsilon$ is the critical eigenvalue.

If $f(u) = u - au^3$ $(a \neq 0)$, the critical manifold was computed in the example in sec. 6.2, and the same computation (through third order terms) is valid whenever f is C^3 with $f'(0) = 1$, $f''(0) = 0$, $f'''(0) \neq 0$: if $f'''(0) < 0$, case (b) of Th. 6.3.2 applies; if $f'''(0) > 0$, case (c) of Th. 6.3.2. applies.

Exercise 1. If $f(u) = u + au^2 + O(u^3)$ as $u \to 0$, with $a \neq 0$, compute the flow in the critical manifold and show case (a) of Th. 6.3.2 applies to $u_t = u_{xx} + (1+\varepsilon)f(u)$, $(0 < x < \pi,\ t > 0)u = 0$ at $x = 0,\pi$.

Example, continued. Let us examine further the case when the origin is asymptotically stable for the critical case $\varepsilon = 0$. For simplicity, suppose $f(u) = u - au^3$, $a > 0$: according to Cor. 4.2.3 there exists a Liapunov function $V(u)$ for $u_t = u_{xx} + u - au^3$, $u = 0$ at $x = 0,\pi$, satisfying,

$$\|v\|_{H_0^1(0,\pi)} \leq V(u) \leq K\|u\|_{H_0^1(0,\pi)}, \quad |V(u_1)-V(u_2)| \leq K\|u_1-u_2\|_{H_0^1(0,\pi)}$$

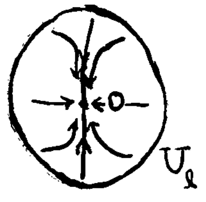
and such that $U_\ell \equiv \{\phi \in H_0^1(0,\pi) \mid V(\phi) < \ell\}$ for sufficiently small $\ell > 0$, is positively invariant with respect to

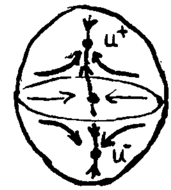
$$u_t = u_{xx} + (1+\varepsilon)(u-au^3), \quad u = 0 \text{ at } x = 0,\pi,$$

for all sufficiently small ε, $|\varepsilon| < \varepsilon_0(\ell)$.

If $u(\cdot,0) \in U_\ell$ and u is a solution (for some sufficiently small ε), then $\{u(\cdot,t)\ t \geq 0\}$ lies in a compact set in U_ℓ. But according to Th. 6.1.4, any invariant subset of U_ℓ (when ℓ is small) must be contained in the local invariant manifold S_ε, and therefore the ω-limit set for the solution u must be $\{0\}$, if $\varepsilon \leq 0$, or one of the equilibria $\{0\}$, $\{u^+(\varepsilon)\}$, or $\{u^-(\varepsilon)\}$ if $\varepsilon > 0$. Now for $\varepsilon > 0$, the domain of attraction of $u^+(\varepsilon)$ (or $u^-(\varepsilon)$) is an open connected set in $H_0^1(0,\pi)$, as is its intersection with U_ℓ; and the region of attraction of $\{0\}$, intersected with U_ℓ, is the stable manifold for the origin which is a C^1 manifold of codimension one.

Thus we have justified the following pictures:





That is, for small $\varepsilon > 0$, a small neighborhood of the origin is split into three sets: the domains of attraction of the two stable equilibria, and their common boundary, the stable manifold for the origin.

Exercise 2. Suppose A is sectorial $f: X^\alpha \to X$ is C^1 near 0, 0 is a simple e.v. of A, $f(0) = 0$, $f'(0) = 0$, and the origin is asymptotically stable for $dx/dt + Ax = f(x)$. If $x = 0$ is an isolated equilibrium for $Ax = f(x) + O(\|x\|_\alpha^N)$, independent of N^{th} order terms, then $x = 0$ is asymptotically stable for $dx/dt + Ax = f(x) + O(\|x\|_\alpha^N)$, independent of N^{th} order terms, and the same is true for the flow in the center (or critical) manifold.

6.4 Bifurcation of a periodic orbit from an equilibrium point
[12, 41, 52, 15, 88]

Consider A, $f(x,\varepsilon)$ as in the previous section, except that we now assume the zero solution loses stability when a pair of complex conjugate eigenvalues pass through the imaginary axis. More precisely, assume $f(0,\varepsilon) = 0$ and $L_\varepsilon = A - f_x(0,\varepsilon)$ has

$$\sigma(L_\varepsilon) \subseteq \{r(\varepsilon) \pm i\omega(\varepsilon)\} \cap \{\text{Re}\,\lambda > \beta > 0\}$$

for small ε, where $r(\varepsilon) \pm i\omega(\varepsilon)$ are simple eigenvalues of L_ε with imaginary part $\omega(0) > 0$, and $r(0) = 0$, $\frac{dr}{d\varepsilon}(0) < 0$.

Observe that there can be no bifurcation of equilibrium points from the origin, for small ε, by the implicit function theorem.

Let S_ε be the two-dimensional local invariant manifold through 0 corresponding to these eigenvalues, constructed as in Th. 6.1.2 and 6.1.4. The flow in S_ε is described by

$$\Sigma_\varepsilon : \frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} r(\varepsilon) & \omega(\varepsilon) \\ -\omega(\varepsilon) & r(\varepsilon) \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \phi_1(y_1,y_2;\varepsilon) \\ \phi_2(y_1,y_2;\varepsilon) \end{pmatrix}$$

with $\phi_{1,2}(y_1,y_2;\varepsilon) = O(y_1^2 + y_2^2)$ as $(y_1,y_2) \to (0,0)$. According to [11,9.1.iv], it follows that when $\varepsilon = 0$, the origin is either a center or a spiral point for the flow in S_0. We assume the origin is not a center for Σ_0, and in fact that this is true independent of terms of order $|y_1|^N + |y_2|^N$, for some $N \geq 2$.

If we introduce polar coordinates in the equation Σ_ε, $y_1 = R\cos\theta$, $y_2 = R\sin\theta$, we find that θ may be taken as a new "time" variable and Σ_ε implies

$$\frac{1}{R}\frac{dR}{d\theta} = -\frac{r(\varepsilon)}{\omega(\varepsilon)} + \phi_3(R,\theta,\varepsilon),$$

$\phi_3(R,\theta,\varepsilon) = O(R)$ as $R \to 0$. A periodic solution of Σ_ε near the origin would have $R(2\pi) = R(0)$. If $R(\theta;\rho,\varepsilon)$ is the solution with initial value $R(0;\rho,\varepsilon) = \rho$, then $J(\rho) \equiv \int_0^{2\pi} \phi_3(R(\theta;\rho,0),\theta,0)d\theta \neq 0$ for small $\rho > 0$, since the origin is not a center for Σ_0. In fact, since this is true independent of N^{th} order we must have $J(\rho) = \beta\rho^m(1 + o(1))$ for some $\beta \neq 0$ and some integer m, $1 \leq m \leq N-2$.

Arguing along the lines of lemma 6.3.1 for $R(2\pi;\rho,\varepsilon)-\rho = 0$, and restricting attention to $\rho \geq 0$, we see there are only two possibilities:

(a) The origin is asymptotically stable for $\frac{dx}{dt} + Ax = f(x,0)$, there is no periodic orbit in $\{0 < \|x\|_\alpha \leq r\}$ for $-\varepsilon_0 \leq \varepsilon \leq 0$, but a unique asymptotically orbitally stable periodic orbit grows out of the origin for $0 < \varepsilon \leq \varepsilon_0$;

or (b) The origin is unstable for $\frac{dx}{dt} + Ax = f(x,0)$ and there is a unique orbitally unstable periodic orbit for $-\varepsilon_0 \leq \varepsilon < 0$, which shrinks to the origin as $\varepsilon \to 0-$, but there is no periodic orbit in $\{0 < \|x\|_\alpha \leq r\}$ for $0 \leq \varepsilon \leq \varepsilon_0$.

In either case, the nontrivial periodic orbits have period approaching $2\pi/\omega(0)$ as $\varepsilon \to 0$.

Remark. Chafee [12], and more generally Hale [41], study this problem for ordinary and functional differential equations without assuming $\frac{dr}{d\varepsilon}(0) < 0$, and in that case several periodic orbits might bifurcate from the origin for small $\varepsilon > 0$. These authors deal only with the case when the origin is asymptotically stable at $\varepsilon = 0$. The argument in this case is an application of the Poincaré-Bendixson theorem in the center manifold.

Example. Bifurcation of a periodic orbit from an equilibrium point. The equations considered are simplified versions of equations of chemical reactor theory [4, 15, 100].

For $0 < x < \pi$, $t > 0$

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + au + bv + f(u,v)$$

$$N\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + cu + dv + g(u,v),$$

and $u = v = 0$ at $x = 0$, $x = \pi$. Here a,b,c,d,N are constants, $N > 0$, and f,g are smooth and $O(|u|^3 + |v|^3)$ as $(u,v) \to (0,0)$. We examine the neighborhood of the zero solution as N varies.

Assume $a < 1 < d$ and $n^4 - n^2\tau + \Delta > 0$ for integers $n \geq 1$ ($\tau = a + d$, $\Delta = ad - bc$).

Exercise 1. The eigenvalues λ of

$$u_{xx} + (\lambda + a)u + bv = 0,$$

$$v_{xx} + cu + (\lambda N + d)v = 0 \qquad (0 < x < \pi)$$

$$u = v = 0 \quad \text{at} \quad x = 0,\pi$$

are the numbers λ satisfying one of the following equations.

$$N\lambda^2 + \lambda(d - n^2 + N(a-n^2)) + (a-n^2)(d-n^2) - bc = 0$$

$$(n = 1,2,3,\cdots)$$

Hint: put $\begin{pmatrix} u \\ v \end{pmatrix} = \sum_1^\infty \begin{pmatrix} u_n \\ v_n \end{pmatrix} \sin nx.$

Under the above assumptions, for $N = d-1/1-a > 0$, there is a pair of pure imaginary eigenvalues $\lambda_1^\pm = \pm i\omega$ $(\omega > 0)$, while all other eigenvalues have $\operatorname{Re}\lambda > 0$. Further, at this point

$$\operatorname{Re}\{d\lambda_1^\pm/dN\} = \frac{(1-a)^2}{2(d-1)} > 0$$

Thus the zero solution is asymptotically stable for N slightly greater than $N^* \equiv (d-1)/(1-a)$, but becomes unstable as N decreases through N^*.

We examine the critical (center) manifold for $N = N^*$, which must have the form

$$\binom{u}{v} = \zeta \sin x + \sum_2^\infty \sigma_n(\zeta)\sin nx,\ \zeta = \binom{\xi}{\eta} \in \mathbb{R}^2,\ \sigma_n(\zeta) = O(|\zeta|^2) \text{ as } \zeta \to 0.$$

Putting $\sigma_1(\zeta) = \zeta$ and substituting in the original equation,

$$\begin{pmatrix}1 & 0\\ 0 & N\end{pmatrix} \sum_1^\infty \sigma_n'(\zeta)\frac{d\zeta}{dt} \sin nx + \sum_1^\infty (n^2 1 - M)\sigma_n(\zeta)\sin nx = F(\sum_1^\infty \sigma_p(\zeta)\sin px) = \begin{pmatrix}f(\cdots)\\ g(\cdots)\end{pmatrix}$$

where $M = \begin{pmatrix}a & b\\ c & d\end{pmatrix}$ and $N = N^*$.

If $F(\zeta) = F_3(\zeta) + O(|\zeta|^4)$ where F_3 is a homogeneous cubic, then

$$F(\sum_1^\infty \sigma_p(\zeta) \sin px) = F_3(\zeta)\sin^3 x + O(|\zeta|^4)$$

and so

$$\begin{pmatrix}1 & 0\\ 0 & N\end{pmatrix} \frac{d\zeta}{dt} + (1-M)\zeta = \frac{3}{4} F_3(\zeta) + O(|\zeta|^4)$$

which describes the flow on the critical manifold.

Let $L = \begin{pmatrix}1 & 0\\ 0 & N^{-1}\end{pmatrix}(1-M) = \begin{pmatrix}1-a & -b\\ -cN^{-1} & (1-d)N^{-1}\end{pmatrix}$ with $N = N^*$; L has eigenvalues $\pm i\omega$, and $p,q \in \mathbb{R}^2$, $L(p+iq) = i\omega(p+iq)$, $p + iq \neq 0$, then with $P = (p,q)$,

$$P^{-1}LP = \begin{pmatrix}0 & \omega\\ -\omega & 0\end{pmatrix}.$$

Let $\zeta = \begin{pmatrix}\xi\\ \eta\end{pmatrix} = P\begin{pmatrix}r\cos\theta\\ r\sin\theta\end{pmatrix}$; then

$$\begin{pmatrix}\dot r\\ r\dot\theta\end{pmatrix} = \begin{pmatrix}0\\ \omega r\end{pmatrix} + \begin{pmatrix}\cos\theta & \sin\\ -\sin\theta & \cos\end{pmatrix} P^{-1}\left\{\frac{3}{4} r^3 \begin{pmatrix}1 & 0\\ 0 & N^{-1}\end{pmatrix} F_3(P\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}) + O(r^4)\right\}$$

Let $G(\theta) = \begin{pmatrix}G_1(\theta)\\ G_2(\theta)\end{pmatrix} = \begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix} P^{-1} \begin{pmatrix}1 & 0\\ 0 & N^{-1}\end{pmatrix} F_3(P\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix})$ then

$$\frac{dr}{d\theta} = \frac{3}{4\omega} r^3\{G_1(\theta) + O(r)\}$$

and if $\gamma \equiv \int_0^{2\pi} G_1(\theta)d\theta \neq 0$, we see:

if $\gamma < 0$, the origin is asymptotically stable, independent of fourth order terms;

if $\gamma > 0$, the origin is unstable independent of fourth order terms.

Thus if $\gamma \neq 0$, a small periodic solution exists for small $\gamma(N-N^*) > 0$, and no small periodic solutions for small $\gamma(N-N^*) < 0$.

We sketch the corresponding result when $\underset{\sim}{F}(\zeta) = F_2(\zeta) + F_3(\zeta) + O(|\zeta|^4)$ as $\zeta \to 0$. In this case $u = \sum_{n=1}^{\infty} \sigma_n(\zeta)\sin nx$, $\sigma_1(\zeta) = \zeta$, $\sigma_n(\zeta) = O(|\zeta|^2)$ for $n > 1$,

$$\begin{pmatrix}1 & 0\\ 0 & N\end{pmatrix}\frac{d\zeta}{dt} + (1-M)\zeta = \frac{8}{3\pi}F_2(\zeta) + \frac{3}{4}F_3(\zeta) - \frac{8}{\pi}\sum_{p\ \text{odd}\geq 3}\frac{1}{p(p^2-4)}F_2'(\sigma_p(\zeta),\zeta) + O(|\zeta|^4)$$

where

$$\sigma_p(\zeta) = \frac{8}{3\pi}\int_0^\infty \exp(-A_p s)\begin{pmatrix}1 & 0\\ 0 & N^{-1}\end{pmatrix}F_2(e^{A_1 s}\zeta)ds + O(|\zeta|^3),\ (p = 2,3,\cdots)$$

$$A_p = \begin{pmatrix}1 & 0\\ 0 & N^{-1}\end{pmatrix}\{p^2 1 - M\}.$$

Substituting $\zeta = rP\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}$, we argue as before with

$$G(\theta) = \begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}P^{-1}\begin{pmatrix}1 & 0\\ 0 & N^{-1}\end{pmatrix}\ F_3(P\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix})$$

$$-\frac{8}{\pi}\sum_{\text{odd } n\geq 3}\frac{1}{n(n^2-4)}F_2'(\sigma_n^{(2)}(P\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}),\ P\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}).$$

(Note: $\begin{pmatrix}\cos\theta & \sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}P^{-1}\begin{pmatrix}1 & 0\\ 0 & N^{-1}\end{pmatrix}\ F_2(P\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix})$ is a homogeneous third order polynomial in $\cos\theta$, $\sin\theta$, and it has average value zero.)

Exercise. (E. Hopf [50])

$$\begin{cases} u_t = -v*v - w*w - u*1 + \mu u_{xx} \\ v_t = v*u + v*a + w*b + \mu v_{xx} \\ w_t = w*u - v*b + w*a + \mu w_{xx} \end{cases}$$

where it is required that u,v,w be even 2π-periodic functions of x, and $f*g(x) = \frac{1}{2\pi}\int_0^{2\pi} f(x+y)g(y)dy$. Here $a(x)$, $b(x)$ are given smooth even 2π-periodic functions and μ is a positive constant. Prove this defines a (local) dynamical system in $X^{\frac{1}{2}} = \{\phi \in H^1_{loc}(\mathbb{R}) \mid \phi(-x) = \phi(x) = \phi(x+2\pi)$ for all $x\}$. This system is easily analyzed by introducing $(u,v,w) = \sum_{n=0}^{\infty} (u_n(t), v_n(t), w_n(t)) \cos nx$; the various components then uncouple

$$\left(\frac{d}{dt} + \mu n^2\right)\begin{pmatrix} u_n \\ v_n \\ w_n \end{pmatrix} = \begin{pmatrix} -v_n^2 - w_n^2 - u_0\delta_{n0} \\ v_n u_n + v_n a_n + w_n b_n \\ w_n u_n - v_n b_n + w_n a_n \end{pmatrix}$$

for $n = 0,1,2,\cdots$, where $a(x) = \sum_0^\infty a_n \cos nx$, etc. Substutiting $v_n + iw_n = e^{R_n + i\theta_n}$, obtain

$$\ddot{R}_n + \mu n^2 \dot{R}_n + e^{2R_n} + \mu n^2(\mu n^2 - a_n) = 0$$

$$u_n = \dot{R}_n + \mu n^2 - a_n$$

$$\dot{\theta}_n = -b_n$$

The standard Liapunov function $(\frac{1}{2}\dot{R}_n^2 + \frac{1}{2}e^{2R_n} + \mu n^2(\mu n^2 - a_n)R_n)$ enables us to conclude:

if $\mu n^2 < a_n$, then $(v_n + iw_n)e^{ib_n t} \to e^{ic_n}\sqrt{\mu n^2(a_n - \mu n^2)}$,

and

if $\mu n^2 \geq a_n$, then $(v_n + iw_n) \to 0$ as $t \to +\infty$.

Thus, for each $\mu > 0$, this system has an attracting finite dimensional invariant manifold of dimension $= \#\{n \quad 0 < \mu n^2 < a_n\}$ namely

$$\Sigma_\mu = \begin{cases} u = \sum_{0 < \mu n^2 < a_n} (\mu n^2 - a_n)\cos nx, \\ v + iw = \sum_{0 < \mu n^2 < a_n} \sqrt{\mu n^2(a_n - \mu n^2)}\; e^{i(c_n - ib_n t)} \cos nx \end{cases}$$

where the c_n are arbitrary real numbers. Observe that, in general, $\dim \Sigma_\mu \to +\infty$ as $\mu \to 0+$. (See sec. 8.5 for the bifurcation of a two-dimensional invariant torus from a periodic orbit.)

Chapter 7

Linear Nonautonomous Equations

7.1 Evolution operators and estimates

Throughout this chapter, we will use the following estimates.

Lemma 7.1.1. Suppose $b \geq 0, \beta > 0$ and $a(t)$ is a nonnegative function locally integrable on $0 \leq t < T$ (some $T \leq +\infty$), and suppose $u(t)$ is nonnegative and locally integrable on $0 \leq t < T$ with

$$u(t) \leq a(t) + b \int_0^t (t-s)^{\beta-1} u(s)ds$$

on this interval;
then

$$u(t) \leq a(t) + \theta \int_0^t E'_\beta(\theta(t-s))a(s)ds, \qquad 0 \leq t < T,$$

where

$$\theta = (b\Gamma(\beta))^{1/\beta}, \; E_\beta(z) = \sum_{n=0}^{\infty} z^{n\beta}/\Gamma(n\beta+1), \; E'_\beta(z) = \frac{d}{dz} E_\beta(z),$$

$E'_\beta(z) \simeq z^{\beta-1}/\Gamma(\beta)$ as $z \to 0+$, $E'_\beta(z) \simeq \frac{1}{\beta} e^z$ as $z \to +\infty$ (and $E_\beta(z) \simeq \frac{1}{\beta} e^z$ as $z \to +\infty$). If $a(t) \equiv a$, constant, then $u(t) \leq aE_\beta(\theta t)$.

Proof. Let $B\phi(t) = b\int_0^t (t-s)^{\beta-1}\phi(s)ds$, $t \geq 0$, for locally integrable functions ϕ. Then $u \leq a + Bu$ implies $u \leq \sum_{k=0}^{n-1} B^k a + B^n u$, and $B^n u(t) = \int_0^t (b\Gamma(\beta))^n (t-s)^{n\beta-1} u(s)ds/\Gamma(n\beta) \to 0$ as $n \to +\infty$ for each t in $0 \leq t < T$. Thus $u(t) \leq a(t) + \int_0^t \{\sum_{n=1}^{\infty} (b\Gamma(\beta))^n (t-s)^{n\beta-1}/\Gamma(n\beta)\}\, a(s)ds$.

The estimates of $E_\beta(z)$ and $E'_\beta(z)$ as $z \to +\infty$ follow from the fact that the Laplace transform $\int_0^\infty e^{-\lambda z} E_\beta(z)dz = \lambda^{-1}/(1-\lambda^{-\beta})$ has a simple pole at $\lambda = 1$. See [24] for details.

For example, we can choose $0 < \gamma < 1$ so $1-\lambda^{-\beta} \neq 0$ for $\mathrm{Re}\, \lambda \geq \gamma$, $\lambda \neq 1$, and then for $z > 0$

$$E_\beta(z) = \frac{1}{\beta}e^z + \frac{1}{2\pi i} \lim_{N\to\infty} \int_{\gamma-iN}^{\gamma+iN} e^{\lambda z}\lambda^{-1}/(1-\lambda^{-\beta})d\lambda,$$

where the shift in the line of integration is justified by the fact $e^{\lambda z}\lambda^{-1}/(1-\lambda^{-\beta}) \to 0$ as $\mathrm{Im}\,\lambda \to \pm\infty$ with $\mathrm{Re}\,\lambda$ bounded. Integration by parts in the last integral shows

$$|E_\beta(z) - \frac{1}{\beta} e^z| = O(\frac{1}{z} e^{\gamma z}) = o(e^z) \quad \text{as} \quad z \to +\infty.$$

The next lemma is used only in Thm. 7.4.2.

Lemma 7.1.2. Suppose $\beta > 0$, $\gamma > 0$, $\beta+\gamma > 1$ and $a \geq 0$, $b \geq 0$, u is nonnegative and $t^{\gamma-1}u(t)$ is locally integrable on $0 \leq t < T$, and

$$u(t) \leq a + b\int_0^t (t-s)^{\beta-1}s^{\gamma-1}u(s)ds$$

a.e. in $(0,T)$; then

$$u(t) \leq aE_{\beta,\gamma}((b\Gamma(\beta))^{1/\nu}t)$$

where $\nu = \beta+\gamma-1 > 0$, $E_{\beta,\gamma}(s) = \sum_{m=0}^{\infty} c_m s^{m\nu}$ with $c_0 = 1$, $c_{m+1}/c_m = \Gamma(m\nu+\gamma)/\Gamma(m\nu+\gamma+\beta)$ for $m \geq 0$. As $s \to +\infty$

$$E_{\beta,\gamma}(s) = O(s^{\frac{1}{2}(\nu/\beta-\gamma)}\exp(\frac{\beta}{\nu} s^{\nu/\beta})).$$

Proof. If $B\phi(t) = b\int_0^t(t-s)^{\beta-1}s^{\gamma-1}\phi(s)ds$, an easy induction shows

$$u(t) \leq a\sum_{m=0}^{n} c_m(b\Gamma(\beta))^m t^{m\nu} + B^{n+1}u(t).$$

Also $B^n u(t) = \int_0^t K_n(t,s)s^{\gamma-1}u(s)ds$ where

$$K_n(t,s) \leq Q_n t^{(n-1)(\gamma-1)}(t-s)^{n\beta-1}$$

$$Q_1 = b,\ Q_{n+1}/Q_n = b\Gamma(\beta)\Gamma(n\beta)/\Gamma(n\beta+\beta)$$

when $\gamma \geq 1$, while for $0 < \gamma < 1$

$$K_n(t,s) \leq Q_n(t-s)^{n\nu-\gamma}$$

$$Q_1 = b,\ Q_{n+1}/Q_n = b\Gamma(\beta)\Gamma(n\nu)/\Gamma(n\nu+\beta).$$

In either case, $Q_{n+1}/Q_n = O(n^{-\beta})$ as $n \to \infty$ so $B^n u(t) \to 0$ as $n \to \infty$ and

$$u(t) \leq aE_{\beta,\gamma}((b\Gamma(\beta))^{1/\nu}t).$$

Now $\Gamma(z+p)/\Gamma(z+q) = z^{p-q}\{1 + (p-q)(p+q-1)/2z + O(z^{-2})\}$ as $z \to +\infty$ so if $\delta = (\beta\gamma+\nu)/2\nu$

$$\frac{\Gamma((n+1)\beta+\delta)\ C_{n+1}}{\Gamma(n\beta+\delta)C_n} = (\beta/\nu)^{\beta}[1 + O(n^{-2})]$$

so $C_n\Gamma(n\beta+\delta)(\beta/\nu)^{-n\beta}$ converges as $n \to +\infty$ and has an upper bound K for all $n \geq 0$. Then

$$E_{\beta,\gamma}(s^{\beta/\nu})s^{\delta-1} \leq K \sum_{n=0}^{\infty} \frac{(\beta/\nu)^{n\beta}}{\Gamma(n\beta+\delta)}\ s^{n\beta+\delta-1}$$

for $s > 0$. The Laplace transform of the right side is $K\lambda^{-\delta}/(1 - (\beta/\nu\lambda)^{\beta})$, so the series is $O(\exp(\beta s/\nu))$ as $s \to \infty$ which proves the result.

Exercise 1. Show $\frac{d}{dt}E_{1/2}(t) = E_{1/2}(t) + (\pi t)^{-1/2}$ and so $e^t \leq E_{1/2}(t) \leq 2e^t$ for $t \geq 0$.

Exercise 2. Show for $\beta = \frac{1}{2}$, $\gamma = 2$ in $E_{1/2,2}(t)$ that

$$3^{k+1}(k+1)!C_{n+2}/3^k k!C_n < 1$$

when $n = 2k+1$ or $2k+2$, $k \geq 0$, and so

$$E_{1/2,2}(t) \leq 1.886 \max(1,t^3)\exp(t^3/3).$$

Exercise 3. If α,β,γ are positive with $\beta+\gamma-1 = \nu > 0$, $\delta = \alpha+\gamma-1 > 0$, and

$$u(t) \leq at^{\alpha-1} + b\int_0^t (t-s)^{\beta-1}s^{\gamma-1}u(s)ds \quad \text{for } t > 0,$$

then

$$u(t) \leq at^{\alpha-1} \sum_{m=0}^{\infty} C'_m(b\Gamma(\beta))^m t^{m\nu}$$

where $C'_0 = 1$, $C'_{m+1}/C'_m = \Gamma(m\nu+\delta)/\Gamma(m\nu+\delta+\beta)$.

Exercise 4*. If $0 \leq \alpha$, $\beta < 1$, $0 < T < \infty$, show there is a constant $C(\beta,b,T) < \infty$ so $u(t) \leq at^{-\alpha} + b\int_0^t (t-s)^{-\beta}u(s)ds$ on $(0,T)$ implies $u(t) \leq \frac{at^{-\alpha}}{1-\alpha}\ C(\beta,b,T)$.

Theorem 7.1.3. Suppose A is sectorial in X, $0 \leq \alpha < 1$ and $t \to B(t): [t_0,t_1] \to \mathscr{L}(X^{\alpha},X)$ is Hölder continuous.

For any $x_0 \in X$ and $t_0 \le \tau < t_1$ there exists a unique solution $x(t) = x(t;\tau,x_0)$ of

$$dx/dt + Ax = B(t)x \quad \text{on} \quad \tau < t \le t_1, \quad x(\tau) = x_0,$$

and $x_0 \to x(t;\tau,x_0)$ is linear and bounded on X, so we write $x(t;\tau,x_0) = T(t,\tau)x_0$, $t \ge \tau$. This family of *evolution operators* $\{T(t,\tau),\ t_0 \le \tau \le t \le t_1\}$ has the following properties.

a) $T(\tau,\tau) = I$, $T(t,s)T(s,\tau) = T(t,\tau)$ if $t \ge s \ge \tau$.

b) $\{T(t,\tau),\ t \ge \tau\}$ is strongly continuous in (t,τ) with values in $\mathscr{L}(X^\beta)$ for any $0 \le \beta < 1$.

c) There is a constant C, depending only on A,α,t_1-t_0 and $\sup\|B(t)\|$, such that for x in $D(A^2)$ and $0 \le \beta,\gamma,\theta \le 1$,

(i) $\|T(t,\tau)x\|_\beta \le \frac{C}{1-\beta}(t-\tau)^{(\gamma-\beta)_-}\|x\|_\gamma$

when $\beta < 1$, $(\gamma-\beta)_- = \min\{\gamma-\beta,0\}$, and $t > \tau$ in $[t_0,t_1]$;

(ii) $\|T(t,\tau)x-x\|_\beta \le \frac{C}{(1-\beta)\theta}(t-\tau)^\theta\|x\|_{\beta+\theta}$

when $\theta > 0$, $\beta+\theta \le 1$;

(iii) $\|T(t+h,\tau)x-T(t,\tau)x\|_\beta \le \frac{C}{1-\beta-\theta}(\frac{1}{\theta} + \frac{1}{1-\beta})h^\theta(t-\tau)^{(\gamma-\beta-\theta)_-}\cdot \|x\|_\gamma$

when $t+h > t > \tau$, $\theta > 0$, $\beta+\theta < 1$;

(iv) $\|T(t,\tau)x-T(t,\tau-h)x\|_\beta$

$$\le C(\frac{1}{\theta(1-\beta)} + \frac{1}{1+\gamma-\alpha-\theta})h^\theta(t-\tau)^{(\gamma-\beta-\theta)_-}\|x\|_\gamma$$

when $t > \tau > \tau-h$, $\theta > 0$, $\beta < 1$ and $1+\gamma > \alpha+\theta$.

(d) If $\|B(t)-B(s)\|_{\mathscr{L}(X^\alpha,X)} \le Q|t-s|^q$ on $[t_0,t_1]$, $0 < q \le 1$, then for $0 \le \beta < q < 1-\alpha$, $t > \tau$ and C as in (c) above,

$$\|\frac{\partial}{\partial t}T(t,\tau)x\|_\beta = \|(A-B(t))T(t,\tau)x\|_\beta$$

$$\le C(\frac{1}{q(1-\alpha-q)^2} + \frac{Q}{q-\beta})(t-\tau)^{\gamma-\beta-1}\|x\|_\gamma.$$

Proof. There is a constant M, depending only on A and t_1-t_0, such that

$$\|e^{-At}x\|_\beta \le Mt^{(\gamma-\beta)_-}\|x\|_\gamma$$

and

$$\|e^{-At}x-x\|_{\beta} \leq \frac{M}{\theta} t^{\theta} \|x\|_{\beta+\theta}$$

for $0 < \theta \leq 1$ and $0 \leq \beta,\gamma \leq 2$. Constants depending only on α, M, t_1-t_0 and $\sup\|B(t)\|$ will be written $C_1, C_2, \ldots$.

Suppose $t > \tau$, $[\tau,t] \subset [t_0,t_1]$, $x_0 \in X^{\alpha}$; then the solution $x(t;\tau,x_0)$ is certainly well-defined in X^{α} and, by uniqueness, a linear function of x_0. Writing $x(t;\tau,x_0) = T(t,\tau)x_0$, the properties (a) also follow by uniqueness, at least on X^{α}. If $\alpha \leq \beta < 1$, the initial value problem is also well-posed in X^{β}, so the restriction $T(t,\tau)|X^{\beta}$ is in $\mathscr{L}(X^{\beta})$, and is strongly continuous in (t,τ). When $0 \leq \beta < \alpha$, the corresponding results follow from estimates (c).

If $\dot{x} + Ax = B(t)x$ on $(\tau,t_1]$, $x(\tau) \in D(A)$, then for $t > \tau$ and $0 \leq \beta < 1$, $0 \leq \gamma \leq 1$,

$$\|x(t)\|_{\beta} \leq M(t-\tau)^{(\gamma-\beta)-} \|x(\tau)\|_{\gamma} + MC_1 \int_{\tau}^{t} (t-s)^{-\beta}\|x(s)\|_{\alpha}\, ds.$$

Taking first $\beta = \alpha$ and then the general case, we find $\|x(t)\| \leq \frac{C_2}{1-\beta}(t-\tau)^{(\gamma-\beta)-} \|x(\tau)\|_{\gamma}$. If $\gamma = \beta$, $T(t,\tau)|X^{\beta}$ is uniformly bounded in $\mathscr{L}(X^{\beta})$ and it is strongly continuous when $\alpha \leq \beta < 1$, hence also for $0 \leq \beta < \alpha$.

Next observe, for $\theta > 0$, $\beta+\theta \leq 1$,

$$\|x(t)-x(\tau)\|_{\beta} \leq \frac{M}{\theta}(t-\tau)^{\theta} \|x(\tau)\|_{\beta+\theta} + MC_1 \int_{\tau}^{t} (t-s)^{-\beta}\|x(s)\|_{\alpha} ds$$

$$\leq C_3\left(\frac{1}{\theta} + \frac{1}{1-\beta}\right)(t-\tau)^{\theta} \|x(\tau)\|_{\beta+\theta}, \text{ using (c)(i).}$$

If $\theta > 0$, $\beta+\theta < 1$, $t+h > t > \tau$,

$$\|x(t+h)-x(t)\|_{\beta} = \|(T(t+h,t)-I)x(t)\|_{\beta}$$

$$\leq C_3\left(\frac{1}{\theta} + \frac{1}{1-\beta}\right)h^{\theta} \cdot \frac{C_2}{1-\beta-\theta}(t-\tau)^{(\gamma-\beta-\theta)-} \|x(\tau)\|_{\gamma},$$

which proves c(iii).

Let $\beta < 1$, $1+\gamma > \alpha+\theta$, $\theta > 0$, $t > \tau > \tau-h$, and

$$z(t) = T(t,\tau)x - T(t,\tau-h)x,$$

so

$$z(t) = (1-e^{-Ah})e^{-A(t-\tau)}x + \int_\tau^t e^{-A(t-s)}B(s)z(s)ds$$
$$- \int_{\tau-h}^{\tau} e^{-A(t-s)}B(s)T(s, -h)x\ ds.$$

Since $t-s \geq \min\{t-\tau, \tau-s\}$ when $\tau-h \leq s \leq \tau$,

$$\left\|\int_{\tau-h}^{\tau} e^{-A(t-s)}B(s)T(s,\tau-h)x\ ds\right\|_\beta$$
$$\leq \frac{C_4}{1-\beta}\min\{(t-\tau)^{-\beta}h^{1+(\gamma-\alpha)_-},\ h^{1-\beta+(\gamma-\alpha)_-}\}\|x\|_\gamma$$
$$\leq \frac{C_5}{1-\beta}h^\theta(t-\tau)^{(\gamma-\beta-\theta)_-}\|x\|_\gamma,$$

considering the various cases and using $\min\{a,b\} \leq a^\lambda b^{1-\lambda}$ for $0 \leq \lambda \leq 1$, $a \geq 0$, $b \geq 0$. Using this to estimate first $\|z(t)\|_\alpha$ and then $\|z(t)\|_\beta$ proves c(iv).

If $x(t) = T(t,\tau)x$, $t > \tau$, then (see Lemma 3.5.1)

$$-dx/dt = Ae^{-A(t-\tau)}x + e^{-A(t-\tau)}B(t)x(t)$$
$$+ \int_\tau^t Ae^{-A(t-s)}(B(t)x(t) - B(s)x(s))ds.$$

Thus

$$\|dx(t)/dt\|_\beta \leq C_6(t-\tau)^{\gamma-\beta-1}\|x\|_\gamma$$
$$+ M\int_\tau^t (t-s)^{\beta-1}[Q(t-s)^q\|x(t)\|_\alpha + C_1\|x(t)-x(s)\|_\alpha]ds$$
$$\leq C_6(t-\tau)^{\gamma-\beta-1}\|x\|_\gamma + \frac{MQC_2}{q-\beta}(t-\tau)^{q-\beta+(\gamma-\alpha)_-}\|x\|_\gamma$$
$$+ \frac{MC_1C_2}{q(1-\alpha-q)}\left(\frac{1}{q-\beta} + \frac{1}{1+(\gamma-\alpha-q)_-}\right)\|x\|_\gamma(t-\tau)^{q-\beta+(\gamma-\alpha-q)_-}$$

and estimate (d) follows.

<u>Theorem 7.1.4</u>. Assume A, $B(t)$ satisfy the hypotheses of Thm. 7.1.3 on $t_0 \leq t \leq t_1$, and suppose $t_0 \leq \tau < t_1$, $x_0 \in X$, $f\colon (\tau,t_1) \to X$ is locally Hölder continuous with $\int_\tau^{\tau+\rho}\|f(s)\|ds < \infty$ for some $\rho > 0$. Then there exists a unique solution of

$$dx/dt + A(t)x = f(t), \quad \tau < t < t_1, \quad x(\tau) = x_0,$$

(where $A(t) = A-B(t)$), namely

$$x(t) = T(t,\tau)x_0 + \int_\tau^t T(t,s)f(s)ds.$$

Remark. If $0 \le \beta < 1$, $x_0 \in X^\beta$, and we want a solution with $x(t) \to x_0$ in X^β as $t \to \tau+$, require

$$\int_\tau^t (t-s)^{-\beta}\|f(s)\|ds \to 0 \quad \text{as} \quad t \to \tau+.$$

Proof. It is sufficient to prove this with $x_0 = 0$. For $\rho \ge 0$, let $F_\rho(t) = \int_\tau^{t-\rho} T(t,s)f(s)ds$ when $t \ge \tau+\rho$, $F_\rho(t) = 0$ for $t < \tau+\rho$. Choose any fixed t^* in (τ,t_1); for small $\rho > 0$ and for t with $\tau < t-\rho < t^* < t$ we have

$$F_\rho(t) = T(t,t^*)\int_\tau^{t-\rho} T(t^*,s)f(s)ds$$

so $F_\rho(t) \in D(A)$, $t \to F_\rho(t)$ is differentiable and

$$\frac{d}{dt} F_\rho(t) + A(t)F_\rho(t) = T(t,t-\rho)f(t-\rho).$$

Also

$$\|F_\rho(t)\|_\alpha \le M\int_\tau^{t-\rho} (t-s)^{-\alpha}\|f(s)\|ds \to 0$$

as $t \to \tau+\rho$. It follows from Thm. 3.2.2 that

$$\begin{aligned} F_\rho(t) &= \int_\tau^{t-\rho} e^{-A(t-s-\rho)}T(s+\rho,\rho)f(s)ds \\ &\quad + \int_{\tau+\rho}^t e^{-A(t-s)}B(s)F_\rho(s)ds, \qquad t \ge \tau+\rho. \end{aligned}$$

Now $F_\rho(t) \to F_0(t)$ in X as $\rho \to 0+$, and if $t > \tau+\rho$

$$\begin{aligned} \Big\|F_\rho(t) - \int_\tau^t e^{-A(t-s)}(B(s)F_0(s) + f(s))ds\Big\| &\le \Big\|\int_{t-\rho}^t e^{-A(t-s)}f(s)ds\Big\| \\ &+ M\int_\tau^{t-\rho} (t-\rho-s)^{-\alpha}\|(e^{-A\rho}-T(s+\rho,s))f(s)\|ds \\ &+ M\sup\|B\|\frac{\rho^{1-\alpha}}{1-\alpha}\int_\tau^t \|f(s)\|ds \end{aligned}$$

which tends to 0 as $\rho \to 0+$. It is easily seen that $t \to F_0(t) \in X^\alpha$ is locally Hölder continuous on (τ,t_1) and $F_0(t) \to 0$ in X as $t \to \tau+$, so $F_0(t)$ is a mild solution -- hence, a strong solution -- of

$$dx/dt + Ax = B(t)F_0(t) + f(t), \quad t > \tau$$

$$x(\tau) = 0$$

and the theorem is proved.

Exercise 5. The frequent hypothesis of local Hölder continuity in time is made to ensure that we have differentiable solutions, but is unnecessary for mild solutions (continuous solutions of the corresponding integral equation). Suppose A is sectorial in X, $0 \le \alpha < 1$, $\mathrm{Re}\,\sigma(A) > 0$, $\|A^\gamma e^{-At}\| \le Mt^{-\gamma}$ for $t > 0$, $0 \le \gamma \le 1$, and $B: [t_0,t_1] \to \mathscr{L}(X^\alpha,X)$ is measurable and $\left(\int_{t_0}^{t_1} \|B(s)\|^p_{\mathscr{L}(X^\alpha,X)} ds\right)^{1/p} < \infty$ for some $p > 1/1-\alpha$. Prove there is a unique mild solution of $\dot{x} + Ax = B(t)x$, $x(\tau) = \xi$, i.e., a unique continuous solution $x: [\tau,t_1] \to X^\alpha$ of

$$x(t) = e^{-A(t-\tau)}\xi + \int_\tau^t e^{-A(t-s)}B(s)x(s)ds, \quad \tau \le t \le t_1,$$

for any $\xi \in X^\alpha$ and $\tau \in [t_0,t_1)$. Writing this $x(t) = T(t,\tau)\xi$ as before, prove there is a constant $C_{\beta,\gamma}$ so that

$$\|T(t,\tau)\xi\|_\beta \le C_{\beta,\gamma}(t-\tau)^{(\gamma-\beta)-} \|\xi\|_\gamma, \quad t_0 \le \tau < t \le t, \ \xi \in X^\gamma,$$

provided $1 - \frac{1}{p} > \alpha$, $1 - \frac{1}{p} \ge \beta$ and $0 \le \beta < 1$, $0 \le \gamma \le 1$.
Hint: use the Hölder inequality on short time intervals.

Exercise 6. If A is sectorial in X with $\|e^{-At}x\|_\alpha \le Me^{\beta t}\|x\|_\alpha$, $\|e^{-At}x\|_\alpha \le Me^{\beta t}\max(t^{-\alpha},1)\|x\|$ for $t > 0$, $x \in X^\alpha$, and if $B: [t_0,\infty) \to \mathscr{L}(X^\alpha,X)$ is bounded and locally Hölder continuous, with $\int_{t_0}^\infty \|B(t)\|_{\mathscr{L}(X^\alpha,X)} dt < \infty$, then there exists $M_1 > 0$ such that the evolution operator $T(t,s)$ for $dx/dt + (A-B(t))x = 0$ satisfies

$$\|T(t,s)x\|_\alpha \le M_1 e^{\beta(t-s)}\|x\|_\alpha$$
$$\|T(t,s)x\|_\alpha \le M_1 e^{\beta(t-s)}\max(1,(t-s)^{-\alpha})\|x\|$$

for all $t > s \ge t_0$, $x \in X^\alpha$.

Exercise 7. Assume A,B satisfy the hypotheses of ex. 6 with $\beta = 0$ and $\sigma(A) \cap \{\mathrm{Re}\,\lambda = 0\}$ is a spectral set. Let $X = X_1 \oplus X_2$ be the corresponding decomposition, with $e^{-A_1 t}$ bounded on $-\infty < t \le 0$, while $\|e^{-A_2 t}\| \le Me^{-\beta t}$, $t \ge 0$, for some $\beta > 0$. Prove that, for all $s \ge t_0$, there exists a linear continuous map $L(s): X \to X_1$ so for

$t \geq s \geq t_0$,

$$T(t,s) = e^{-A_1(t-s)} L(s) + T_2(t,s)$$

and

$$\|T_2(t,s)\| \to 0 \quad \text{as} \quad t-s \to +\infty \quad (t \geq s \geq t_0).$$

Also, $\|L(s)-E_1\| \to 0$ as $s \to +\infty$, where E_1 is the projection of X onto X_1, along X_2.

Exercise 8. Suppose A is sectorial, $\|e^{-At}\| \leq Me^{\beta t}$ and $\|e^{-At}x\|_\alpha \leq Mt^{-\alpha}e^{\beta t}\|x\|_\alpha$ for $t > 0$, $x \in X^\alpha$. Suppose also that $B: [t_0,\infty) \to (X^\alpha, X)$ is locally Hölder continuous with $\|B(t)\|_{\mathscr{L}(X^\alpha,X)} \leq \gamma$ for all $t \geq t_0$. Prove $\|T(t,s)\| \leq M_1 e^{\delta(t-s)}$, $t \geq s \geq t_0$, if $\delta = \beta + (\gamma M\Gamma(1-\alpha))^{1/(1-\alpha)}$.

Exercise 9. If A, $B(\cdot)$ satisfy the hypotheses of Th. 7.1.3 and if the evolution operator $T(t,s)$ for $dx/dt + (A-B(t))x = 0$ has $\|T(t,s)\| \leq Me^{-\beta(t-s)}$ for $t \geq s \geq t_0$ and some $\beta > 0$, $M > 0$, and if $g(t,x)$ is locally Hölder continuous in t, locally Lipschitz in x, from a neighborhood of $(t_0,\infty) \times \{0\}$ in $R \times X^\alpha$ into X, with $\|g(t,x)\| = o(\|x\|_\alpha)$ as $x \to 0$ in X^α, uniformly in $t \geq t_0$, prove uniform asymptotic stability of the origin for

$$dx/dt + (A-B(t))x = g(t,x).$$

Exercise 10*. Suppose A, $B(\cdot)$ satisfy assumptions of Th. 7.1.3 and assume A has compact resolvent; prove $T(t,s) \in \mathscr{L}(X^\beta, X^\beta)$ is compact for each $t > s$ and $\beta < 1$.

Remark. Tanabe [134], Sobolevskii [93] and Kato [57] have studied evolution operators for linear nonautonomous equations of much more general form, where the highest-order terms vary with time and even their domains are nonconstant. S. G. Krein [63, Ch. II, 5] also treats this problem.

7.2 Linear periodic systems (cf. [98], [39]).

Suppose A is sectorial, $t \to B(t) \in \mathscr{L}(X^\alpha, X)$ is Hölder continuous and p-periodic for some $p > 0$:

$$B(t + p) = B(t) \quad \text{for all } t.$$

We consider first the homogeneous equation

$$\frac{dx}{dt} + (A - B(t))x = 0$$

whose solutions satisfy

$$x(t) = T(t,s)x(s), \quad t \geq s.$$

Exercise 1. $T(t+p, s+p) = T(t,s)$ for all $t \geq s$.
Since $\|B(t)\|_{\mathscr{L}(X^\alpha, X)}$ is bounded, there exist $M \geq 0$ and m such that for $t > s$ and $x \in X^\beta$

$$\|T(t,s)x\|_\beta \leq \frac{M}{1-\beta}(t-s)^{\gamma-\beta} e^{m(t-s)} \|x\|_\gamma \quad (0 \leq \gamma < \beta < 1)$$

$$\|T(t,s)x - x\|_\gamma \leq \frac{M}{\beta-\gamma}(t-s)^{\beta-\gamma} e^{m(t-s)} \|x\|_\beta .$$

To study stability problems we need more precise estimates of the growth of the evolution operator $T(t,s)$ as $t-s \to \infty$. Observe that, if $t \geq s$, $n = 1,2,3,\ldots$

$$T(t+np, s) = \{T(t+p,t)\}^n T(t,s) = T(t,s)\{T(s+p,s)\}^n$$

so the problem is to estimate powers of $T(t+p,t)$.

Definition 7.2.1. The *period map* (Poincaré map) is

$$U(t) = T(t+p,t).$$

The nonzero eigenvalues of $U(t)$ are called *characteristic multipliers*.

Lemma 7.2.2. $U(t+p) = U(t)$ for all t; the characteristic multipliers are independent of time, i.e. the nonzero eigenvalues of $U(t)$ coincide with those of $U(s)$. In fact, $\sigma(U(t)) \setminus \{0\}$ is independent of t. If A has compact resolvent, then $U(t)$ is compact, so

$\sigma(U(t))\setminus\{0\}$ consists entirely of characteristic multipliers.

Proof. Suppose $\mu \neq 0$, $U(s)x = \mu x \neq 0$, and $s \leq t \leq s+p$; let $y = T(t,s)x$, so $T(s+p,t)y = \mu x \neq 0$, $y \neq 0$ and $U(t)y = \mu y$. Since $U(\cdot)$ is p-periodic, this proves the multipliers are independent of time.

If $\mu \neq 0$, $\mu \in \rho(U(s))$, $s \leq t \leq s+p$, and $(\mu - U(t))x = y$, then $\mu x = y + w$, $w = T(t,s)(\mu - U(s))^{-1}T(s+p,t)y$. Conversely defining x by this equation, it follows that $(\mu - U(t))x = y$ so $\rho(U(t)) \supset \rho(U(s))\setminus\{0\}$, and the result follows by periodicity.

If A has compact resolvent, then e^{-At} is compact for $t > 0$, and it follows that $T(t,s)$ is compact for $t > s$.

Theorem 7.2.3. Suppose σ_1 is a spectral set for $\sigma(U(t))$ for all t; the usual case is when σ_1 is a finite collection of isolated eigenvalues, or the complement of such a set. Then for each t, the space X may be decomposed as $X = X_1(t) \oplus X_2(t)$, the direct sum of closed subspaces invariant under $U(t)$, $\sigma(U(t)|_{X_1(t)}) = \sigma_1$, $\sigma(U(t)|_{X_2(t)} = \sigma(U(t)) \setminus \sigma_1$. If $t \geq s$, $T(t,s)$ maps $X_1(s)$ into $X_1(t)$, and is a one-one map onto $X_1(t)$ if $0 \notin \sigma_1$.

Let $e^{\beta p} = \sup\{|\mu|, \mu \in \sigma_1\}$; then for any $\varepsilon > 0$, there exists $M_\varepsilon > 0$ such that

$$\|T(t,s)x\| \leq M_\varepsilon e^{(\beta+\varepsilon)(t-s)} \|x\|$$

for $t \geq s$ and $x \in X_1(s)$.

Now suppose $0 \notin \sigma_1$, and let $e^{\gamma p} = \inf\{|\mu|, \mu \in \sigma_1\} > 0$. Then $T(t,s)x$, $x \in X_1(s)$, may be defined also for $t \leq s$, still satisfying (a) of Th. 7.1.3, and for $x \in X_1(s)$, small $\varepsilon > 0$,

$$\|T(t,s)x\| \leq M_\varepsilon e^{(\gamma-\varepsilon)(t-s)}\|x\|, \quad t \leq s.$$

In fact, provided there is a path γ in the complex plane disjoint from σ_1, joining 0 to ∞, we have a kind of Floquet representation. There exists a family of bounded invertible operators $P(t): X_1(s_0) \to X_1(t)$ for all t with $P(t+p) = P(t)$, $P(s_0) = 1$, and a bounded operator C on $X_1(s_0)$ with spectrum $\sigma(C) = \frac{1}{p} \ln \sigma_1$ and for $x \in X_1(s)$ and all t,s

$$T(t,s)x = P(t)e^{C(t-s)}P^{-1}(s)x.$$

Proof. If $t \geq s$, $T(t,s)U(s) = U(t)T(t,s)$ so

$$T(t,s)(\lambda - U(s))^{-1} = (\lambda - U(t))^{-1} T(t,s)$$

for $\lambda \in \gamma$, a contour disjoint from $\sigma(U(t))$ enclosing σ_1 and excluding σ_2. Then integrating around γ,

$$T(t,s)E_1(s) = E_1(t)T(t,s)$$

where $E_1(t)$ is the projection onto $X_1(t)$, along $X_2(t)$ [23, I, Ch. 7]. This says $T(t,s)$ maps $X_1(s)$ into $X_1(t)$. If $T(t,s)x = 0$ and $x \in X_1(s)$, $x \neq 0$, then for integer n so that $s \leq t \leq s + np$, $(U(s))^n x = 0$ and so $0 \in \sigma_1$.

Suppose $e^{\beta p}$ = spectral radius of $U(s)|_{X_1(s)}$; then for any $\varepsilon > 0$,

$$e^{-(\beta+\varepsilon)(t-s)} \|T(t,s)|_{X_1(s)}\| \to 0 \quad \text{as} \quad t \to +\infty.$$

Indeed, if this failed, there exist $C_0 > 0$ and $t_\nu \to +\infty$, $t_\nu - s = n_\nu p + \theta_\nu$ $(0 \leq \theta_\nu < p, n_\nu = \text{integer})$ so

$$e^{-(\beta+\varepsilon)(t_\nu - s)} \|T(t_\nu,s)|_{X_1(s)}\| \geq C_0 > 0$$

i.e.

$$e^{-(\beta+\varepsilon)pn_\nu} \|T(s+\theta_\nu,s)|_{X_1(s)}\| \, \|U(s)|_{X_1(s)})^{n_\nu}\| \geq C_0 e^{(\beta+\varepsilon)\theta_\nu}$$

Taking n_ν-roots and letting $n_\nu \to \infty$,

$$e^{-\varepsilon p} = e^{-(\beta+\varepsilon)p} \lim_{\nu\to\infty} \|(U(s)|_{X_1(s)})^{n_\nu}\|^{1/n_\nu} \geq 1,$$

a contradiction. If $0 \notin \sigma_1$, $U(s)|_{X_1(s)}$ is invertible as is $T(t,s)|_{X_1(s)}$ and the "lower" estimate is made similarly.

Now assume there is a path γ from 0 to ∞, disjoint from σ_1, and choose any s_0: then $U(s_0)|_{X_1(s_0)} = e^{pC}$, where

$$C = \frac{1}{2\pi i p} \int_L (\lambda - U(s_0)|_{X_1(s_0)})^{-1} \log \lambda \, d\lambda,$$

L is a contour surrounding σ_1 but disjoint from γ, and we take the branch cut for $\log \lambda$ along γ. Then $\sigma(C) = \frac{1}{p} \log \sigma_1$, using the same branch of the logarithm, so $\sigma(e^{pC}) = \sigma_1$. Now if $t \geq s$, $T(t,s)|_{X_1(s)}$ maps $X_1(s)$ onto $X_1(t)$ and has a continuous inverse, namely $(U(s)|_{X_1(s)})^{-n}T(s+np,t)|_{X_1(t)}$, with n chosen so that $s+np \geq t$. Define, for $t \geq s_0$,

$$P(t) = T(t,s_0)|_{X_1(s_0)}e^{-C(t-s_0)} : X_1(s_0) \to X_1(t),$$

Then

$$\begin{aligned} P(t+p) &= T(t+p,s_0)|_{X_1(s_0)}e^{-C(t+p,s_0)} \\ &= T(t+p,s_0+p)|_{X_1(s_0)}U(s_0) \ {}_{X_1(s_0)}e^{-Cp}e^{-C(t-s_0)} \\ &= P(t). \end{aligned}$$

If $t \geq s$, it follows that

$$T(t,s)|_{X_1(s)} = P(t)e^{C(t-s)}P(s)^{-1},$$

and we may use this equation to define $T(t,s)|_{X_1(s)}$ for all t, and then for $t \geq s$,

$$T(s,t)|_{X_1(t)}T(t,s)|_{X_1(s)} = \text{identity on } X_1(s).$$

<u>Remark</u>. From an estimate like

$$\|T(t,s)|_{X_1(s)}\| \leq Me^{\delta(t-s)} \quad \text{for } t \geq s,$$

it follows that for $0 \leq \gamma < \beta < 1$, $t > s$, $x \in X_1(s) \cap X^\beta$

$$\|T(t,s)x\|_\beta \leq \frac{M_1}{1-\beta}\max[(t-s)^{\gamma-\beta},1]e^{\delta(t-s)}\|x\|_\gamma$$

$$\|T(t,s)x-x\|_\gamma \leq \frac{M_1}{\beta-\gamma}\min[(t-s)^{\beta-\gamma},1]e^{\delta(t-s)}\|x\|_\beta$$

<u>Corollary 7.2.4</u>. If A has compact resolvent, $B(\cdot)$ as above, then the zero solution of

$$\frac{dx}{dt} + (A - B(t))x = 0$$

is asymptotically stable if and only if all characteristic multipliers have modulus less than one.

<u>Exercise 2</u>. Suppose in the decomposition above we have the Floquet representation: $T(t,s)|_{X_1(s)} = P(t)e^{C(t-s)}P(s)^{-1}$. If

$dx/dt + A(t)x = g(t)$ and $x(t) = x_1(t) + x_2(t) \in X_1(t) \oplus X_2(t)$ where $x_1(t) = P(t)y(t)$, $y(t) \in X_1(s_0) \equiv Y$, then

$$dy/dt = Cy + P^{-1}(t)E_1(t)g(t),$$

$$dx_2/dt + A_2(t)x_2 = E_2(t)g(t)$$

where $A_2(t)x_2 = (E_2(t)A(t) - \dot{E}_2(t))x_2$ for $x_2 \in X_2(t)$. The equation in Y has constant coefficients, and for $t \geq s$

$$x_2(t) = T(t,s)|_{X_2(s)}x_2(s) + \int_s^t T(t,\sigma)|_{X_2(\sigma)}E_2(\sigma)g(\sigma)d\sigma$$

Exercise 3. Suppose $A, B(\cdot)$ satisfy the hypotheses of Th. 7.1.3 and also $B(t+p) = B(t)$ for all t and some $p > 0$. Assume $e^{-p\lambda}$ is an isolated simple eigenvalue of e^{-pA}. (For example, this would be true if λ is a simple e.v. of A, $\lambda + 2\pi in/p$ is not an eigenvalue of A for integer $n \neq 0$, and A has compact resolvent.)

Prove that for small $|\varepsilon|$ there exists a unique multiplier $\mu(\varepsilon)$ near $e^{-p\lambda}$ of $dx/dt + Ax = \varepsilon B(t)x$, $\mu(\varepsilon)$ is a simple isolated eigenvalue of the period map, depends analytically on ε, and $\mu(\varepsilon) \to e^{-p\lambda}$ as $\varepsilon \to 0$. In fact, if $(e^{-pA} - e^{-p\lambda})x_0 = 0$, $(e^{-pA^*} - e^{-p\lambda})y_0 = 0$, $\langle x_0, y_0\rangle = 1$, then

$$\mu(\varepsilon) = e^{-p\lambda} + \varepsilon\langle L_1x_0, y_0\rangle + O(\varepsilon^2),\ L_1 = \int_0^p e^{-A(p-s)}B(s)e^{-As}ds.$$

Exercise 4. A model for the interaction of predator (density u) and prey (density v) populations has the form

$$\partial u/\partial t = D_1\Delta u + f(u,v)$$

$$\partial v/\partial t = D_2\Delta v + g(u,v) \quad \text{in} \quad \Omega \times \mathbb{R}^+,$$

with

$$\partial u/\partial N = 0, \quad \partial v/\partial N = 0 \quad \text{on} \quad \partial\Omega.$$

Here f, g are C^2 functions, D_1, D_2 are positive constants and Ω is a bounded smooth domain in $\mathbb{R}^n$ ($n = 1,2$ or 3).

Suppose the ODE $\dot{p} = f(p,q)$, $\dot{q} = g(p,q)$ has a nonconstant periodic solution $\overline{p}(t)$, $\overline{q}(t)$ with period T; the multipliers of the linearization are 1 and e^{μ} where $\mu = \int_0^T (f_u(\overline{p},\overline{q}) + g_v(\overline{p},\overline{q}))dt$. The PDE has $u(x,t) = \overline{p}(t)$, $v(x,t) = \overline{q}(t)$ as a periodic solution and the multipliers of the linearization are the multipliers of

$$\dot{p} = -\lambda D_1 p + f_u(\bar{p},\bar{q})p + f_v(\bar{p},\bar{q})q$$

$$\dot{q} = -\lambda D_2 q + g_u(\bar{p},\bar{q})p + g_v(\bar{p},\bar{q})q$$

where λ is any eigenvalue of the Neumann problem:

$$\Delta\phi + \lambda\phi = 0 \text{ in } \Omega, \quad \partial\phi/\partial N = 0 \text{ on } \partial\Omega, \quad \phi \not\equiv 0.$$

If $(\bar{p},\bar{q})$ is unstable by the linear approximation ($\mu > 0$) for the ODE, the same is true for the PDE. (See Thm. 8.2.4.)

If $(\bar{p},\bar{q})$ is orbitally asymptotically stable by the linear approximation ($\mu < 0$) <u>and</u> if $D_1 = D_2$, then it is also stable for the PDE. (See Thm. 8.2.3.) The same conclusion holds if $|D_1 - D_2|$ is small.

If $\int_0^T g_v(\bar{p},\bar{q})dt > 0$ but $\mu < 0$, $D_1 = \varepsilon^{-1}$, $D_2 = \varepsilon$ for small $\varepsilon > 0$, then $(\bar{p},\bar{q})$ is stable for the ODE but unstable for the PDE.

7.3. The adjoint system and backward uniqueness

For a Banach space X, let X^* denote the dual space, the space of (real) continuous linear functionals on X with $\langle x,y\rangle$ = value of $y \in X^*$ on $x \in X$. If $L: X \to Y$ is a continuous linear operator, then $L^*: Y^* \to X^*$ is a continuous linear operator with $\langle x,L^*y\rangle = \langle Lx,y\rangle$ for all $x \in X$, $y \in X^*$. If L is linear with domain $D(L)$ dense in X, $L: D(L) \subset X \to Y$, then L^* is not generally continuous and $D(L^*)$ consists of all $y \in Y^*$ such that $x \to \langle Lx,y\rangle: D(L) \to \mathbb{R}$ is bounded in the X-norm, and L^*y is the element of X^* such that

$$\langle x,L^*y\rangle = \langle Lx,y\rangle \quad \text{for all } x \in D(L).$$

If Y is a reflexive space and L is closeable, then L^* is densely defined in Y^* (see e.g. Kato [56]).

<u>Example 1</u>. Suppose V and H are real Hilbert spaces, V is a dense subspace of H and the inclusion of V in H is continuous (so $\|x\|_H \le C\,\|x\|_V$ for a constant C). Assume $a: V \times V \to \mathbb{R}$ is a continuous bilinear map and

$$\lambda_0\|x\|_H^2 + a(x,x) \ge \alpha\|x\|_V^2, \quad x \in V,$$

for some constants $\alpha > 0$ and $\lambda_0 \geq 0$.

Define $D(A)$ as the collection of $x \in V$ such that $y \to a(x,y): V \to \mathbb{R}$ is continuous in the H-norm, and for such x, $Ax \in H$ is defined by

$$\langle Ax,y\rangle_H = a(x,y), \quad \text{all } y \in V.$$

Then $A: D(A) \subset H \to H$ is closed, sectorial in H, and A^* is the operator (identifying H^* with H) defined similarly using the bilinear form a^*, $a^*(x,y) = a(y,x)$. The first remarks are proved, for example, by Kato [56], and the statement about the adjoint follows easily.

<u>Example 2</u>. Let Ω be a bounded open set in $\mathbb{R}^n$ with C^2 boundary, $H = L_2(\Omega)$, $V = H_0^1(\Omega)$ [or $H^1(\Omega)$] and for u,v in V

$$a(u,v) = \int_\Omega \{\sum_{i,j=1}^n a_{ij}(x)u_{x_i}v_{x_j} + \Sigma b_j(x)u_{x_j}v + c(x)uv\}dx$$
$$+ [\int_{\partial\Omega} \beta(x)uv\, ds, \text{ in case } V = H^1(\Omega)]$$

where $a_{ij}(x) = a_{ji}(x)$ and $b_j(x)$ are Lipschitzian and $c(x)$ [and $\beta(x)$] are continuous, and for some $\alpha > 0$,

$$\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \geq 2\alpha(\sum_1^n \xi_k^2) \quad \text{for } x \in \Omega,\ \xi \in \mathbb{R}^n.$$

In either case, a satisfies the hypotheses of example 1.

Then

$$D(A) = \{u \in H^2(\Omega) \mid u = 0 \text{ on } \partial\Omega \ [\text{or } \frac{\partial u}{\partial\nu} + \beta u = 0 \text{ on } \partial\Omega]\}$$

where $\partial u/\partial\nu = \sum_{i,j=1}^n a_{ij}N_i u_{x_j}$ and N is the unit outward normal, and in Ω

$$Au = -\sum_{i,j=1}^n (a_{ij}u_{x_i})_{x_j} + \sum_1^n b_j u_{x_j} + cu.$$

Also

$$D(A^*) = \{v \in H^2(\Omega) \mid v = 0 \text{ on } \partial\Omega \ [\text{or } \frac{\partial u}{\partial\nu} + \beta v = (\sum_1^n b_j N_j)v \text{ on } \partial\Omega]\}$$

with

$$A^*v = -\sum_{i,j=1}^n (a_{ij}v_{x_i})_{x_j} - \sum_1^n (b_j v)_{x_j} + cv.$$

Example 3. If $X = L_p(\Omega)$, $1 < p < \infty$, and $X^* = L_{p'}(\Omega)$ $(1/p + 1/p' = 1)$ and A is the operator formally given in ex. 2 above, on

$$D(A) = \{u \in W^{2,p}(\Omega) \mid u = 0 \text{ on } \partial\Omega \quad [\text{or } \frac{\partial u}{\partial \nu} + \beta u = 0 \text{ on } \partial\Omega]\}$$

then A^* has domain

$$D(A^*) = \{u \in W^{2,p'}(\Omega) \mid u = 0 \text{ on } \partial\Omega \quad [\text{or } \frac{\partial u}{\partial \nu} + \beta u = b\cdot Nu \text{ on } \partial\Omega]\}$$

and A^* has the same form as in ex. 2 above. This follows from standard results on elliptic boundary value problems in $L_{p'}(\Omega)$ [2, 8,30], and ex. 3 below.

Exercise 1. If A is sectorial in X, $D(A^*)$ is dense in X^*, then A^* is sectorial in X^*.

Exercise 2. If A is sectorial in X, $\alpha > 0$, $\operatorname{Re} \sigma(A) > 0$, and $L \in \mathscr{L}(X) \cap \mathscr{L}(X^\alpha)$, then $L(X^\alpha)$ is dense in X^α if and only if for $y \in X^*$, $\langle A^\alpha Lx, y\rangle = 0$ for all $x \in X^\alpha$ implies $y = 0$.

Exercise 3. Suppose $T: D(T) \subset X \to Y$, $S: D(S) \subset Y^* \to X^*$ are linear, $D(T)$ is dense, and for some λ_0, $R(T-\lambda_0)$ is dense in Y and $R(S-\lambda_0) = X^*$, and $\langle Tx,y\rangle = \langle x,Sy\rangle$ for $x \in D(T)$, $y \in D(S)$. Prove $S = T^*$.

Theorem 7.3.1. Suppose A_0 is sectorial in X, $0 \le \alpha < 1$, $t \to A(t) - A_0: [t_0,t_1] \to \mathscr{L}(X^\alpha, X)$ is Hölder continuous with exponent $\theta > 0$, and let $T(t,s)$ be the evolution operator for $\dot{x} + A(t)x = 0$.

If $y \in X^*$, and $y(s) = T(t_1,s)^*y$ for $t_0 \le s \le t_1$, then $s \to y(s): [t_0,t_1) \to X^*$ is locally Hölder continuous, $y(s) \to y$ weakly* as $s \to t_1-$ (i.e. $\langle x,y(s)\rangle \to \langle x,y\rangle$ as $s \to t_1-$ for any $x \in X$) and for each $x \in D(A_0)$, $s \to \langle x,y(s)\rangle$ is differentiable on $[t_0,t_1)$ with $\frac{d}{ds}\langle x,y(s)\rangle = \langle A(s)x,y(s)\rangle$. Also for $t_0 \le s \le t \le t_1$, $y(s) = T(t,s)^*y(t)$.

If $\theta > \alpha$ then $y: [t_0,t_1) \to X^*$ is continuously differentiable, $y(s) \in D(A(s)^*)$ and

$$\frac{dy}{ds}(s) = A(s)^*y(s) \quad \text{for } t_0 \le s < t_1$$

with $\|A(s)^*y(s)\|_{X^*} \le C(t_1-s)^{-1}\|y\|_{X^*}$ for a constant C.

Proof. The results of the first paragraph follow directly from Thm. 7.1.3. For example ($t_0 \le s-h \le s < t_1$)

$$|\langle x, y(s)-y(s-h)\rangle| = |\langle T(t_1,s)x - T(t_1,s-h)x_1, y\rangle|$$

$$\le \|y\|_{X^*} C_\delta h^\delta (t_1-s)^{-\delta}\|x\| \qquad (0 < \delta < 1-\alpha)$$

for any $x \in X$, so

$$\|y(s)-y(s-h)\|_{X^*} \le C_\delta h^\delta (t_1-s)^{-\delta} \|y\|_{X^*}.$$

Now suppose $\theta > \alpha$ and without loss of generality, $\alpha < \theta < 1$. We prove there is a constant C such that

$$\|T(t,s)A(s)x\| \le C(t-s)^{-1}\|x\|$$

for $t_0 \le s < t \le t_1$ and $x \in D(A_0)$. This implies

$$|\langle A(s)x, y(s)\rangle| = |\langle T(t_1,s)A(s)x, y\rangle| \le C(t_1-s)^{-1}\|x\| \, \|y\|_{X^*}$$

for $x \in D(A_0)$, $t_0 \le s < t_1$, so $y(s) \in D(A^*(s))$ with

$$\|A^*(s)y(s)\|_{X^*} \le C(t_1-s)^{-1}\|y\|_{X^*}.$$

Similarly for any ε in $0 < \varepsilon < \theta-\alpha$ we have

$$\|T(t,s)A(s)x - T(t,s-h)A(s-h)x\| \le C_\varepsilon h^\varepsilon (t-s)^{-1-\varepsilon}\|x\|$$

for $t_0 \le s-h \le s < t \le t_1$, $x \in D(A_0)$ and a constant C_ε (see ex. 4) so

$$\|A^*(s)y(s) - A^*(s-h)y(s-h)\|_{X^*} \le C_\varepsilon (t_1-s)^{-1-\varepsilon} h^\varepsilon \|y\|_{X^*}.$$

Now $x(t) = e^{-A_0(t-\tau)}x(\tau)$ solves $\dot{x} + A(t)x = (A(t)-A_0)x$ for $t > \tau$, so

$$T(t,\tau) = e^{-A_0(t-\tau)} - \int_\tau^t T(t,s)(A(s)-A_0)e^{-A_0(s-\tau)}\,ds$$

for $t_0 \le \tau \le t \le t_1$. If $x \in D(A_0)$, this implies

$$T(t,\tau)A(\tau)x = A_0 e^{-A_0(t-\tau)} x + T(t,\tau)(A(\tau)-A_0)e^{-A_0(t-\tau)} x$$
$$- \int_\tau^t (T(t,s)B(s)-T(t,\tau)B(\tau))A_0 e^{-A_0(s-\tau)} x$$

where $B(s) = A(s)-A_0$. If $\alpha < 1/2$ (so $1-\alpha > \theta > \alpha$ may be assumed) the estimate follows from Thm. 7.1.3(c). In general

$$T(t,\tau)-T(t,\tau-h) = (I-e^{-A_0 h})e^{-A_0(t-\tau)}$$
$$+ \int_t^{t+h} T(t,s-h)B(s-h)e^{-A_0(s-\tau)}$$
$$- \int_\tau^t [T(t,s)B(s)-T(t,s-h)B(s-h)]e^{-A_0(s-\tau)} ds$$

implies

$$\|T(t,\tau)-T(t,\tau-h)\|_{\mathscr{L}(X)} \leq Ch^\theta (t-\tau)^{-\theta},$$

and substitution above completes the proof.

Exercise 4. Prove Hölder continuity of $\tau \to T(t,\tau)A(\tau)x \in X$ for $t_0 \leq \tau < t \leq t_1$, uniformly in $x \in D(A_0)$, $\|x\| \leq 1$.

A simple application of the adjoint equation is the following:

Theorem 7.3.2. Suppose A_0 is sectorial in X, $0 \leq \alpha < 1$, $t \to A(t)-A_0: \mathbb{R} \to \mathscr{L}(X^\alpha, X)$ is Hölder continuous with exponent greater than α, and

$$A(t+p) = A(t) \quad \text{for all } t,$$

for some constant $p > 0$. Assume also, for some t_0, that $I-T(t_0+p,t_0)$ has closed range (as an operator in $\mathscr{L}(X)$); this is certainly true when A_0 has compact resolvent.

If $f: \mathbb{R} \to X$ is Hölder continuous and periodic with period p, the equation

$$dx/dt + A(t)x = f(t)$$

has a p-periodic solution if and only if

$$\int_0^p \langle y(s), f(s)\rangle ds = 0$$

for every p-periodic solution $y(s)$ of the adjoint equation $\dot{y}(s) = A(s)^* y(s)$.

Proof. A solution $x(t)$ of $\dot{x} + A(t)x = f(t)$ is p-periodic if and only if $x(t_0+p)-x(t_0) = 0$, i.e. if and only if

$$z \equiv \int_{t_0}^{t_0+p} T(t_0+p,s)f(s)ds = (T(t_0+p,t_0)-I)x(t_0)$$
$$\in R(T(t_0+p,t_0)-I).$$

Since $L = T(t_0+p,t_0)-I$ has closed range, $R(L) = N(L^*)^{\perp}$ and $z \in R(L)$ if and only if $\langle y,z\rangle = 0$ whenever $L^*y = 0$. Let $y(s) = T(t_0+p,s)^*y$; then $\dot{y}(s) = A(s)^*y(s)$, $y(s+p) = y(s)$, and the condition is

$$\langle y,z\rangle = \int_{t_0}^{t_0+p} \langle y(s),f(s)\rangle ds = 0.$$

Our main use of the adjoint equation is in application of Thm. 6.1.9, giving conditions which imply $T(t_1,t_0)$ has dense range. But $T(t_1,t_0)X$ is dense in X if and only if $T(t_1,t_0)^*y = 0$, $y \in X^*$, implies $y = 0$. Generally we need instead to prove $T(t_1,t_0)X^{\alpha}$ is dense in X^{α}; but the conditions are essentially the same.

Theorem 7.3.3. Suppose A_0 is sectorial in X, $0 \le \alpha < 1$, $t \to A(t)-A_0$: $[t_0,t_1] \to \mathscr{L}(X^{\alpha},X)$ is Hölder continuous with exponent greater than α. Assume any continuously differentiable z: $(t_0,t_1) \to X^*$ with $z(s) \in D(A(s)^*)$ and $dz/ds = A(s)^*z$ on (t_0,t_1) with $z(s) \to 0$ in X^* as $s \to t_0+$ must vanish on (t_0,t_1). Then for any $0 \le \beta \le 1$, $T(t_1,t_0)X^{\beta}$ is dense in X^{β}.

Proof. Suppose $T(t_1,t_0)X^{\beta}$ is not dense in X^{β}, so (assuming $\operatorname{Re}\sigma(A_0) > 0$) $A_0^{\beta}T(t_1,t_0)X^{\beta}$ is not dense in X; then there exists $\eta \neq 0$ in X^* with $\langle \eta, A_0^{\beta}T(t_1,t_0)x\rangle = 0$ for all $x \in X^{\beta}$. Define $y(s) = (A_0^{\beta}T(t_1,s))^*\eta \in X^*$ when $t_0 \le s < t_1$; observe $y(s) = T(t,s)^*y(t)$ for $t_0 \le s \le t < t_1$ so y is a continuously differentiable solution of $\dot{y}(s) = A(s)^*y(s)$ on (t_0,t_1) and $y(s) \to y(t_0)$ in X^* as $s \to t_0+$. Since

$$\langle y(t_0),x\rangle = \langle \eta, A_0^{\beta}T(t_1,t_0)x\rangle = 0 \quad \text{for all } x \in X^{\beta},$$

$y(t_0) = 0$, so $y(s) \equiv 0$, and $0 = \langle y(s),x\rangle = \langle \eta, A_0^{\beta}T(t_1,s)x\rangle$ for $s < t_1$, so $\langle \eta, A_0^{\beta}x\rangle = 0$ for all $x \in X^{\beta}$, hence $\eta = 0$ contrary to hypothesis.

Thus we prove denseness of the range by proving backward uniqueness for the adjoint equation. One simple case is when $t \to A(t)-A_0$ is analytic from (t_0,t_1) to $\mathscr{L}(X^\alpha,X)$; it follows $s \to T(t,s)$ is analytic from $t_0 < s < t < t_1$, into $\mathscr{L}(X)$ and $z(s) = T(t,s)^*z(t)$ is analytic on (t_0,t). By Thm. 7.3.3, $T(t,s)$ is injective and $T(t,s)X^\beta$ is dense in X^β whenever $t_0 < s \leq t < t_1$ and $0 \leq \beta \leq 1$. This may be applied, for example, to the Navier -Stokes equation in a bounded domain in $\mathbb{R}^3$ (as discussed in sec. 3.8 above). If the forcing function f is independent of time, solutions $x(t;t_0,x_0)$ will depend analytically on t,t_0 and $x_0 \in H_\sigma^\alpha$ for $t > t_0$ on the domain of existence. If u_0 is an equilibrium point which is a saddle point (unstable and the linearization has no eigenvalues on the imaginary axis), the local stable and unstable manifolds for u_0 can be extended globally to give the injectively immersed manifolds $W^s(u_0)$, $W^u(u_0)$. In particular $W^s(u_0)$ is of Baire category I in H_σ^α so an open dense set (in H_σ^α) of initial conditions yield solutions which do not tend to u_0 as $t \to +\infty$. To see this, note first any solutions $x(t;t_0,x_j)$ $(j = 1,2)$ with $x(t_1;t_0,x_1) = x(t_1;t_0,x_2)$ for some $t_1 > t_0$ must satisfy the same condition for larger values of t (since the initial value problem is well posed), and so they must agree for all $t_0 < t \leq t_1$, by analyticity, so $x_1 = x_2$. The linearization about such a solution has coefficients analytic in t (for $t > t_0$) and the result follows from remarks above.

Similarly, if $u_0(t)$ is a periodic orbit of the (autonomous) Navier-Stokes equation and the period map of the linearization has no multipliers on the unit circle except the simple multiplier 1, we may construct the (analytic) Poincaré map on a surface of section through the orbit (see sec. 8.4) and construct the global stable and unstable manifolds for the orbit. If $u_0(\cdot)$ is unstable, its stable manifold is again of category I. In fact, the region of attraction of any countable collection of such unstable equilibria or periodic orbits is of category I, which is not to deny that solutions may be attracted to an invariant set containing them, without approaching any individual orbits. This may indeed be the phenomenon of turbulence.

The nonanalytic case is also important, but the required backward uniqueness results are available only when A_0 is a second order elliptic operator with $A(t)-A_0$ first order, or when A_0 is self adjoint in some Hilbert space with $\alpha = 1/2$. In the applications, these conditions are not generally oppressive. We quote a result of Lions [126] and use it to discuss a fairly general example (a reaction-diffusion system). Other backward uniqueness results may be found in [29, 64] and especially Agmon [105] and Lees and Protter [125].

Theorem 7.3.4. [126] Suppose V, H are real Hilbert spaces, $V \subset H$ is a dense subspace with continuous inclusion, and for each t in $0 \le t \le T$ $a(t,\cdot,\cdot) : V \times V \to \mathbb{R}$ is a continuous bilinear form. Define $A(t)$ by

$$u \in D(A(t)) \text{ and } A(t)u \in H \text{ if}$$
$$a(t;u,v) = \langle A(t)u,v\rangle_H \text{ for all } v \in V,$$

where $\langle\cdot,\cdot\rangle_H$ is the inner product in H. Assume

(i) $a = a_0+a_1$ where $a_j(t;\cdot,\cdot)$ is a continuous bilinear form for each t and $j = 0,1$, and $t \to a_j(t;u,v)$ is continuously differentiable on $[0,T]$ for each u,v in V and $j = 0,1$.

(ii) $a_0(t;u,v) = a_0(t;v,u)$

(iii) There are positive constants λ,μ,C such that

$$a_0(t;u,u) \ge \mu\|u\|_V^2 - \lambda\|u\|_H^2$$

$$|a_1(t;u,v)| \le C\,\|u\|_V\,\|v\|_H$$

for all $0 \le t \le T$ and u,v in V.

If $u \in L_2(0,T;V)$ has time derivative $\dot{u} \in L_2(0,T;H)$ with $u(t) \in D(A(t))$ and $\dot{u}(t) + A(t)u(t) = 0$ for a.e. t in $(0,T)$, and $\|u(t)\|_H \to 0$ as $t \to T-$, then $u(t) \equiv 0$ in $0 \le t \le T$.
(Note: $D(A(t)) = D(A_0(t))$ where $A_0(t)$ is the operator defined by the form $a_0(t;\cdot,\cdot)$.

Example. Suppose Ω is a bounded open set in $\mathbb{R}^n$ with C^2 boundary, $D(x) = \mathrm{diag}(D_1(x),\ldots,D_m(x)) > 0$ in $\bar{\Omega}$ and the $m \times m$ matrix functions $D(x)$, $\partial D(x)/\partial x_j$, $B_k(x,t)$, $\partial B_k(x,t)/\partial x_j$, $C(x,t)$, $\partial B_k(x,t)/\partial t$, $\partial C(x,t)/\partial t$ are continuous and bounded on $\bar{\Omega} \times [t_0,t_1]$, and the symmetric $r \times r$-matrix $\beta(x)$ and its derivatives $\partial\beta(x)/\partial x_j$ are also continuous. If $u = \mathrm{col}(u_1,\ldots,u_m)$, consider the initial boundary-value problem

$$\partial u/\partial t = D(x)\Delta u + \sum_{k=1}^{n} B_k(x,t)\partial u/\partial x_k + C(x,t)u \quad \text{on } \Omega \times (t_0,t_1)$$

with

$$D_j\partial u_j/\partial N + \sum_{k=1}^{r} \beta_{jk}(x)u_k = 0 \quad \text{on } \partial\Omega \qquad (1 \le j \le r)$$

and $u_j = 0$ on $\partial\Omega$ for $r+1 \le j \le m$. Here N is the unit outward normal to $\partial\Omega$ and one of the boundary conditions is dropped if $r = 0$ or $r = m$.

Let $H = L_2(\Omega,\mathbb{R}^m)$, $V = H^1(\Omega,\mathbb{R}^r) \times H_0^1(\Omega,\mathbb{R}^{m-r})$.

Choose C^1 symmetric $r \times r$-matrix functions $\lambda^i(x)$ $(1 \le i \le n)$ such that $\sum_{i=1}^{n} \lambda^i N_i = \beta$ on $\partial\Omega$. Define $\lambda^i_{jk}(x) = 0$ when $j > r$ or $k > r$ $(1 \le j,k \le m)$, and let

$$a_0(u,v) = \int_\Omega \{\sum_1^m D_k(x)\nabla u_k\cdot\nabla v_k + \sum_1^n \frac{\partial}{\partial x_k}(u\cdot\lambda^k v)\}dx$$

$$a_1(t;u,v) = -\int_\Omega v\{\sum_1^m \tilde{B}_k \frac{\partial u}{\partial x_k} + Cu\}dx$$

for $u,v \in V$, with $\tilde{B}_k(x,t) = B_k(x,t) - \partial D(x)/\partial x_k$. It is not difficult to verify the hypotheses of Thm. 7.3.4 for this example, and if $A(t)$ is the operator defined by the form $a = a_0+a_1$ we have

$$D(A(t)) = \{u \in V \cap H^2(\Omega,\mathbb{R}^m)\,|\,D_j\ \partial u_j/\partial N + \sum_{k=1}^{r} \beta_{jk}(x)u_k = 0 \text{ on } \partial\Omega\}$$
$$\text{for } 1 \le j \le r$$

and for $u \in D(A(t))$,

$$-A(t)u = D\Delta u + \sum_1^n B_k(\cdot,t)\ \partial u/\partial x_k + C(\cdot,t)u,$$

so the abstract form of our equation is

$$du/dt + A(t)u = 0, \quad 0 < t < T.$$

The hypotheses of Thm. 7.1.3 are also satisfied with $X = H$, A_0 the operator defined by the form a_0, $\alpha = 1/2$, and $V = H^{\frac{1}{2}} = D((A_0+\lambda)^{\frac{1}{2}})$. If $T(t,s)$ is the evolution operator, $u_0 \in L_2(\Omega,\mathbb{R}^m)$, and $T(t,s)u_0 = 0$ for some $t > s$ $(0 \le s < t \le T)$ then $u_0 = 0$.

The adjoint equation may be treated similarly provided $D(x)$ is twice continuously differentiable and the $m \times m$ matrix $\sum_{j=1}^{n} B_j(x,t)N_j(x) = M$ is symmetric in its first r indices ($M_{k\ell} = M_{\ell k}$ when $1 \le k,\ell \le r$) whenever $x \in \partial\Omega$. The domain of the adjoint $A(t)^*$ consists of all v in $H^2(\Omega,\mathbb{R}^m) \cap V$ such that, for $1 \le j \le r$ and x in $\partial\Omega$,

$$\frac{\partial}{\partial N}(D_j v_j) + \sum_{k=1}^{r} \beta_{kj}(x)v_k = \sum_{k=1}^{r} M_{kj}(x,t)v_k.$$

Observe that $D(A(t)^*)$ depends on t. With these hypotheses, $T(t,s)H^\beta$ is dense in H^β $(0 \le \beta \le 1)$ whenever $0 \le s \le t \le T$, but it would certainly be helpful if we could eliminate the above symmetry conditions.

Now consider this equation in $X = L_p(\Omega,\mathbb{R}^m)$ for some $1 < p < \infty$. If we define

$$D(A_0) = \{u \in W^{2,p}(\Omega,\mathbb{R}^m) \,|\, D_j\ \partial u_j/\partial N + \sum_1^r \beta_{jk}u_k = 0 \text{ on } \partial\Omega \quad (1 \le j \le r)$$
$$u_j = 0 \text{ on } \partial\Omega \quad (r+1 \le j \le m)\ ,$$

$D(A(t)) = D(A_0)$ and for $u \in D(A_0)$,

$$-A(t)u = D\Delta u + \sum_1^n B_j(\cdot,t)\frac{\partial u}{\partial x_j} + C(\cdot,t)u,$$

then the hypotheses of Thm. 7.1.3 hold in X for any $\alpha > 1/2$. If $T(t,s)$ is the evolution operator in $\mathscr{L}(X)$, observe that when $u \in X$, $T(t,s)u \in L_2(\Omega,\mathbb{R}^m)$ for $t > s$ and by the arguments in the case $p = 2$ we see $T(t,s)$ is injective. Similarly, if $v \in X^* = L_{p'}(\Omega,\mathbb{R}^m)$ then $T(t,s)^*v \in L_2(\Omega,\mathbb{R}^m)$ when $s < t$ so we conclude $T(t,s)X^\beta$ is dense in X^β for any $0 \le \beta \le 1$.

<u>Query</u>. Can the symmetry requirements of the backward uniqueness theorem be relaxed or removed?

Suppose $(t,x,u,v) \to f(t,x,u,v)$ is C^2 from $\mathbb{R} \times \bar{\Omega} \times \mathbb{R}^m \times \mathbb{R}^{mn}$ to $\mathbb{R}^m$ and consider the problem

$$u_t = D(x)\Delta u + f(t,x,u,\nabla u) \quad \text{in } \Omega \times \mathbb{R},$$
$$D_j \frac{\partial u_j}{\partial N} + \sum_{k=1}^r \beta_{jk}(x)u_k = 0 \quad \text{on } \partial\Omega \quad (1 \le j \le r)$$
$$u_j = 0 \quad \text{on } \partial\Omega \quad (r+1 \le j \le m).$$

If $p > n$, $X = L_p(\Omega,\mathbb{R}^m)$ and $\alpha > 1/2$, the initial value problem is locally well-posed in X^α and if $u(x,t)$ is a solution on (t_0,t_1), the coefficients of the linearization are C^1 in x and t on $\bar{\Omega} \times [t_2,t_3]$ if $[t_2,t_3] \subset (t_0,t_1)$. If $\beta_{jk}(x) = \beta_{kj}(x)$ as before, the equation and its linearization define injective maps. In order that the evolution operator for the linearization also has dense range, we impose a symmetry condition:

$$\sum_{k=1}^n \frac{\partial f}{\partial \gamma_k}(t,x,u(x,t),\nabla u(x,t))N_k(x)$$

is symmetric in its first r indices when $x \in \partial\Omega$. Here we have written f in the form $f(t,x,u,\gamma_1,\ldots,\gamma_n)$ where each $\gamma_j \in \mathbb{R}^m$ and $\nabla u = (\frac{\partial u}{\partial x_1},\ldots,\frac{\partial u}{\partial x_n})$, so $\frac{\partial f}{\partial \gamma_j}$ is an $m \times m$ matrix at each point. The only practical way to impose this requirement is to assume it holds whenever $x \in \partial\Omega$ and $(u,\nabla u)$ are replaced by arbitrary elements $(u,\gamma_1,\gamma_2,\ldots,\gamma_n)$ of $\mathbb{R}^m \times \mathbb{R}^{mn}$ with $u_j = 0$ for $r+1 \le j \le m$.

7.4. Slowly varying coefficients

Theorem 7.4.1. Suppose A_0 is sectorial in X, $0 \le \alpha < 1$, Λ is a set and $A(\lambda)-A_0 \in \mathscr{L}(X^\alpha,X)$ for $\lambda \in \Lambda$,

$$\|A(\lambda)-A_0\|_{\mathscr{L}(X^\alpha,X)} \le M \quad \text{for all } \lambda,$$

$$\{A(\lambda)A_0^{-1} \mid \lambda \in \Lambda\} \text{ is in a compact set in } \mathscr{L}(X)$$

$$\operatorname{Re}\sigma\,(A(\lambda)) \ge \beta+\varepsilon > \beta \quad \text{for all } \lambda \in \Lambda,$$

and whenever $\{\lambda_\nu\}_{\nu=1}^\infty$ is a sequence in Λ such that $A(\lambda_\nu) \to \tilde{A}$ in $\mathscr{L}(X^1,X)$, the essential spectrum of $\tilde{A}$ is in $\operatorname{Re} z \ge \beta+\varepsilon$. (The last condition is trivially true if A_0 has compact resolvent, since there is no essential spectrum.) Then there exists $M_1 > 0$ such that, for all $\lambda \in \Lambda$ and $x \in D(A_0)$

$$\|e^{-A(\lambda)t}x\|_\gamma \le M_1 e^{-\beta t}\|x\|_\gamma$$

$$\|e^{-A(\lambda)t}x\|_\gamma \le M_1 t^{-\gamma} e^{-\beta t}\|x\|$$

when $t > 0$ and $0 \le \gamma \le 1$.

Remark. The compactness criterion is satisfied, if $\Lambda \subset \mathbb{R}$, $\lambda \to A(\lambda)-A_0$ is continuous, and Λ is compact or $\lambda \to A(\lambda)$ is periodic, or asymptotically periodic, or almost periodic.

Proof. Without loss of generality, suppose $\beta = 0$. By the proof of Thm. 1.3.4, it suffices to establish uniform estimates

$$\|(z-A(\lambda))^{-1}\| \le C/|z|$$

for $\lambda \in \Lambda$ and $|\arg z| \geq \phi$ for some $0 < \phi < \pi/2$.

We may suppose $\|(z-A_0)^{-1}\| \leq C_0/|\lambda|$ for $|\arg z| \geq \phi_0$ $(0 < \phi_0 < \pi/2)$, and $\|A_0^\alpha(z-A_0)^{-1}\| \leq C_0|z|^{\alpha-1}$ in this sector. If R_0 is so large that $R_0^{1-\alpha} \geq 2MC_0$ then for $|\arg z| \geq \phi_0$, $|z| > R_0$, we have

$$\|(z-A(\lambda))^{-1}\| \leq \|(z-A_0)^{-1}\| \, \|\{I-(A(\lambda)-A_0)(z-A_0)^{-1}\}^{-1}\|$$

$$\leq 2C_0/|z|.$$

There exists ϕ, $\phi_0 \leq \phi < \pi/2$, so $K = \{z\colon |\arg z| \geq \phi, |z| \leq R_0\}$ is disjoint from $\{\mathrm{Re}\, z \geq \varepsilon\}$, hence bounded away from $\sigma(A(\lambda))$ for all λ. Suppose $\tilde{A} = \lim_{\nu\to\infty} A(\lambda_\nu)$, $\lambda_\nu \in \Lambda$ and $\sigma(\tilde{A})$ meets K; then any $z_0 \in \sigma(\tilde{A}) \cap K$ is an isolated eigenvalue of finite multiplicity, so $A(\lambda_\nu)$ has spectrum near z_0 for large ν, contrary to hypothesis. Thus $(z-\tilde{A})^{-1}$ is defined for all $z \in K$ and $\tilde{A} \in F$ = closure in $\mathscr{L}(X^1,X)$ of $\{A(\lambda), \lambda \in \Lambda\}$. But then the continuous function $(z,\tilde{A}) \to \|(z-\tilde{A})^{-1}\|$ is bounded on the compact set $K \times F$, so there is a constant C_1 with

$$\|(z-A(\lambda))^{-1}\| \leq C_1 \quad \text{for} \quad z \in K,\ \lambda \in \Lambda.$$

This completes the proof.

Remark. The essential spectrum is defined in the appendix to Ch. 5. A more stringent definition -- apparently the most stringent version -- is that of Kato [56]: the set of complex z where $z-\tilde{A}$ is not semi-Fredholm. The theorem also holds under this definition.

Theorem 7.4.2. Assume A is sectorial in X, $0 \leq \alpha < 1$, and $B\colon [t_0,\infty) \to \mathscr{L}(X^\alpha,X)$ satisfies

$$\|B(t)x-B(s)x\| \leq \mu|t-s|^\theta\|x\|_\alpha$$

for all $t,s \geq t_0$ and $x \in X^\alpha$ for some positive constants μ,θ. Assume also, for some constants M,β, with $A(t) = A-B(t)$,

$$\|e^{-A(t)s}x\|_\alpha \leq Me^{-\beta s}\|x\|_\alpha$$

$$\|e^{-A(t)s}x\|_\alpha \leq Ms^{-\alpha}e^{-\beta s}\|x\|_\alpha$$

for $s > 0$, $t \geq t_0$ and x in X^α. We assume $M > 1$ for simplicity.

If $T(t,s)$ is the evolution operator for $dx/dt + A(t)x = 0$, then for $t > s \geq t_0$ and $x \in X^\alpha$,

$$\|T(t,s)x\|_\alpha \leq M_2 e^{-\delta(t-s)} \|x\|_\alpha$$

$$\|T(t,s)x\|_\alpha \leq M_2 \max((t-s)^{-\alpha},1) e^{-\delta(t-s)} \|x\|$$

where

$$\delta = \beta - m(Q + \max(0, q \frac{\ell n\ Q^{r-1}}{Q^{r-1}}))$$

or (more crudely) $\delta = \beta - mQ(1 + q\frac{(r-1)}{r} e^{-1})$, $m = (\mu M\Gamma(1-\alpha))^{1/\nu}$, $\nu = 1+\theta-\alpha$, $r = \nu/\theta$, $Q = (r\ \ell n\ MC)^{1/r}$ and q, $C \geq 1$ are constants such that

$$E_{1-\alpha,1+\theta}(s) \leq C \max(1,s^q)\exp(\frac{s^{\nu/(1-\alpha)}}{\nu/(1-\alpha)})$$

(see Lemma 7.1.2). Such an estimate holds with $q = \alpha\theta/2\nu$.

Note that $\beta-\delta = O(\mu^{1/\nu})$ as $\mu \to 0$.

<u>Remark</u>. If $\alpha = 0$ we can let $C = 1$, $q = 0$; and if $\alpha = 1/2$, $\theta = 1$ we have $E_{\frac{1}{2},2}(s) \leq 1.886 \max(1,s^3)\exp(s^3/3)$ (see ex. 2, sec. 7.). Thus we can take

$$\delta = \beta - (\mu M)^{1-p}(\frac{1}{p}\ \ell n\ M)^p, \quad p = \frac{\theta}{1+\theta}, \text{ when } \alpha = 0,$$

and

$$\delta = \beta - (4.254\ \mu M\ \ell n\ 1.886\ M)^{2/3} \quad \text{when } \alpha = \tfrac{1}{2},\ \theta = 1.$$

<u>Proof</u>. Let $t \geq \tau \geq t_0$, $x(t) = T(t,\tau)x(\tau)$; then

$$x(t) = e^{-A(\tau)(t-\tau)}x(\tau) + \int_\tau^t e^{-A(\tau)(t-s)}(B(s)-B(\tau))x(s)ds$$

so

$$e^{\beta(t-\tau)}\|x(t)\|_\alpha \leq M\|x(\tau)\|_\alpha + \mu M \int_\tau^t (t-s)^{-\alpha}(s-\tau)^\theta \|x(s)\|_\alpha e^{\beta(s-\tau)} ds$$

and, according to Lemma 7.1.2,

$$e^{\beta(t-\tau)}\|x(t)\|_\alpha \leq M\,\|x(\tau)\|_\alpha\, E_{1-\alpha,1+\theta}(m(t-\tau))$$

where $m = (\mu M\Gamma(1-\alpha))^{1/\nu}$, $\nu = 1+\theta-\alpha$. If

$$E_{1-\alpha,1+\theta}(s) \leq C\max(1,s^q)\exp(s^\lambda/\lambda), \quad \lambda = \nu/(1-\alpha),$$

we have for any $\sigma > 0$ and $\tau \geq t_0$,

$$\frac{1}{\sigma} \ln \|T(\tau+\sigma,\tau)\|_{\mathscr{L}(X^\alpha)} \leq \frac{1}{\sigma} \ln MC - \beta + \frac{m^\lambda}{\lambda} \sigma^{\lambda-1} + \max(0, q \frac{\ln m\sigma}{\sigma}).$$

Choose σ so $(m\sigma)^\lambda = \frac{\lambda}{\lambda-1} \ln MC = Q^r$ (note $MC > 1$) so $\frac{1}{\sigma} \ln \|T(\tau+\sigma,\tau)\|_{\mathscr{L}(X^\alpha)} \leq -\delta$, with δ as described above. If $M_2 = e^{\delta\sigma} \sup_{0 \leq t-\tau \leq \sigma} \|T(t,\tau)\|_{\mathscr{L}(X^\alpha)}$ (which is finite, by estimates above) then for $t = \tau+n\sigma+\sigma'$ $(0 \leq \sigma' < \sigma$, n = integer $\geq 0)$

$$\|T(t,\tau)\|_{\mathscr{L}(X^\alpha)} \leq M_2 e^{-\delta\sigma} \|T(\tau+n\sigma,\tau)\|_{\mathscr{L}(X^\alpha)} \leq M_2 e^{-\delta\sigma - n\sigma\delta} \leq M_2 e^{-\delta(t-\tau)}.$$

Example. Consider, for small $\varepsilon > 0$,

$$\varepsilon \frac{dx}{dt} + A(t)x = 0, \quad A(t) = A-B(t),$$

where $\mathrm{Re}\, \sigma(A(t)) > \beta > 0$ and $t \to B(t) \in \mathscr{L}(X^\alpha,X)$ is Hölder continuous for $t_0 \leq t \leq t_1$. Change the time variable to $\tau = t/\varepsilon$; then

$$\frac{dx}{d\tau} + A(\varepsilon\tau)x = 0$$

and

$$\|B(\varepsilon\tau)x - B(\varepsilon\sigma)x\| \leq \mu\varepsilon^\theta |\tau-\sigma|^\theta \|x\|_\alpha .$$

Thus for $\varepsilon > 0$ sufficiently small

$$\|x(t;\varepsilon)\|_\alpha \leq M_2\, e^{-\beta(t-t_0)/\varepsilon} \|x(t_0;\varepsilon)\|_\alpha$$

for $t_0 \leq t \leq t_1$.

This sort of rapid decay is a familiar phenomenon in singular perturbation problems. (See in particular the paper of Hoppenstadt [51].)

Exercise 1. Suppose $A(t)$ is sectorial in X with $A(t)-A(t_0) \in \mathscr{L}(X^\alpha,X)$ and assume for each B in a certain set $U \subset \mathscr{L}(X^\alpha,X)$, the evolution operator $T_B(t,s)$ for $\dot{x} + (A(t)+B)x = 0$ satisfies $\|T_B(t,s)\|_{\mathscr{L}(X^\alpha)} \leq Me^{-\beta(t-s)}$ and

$$\|T_B(t,s)\|_{\mathscr{L}(X,X^\alpha)} \le Me^{-\beta(t-s)}(t-s)^{-\alpha} \quad \text{for} \quad t > s.$$

If $t \to B(t): \mathbb{R} \to U \subset \mathscr{L}(X^\alpha, X)$ satisfies

$$\|B(t)-B(s)\|_{\mathscr{L}(X^\alpha,X)} \le L|t-s|^\theta,$$

$0 < \theta \le 1$, and if $\beta_1 < \beta$, then for sufficiently small $\varepsilon > 0$, any solution x of

$$\dot{x} + (A(t)+B(\varepsilon t))x = 0$$

satisfies $\|x(t)\|_\alpha \le M_1 e^{-\beta_1(t-s)}\|x(s)\|_\alpha$ for $t \ge s$, and a constant $M_1 < \infty$.

Exercise 2. Suppose A is sectorial in X with compact resolvent $0 \le \alpha < 1$, U is open in X^α, $(f,g): \mathbb{R} \times U \times Y \to X \times Y$ are uniformly C^1_{Lip}. Suppose there exists $\xi: \mathbb{R} \times Y \to U$, uniformly C^1_{Lip}, so $x = \xi(t,y)$ solves $Ax = f(t,x,y)$, and $A(t,y) \equiv A - f_x(t,\xi(t,y),y)$ has $\operatorname{Re}\sigma(A(t,y)) \ge 2\beta > 0$, for all (t,y), and $\{A(t,y)(\lambda_0+A)^{-1}\}$ is in a compact set in $\mathscr{L}(X)$. For small $\varepsilon > 0$, prove the system

$$\varepsilon \frac{dx}{dt} + Ax = f(t,x,y), \qquad \frac{dy}{dt} = g(t,x,y)$$

has an attracting invariant manifold near $x = \xi(t,y)$,

$$S_\varepsilon = \{(t,x,y) \mid x = \xi(t,y) + \sigma_\varepsilon(t,y), \ (t,y) \in \mathbb{R} \times Y\}$$

which attracts nearby solutions like $O(e^{-\beta(t-t_0)/\varepsilon})$.

(Hint: let $t = \varepsilon\tau$, $x = \xi + z$, and apply Thm. 7.4.1-2 and exercise 7, sec. 6.1.)

Exercise 3. Consider the special case when all $A(t)$ are self adjoint and obtain the estimates of Thm. 7.4.1 without the compactness assumption.

Exercise 4. Suppose $\{A(t),\ t_0 \le t < t_1\}$ is a collection of $n \times n$ matrices with $\|A(t)\| \le M$, $\operatorname{Re}\sigma(A(t)) \ge \beta+\varepsilon$, $1 \ge \varepsilon > 0$; prove for $s \ge 0$, $t_0 \le t < t_1$,

$$\|e^{-A(t)s}\| \le C_n \frac{(M+1)^{n-1}}{\varepsilon^n} e^{-\beta s}$$

where C_n depends only on n.
(Hint: use the contour integral, and the representation
$M^{-1} = \frac{\text{matrix of cofactor}}{\det M}$.)

7.5. Rapidly varying coefficients

We discuss, more precisely, integrally small perturbations, but these generally arise from rapidly varying coefficients with small mean value.

Lemma 7.5.1. Suppose X,Y,Z are Banach spaces and $A(t,s)$, $B(s)$, $C(s)$ map $\{t_0 < s < t < t_0+\ell\}$ or $\{t_0 < s < t_0+\ell\}$ into $\mathscr{L}(Y,X)$, $\mathscr{L}(Z,Y)$ and Z respectively, with

$$\|A(t,s)\| \leq a(t-s)^{-\alpha}$$
$$\|A(t,s)-A(t,s-h)\| \leq ah^{\delta}(t-s)^{-\alpha-\delta}$$
$$\|B(s)\| \leq b$$
$$\left\| \int_{t_1}^{t_2} B(s)ds \right\| \leq q$$
$$\|C(s)\| \leq C(s-t_0)^{-\gamma}$$
$$\|C(s+h)-C(s)\| \leq ch^{\delta}(s-t_0)^{-\gamma-\delta}$$

on $(t_0,t_0+\ell)$ when $0 \leq h \leq q/b \leq \ell$ and $0 \leq \alpha,\gamma < 1-\delta < 1$. Then for $0 < t-t_0 \leq \ell$

$$\left\|\int_{t_0}^{t} A(t,s)B(s)C(s)ds\right\| \leq 3acb^{1-\delta}q^{\delta}\int_{t_0}^{t} \{(t-s)^{-\alpha-\delta}(s-t_0)^{-\gamma} + (t-s)^{-\alpha}(s-t_0)^{-\gamma-\delta}\}ds.$$

If $\alpha+\gamma+\delta \leq 1$, the integral on the right side is uniformly bounded on $[t_0,t_0+\ell]$.

Proof. Let $h = q/b$. If $t-t_0 \leq h$ then

$$\left\|\int_{t_0}^{t} A(t,s)B(s)C(s)ds\right\| \leq abc \int_{t_0}^{t} (t-s)^{-\alpha}(s-t_0)^{-\gamma}ds$$
$$\leq abc\, h^{\delta} \int_{t_0}^{t} (t-s)^{-\alpha-\delta}(s-t_0)^{-\gamma}ds.$$

Note $bh^{\delta} = qh^{\delta-1} = q^{\delta}b^{1-\delta}$.

If $t-t_0 > h$ let $t_j = t_0+jh$ $(j = 0,1,\ldots,n)$ with $0 < t-t_n \leq h$. Writing $A_j = A(t,t_j)$, $C_j = C(t_j)$,

$$\int_{t_0}^{t} ABC = \int_{t_n}^{t} ABC + \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} (ABC - A_{j-1}BC_j)$$
$$+ \sum_{j=1}^{n} A_{j-1}\left(\int_{t_{j-1}}^{t_n} B - \int_{t_j}^{t_n} B\right)C_j$$

and the last sum may be rearranged as

$$A_0 \int_{t_0}^{t_n} BC_1 + \sum_{1}^{n-1} \left(A_j \int_{t_j}^{t_n} BC_{j+1} - A_{j-1} \int_{t_j}^{t_n} BC_j\right).$$

Therefore

$$\left\|\int_{t_0}^{t} ABC\right\| \leq abch^{\delta} \int_{t_n}^{t} (t-s)^{-\alpha-\delta}(s-t_0)^{-\gamma}$$
$$+ abch^{\delta} \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \{(t-s)^{-\alpha-\delta}(s-t_0)^{-\gamma} + (t-t_{j-1})^{-\alpha}(s-t_0)^{-\gamma-\delta}\}$$
$$+ acq(t-t_0)^{-\alpha}(t_1-t_0)^{-\gamma-\delta}h^{\delta}$$
$$+ acqh^{\delta-1} \sum_{j=1}^{n-1} \{h(t-t_j)^{-\alpha-\delta}(t_{j+1}-t_0)^{-\gamma}+h(t-t_{j-1})^{-\alpha}(t_j-t_0)^{-\gamma-\delta}\}$$
$$\leq K \int_{t_n}^{t} (t-s)^{-\alpha-\delta}(s-t_0)^{-\gamma} + K \int_{t_0}^{t_1} (t-s)^{-\alpha}(s-t_0)^{-\gamma-\delta}$$
$$+ K \int_{t_0}^{t_n} \{(t-s)^{-\alpha-\delta}(s-t_0)^{-\gamma} + (t-s)^{-\alpha}(s-t_0)^{-\gamma-\delta}\}$$
$$+ K \int_{t_1}^{t_n} (t-s)^{-\alpha-\delta}(s-t_0)^{-\gamma} + K \int_{t_0}^{t_{n-1}} (t-s)^{-\alpha}(s-t_0)^{-\gamma-\delta}$$
$$\leq 3K \int_{t_0}^{t} \{(t-s)^{-\alpha-\delta}(s-t_0)^{-\gamma}+(t-s)^{-\alpha}(s-t_0)^{-\gamma-\delta}\}ds,$$

where $K = acq^{\delta}b^{1-\delta} = abch^{\delta} = acqh^{\delta-1}$.

Theorem 7.5.2. Suppose A_0 is sectorial in X, $0 \leq \alpha < 1$, $0 < \delta < (1-\alpha)/2$, $\ell > 0$, J is an interval in $\mathbb{R}$ and $t \to A(t)-A_0$: $J \to \mathscr{L}(X^{\alpha},X)$ is bounded and locally Hölder continuous. There exist positive constants ε_1 and K_1 such that, if B: $J \to \mathscr{L}(X^{\alpha},X)$ is locally Hölder continuous with

$$\|B(t)\|_{\mathscr{L}(X^\alpha,X)} \le b$$

$$\left\| \int_{t_1}^{t_2} B(t)\,dt \right\|_{\mathscr{L}(X^\alpha,X)} \le q \quad \text{for} \quad |t_1-t_2| \le \ell$$

on J and $q^\delta b^{1-\delta} \le \varepsilon_1$, then

$$\|T_B(t,s)-T_0(t,s)\|_{\mathscr{L}(X^\alpha)} \le K_1 q^\delta b^{1-\delta}$$

for $0 \le t-s \le \ell$ on J, where $T_B(t,s)$ is the evolution operator for $\dot{x} + A(t)x = B(t)x$, and T_0 is the case $B = 0$. Also if $\|T_0(t,s)\|_{\mathscr{L}(X^\alpha)} \le Me^{\beta(t-s)}$ for $t \ge s$ on J, then for $q^\delta b^{1-\delta} \le \varepsilon \le \varepsilon_1$ as above

$$\|T_B(t,s)\|_{\mathscr{L}(X^\alpha)} \le M_\varepsilon e^{\beta_\varepsilon(t-s)}$$

for $t \ge s$ in J, where $\beta_\varepsilon \to \beta$ and $M_\varepsilon \to M$ as $\varepsilon \to 0$.

Remark. ε_1 and K_1 depend only on A_0, $\sup_J \|A(t)-A_0\|_{\mathscr{L}(X^\alpha,X)}$, α, δ, ℓ.

Proof. There is a constant M_B (or M_0, when $B = 0$) so

$$\|T_B(t,s)\|_{\mathscr{L}(X^\alpha)} \le M_B$$

and

$$\|T_B(t+h,s) - T_B(t,s)\|_{\mathscr{L}(X^\alpha)} \le M_B h^\delta (t-s)^{-\delta}$$

for $t > s$, $h \ge 0$ with $t,s,t+h$ in $[t_0,t_0+\ell]$. These estimates are uniform in t_0 for $[t_0,t_0+\ell] \subset J$. The required estimates of $T_0(t,s)$ (from Thm. 7.1.3(c)) will be used with constant K. Now for $t_0 \le t \le t_0+\ell$,

$$T_B(t,t_0)-T_0(t,t_0) = \int_{t_0}^{t} T_0(t,s)B(s)T_B(s,t_0)\,ds$$

so by the Lemma 7.5.1 (with $Y = X^\alpha$, $Z = \mathscr{L}(X^\alpha)$, $\gamma = 0$)

$$\begin{aligned}\|T_B(t,t_0)-T_0(t,t_0)\|_{\mathscr{L}(X^\alpha)} &\le 3KM_B q^\delta b^{1-\delta} \int_{t_0}^{t} \{(t-s)^{-\alpha-\delta} + (t-s)^{-\alpha}(s-t_0)^{-\delta}\} \\ &\le \tfrac{1}{2} M_B \quad \text{for } 0 \le t-t_0 \le \ell \text{ and } q^\delta b^{1-\delta} \text{ small.}\end{aligned}$$

Also if $h_0 > 0$ and $A(t,s) = T_0(t+h_0,s)-T_0(t,s)$, we have

$$\|A(t,s)\|_{\mathscr{L}(X,X^\alpha)} \le Kh_0^\delta (t-s)^{-\alpha-\delta}$$

$$\begin{aligned}\|A(t,s)-A(t,s-h)\|_{\mathscr{L}(X,X^\alpha)} &\le 2K\min\{h^{2\delta}(t-s)^{-\alpha-2\delta}, h_0^{2\delta}(t-s)^{-\alpha-2\delta}\} \\ &\le 2Kh_0^\delta\, h^\delta (t-s)^{-\alpha-2\delta}\end{aligned}$$

and so (with $a = 2Kh_0^\delta$ in the lemma)

$$\begin{aligned}&\|T_B(t+h_0,t_0)-T_B(t,t_0)\|_{\mathscr{L}(X^\alpha)} \\ &\quad = \|T_0(t+h_0,t_0)-T_0(t,t_0) + \int_{t_0}^{t} A(t,s)B(s)T_B(s,t_0)ds \\ &\qquad\qquad + \int_t^{t+h_0} T_0(t+h_0,s)B(s)T_B(s,t_0)ds\|_{\mathscr{L}(X^\alpha)} \\ &\quad \le M_0 h_0^\delta (t-t_0)^{-\delta} \\ &\qquad + 3(2Kh_0^\delta)M_B q^\delta b^{1-\delta}\int_{t_0}^{t} \{(t-s)^{-\alpha-2\delta}+(t-s)^{-\alpha-\delta}(s-t_0)^{-\delta}\} \\ &\qquad + 3KM_B q^\delta b^{1-\delta}\int_t^{t+h_0}\{(t+h_0-s)^{-\alpha-\delta}+(t+h_0-s)^{-\alpha}(s-t)^{-\delta}\}.\end{aligned}$$

If $q^\delta b^{1-\delta} \le \varepsilon_1$ with ε_1 sufficiently small, it follows that we can take $M_B = 2M_0$ and so when $0 \le t-t_0 \le \ell$,

$$\begin{aligned}\|T_B(t,t_0)-T(t,t_0)\|_{\mathscr{L}(X^\alpha)} &\le 6KM_0 q^\delta\, b^{1-\delta}\int_0^\ell \{(\ell-s)^{-\alpha-\delta}+(\ell-s)^{-\alpha}s^{-\delta}\}ds \\ &= K_1 q^\delta b^{1-\delta}\ .\end{aligned}$$

Now suppose $\|T(t,s)\|_{\mathscr{L}(X^\alpha)} \le Me^{\beta(t-s)}$ for $t \ge s$ in J. If $q^\delta b^{1-\delta} \le \varepsilon \le \varepsilon_1$ then according to exercise 3 below,

$$\|T_B(t,s)\|_{\mathscr{L}(X^\alpha)} \le M_\varepsilon e^{\beta_\varepsilon(t-s)}, \quad t \ge s \ \text{ in } J,$$

where $M_\varepsilon = M(1+K_1\varepsilon c)$, $\beta_\varepsilon = \beta + \frac{1}{\ell}\ln(1+MK_1\varepsilon c)$ with $c = \max(e^{-\beta\ell},1)$.

<u>Corollary 7.5.3</u>. With the assumptions of Thm. 7.5.2 above, $\ell = 1$ and $J = [t_0,\infty)$, if the zero solution of $\dot{x} + A(t)x = 0$ is uniformly asymptotically stable, the same is true for $\dot{x} + A(t)x = B(t)x$ provided $q^\delta b^{1-\delta}$ is sufficiently small.

Example. Suppose $b(x,t)$, $c(x,t)$ are uniformly bounded and locally Hölder continuous in t for $0 \le x \le 1$, and suppose $t \to b(x,t)$, $c(x,t)$ are periodic with period $p > 0$. Let $b_0(x)$, $c_0(x)$ be the averaged functions, e.g.

$$b_0(x) = \frac{1}{p}\int_0^p b(x,t)\,dt.$$

Then as $\omega \to +\infty$, the evolution operator in $H_0^1(0,1)$ for

$$u_t = u_{xx} + b(x,\omega t)u_x + c(x,\omega t)u, \quad 0 < x < 1$$
$$u = 0 \quad \text{at} \quad x = 0,1$$

converges to that of

$$u_t = u_{xx} + b_0(x)u_x + c_0(x)u, \qquad 0 < x < 1$$
$$u = 0 \quad \text{at} \quad x = 0,1$$

uniformly on compact time intervals.

To see this, suppose $|b(x,t)| \le N$, $|c(x,t)| \le N$ and $B(t)u(x) = (b(x,\omega t)-b_0(x))u'(x) + (c(x,\omega t)-c_0(x))u(x)$ so

$$\|B(t)u\|_{L_2(0,1)} \le 4N\,\|u\|_{H_0^1(0,1)}$$

$$\left\|\int_{t_1}^{t_2} B(t)u\,dt\right\|_{L_2(0,1)} = \left\|\int_{t_1+np/\omega}^{t_2} B(t)u\,dt\right\|_{L_2}$$
$$\le \frac{4Np}{|\omega|}\,\|u\|_{H_0^1(0,1)}$$

where the integer n is chosen so $|t_2 - t_1 - \frac{np}{\omega}| \le \frac{p}{|\omega|}$. We may take $\ell = 1$, $b = 4N$, $q = 4Np/|\omega|$ and $\alpha = 1/2$, and Thm. 7.5.2 gives the result.

Exercise 1. In the example above, in place of periodicity assume the limit

$$\lim_{T\to\infty} \frac{1}{T}\int_t^{t+T} b(x,s)\,ds = b_0(x)$$

exists uniformly in $t \ge \tau$, $0 \le x \le 1$, and is independent of t, and similarly for $c(x,t)$. Obtain the same conclusion for this case.

Exercise 2. Assume A is sectorial in X, $0 \le \alpha < 1$, U is a neighborhood of 0 in X^α, $f: \mathbb{R} \times U \to X$ is Hölder continuous in t and uniformly C^1 in $x \in U$, and $f(t+p,x) = f(t,x)$ for some constant $p > 0$. Let

$$f_0(x) = \frac{1}{p}\int_0^p f(t,x)\,dt$$

and assume $f_0(0) = 0$ and $\operatorname{Re}\sigma(A - f_0'(0)) > 0$. Thus the averaged equation $\dot{x} + Ax = f_0(x)$ has $x = 0$ as a stable equilibrium point, although $f(t,0) \neq 0$ in general.

Prove there exist positive constants r, ω_0, M, β, such that

(i) If $\omega \geq \omega_0$, there is a unique solution $x_\omega(t)$ of

$$\dot{x} + Ax = f(\omega t, x)$$

on $-\infty < t < \infty$ with $\sup\|x(t)\|_\alpha \leq r$; x_ω is periodic with period p/ω and

$$\sup\|x_\omega(t)\|_\alpha \to 0 \quad \text{as} \quad \omega \to \infty.$$

(ii) If $\omega \geq \omega_0$ and $x(t)$ is any solution of the equation for $t > t_0$ with $\|x(t_0)\|_\alpha \leq r/4M$ then

$$\|x(t) - x_\omega(t)\|_\alpha \leq 2Me^{-\beta(t-t_0)}\|x(t_0) - x_\omega(t_0)\|_\alpha,$$

$\|x(t)\|_\alpha \leq r$ for all $t \geq t_0$.

(iii) If $\sigma(A - f_0'(0))$ does not intersect the imaginary axis but meets the left half-plane, prove there is a p/ω-periodic solution near $x = 0$ for large ω, but the solution x_ω is unstable.

(Hint: apply Thm. 7.5.2 to $\dot{x} + Ax = f_x(\omega t, 0)x$.)

Exercise 3*. Suppose $T(t,s)$, $T_B(t,s)$ are evolution operators for $t \geq s \geq \tau$, $\ell > 0$, and

$$\|T(t,s)\| \leq Me^{\beta(t-s)}$$

$$\|T_B(t,s) - T(t,s)\| \leq \varepsilon \quad \text{when} \quad 0 \leq t-s \leq \ell .$$

Prove

$$\|T_B(t,s)\| \leq M_\varepsilon e^{\beta_\varepsilon(t-s)} \qquad \text{for} \quad t \geq s \geq \tau$$

where

$$M_\varepsilon = M(1+\varepsilon b), \quad \beta_\varepsilon = \beta + \frac{1}{\ell}\ln(1+\varepsilon Mb),$$

$b = \max(e^{-\beta\ell}, 1)$, so $M_\varepsilon \to M$, $\beta_\varepsilon \to \beta$ as $\varepsilon \to 0+$.

Hint: use the identity

$$T(t_n,t_0) - T_B(t_n,t_0) = \sum_{k=0}^{n-1} T(t_h,t_{k+1})(T(t_{k+1},t_k)-T_B(t_{k+1},t_k))\cdot \cdot T_B(t_k,t_0)$$

where $0 \leq t_{k+1}-t_k \leq \ell$.

Exercise 4. Suppose $\|T(t,s)\|_{\mathscr{L}(X^\alpha)} \leq Me^{-\beta(t-s)}$ for $t > s$, and

$$\|T(t,s)\|_{\mathscr{L}(X,X^\alpha)} \leq K(t-s)^{-\alpha}, \quad \|T(t,s)-T(t,s-h)\|_{\mathscr{L}(X,X^\alpha)} \leq Kh^\delta(t-s)^{-\alpha-\delta}$$

for $0 < h, t-s \leq 1$ $(\beta > 0,\ 0 < \delta < 1-\alpha \leq 1)$ and $f\colon \mathbb{R} \to X$ has $\|f(t)\| \leq b$, $\|\int_{t_1}^{t_2} f(s)ds\| \leq q$ for $|t_1-t_2| \leq 1$. Prove

$$\|\int_{-\infty}^{t} T(t,s)f(s)ds\|_\alpha \leq 3MKq^\delta b^{1-\delta}/(1-\alpha-\delta)(1-e^{-\beta}).$$

Hint: write the integral as $\int_{-\infty}^{t} = \sum_{n=0}^{\infty} \int_{t-n-1}^{t-n}$.

Exercise 5. Suppose for each x, $\|\int_{t_1}^{t_2} f(t,x)dt\| \leq q$ for t_1,t_2 in $[0,1]$, $\|f(t,x)-f(t,y)\| \leq L\|x-y\|^\alpha$; $0 \leq t \leq 1$, and $\|x(t_1)-x(t_2)\| \leq M|t_1-t_2|^\beta$ $(0 < \alpha,\beta \leq 1)$ on $[0,1]$ and $q \leq LM^\alpha$; then show

$$\|\int_0^1 f(t,x(t))dt\| \leq 3q^\nu(LM^\alpha)^{1-\nu}, \quad \nu = \frac{\alpha\beta}{1+\alpha\beta}$$

(Hint: $\|\int_t^{t+\Delta t} f(s,x(s))\| \leq q + LM^\alpha(\Delta t)^{1+\alpha\beta}$, if $0 \leq t < t+\Delta t \leq 1$.)

Exercise 6. Suppose $0 < \delta \leq 1$, and for $0 < s < t < T$

$$\|A(s)-A(0)\| \leq a_0(s), \quad \|A(t)-A(s)\| \leq a_1(t)(t-s)^\delta$$

$$\|C(s)\| \leq c_0(s), \quad \|C(t)-C(s)\| \leq c_1(s)(t-s)^\delta$$

where a_0, a_1 are increasing and c_0, c_1 decreasing on $(0,T)$, and with $\overline{B} = \frac{1}{T}\int_0^T B(\sigma)d\sigma$,

$$\|\int_s^{t_0} (B(\sigma)-\overline{B})d\sigma\| \leq q \quad \text{for } 0 \leq s \leq t \leq T.$$

Prove

$$\left\| \int_0^T A(s)B(s)C(s)ds \right\| \le \left\| \int_0^T A(s)\bar{B}C(s)ds \right\|$$
$$+ qT^{\delta-1} \int_0^T \alpha(s)ds$$
$$+ 2\left(q \int_0^T \alpha(s)ds\right)^{\delta}\left(\int_0^T \alpha(s)\|B(s)-\bar{B}\|\right)^{1-\delta}$$

where

$$\alpha(s) = (a_0(s) + \|A(0)\|)c_1(s) + a_1(s)c_0(s).$$

Hint. first consider the case $\bar{B} = 0$ and $A(0) = 0$; then the case $\bar{B} = 0$, $A(s) \equiv A(0)$ so

$$\int_0^T A(0)B(s)C(s)ds = \int_0^T A(0)B(s)(C(s)-C(T))ds;$$

and then the general case.

Note: the factor "2" in this result may be replaced by $(\delta/1-\delta)^{-\delta} + (\delta/1-\delta)^{1-\delta}$, which takes values between 1 and 2 as δ varies between 0 and 1.

The order of the terms (ABC) is unimportant, and we may replace the product by any trilinear map with norm ≤ 1.

7.6. Exponential dichotomies

Suppose the evolution operators $T(t,s) \in \mathscr{L}(X)$ $(t \ge s)$ for $\dot{x} + A(t)x = 0$ are defined on an interval $J \subset \mathbb{R}$; ordinarily $J = \mathbb{R}$ or $[\tau,\infty)$ or $(-\infty,\tau]$.

Definition 7.6.1. The equation $dx/dt + A(t)x = 0$ has an exponential dichtomy on J with exponent $\beta > 0$ and bound M (with respect to X) if there exist projections $P(t)$, $t \in J$, such that

(i) $T(t,s)P(s) = P(t)T(t,s)$, $t \ge s$ in J;

(ii) the restriction $T(t,s)|R(P(s))$, $t \ge s$, is an isomorphism of $R(P(s))$ onto $R(P(t))$, and we define $T(s,t)$ as the inverse map from $R(P(t))$ to $R(P(s))$;

(iii) $\|T(t,s)(1-P(s))\| \le Me^{-\beta(t-s)}$ for $t \ge s$ in J;

(iv) $\|T(t,s)P(s)\| \le Me^{-\beta(s-t)}$ for $s \ge t$ in J, where this operator is defined in (ii).

The norms here are those of $\mathscr{L}(X)$.

Examples.

(1) If $A(t) = A$ is a constant sectorial operator in Y and for some $\beta > 0$, $\sigma(A)$ is disjoint from the strip $\{\lambda: -\beta \leq \text{Re}\,\lambda \leq \beta\}$, then $\dot{x} + Ax = 0$ has an exponential dichotomy on $\mathbb{R}$ (or $\mathbb{R}^+$ or $\mathbb{R}^-$) with exponent β with respect to $X = Y^\alpha$, $\alpha \geq 0$. The projections are constant.

(2) If $A(t)$ is periodic with period $p > 0$ and satisfies the conditions of Thm. 7.2.3, and if for some $\beta > 0$ the spectrum of the period map is disjoint from the annulus $\{\lambda: e^{-\beta p} \leq |\lambda| \leq e^{\beta p}\}$, then $\dot{x} + A(t)x = 0$ has an exponential dichotomy on $\mathbb{R}$ (or $\mathbb{R}^+$ or $\mathbb{R}^-$) with exponent β. The projections are periodic with period p.

(3) If $\|T(t,s)\| \leq Me^{-\beta(t-s)}$ for $t \geq s > \tau$, $\beta > 0$, we have a trivial "dichotomy" on (τ,∞) with exponent β and $P(t) = 0$.

We will study primarily dichotomies on $\mathbb{R}$, since in this case the projections are uniquely determined (ex. 4 below), but many of the results also apply to dichotomies on $\mathbb{R}^+$ or $\mathbb{R}^-$. Some of our results seem to be new, even in finite dimensions, and the method of proof -- using a kind of difference equation -- is simpler than the usual arguments [112 ,114].

Exercise 1*. When $T(t,s)P(s)$ is defined for $t < s$ by (ii) in the above definition, prove $T(t,s)T(s,\tau)P(\tau) = T(t,\tau)P(\tau)$ and $T(t,s)P(s) = P(t)T(t,s)$ on $X_1(s)$ for all t,s,τ in J.

Exercise 2. In example (1) [or (2)] above, if the equation has an exponential dichotomy on $\mathbb{R}$ or $\mathbb{R}^+$ or $\mathbb{R}^-$ with exponent β then the spectrum of A [or, of the period map] must be disjoint from the strip $\{|\text{Re}\,\lambda| < \beta\}$ [or, the annulus $\{e^{-p\beta} < |\lambda| < e^{p\beta}\}$].

Exercise 3. Suppose $t \to A(t) \in \mathscr{L}(X)$ is continuous and the ODE $\dot{x} + A(t)x = 0$ has an exponential dichotomy on J. If $M(t)$ is an $\mathscr{L}(X)$-valued solution of $\dot{M} + A(t)M = 0$ and $M(t_0)$ has a bounded inverse for some $t_0 \in J$, show $M(t)^{-1}$ exists in $\mathscr{L}(X)$ for all t and the evolution operator is $T(t,s) = M(t)M(s)^{-1}$. Prove there is a constant projection E and constant C such that

$$\|M(t)EM(s)^{-1}\| \leq Ce^{-\beta(s-t)}, \quad s \geq t \text{ in } J,$$

$$\|M(t)(1-E)M(s)^{-1}\| \leq Ce^{-\beta(t-s)}, \quad t \geq s \text{ in } J.$$

Conversely if such a projection E exists, define projections $P(t)$

to verify definition 7.6.1 for this equation.

Exercise 4*. If $\dot{x} + A(t)x = 0$ has an exponential dichotomy on $\mathbb{R}$ with exponent $\beta > 0$, prove the only solution $x(t)$ bounded on $-\infty < t < \infty$ is zero. In fact, if $\|x(t)\| = o(e^{\beta|t|})$ as $|t| \to +\infty$, then $x(t) = 0$. Also show the projections $P(t)$ are uniquely determined.

Exercise 4½. If $\{T(t,s),\ t \geq s\}$ is a family of evolution operators with an exponential dichotomy on an interval $J \subset \mathbb{R}$, and if $T(t_2,t_1)$ is a compact operator for some $t_2 \geq t_1$ in J, then prove $R(P(t))$ is finite dimensional with the same dimension for all t.

Lemma 7.6.2. Suppose A_0 is sectorial in a Banach space X, $0 \leq \alpha < 1$, $t \to A(t)-A_0: \mathbb{R} \to \mathscr{L}(X^\alpha, X)$ is uniformly bounded and locally Hölder continuous and suppose $\dot{x} + A(t)x = 0$ has an exponential dichotomy on $\mathbb{R}$ with exponent β (with respect to X or X^α) and $P(t)$ is the corresponding projection. Then there are constants $M_1,\ldots,M_5$ (depending on α,δ,γ) such that

(i) $\|T(t,s)P(s)x\|_\gamma \leq M_1 e^{-\beta(s-t)}\|x\|_\delta$ for $s \geq t$ when $0 \leq \gamma < 1$ and $\delta \geq 0$;

(ii) $\|T(t,s)(1-P(s))x\|_\gamma \leq M_2 e^{-\beta(t-s)}\max\{1,(t-s)^{\delta-\gamma}\}\,\|x\|_\delta$ for $t > s$ and $0 \leq \delta \leq \gamma < 1$

(iii) $\|P(t_1)x-P(t_2)x\|_\gamma \leq M_3\|x\|_\gamma\,|t_1-t_2|^\delta$ if $0 < \delta < 1-\gamma \leq 1$;

(iv) If $G(t,s) = T(t,s)(1-P(s))$ for $t > s$,

$$G(t,s) = -T(t,s)P(s) \quad \text{for } t < s,$$

and $\psi_\gamma(\sigma) = \sigma^{-\gamma}$ on $0 < \sigma \leq 1$, $\psi_\gamma(\sigma) = 1$ otherwise, then

$$\|G(t,s)x\|_\gamma \leq M_1\psi_\gamma(t-s)e^{-\beta|t-s|}\|x\|$$

$(0 \leq \gamma < 1)$;

(v) $\|G(t+h,s)x-G(t,s)x\|_\gamma \leq M_4|h|^\delta\psi_{\gamma+\delta}(t-s)e^{-\beta|t-s|}\|x\|$ if $0 < \delta < 1-\alpha \leq 1$, $|h| \leq 1$, $|t-s| \leq |t+h-s|$ and s is not between t and $t+h$;

(vi) $\|G(t,s-h)x-G(t,s)x\|_\gamma \leq M_5|h|^\delta\,\psi_{\gamma+\delta}(t-s)e^{-\beta(t-s)}\|x\|$ if $0 < \delta < 1-\alpha \leq 1$, $0 \leq \gamma < 1$, $|h| \leq 1$, $|t-s| \leq |t+h-s|$ and t is not between s and $s-h$.

There is a constant C, depending only on A_0,α, the constants M,β of the dichotomy and $\sup\|A(t)-A_0\|_{\mathscr{L}(X^\alpha,X)}$, such that

$$M_1 = C/(1-\gamma),\ M_2 = C/(1-\gamma)(1-\delta),\ M_3 = \delta^{-1}(M_2 + C(1-\gamma-\delta)^{-2}),$$

$$M_4 = C/\{\delta(1-\delta)(1-\gamma-\delta)\},\ \text{and}\ \ M_5 = C/\{\delta(1-\delta)(1-\gamma)^2(1-\alpha-\delta)^2\}.$$

Proof. These follow directly from Thm. 7.1.3 and definition 7.6.1. We use notation from these results and prove (i)-(iii); the other cases are similar. We assume the dichotomy is in X (see Ex. 5).

If $t \le s$ then

$$\|T(t,s)P(s)x\|_\gamma = \|T(t,-1)T(t-1,s)P(s)x\|_\gamma$$

$$\le \frac{MC}{1-\gamma}e^{-\beta(s-t+1)}\|x\| \le M_1 e^{-\beta(s-t)}\|x\|_\delta.$$

If $s < t \le s+1$, $0 \le \delta \le \gamma < 1$ then

$$\|T(t,s)(1-P(s))x\|_\gamma \le \frac{C}{1-\gamma}(t-s)^{\delta-\gamma}\|(1-P(s))x\|_\delta$$

$$\le \frac{C}{1-\gamma}(1 + \frac{MC}{1-\delta})(t-s)^{\delta-\gamma}\|x\|_\delta.$$

If $t-s > 1$,

$$\|T(t,s)(1-P(s))x\|_\gamma \le \frac{C}{1-\gamma}(1 + MC)Me^{-\beta(t-1-s)}\|x\|.$$

Since $\|x\| \le \|x\|_\delta$, these together prove (ii).

Now suppose $0 \le h \le 1$, $0 < \delta < 1-\gamma \le 1$; then

$$\|P(t+h)x-P(t)x\|_\gamma \le \|(T(t+h,t)-1)P(t)x\|_\gamma$$

$$+ \|P(t+h)(T(t+h,t)x-x)\|_\gamma$$

$$\le \frac{Ch^\delta}{\delta(1-\gamma-\delta)}\|P(t)x\|_{\gamma+\delta} + \frac{MC}{1-\gamma}\|T(t+h,t)x-x\|$$

which implies (iii).

Exercise 5*. With the assumptions of the lemma, we have a dichotomy on $\mathbb{R}$ in X if and only if we have a dichotomy with respect to X^γ, for any $0 \le \gamma < 1$.

Theorem 7.6.3. Suppose A_0 is sectorial in X, $0 \le \alpha < 1$, $t \to A(t)-A_0 : \mathbb{R} \to \mathscr{L}(X^\alpha, X)$ is bounded and locally Hölder continuous, and $\dot{x} + A(t)x = 0$ has an exponential dichotomy on $\mathbb{R}$ with exponent $\beta > 0$.

For any bounded locally Hölder continuous $f: \mathbb{R} \to X$ there is a unique bounded solution x of

$$\dot{x} + A(t)x = f(t), \quad -\infty < t < \infty ,$$

namely

$$x(t) = \int_{-\infty}^{\infty} G(t,s)f(s)ds.$$

More generally if $-\beta < \gamma < \beta$ and $\|f(t)\| = O(e^{\gamma|t|})$ as $|t| \to \infty$, there is a unique solution x with $\|x(t)\|_\alpha = o(e^{\beta|t|})$ and it satisfies

$$\sup\{\|x(t)\|_\alpha e^{-\gamma|t|}\} \le \frac{C}{\beta-|\gamma|} \sup\{\|f(t)\| \; e^{-\gamma|t|}\}$$

for a constant C independent of x, f and γ.

Proof. Uniqueness is clear (see ex. 4) and the estimates of $x(t) = \int_{-\infty}^{\infty} G(t,s)f(s)ds$ follow from Lemma 7.6.2; in particular, the integral converges. Choose any real t_0. By definition of G and an easy calculation, for $t > t_0$

$$x(t) = \int_{-\infty}^{\infty} G(t,s)f(s)ds = T(t,t_0)x(t_0) + \int_{t_0}^{t} T(t,s)f(s)ds$$

so $\dot{x} + A(t)x = f(t)$ for $t > t_0$ (Thm. 7.1.4).

Remark. A converse for the corresponding discrete problem is proved below (Thm. 7.6.5).

Exercise 6. If $-\beta < \gamma_+, \gamma_- < \beta$ and $e_\gamma(t) = e^{\gamma_+ t}$ for $t \ge 0$, $e_\gamma(t) = e^{\gamma_-|t|}$ for $t \le 0$, prove $x(t) = \int_{-\infty}^{\infty} G(t,s)f(s)ds$ has

$$\sup\{\|x(t)\|_\alpha / e_\gamma(t)\} \le C\left(\frac{1}{\beta-|\gamma_+|} + \frac{1}{\beta-|\gamma_-|}\right)\sup\{\|f(t)\|/e_\gamma(t)\}.$$

Exercise 7*. If $A(t)$ satisfies the conditions of Thm. 7.6.3 on $[t_0,\infty)$, prove for any bounded $f: [t_0,\infty) \to X$, x is a bounded solution of $\dot{x} + A(t)x = f(t)$ for $t > t_0$ if and only if

$$x(t) = T(t,t_0)(1-P(t_0))x(t_0) + \int_{t_0}^{\infty} G(t,s)f(s)ds, \quad t \ge t_0.$$

Hint: if x is a bounded solution and $t_1 > t > t_0$, then $T(t_1,t)P(t)\{P(t)x(t) + \int_t^{t_1} T(t,s)P(s)f(s)ds\}$ is bounded as $t_1 \to +\infty$ so $\{\cdots\} \to 0$ as $t_1 \to +\infty$.

Exercise 8*. If $A(t)$ satisfies the conditions of Thm. 7.6.3 on $(-\infty,t_0]$, $f: (-\infty,t_0] \to X$ is bounded, then x is a bounded solution of $\dot{x} + A(t)x = f(t)$, $t < t_0$, if and only if

$$x(t) = T(t,t_0)P(t_0)x(t_0) + \int_{-\infty}^{t_0} G(t,s)f(s)ds, \quad t \le t_0.$$

Exercise 9. Suppose $\{T_n\}_{n=-\infty}^{\infty}$ is a sequence in $\mathscr{L}(X)$; if $x_{n+1} = T_n x_n + y_n$ for $m \le n < p$, show

$$x_n = T_{n,m}x_m + \sum_{k=m}^{n-1} T_{n,k+1}y_k, \quad m \le n \le p$$

where $T_{n,m} = T_{n-1} \cdots T_{m+1}T_m$ for $n > m$, $T_{m,m} = I$. If $\{T_n\}$ has a discrete dichotomy (def. 7.6.4 below), prove results analogous to ex. 7 and 8 for this difference equation. Note $T_{n,m}T_{m,k} = T_{n,k}$ for $n \ge m \ge k$ and if $T_nP_n = P_{n+1}T_n$, then $T_{n,m}P_m = P_nT_{n,m}$ for $n \ge m$.

Definition 7.6.4. If X is a Banach space and $\{T_n\}_{n=-\infty}^{\infty}$ is a sequence in $\mathscr{L}(X)$, we say $\{T_n\}$ has a *discrete dichotomy* (with constants M,θ) if there exist positive constants M, $\theta < 1$ and a sequence of projections $\{P_n\}_{-\infty}^{\infty}$ in $\mathscr{L}(X)$ such that

(i) $T_nP_n = P_{n+1}T_n$

(ii) $T_n(R(P_n))$ is an isomorphism of $R(P_n)$ onto $R(P_{n+1})$

(iii) If $T_{n,m} = T_{n-1}\cdot\ldots\cdot T_{m+1}\cdot T_m$ for $n > m$, $T_{m,m} = I$, then

$$\|T_{n,m}(1-P_m)x\| \le M\theta^{n-m}\|x\| \quad \text{for} \quad n \ge m$$

(iv) $\|T_{n,m}P_mx\| \le M\theta^{m-n}\|x\|$ for $n < m$

where $T_{n,m}P_mx = y \in R(P_n)$ if and only if $P_mx = T_{m,n}y$, which is well-defined by (ii).

Remark. If $\{T(t,s)\ t \ge s\}$ is a family of evolution operators with an exponential dichotomy (exponent β, bound M) and $\ell > 0$, then for any real t_0, $\{T(t_0+(n+1)\ell,\ t_0+n\ell)\}_{-\infty}^{\infty}$ has a discrete dichotomy with constants M, $\theta = e^{-\beta\ell}$.

Exercise 10. Suppose $\{T(t,s),\ t \ge s\}$ is a family of evolution operators with $\sup_{0\le t-s\le\ell} \|T(t,s)\| < \infty$ and ℓ,M,θ are positive constants with $\theta = e^{-\beta\ell} < 1$ (so $\beta > 0$). For each real t_0, assume

$\{T(t_0+(n+1)\ell, t_0+n\ell)\}_{-\infty}^{\infty}$ has a discrete dichotomy with constants M, θ. Prove $\{T(t,s),\ t \geq s\}$ has an exponential dichotomy on $\mathbb{R}$ with exponent β and bound KM, where

$$K = \sup\{\|T(t,s)\| e^{\beta(t-s)},\quad 0 \leq t-s \leq \ell\}.$$

<u>Hint</u>: If $\{P_n(t_0)\}_{-\infty}^{\infty}$ is the sequence of projections let $X_+(t_0) = N(P_0(t_0))$, $X_-(t_0) = R(P_0(t_0))$, so $X = X_+(t_0) \oplus X_-(t_0)$ for each t_0. Show $T(t,t_0)X_+(t_0) \subset X_+(t)$ and $T(t,t_0)\,X_-(t_0)$ is an isomorphism of $X_-(t_0)$ onto $X_-(t)$ for each $t \geq t_0$.

<u>Theorem 7.6.5</u>. Assume $\{T_n\}_{-\infty}^{\infty}$ is a sequence in $\mathscr{L}(X)$. The following are equivalent:

(i) $\{T_n\}_{-\infty}^{\infty}$ has a discrete dichotomy.

(ii) For each bounded sequence $\{f_n\}_{-\infty}^{\infty} \subset X$, there is a unique bounded solution $\{x_n\}_{-\infty}^{\infty}$ of $x_{n+1} = T_n x_n + f_n$, $-\infty < n < \infty$.

<u>Proof</u>. If (i) holds, using the notation of the definition, it is easily proved (ii) holds with $x_n = \sum_{-\infty}^{\infty} G_{n,k+1} f_k$ the unique bounded solution of $x_{n+1} = T_n x_n + f_n$, where $G_{n,m} = T_{n,m}(1-P_m)$ for $n \geq m$, $G_{n,m} = -T_{n,m}P_m$ for $n < m$. We call the sequence $\{G_{n,m}\}$ or the corresponding operator the Green's function for $\{T_n\}$.

Assume (ii) holds. Let $B = \ell_\infty(\mathbb{Z}, X)$, the Banach space of bounded sequences $x = \{x_n\}_{-\infty}^{\infty}$ in X, with norm $\sup_n \|x_n\|$. Let L be the linear operator $\{x_n\}_{-\infty}^{\infty} \to \{x_{n+1} - T_n x_n\}_{-\infty}^{\infty}$ with domain consisting of all $x \in B$ such that $Lx \in B$. Then L is a closed linear operator which (by (ii)) maps its domain one-one onto B. By the closed graph theorem, L has a bounded inverse G in $\mathscr{L}(B)$ which may be represented in the form

$$(Gf)_n = \sum_{-\infty}^{\infty} G_{n,k+1} f_k, \quad -\infty < n < \infty,$$

at least for sequences $\{f_k\}$ with $f_k = 0$ for all large $|k|$. Here each $G_{n,m} \in \mathscr{L}(X)$ with $\|G_{n,m}\| \leq \|G\|_{\mathscr{L}(B)}$, and $G_{n+1,k+1} - T_n G_{n,k+1} = 0$ if $n \neq k$, I if $n = k$. By induction, if we define $P_m = I - G_{m,m}$,

$$G_{n,m} = T_{n,m}(1-P_m) \text{ for } n \geq m, \quad T_{m,n}G_{n,m} = -P_m \text{ for } n < m.$$

We will prove P_m is a projection and satisfies the other conditions of the definition with $M = (1+\|G\|)^2$ and $\theta = \|G\|/(1+\|G\|)$. It follows that the above representation of G holds for all $f \in B$, but in the course of the proof we use it only for finite sequences.

Suppose $x_{n+1} = T_n x_n$, $n \geq m$, defines a bounded sequence; then $P_m x_m = 0$. For we can set $x_n = 0$ when $n < m$, and then $x_{n+1} - T_n x_n = 0$ for $n \neq m-1$, $x_m - T_{m-1}x_{m-1} = x_m$, so $x_n = G_{n,m}x_m$ for all n; in particular, $x_m = (1-P_m)x_m$, $P_m x_m = 0$.

For any $x \in X$, let $x_n = G_{n,m}x$; then $x_{n+1} = T_n x_n$ for $n \geq m$, x_n is bounded, so $P_m x_m = 0 = P_m(1-P_m)x$, so $P_m = P_m^2$. Note also that, if $P_m x = 0$, then $x_m = x$ and $P_{m+1}x_{m+1} = 0$ so

$$P_{m+1}T_m x = T_m P_m x \quad \text{when} \quad P_m x = 0.$$

Now suppose $\{x_n, n \leq m\}$ is bounded and $x_{n+1} = T_n x_n$ for $n < m$; set $x_n = 0$ for $n > m$, and then $x_n = -G_{n,m+1}T_m x_m$ for all n, so $T_m x_m \in R(P_{m+1})$.

If $x \in R(P_m)$, let $y_n = G_{n,m}x$; then y_n is bounded, $y_{n+1} = T_n y_n$ for $n < m-1$, hence for $n < m-2$, so $T_{m-2}y_{m-2} = y_{m-1} \in R(P_{m-1})$ and $y_m = (1-P_m)x = 0$, so $-x = T_{m-1}y_{m-1} \in T_{m-1}R(P_{m-1})$. Thus $T_{m-1}R(P_{m-1}) \supset R(P_m)$. Let $\bar{y}_n = y_n$ for $n < m$, $\bar{y}_{n+1} = T_n\bar{y}_n$ for $n+1 \geq m$; then $\bar{y}_m = T_{m-1}y_{m-1} = -x$ and $\bar{y}_n$ is bounded as $n \to -\infty$, so $T_m x \in R(P_{m+1})$. Also if $T_m x = 0$, then $\bar{y}_n = 0$ for $n > m$ so $P_m\bar{y}_m = 0$, i.e. $x = P_m x = 0$. Thus $T_m|R(P_m)$ is an isomorphism of $R(P_m)$ onto $R(P_{m+1})$, and we may write

$$G_{n,m} = -T_{n,m}P_m \quad \text{for} \quad n < m, \qquad G_{n,m} = T_{n,m}(1-P_m) \quad \text{for} \quad n \geq m.$$

Note $T_n P_n x = P_{n+1}T_n x$ if $P_n x = x$, and this is also true when $P_n x = 0$, hence by linearity, $T_n P_n = P_{n+1}T_n$.

Now choose $x \in X$. If $T_{n,m}x = 0$ for some $n \geq m$ then $T_{p,m}x = 0$ for all $p \geq n$, and so if $T_{n,m}(1-P_m)x \neq 0$

$$\phi_k^{-1} = \|T_{k,m}(1-P_m)x\| > 0, \qquad m \leq k \leq n$$

and

$$\sum_{k=m}^{n} T_{n,k}(1-P_k)T_{k,m}(1-P_m)x\phi_k = T_{n,m}(1-P_m)x\sum_{k=m}^{n}\phi_k$$

so

$$\phi_n^{-1}\sum_{k=m}^{n}\phi_k \leq \|G\|.$$

If $\psi_n = \sum_m^n \phi_k$ then $\psi_{n-1} \leq (1-\|G\|^{-1})\psi_n$ and so

$$\phi_n \geq \|G\|^{-1}(1 - \|G\|^{-1})^{m-n}\phi_m$$

$$\|T_{n,m}(1-P_m)x\| \leq \|G\|^2(1 - \|G\|^{-1})^{n-m}\|x\|, \quad n \geq m.$$

This was proved assuming the left-side is positive, but it is trivially true when the left side vanishes. Similarly if $\rho_n^{-1} = \|T_{n,m}P_m x\| > 0$, $n < m$, then

$$\rho_n^{-1} \sum_{n+1}^{m} \rho_k \leq \|G\|$$

so

$$\rho_n \geq \|G\|^{-1}(1 + \|G\|^{-1})^{m-n-1}\rho_m$$

$$\|T_{n,m}P_m x\| \leq (1 + \|G\|)^2(\|G\|/1 + \|G\|)^{m-n}\|x\|, \quad n < m.$$

This completes the proof.

<u>Corollary 7.6.6</u>. For $b > 0$, let B_b = all sequences $\{x_n\}_{-\infty}^{\infty}$ in X with $\sup\{\|x_n\| b^n\} < \infty$. Suppose $0 < \theta_1 < 1$ and for $\theta_1 \leq b \leq 1/\theta_1$ and any $\{f_n\}_{-\infty}^{\infty}$ in B_b, there is a unique solution $\{x_n\}$ in B_b of $x_{n+1} = T_n x_n + f_n$, $-\infty < n < \infty$. If $\{G_{n,m}\}$ is the corresponding Green's function,

$$\|G_{n,m}\| \leq M\,\theta_1^{|n-m|}$$

for some constant M.

<u>Proof</u>. If $x, f \in B_b$, $x_{n+1} = T_n x_n + f_n$, and $y_n = x_n b^n$, $g_n = f_n b^n$, then $y, g \in B_1 = B$ and $y_{n+1} = bT_n y_n + bg_n$, and $\{bT_n\}_{-\infty}^{\infty}$ has a discrete dichotomy whose Green's function is $\{b^{n-m}G_{n,m}\}$, so $\|G_{n,m}\| \leq Cb^{m-n}$ with $b = \theta_1$ or θ_1^{-1}.

<u>Theorem 7.6.7</u>. Suppose $\{T_n\}_{-\infty}^{\infty} \subset \mathscr{L}(X)$ has a discrete dichotomy with constants M, $\theta < 1$. If $M_1 > M$ and $\theta < \theta_1 < 1$, there exists $\varepsilon > 0$ (depending only on M, M_1, θ, θ_1) so any sequence $\{S_n\}_{-\infty}^{\infty} \subset \mathscr{L}(X)$ with $\sup_n \|T_n - S_n\| \leq \varepsilon$ has a discrete dichotomy with constants M_1, θ_1.

<u>Remark</u>. The requirement on ε is

$$\varepsilon M < (\theta_1 - \theta)/(1 + \theta\theta_1) \quad \text{and} \quad M_1\left(1 - \frac{\varepsilon M(1+\theta\theta_1)}{\theta_1 - \theta}\right) \geq M.$$

(See ex. 11.)

<u>Proof</u>. $x_{n+1} = S_n x_n + f_n$ has a unique bounded solution x for each bounded f if and only if

$$x_n = \sum_{-\infty}^{\infty} G_{n,k+1}\{(S_k - T_k)x_k + f_k\}$$

is also solvable for each bounded f, and this is true provided

$$\sup_n \sum_{-\infty}^{\infty} \|G_{n,k+1}(S_k - T_k)\| \le \varepsilon M \frac{1+\theta}{1-\theta} < 1.$$

In this case, Thm. 7.6.5 shows $\{S_n\}$ has a discrete dichotomy; let $\{\tilde{G}_{n,m}\}$ be the corresponding Green's function, and then for all n,m

$$\tilde{G}_{n,m} = G_{n,m} + \sum_{-\infty}^{\infty} G_{n,k+1}(S_k - T_k)\tilde{G}_{k,m}$$

so

$$\|\tilde{G}_{n,m}\| \le M\theta^{|n-m|} + \varepsilon M \sum_{-\infty}^{\infty} \theta^{|n-k-1|} \|\tilde{G}_{k,m}\|$$

and $\|\tilde{G}_{n,m}\|$ is bounded. By ex. 11 below, if $\theta < \theta_1 < 1$ and $\varepsilon M < (\theta_1 - \theta)/(1+\theta\theta_1)$ (which is $< (1-\theta)/(1+\theta)$)

$$\|\tilde{G}_{n,m}\| \le M\theta_1^{|n-m|}/\{1 - \varepsilon M(1+\theta\theta_1)/(\theta_1 - \theta)\} \le M_1\theta_1^{|n-m|}$$

for small $\varepsilon > 0$.

<u>Exercise 11*</u>. If $a \ge 0$, $b \ge 0$, $0 < r < r_1, r_2 \le 1$

$$b < (r_j - r)/(1+rr_j) \quad \text{for} \quad j = 1,2$$

and $\{g_n\}_{-\infty}^{\infty}$ is a nonnegative sequence in $\mathbb{R}$ with $g_n = O(r_2^{-|n|})$ as $|n| \to \infty$, and

$$g_n \le ar_1^{|n|} + b\sum_{-\infty}^{\infty} r^{|n-k-1|} g_k \quad \text{for all } n,$$

then

$$g_n \le ar_1^{|n|}/[1 - b(1+rr_1)/r_1 - r)] \quad \text{for all } n.$$

<u>Hint</u>: Show the map of sequences

$$\{f_n\} \to \{b\sum_{-\infty}^{\infty} r^{|n-k-1|} f_k\}$$

is a contraction in the norm $\sup\{|f_n|q^{|n|}\}$ whenever $r_{1,2} \le q \le 1/r_{1,2}$.

<u>Remark</u>. The result of ex. 11 can be improved slightly by solving explicitly in the case of equality (using the generating function $\sum_{-\infty}^{\infty} g_n t^n$ for $|t|$ near 1) and we find $g_n = O(r_1^{|n|})$ if

$b < b^* = (r_1 - r)(1 - rr_1)/(1 - r^2)$, $r < r_1 < 1$, but this bound does not hold for $b^* < b < 1 - r/1 + r$.

Exercise 12. If $a \geq 0$, $b \geq 0$, $\beta > 2b$, $|\alpha| < \beta - 2b$,

$$0 \leq u(t) \leq ae^{\alpha|t|} + b \int_{-\infty}^{\infty} e^{-\beta(t-s)} u(s)ds \quad \text{for all } t,$$

and $u(t) = O(e^{\gamma|t|})$ as $|t| \to \infty$ where $0 \leq \gamma < \beta - 2b$, then $u(t) \leq ae^{\alpha|t|}/\{1 - 2b/(\beta - |\alpha|)\}$. Similarly if $0 \leq \delta < 1$ and $2b(\frac{1}{1-\delta} + \frac{1}{\beta - |\sigma|}) < 1$ for $\sigma = \alpha$ or γ, $u(t) = O(e^{\gamma|t|})$ and

$$u(t) \leq ae^{\alpha|t|} + b \int_{-\infty}^{\infty} \max\{|t-s|^{-\delta}, 1\} e^{-\beta|t-s|} u(s)ds$$

for all t, then

$$u(t) \leq ae^{\alpha|t|}/\{1 - 2b(\frac{1}{1-\delta} + \frac{1}{\beta - |\alpha|})\}.$$

Theorem 7.6.8. Suppose $\{P_n\}$, $\{\tilde{P}_n\}$ are bounded sequences of projections in $\mathscr{L}(X)$, $\{T_n\}_{-\infty}^{\infty}$ is a bounded sequence in $\mathscr{L}(X)$ with

$$T_n P_n = \tilde{P}_{n+1} T_n, \quad R(T_n P_n) = R(\tilde{P}_{n+1})$$

$$\|T_n x\| \leq \theta \|x\| \quad \text{when} \quad P_n x = 0$$

$$\|T_n x\| \geq \theta^{-1} \|x\| \quad \text{when} \quad P_n x = x$$

for a constant θ in $0 < \theta < 1$, and $\|P_n\| \leq M$, $\|\tilde{P}_n\| \leq M$, $\|1 - \tilde{P}_n\| \leq M$, for all n. If $\theta < \theta_1 < 1$ and $M_1 > M$, there exists $\varepsilon > 0$ depending only on θ, θ_1, M, M_1 and $\sup_k \|T_k\|$, such that for any $\{S_n\}_{-\infty}^{\infty}$ in $\mathscr{L}(X)$, if $\|P_n - \tilde{P}_n\| \leq \varepsilon$ and $\|T_n - S_n\| \leq \varepsilon$ for all n then $\{S_n\}_{-\infty}^{\infty}$ has a discrete dichotomy with constants M_1, θ_1.

Proof. Let $W_n = \tilde{P}_n P_n + (1 - \tilde{P}_n)(1 - P_n)$; then

$$\tilde{P}_n W_n = \tilde{P}_n P_n = W_n P_n,$$

$$\|I - W_n\| = \|\tilde{P}_n(\tilde{P}_n - P_n) + (P_n - \tilde{P}_n)P_n\| \leq 2\varepsilon M.$$

If $2\varepsilon M < 1$ then W_n has a bounded inverse with $\|W_n^{-1}\| \leq (1 - 2\varepsilon M)^{-1}$. Define $\tilde{T}_n = T_n W_n^{-1}$; then $\tilde{T}_n \tilde{P}_n = T_n P_n W_n^{-1} = \tilde{P}_{n+1} \tilde{T}_n$ and $R(\tilde{T}_n \tilde{P}_n) = R(\tilde{P}_{n+1})$. If $\tilde{P}_n x = 0$ then $P_n W_n^{-1} x = 0$ and

$$\|\tilde{T}_n x\| \le \theta \|W_n^{-1} x\| \le \frac{\theta}{1-2\varepsilon M} \|x\|$$

and if $\tilde{P}_n x = x$, then $(1-P_n)W_n^{-1}x = 0$ and

$$\|\tilde{T}_n x\| \ge (\theta(1+2\varepsilon M))^{-1} \|x\| .$$

If $\theta_2 = \theta/(1-2\varepsilon M) < 1$, we have for $n \ge m$

$$\|\tilde{T}_{n,m}(1-\tilde{P}_m)x\| \le \theta_2^{n-m} \|(1-\tilde{P}_m)x\| \le M\theta_2^{n-m} \|x\| ,$$

and for $n < m$,

$$\|\tilde{T}_{n,m}\tilde{P}_m x\| \le \theta_2^{m-n} \|\tilde{P}_m x\| \le M\theta_2^{m-n} \|x\| .$$

Thus $\{\tilde{T}_n\}$ has a discrete dichotomy with constants M, θ_2, and

$$\|\tilde{T}_n - S_n\| \le \varepsilon + \|W_n^{-1} - 1\| \|T_n\| \le \varepsilon\{1 + \frac{2M}{1-2\varepsilon M} \sup_k \|T_k\|\}.$$

If ε is sufficiently small, $\theta_2 = \theta + O(\varepsilon) < \theta_1 < 1$ and $\{S_n\}$ also has a discrete dichotomy with constants M_1 and θ_1, by Thm. 7.6.7.

<u>Theorem 7.6.9</u>. Suppose $\{T_n\}_{-\infty}^{\infty}$ is a sequence in $\mathscr{L}(X)$ which has a discrete dichotomy and suppose $\{C_n\}_{-\infty}^{\infty}$ is a sequence of compact operators in $\mathscr{L}(X)$ with $\|C_n\| \to 0$ as $n \to \infty$, and let $S_n = T_n + C_n$. One of the following is true.

Either (i) $\{S_n\}_{-\infty}^{\infty}$ has a discrete dichotomy

or (ii) $x_{n+1} = S_n x_n$, $-\infty < n < \infty$, has a nontrivial bounded solution.

If the dichotomy for $\{T_n\}$ has constants M, θ and if $\theta < \theta_1 < 1$, then in case (i) the constants for the dichotomy are M_1, $\theta_1 < 1$ for some $M_1 < \infty$. In case (ii), any solution of $x_{n+1} = S_n x_n$ with $\|x_n\| = O(\theta_1^{-|n|})$ actually satisfies $\|x_n\| = O(\theta_1^{|n|})$, and the space of such bounded solution sequences is finite dimensional.

<u>Proof</u>. Let $B = \ell_\infty(\mathbb{Z}, X)$ as before and note $G \in \mathscr{L}(B)$. Define $C \in \mathscr{L}(B)$ by $(Cx)_n = C_n x_n$ for all n, and let $(C^N x)_n = C_n x_n$ when $|n| \le N$, $(C^N x)_n = 0$ for $|n| > N$. Clearly C^N is compact in $\mathscr{L}(B)$ for each N and $\|C^N - C\|_{\mathscr{L}(B)} \le \sup_{|m|>N} \|C_m\| \to 0$ as $N \to \infty$, so C is also compact, as is GC. If $x = GCx$, $x \in B$, then $x_{n+1} = S_n x_n$ for all n. Thus if (ii) fails, $I-GC$ is injective on B, and so it has a

bounded inverse. Thus for each $f \in B$ there is a unique $x \in B$ with $(1-GC)x = Gf$, i.e. $x_{n+1} = S_n x_n + f_n$ has a unique bounded solution for each bounded $\{f_n\}$, and $\{S_n\}$ has a discrete dichotomy.

Suppose $\theta < \theta_1 < 1$, $x_{n+1} = S_n x_n$ for all n and $\|x_n\| = O(\theta_1^{-|n|})$. Then $x_{n+1} - T_n x_n = O(\theta_1^{-|n|})$ so $x_n = \sum_{-\infty}^{\infty} G_{n,k+1} C_k x_k$. Choose N so large that

$$\sup_n \sum_{-\infty}^{\infty} \|G_{n,k+1}\| \, \theta_1^{-|n-k|} \cdot \sup_{|k|>N} \|C_k\| < 1/2.$$

For sequences $\{y_n\} = y$ such that $y_n = x_n$ for $|n| \le N$ and $|||y|||_b^N = \sup_{|n|>N} \{\|y_n\| \, b^{|n|}\} < \infty$ (for some $\theta_1 \le b \le 1/\theta_1$) define

$$(F_N(y))_n = x_n \quad \text{for } |n| \le N,$$
$$(F_N(y))_n = \sum_{-\infty}^{\infty} G_{n,k+1} C_k y_k \quad \text{for } |n| > N.$$

Then if $y, \hat{y}$ are any such sequences

$$|||F_N(y) - F_N(\hat{y})|||_b^N \le \tfrac{1}{2} |||y - \hat{y}|||_b^N$$

so F_N has a unique fixed point in this class. When $b = \theta_1$, $x = F_N(x)$ is the fixed point. But there is also a fixed point in the case $b = \theta_1^{-1}$ so, by uniqueness, $|||x|||_{\theta_1^{-1}}^N < \infty$, i.e.

$$\|x_n\| = O(\theta_1^{|n|}) \quad \text{as} \quad |n| \to \infty.$$

If $\theta_1 \le b \le 1/\theta_1$, the argument given in the space B also works in B_b = sequences $\{x_n\}$ with

$$\sup_n \{\|x_n\| \, b^n\} < \infty.$$

If $x = GCx$ is in B_b then $x_{n+1} = S_n x_n$, $x_n = O(b^{-n}) = O(\theta_1^{-|n|})$ so $\|x_n\| = O(\theta_1^{|n|}) = O(1)$, and if (ii) fails, $x = 0$. As before, we obtain a dichotomy but now we see the Green's function $\{\tilde{G}_{n,m}\} = \tilde{G}$ for $\{S_n\}$ satisfies

$$\|\tilde{G}_{n,m}\| \le M_1 \theta_1^{|n-m|}$$

where $M_1 = \max\{\|\tilde{G}\|_{\mathscr{L}(B_{\theta_1})}, \|\tilde{G}\|_{\mathscr{L}(B_{1/\theta_1})}\}$, by Cor. 7.6.6.

Exercise 13. In case (ii) above, if $x_{n+1} = S_n x_n$ has a nontrivial bounded solution $\{x_n\}$ prove

$$\|(1-P_n)x_n\|/\|P_n x_n\| \to \begin{array}{l} \infty \text{ as } n \to +\infty, \\ 0 \text{ as } n \to -\infty. \end{array}$$

Thus we will not have a "trichotomy", the space of bounded solutions becomes tangent to the decreasing subspace $N(P_n)$ as $n \to +\infty$ and tangent to $R(P_n)$ as $n \to -\infty$.

Exercise 14. Suppose $\{T_n^{(k)}\}_{n=-\infty}^{\infty}$ has a discrete dichotomy for $k = 1,2$, and the Green's functions satisfy

$$\|G_{n,m}^{(k)}\| \leq M\theta^{|n-m|} \qquad (\theta < 1).$$

If $\|T_n^{(1)} - T_n^{(2)}\| \leq \varepsilon$ for $|n| \leq N$ and $\|T_n^{(1)} - T_n^{(2)}\| \leq B$ for all n, prove

$$\|P_0^{(1)} - P_0^{(2)}\| \leq \frac{2M^2}{1-\theta^2}(\varepsilon + B\theta^{2N+1}) \to 0$$

as $\varepsilon \to 0$ and $N \to \infty$. If $b < \theta^{-2}$ and $\|T_n^{(1)} - T_n^{(2)}\| \leq \varepsilon b^{|n|}$ for all n, then

$$\|P_0^{(1)} - P_0^{(2)}\| \leq (1+b)M^2\varepsilon/(1-b\theta^2).$$

Hint: $x_n = G_{n,0}^{(1)}z$ satisfies $x_{n+1} - T_n^{(2)}x_n = (T_n^{(1)} - T_n^{(2)})x_n$ for $n \neq -1$, so estimate $\|G_{0,0}^{(1)}z - G_{0,0}^{(2)}z\| = \|P_0^{(1)}z - P_0^{(2)}z\|$.

Exercise 15. Suppose $\{T_n\}_{n\geq 0}$, $\{P_n\}_{n\geq 0}$ satisfy the conditions for a discrete dichotomy for $n \geq 0$ -- in particular $\|G_{n,m}\| \leq M\theta^{|n-m|}$ when $n,m \geq 0$, where $G_{n,m}$ is the corresponding Green's function. Define $P_n = P_0$ and

$$T_n = \theta(1-P_0) + \theta^{-1}P_0 \quad \text{for } n < 0.$$

Show the extended sequence has a discrete dichotomy, and prove a result corresponding to Thm. 7.6.7 for this case.

There is a corresponding result when the conditions for a discrete dichotomy hold for $n \leq 0$, $T_0|R(P_0)$ is an isomorphism onto a space $V \subset X$ and $T_0R(1-P_0) \subset W$ where $V \oplus W = X$.

Theorem 7.6.10. Suppose $\dot{x} + A(t)x = 0$ has an exponential dichotomy on $\mathbb{R}$ with exponent β and bound M, and the corresponding evolution operator $T(t,s)$ has

$$\sup\{\|T(t,s)\| : 0 \le t-s \le 1\} < \infty.$$

If $0 < \beta_1 < \beta$, $M_1 > M$, there exists $\varepsilon > 0$ (depending only on β, β_1, M, M_1 and $\sup_{0 \le t-s \le 1} \|T(t,s)\|$) such that any equation whose evolution operator $S(t,s)$ satisfies

$$\|T(t,s)-S(t,s)\| \le \varepsilon \quad \text{whenever} \quad 0 \le t-s \le 1,$$

has an exponential dichotomy on $\mathbb{R}$ with exponent β_1 and bound M_1.

Proof. Choose $\ell > 0$ so $Me^{-\beta\ell} < e^{-\beta_1\ell}$. For any real t_0, let $t_n = t_0+n\ell$ for integers n and apply Thm. 7.6.7 to show $\{S(t_{n+1},t_n)\}_{-\infty}^{\infty}$ has a discrete dichotomy with constants M_1, $e^{-\beta_1\ell}$ for small $\varepsilon > 0$, uniformly in t_0. According to exercise 10, $\{S(t,s),\ t \ge s\}$ then has an exponential dichotomy on $\mathbb{R}$ with exponent β_1, and the corresponding Green's function $\tilde{G}(t,s)$ has

$$\|\tilde{G}(t,s)\| \le M_1 e^{-\beta_1|t-s|}$$

when $t-s/\ell$ is an integer. To complete the proof, suppose $G(t,s)$ is the original Green's function and observe (cf. proof of Thm. 7.6.7)

$$\|\tilde{G}(t,s)-G(t,s)\| \le K\varepsilon \quad \text{when} \quad \frac{t-s}{\ell} \quad \text{is an integer}$$

(for some constant K) and for $|t-s| \le \ell$,

$$\|\tilde{G}(t,s)-G(t,s)\| \le K_\ell \varepsilon M_1 + \varepsilon K \sup_{0 \le t-s \le \ell} \|T(t,s)\|.$$

where $\sup_{0 \le t-s \le \ell} \|T(t,s)-S(t,s)\| \le K_\ell \varepsilon$. For ε sufficiently small we have

$$e^{\beta_1|t-s|}\|\tilde{G}(t,s)\| \le Me^{-(\beta-\beta_1)|t-s|} + O(\varepsilon) \le M_1$$

for $|t-s| \le \ell$, and $e^{\beta_1\ell}\|\tilde{G}(s\pm\ell,s)\| \le 1$, which imply $\|\tilde{G}(t,s)\| \le M_1 e^{-\beta_1|t-s|}$ for all t,s.

Theorem 7.6.11. Suppose A_0 is sectorial in X, $0 \le \alpha < 1$, $t \to A(t)-A_0 : \mathbb{R} \to \mathscr{L}(X^\alpha, X)$ is bounded and locally Hölder continuous and $\dot{x} + A(t)x = 0$ has an exponential dichotomy on $\mathbb{R}$ with exponent

$\beta > 0$ and bound M.

If $0 < \beta_1 < \beta$, $M_1 > M$, $0 < \delta < (1-\alpha)/2$, there exists $\varepsilon > 0$ such that, for any locally Hölder continuous $B: \mathbb{R} \to \mathscr{L}(X^\alpha, X)$ with

$$\|B(t)x\| \le b\,\|x\|_\alpha$$
$$\left\|\int_{t_1}^{t_2} B(t)x\,dt\right\| \le q\,\|x\|_\alpha \quad \text{for} \quad |t_1 - t_2| \le 1,$$

and $q^\delta b^{1-\delta} \le \varepsilon$, the equation

$$\dot{x} + A(t)x = B(t)x$$

has an exponential dichotomy on $\mathbb{R}$ with exponent β_1 and bound M_1.

Proof. Let $T(t,s)$, $T_B(t,s)$ be the evolution operators for $\dot{x} + A(t)x = 0$, $\dot{x} + A(t)x = B(t)x$ respectively. Theorem 7.5.2 shows

$$\sup_{0 \le t-s \le 1} \|T_B(t,s) - T(t,s)\|_{\mathscr{L}(X^\alpha)} = O(q^\delta b^{1-\delta})$$

as $q^\delta b^{1-\delta} \to 0$. Then Thm. 7.6.10 gives a dichotomy with respect to X^α, and thus a dichotomy in X^γ when $0 \le \gamma < 1$ (see ex. 5).

Exercise 16. Suppose $\dot{x} + A(t)x = 0$ has an exponential dichotomy on $\mathbb{R}$, $t \to A(t) - A_0: \mathbb{R} \to \mathscr{L}(X^\alpha, X)$ is bounded and locally Hölder continuous, $t \to f(t,x) \in X$ is locally Hölder continuous and bounded for $\|x\|_\alpha$ small, $x \to f(t,x)$ is continuously differentiable, $\|f_x(t,x) - f_x(t,0)\| \to 0$ as $\|x\|_\alpha \to 0$ uniformly in t, and

$$\frac{1}{T}\int_{t_0}^{t_0+T} (f, f_x)(t,0)\,dt \to 0 \quad \text{as} \quad T \to \infty$$

uniformly in t_0. Prove for large ω there is a unique solution $x_\omega(t)$ of

$$\dot{x} + A(t)x = f(\omega t, x), \quad -\infty < t < \infty,$$

with $\sup \|x(t)\|_\alpha \le r_1$ (for some small $r_1 > 0$ independent of ω) and $\|x_\omega(t)\|_\alpha \to 0$ as $\omega \to \infty$ uniformly in t. If the zero solution of $\dot{x} + A(t)x = 0$ is stable, prove x_ω is also uniformly asymptotically stable. (A corresponding instability result holds when $\dot{x} + A(t)x = 0$ is unstable: see ex. 21 below.)

Exercise 17. Suppose $A_j(t)-A_0 \in \mathscr{L}(X^\alpha,X)$ is bounded and locally Hölder continuous for $j = 1,2$ and each of the equations $\dot{x} + A_j(t)x = 0$ has an exponential dichotomy on $\mathbb{R}$. If $G_j(t,s)$, $P_j(t)$ are the corresponding Green's functions and projections, show

$$G_1(t,\tau)-G_2(t,\tau) = \int_{-\infty}^{\infty} G_2(t,s)(A_2(s)-A_1(s))G_1(s,\tau)ds.$$

(Hint: $x(t) = G_1(t,\tau)\xi$ satisfies $\dot{x} + A_2(t)x = (A_2(t)-A_1(t))x$ on each interval $(-\infty,\tau)$ and (τ,∞), so use ex. 7, 8 above.) If the dichotomies have exponent β and $p < 2\beta$, there is a constant $C < \infty$ with

$$\|P_2(\tau)-P_1(\tau)\|_{\mathscr{L}(X^\alpha)} \leq C \sup_s\{\|A_2(s)-A_1(s)\|_{\mathscr{L}(X^\alpha,X)} e^{-p|s-\tau|}\}.$$

Exercise 18. The criterion of Besicovitch and Bochner [3, 89] for almost-periodicity of continuous $f\colon \mathbb{R} \to X$, X a Banach space, is that every sequence in $\mathbb{R}$ contains a subsequence $\{t_n\}$ such that $\|f(t+t_n)-f(t+t_m)\| \to 0$ as $n,m \to \infty$ uniformly in t. Suppose $f\colon \mathbb{R} \to X$ is almost periodic and $g\colon \mathbb{R} \to Y$ is continuous and such that, for any sequence $\{t_n\}$ with $\{f(t+t_n)\}$ a Cauchy sequence uniformly in t, we also have $\|g(t+t_n)-g(t+t_m)\| \to 0$ uniformly in t; then g is almost periodic and its frequency module is contained in the module of f [3].

Assume $A(t)$ satisfies the assumptions of Thm. 7.6.3 and $t \to A(t)-A_0\colon \mathbb{R} \to \mathscr{L}(X^\alpha,X)$ is almost periodic. If $G(t,\tau)$, $P(\tau)$ are the corresponding Green's function and projection, $\{t_n\} \subset \mathbb{R}$, $\|A(t+t_n)-A(t+t_m)\|_{\mathscr{L}(X^\alpha,X)} \leq \Delta_m$ for $n \geq m$ with $\Delta_m \to 0$ as $m \to \infty$, prove

$$\|G(t+t_n,\tau+t_n)-G(t+t_m,\tau+t_m)\|_{\mathscr{L}(X)} \leq C\Delta_m e^{-\beta|t-\tau|}$$

for some positive constants C,β, and conclude the projection $P(t)$ is almost periodic with frequency module contained in the module of A. If $f\colon \mathbb{R} \to X$ is almost periodic, prove the bounded solution x of $\dot{x} + A(t)x = f(t)$, $-\infty < t < \infty$, is almost periodic with frequency module contained in the joint module of f and A.

Theorem 7.6.12. Suppose A_0 is sectorial in X, $0 \leq \alpha < 1$, Λ is a set, $A(t,\lambda)-A_0 \in \mathscr{L}(X^\alpha,X)$ is uniformly bounded for $(t,\lambda) \in \mathbb{R} \times \Lambda$, $t \to A(t,\lambda)-A_0\colon \mathbb{R} \to \mathscr{L}(X^\alpha,X)$ is locally Hölder continuous for each $\lambda \in \Lambda$. For each $\lambda \in \Lambda$, $\dot{x} + A(t,\lambda)x = 0$ has an exponential dichotomy on $\mathbb{R}$ with exponent β and bound M independent of λ: in obvious notation,

$$\|T_\lambda(t,s)(1-P_\lambda(s))x\|_\alpha \le Me^{-\beta(t-s)}\|x\|_\alpha \quad \text{for} \quad t \ge s$$

$$\|T_\lambda(t,s)P_\lambda(s)x\|_\alpha \le Me^{-\beta(s-t)}\|x\|_\alpha \quad \text{for} \quad s \ge t.$$

Suppose $0 < \beta_1 < \beta$, $M_1 > M$ and choose $\ell > 0$ so $Me^{-\beta\ell} < e^{-\beta_1\ell}$. Assume

$$t \to A(t)-A_0 : \mathbb{R} \to \mathscr{L}(X^\alpha, X)$$

is bounded and locally Hölder continuous, and there is a function $\lambda(\cdot): \mathbb{R} \to \Lambda$ with

$$\left\|\int_{t_0}^{t_1}(A(t)-A(t,\lambda(t_0)))dt\right\|_{\mathscr{L}(X^\alpha,X)} \le \varepsilon$$

for $t_0 \le t_1 \le t_0+\ell$, and

$$\|P_{\lambda(t_0+\ell)}(t_0)-P_{\lambda(t_0)}(t_0)\|_{\mathscr{L}(X^\alpha)} \le \varepsilon$$

for all t_0.

If ε is sufficiently small (depending on A_0, $A(t,\lambda)$, α, β, M, β_1, M_1, ℓ and $\sup\|A(t)-A_0\|$)

$$\dot{x} + A(t)x = 0$$

has an exponential dichotomy on $\mathbb{R}$ with exponent β_1 and bound M_1. If $P(t)$ is the corresponding projection, then as $\varepsilon \to 0$

$$\|P(t)-P_{\lambda(t)}(t)\| = O(\varepsilon)$$

uniformly in t.

<u>Proof</u>. Let $t_n = t_0+n\ell$ $(n = 0,\pm1,\ldots)$, $\lambda_n = \lambda(t_n)$ with $P_n = P_{\lambda_n}(t_n)$, $\tilde{P}_n = P_{\lambda_{n-1}}(t_n)$, $T_n = T_{\lambda_n}(t_{n+1},t_n)$ and $\theta = Me^{-\beta\ell}$. Then $T_nP_n = \tilde{P}_{n+1}T_n$, $T_n|R(P_n)$ is an isomorphism of $R(P_n)$ onto $R(\tilde{P}_{n+1})$, $\|T_nx\|_\alpha \le \theta\|x\|_\alpha$ when $P_nx = 0$ and $\|x\|_\alpha = \|T_{\lambda_n}(t_n,t_{n+1})T_nx\|_\alpha \le \theta\|T_nx\|_\alpha$ when $P_nx = x$. If $T(t,s)$ is the evolution operator for $\dot{x} + A(t)x = 0$ then (if $0 < \delta < (1-\alpha)/2$)

$$\|T(t_{n+1},t_n)-T_n\|_{\mathscr{L}(X^\alpha)} = O(\varepsilon^\delta)$$

uniformly in n and t_0, and $\|\tilde{P}_n - P_n\| \leq \varepsilon$. Applying Thm. 7.6.8 and exercise 10 we have a dichotomy on $\mathbb{R}$ with exponent β_1. To see the bound is M_1, argue as in the proof of Thm. 7.6.10.

As a simple application of this result, we treat the case of slowly varying coefficients (cf. Coppel [112] for the ODE case.)

Theorem 7.6.13. A_0 is sectorial in X, $0 \leq \alpha < 1$, (Λ, d) is a metric space, $\lambda \to A(\lambda) - A_0 : \Lambda \to \mathscr{L}(X^\alpha, X)$ is locally Hölder continuous, bounded, uniformly continuous and its image is in a compact set of $\mathscr{L}(X^1, X)$. Assume also there exists $\beta > 0$ so

$$\sigma(A(\lambda)) \subset \{Z: |\mathrm{Re}\ z| \geq \beta\}$$

and if $\tilde{A} = \lim_{\nu\to\infty} A(\lambda_\nu)$ in $\mathscr{L}(X^1, X)$ for a sequence $\{\lambda_\nu\} \subset \Lambda$, the essential spectrum of $\tilde{A}$ is in $\{|\mathrm{Re}\ z| \geq \beta\}$. (The second condition is trivial if A_0 has compact resolvent.)

Let $0 < \beta_1 < \beta$; there exists $\varepsilon > 0$ and $M_1 > 0$ such that, for any locally Hölder continuous $\lambda: \mathbb{R} \to \Lambda$ with

$$d(\lambda(t), \lambda(s)) \leq \varepsilon \quad \text{for} \quad |t-s| \leq 1,$$

the equation

$$dx/dt + A(\lambda(t))x = 0$$

has an exponential dichotomy on $\mathbb{R}$ with exponent β_1 and bound M_1.

Remark. Essential spectrum has several definitions, from Kato's definition [56] as the set of complex z for which $z - \tilde{A}$ is not semi-Fredholm (apparently the smallest) to that of Gohberg and Krein [118] which we used in Ch. 5, all points of the spectrum except isolated eigenvalues of finite multiplicity. The theorem holds under either definition, since the strip $\{|\mathrm{Re}\ z| < \beta\}$ meets the resolvent set for large $|\mathrm{Im}\ z|$.

Proof. We apply Thm. 7.6.12 with $A(t,\lambda) = A(\lambda)$. There exist constants C, C_1 and ϕ $(0 < \phi < \pi/2)$ so $\|(z-A_0)^{-1}\| \leq C/|z|$ when $|\arg z| \geq \phi$ and $\|(A(\lambda)-A_0)(z-A_0)^{-1}\| \leq C_1|z|^{\alpha-1}$ when $|\arg z| \geq \phi$, $\lambda \in \Lambda$, since $A(\lambda) - A_0$ is bounded in $\mathscr{L}(X^\alpha, X)$. If $R^{1-\alpha} > 2C_1$ then $\|(z-A(\lambda))^{-1}\| \leq 2C/|z|$ for $|\arg z| \geq \phi$, $|z| \geq R$ and all λ. Let $\beta_2 = \frac{1}{2}(\beta+\beta_1)$ and let γ_- be contour in the left half-plane, consisting of a segment of $\mathrm{Re}\ z = -\beta_2$ closed by the circle $|z| = R$, and γ_+ in the right half-plane formed by $\operatorname{Re} z = \beta_2$ and the rays $\arg z = \pm\theta$, $|z| \geq R$, (with $\phi \leq \theta < \pi/2$, $R \cos\theta = \beta_2$, increasing R if necessary). Then

$$e^{-A(\lambda)t}P_\lambda = \frac{1}{2\pi i}\int_{\gamma_-}(z-A(\lambda))^{-1}e^{-zt}dz, \quad t \leq 0,$$

$$e^{-A(\lambda)t}(1-P_\lambda) = \frac{1}{2\pi i}\int_{\gamma_+}(z-A(\lambda))^{-1}e^{-zt}dz, \quad t \geq 0$$

and the required estimates will hold if we show $\|(z-A(\lambda))^{-1}\|$ is bounded uniformly in λ and $z \in K = \{|\operatorname{Re} z| \leq \beta_2, |z| \leq R\}$. This will follow in turn if $z-\tilde{A}$ is invertible for every $z \in K$ and $\tilde{A}$ in F (F = the closure in $\mathscr{L}(X^1,X)$ of $\{A(\lambda), \lambda \in \Lambda\}$) since the continuous function $(z,\tilde{A}) \to \|(z-\tilde{A})^{-1}\|$ is bounded on the compact set $K \times F$. Certainly $z-\tilde{A}$ is invertible if $\tilde{A} = A(\lambda)$ for some λ, but suppose it is not invertible for some $z \in K$ and $\tilde{A} = \lim_{\nu\to\infty} A(\lambda_\nu)$. Then z is an isolated eigenvalue of $\tilde{A}$, which implies $A(\lambda_\nu)$ also has spectrum arbitrarily near z for large ν, contrary to hypothesis. Thus $\|(z-\tilde{A})^{-1}\|$ is bounded on $K \times F$. Since $\lambda \to P_\lambda$ is uniformly continuous from Λ to $\mathscr{L}(X)$, Thm. 7.6.12 applies and completes the proof.

Exercise 19. Suppose $\dot{x} + A(t)x = 0$ has an exponential dichotomy on $\mathbb{R}$ with exponent β and $t \to A(t)-A_0 \in \mathscr{L}(X^\alpha,X)$ is bounded and uniformly continuous. If $0 < \beta_1 < \beta$, prove there exists $\varepsilon > 0$ such that, for any differentiable $\theta: \mathbb{R} \to \mathbb{R}$ with $\sup|d\theta/dt - 1| \leq \varepsilon$, the equation $\dot{x} + A(\theta(t))x = 0$ has an exponential dichotomy on $\mathbb{R}$ with exponent β_1. If $t \to A(t)-A_0$ is uniformly Lipschitzian and θ_1, θ_2 both satisfy the above hypotheses and $P_{\theta_1}(t)$, $P_{\theta_2}(t)$ are the corresponding projections,

$$\|P_{\theta_1}(t)-P_{\theta_2}(t)\|_{\mathscr{L}(X^\alpha)} \leq C_b \sup_s \{|\theta_1(s)-\theta_2(s)|e^{-b|t-s|}\}$$

for a constant $C_b < \infty$ if $b < 2\beta_1$.

Exercise 20. Suppose A_0 is sectorial in X and has compact resolvent, $0 \leq \alpha < 1$, $t \to A(t)-A_0: \mathbb{R} \to \mathscr{L}(X^\alpha,X)$ is almost periodic and locally Hölder continuous (hence, uniformly continuous and with image in a compact set of $\mathscr{L}(X^\alpha,X)$.) Also there exists $\beta > 0$ such that if $-\infty < t < \infty$ and $\lambda \in \sigma(A(t))$, then $|\operatorname{Re}\lambda| \geq \beta$.

Assume $f: \mathbb{R} \times \{\|x\|_\alpha < \rho_0\} \times \{0 \leq \varepsilon \leq \varepsilon_0\} \to X$ is uniformly almost periodic in t, $f(t,0,0) = 0$, and

$$\|f(t,x_1,\varepsilon) - f(t,x_2,\varepsilon)\| \le \mu(\varepsilon,\rho)\, \|x_1-x_2\|_\alpha$$

for $\|x_1\|_\alpha, \|x_2\|_\alpha \le \rho \le \rho_0$, $0 \le \varepsilon \le \varepsilon_0$, and $\mu(\varepsilon,\rho) \to 0$ as $(\varepsilon,\rho) \to (0,0)$.

Prove there exist $\rho_1 > 0$ and $\varepsilon_1 > 0$ such that, if $0 < \varepsilon \le \varepsilon_1$, there is a unique solution $x_\varepsilon(t)$ of

$$\varepsilon \frac{dx}{dt} + A(t)x = f(t,x,\varepsilon), \qquad -\infty < t < \infty,$$

with $\sup \|x(t)\|_\alpha \le \rho_1$. Also x_ε is almost periodic, its frequency module is contained in the joint module of A and f, and $\sup \|x_\varepsilon(t)\|_\alpha \to 0$ as $\varepsilon \to 0+$. (For the almost-periodicity results, cf. ex. 18 above). If $\mathrm{Re}\,\sigma(A(t)) > 0$ for all t, x_ε is asymptotically stable; otherwise x_ε is unstable by ex. 21. (Hint: change the time variable to $\tau = t/\varepsilon$.)

Exercise 21. Suppose $\dot{x} + A(t)x = 0$ has an exponential dichotomy on $\mathbb{R}$, $t \to A(t)-A_0: \mathbb{R} \to \mathscr{L}(X^\alpha,X)$ is bounded and locally Hölder continuous, $f(t,0) = 0$, $\|f(t,x)-f(t,y)\| \le \eta(\rho)\, \|x-y\|_\alpha$ if $\|x\|_\alpha, \|y\|_\alpha \le \rho$ and $\eta(\rho) \to 0$ as $\rho \to 0$. There exist $\rho > 0$, $M \ge 1$, such that (if $P(\tau)$ is the projection of the dichotomy) the equation

$$(*) \qquad \dot{x} + A(t)x = f(t,x)$$

has local stable and unstable manifolds

$$S(\tau) = \{x_0 \mid \|(1-P(\tau))x_0\|_\alpha \le \rho/2M,\ x(t) \text{ satisfies } (*) \text{ on } (\tau,\infty) \text{ with } x(\tau) = x_0,\ \|x(t)\|_\alpha \le \rho\}$$

$$U(\tau) = \{x_0 \mid \|P(\tau)x_0\|_\alpha \le \rho/2M, \text{ there is a solution } x(t) \text{ of } (*) \text{ on } (-\infty,\tau) \text{ with } x(\tau) = x_0,\ \|x(t)\|_\alpha \le \rho\}$$

which are homeomorphic under projection (by $I-P(\tau)$, $P(\tau)$ respectively) to the closed ball of radius ρ in $X^\alpha_+ = R(1-P(\tau))$, $X^\alpha_- = R(P(\tau))$ respectively. Also $S(\tau)$, $U(\tau)$ are Lipschitzian manifolds tangent at the origin to X^α_+, X^α_-.

If $x \to f(t,x)$ is differentiable and $(t,x) \to f_x(t,x) \in \mathscr{L}(X^\alpha,X)$ is continuous uniformly in $t \in \mathbb{R}$ and $\|x\|_\alpha \le r_0$, then $S(\tau)$, $U(\tau)$ are C^1 graphs over the ρ-balls in X^α_+, X^α_- respectively, if $\rho \le r_0$. In fact, it is sufficient that f_x is continuous and $f_x(t,x) \to 0$ as $\|x\|_\alpha \to 0$ uniformly in t.

Theorem 7.6.14. Suppose A_0 is sectorial in X, $0 \le \alpha < 1$, $t \to A(t)-A_0\colon \mathbb{R} \to \mathscr{L}(X^\alpha, X)$ is bounded and locally Hölder continuous, and $\dot{x} + A(t)x = 0$ has an exponential dichotomy on $\mathbb{R}$ with exponent β. Assume $0 < \beta_1 < \beta$, $B\colon \mathbb{R} \to \mathscr{L}(X^\alpha, X)$ is locally Hölder continuous, $\|\int_{t_0}^{t_0+h} B(t)\|_{\mathscr{L}(X^\alpha,X)} \to 0$ as $|t_0| \to \infty$, for $0 \le h \le 1$, and for some λ_0, $B(t)(\lambda_0+A_0)^{-1}$ is a compact operator in $\mathscr{L}(X)$ for each t. Then one of the following holds.

Either (i) $\dot{x} + A(t)x = B(t)x$ has a nontrivial bounded solution on $-\infty < t < \infty$;

or (ii) $\dot{x} + A(t)x = B(t)x$ has an exponential dichotomy on $\mathbb{R}$ with exponent β_1.

In case (i), the space of such bounded solutions is finite dimensional. Any solution of $\dot{x} + A(t)x = B(t)x$ on $-\infty < t < \infty$ with $\|x(t)\|_\alpha = O(e^{\beta_1|t|})$ actually has $\|x(t)\|_\alpha = O(e^{-\beta_1|t|})$.

Proof. Let $T(t,s)$, $T_B(t,s)$ be the evolution operators for $\dot{x} + A(t)x = 0$, $\dot{x} + A(t)x = B(t)x$ respectively. Let $\ell > 0$ and note

$$T_B(t_0+\ell,t_0)-T(t_0+\ell,t_0) = \int_{t_0}^{t_0+\ell} T_B(t_0+\ell,s)B(s)T(s,t_0)ds.$$

It is easily shown (see Thm. 7.5.2) that

$$\|T_B(t_0+\ell,t_0)-T(t_0+\ell,t_0)\|_{\mathscr{L}(X)} \to 0 \quad \text{as} \quad |t_0| \to \infty.$$

For $\varepsilon \ge 0$, let $I_\varepsilon = \int_{t_0+\varepsilon}^{t_0+\ell} T_B(t_0+\ell,s)B(s)T(s,t_0)ds$; then I_ε is a compact operator in $\mathscr{L}(X)$ for each $0 < \varepsilon < \ell$, and $\|I_\varepsilon - I_0\|_{\mathscr{L}(X)} = O(\varepsilon^{1-\alpha})$ as $\varepsilon \to 0+$, so

$$I_0 = T_B(t_0+\ell,t_0) - T(t_0+\ell,t_0)$$

is also compact. Applying Thm. 7.6.13 and ex. 10 completes the proof.

Exercise 22. Suppose A_+, A_- are sectorial in X and have spectrum disjoint from the imaginary axis, and let $P_\pm = \frac{1}{2\pi i}\int_\gamma (\lambda - A_\pm)^{-1}d\lambda$, where γ encloses the spectrum in $\operatorname{Re}\lambda < 0$. If $A(t) = A_+$ for $t > 0$, $A(t) = A_-$ for $t < 0$, the equation $\dot{x} + A(t)x = 0$ has an exponential dichotomy on $\mathbb{R}$ if and only if $X = R(P_-) \oplus N(P_+)$. If

$R(P_\pm)$ are finite dimensional, the requirement is $\dim R(P_+) = \dim R(P_-)$ and $\dot{x} + A(t)x = 0$ has no bounded nontrivial solution on $(-\infty,\infty)$.

Consider also the case when $\dot{x} + A_+(t)x = 0$, $\dot{x} + A_-(t)x = 0$ both have exponential dichotomies on $\mathbb{R}$ and $A(t) = A_+(t)$ for $t > 0$, $A(t) = A_-(t)$ for $t < 0$.

Note: Because of the discontinuity at $t = 0$, the solutions will be strict solutions for $t \neq 0$ and continuous at $t = 0$ -- they may be interpreted as mild solutions of the equation if $A_\pm - A_0 \in \mathscr{L}(X^\alpha, X)$ where A_0 is a fixed sectorial operator.

The evolution operators are unambiguously defined and the discontinuity may be eliminated by an integrally-small perturbation.

Chapter 8

Neighborhood of a Periodic Solution

8.1 Stability and instability for nonautonomous systems

Let A be sectorial in a Banach space X and let $f(t,x)$ be, say continuously differentiable from $\mathbb{R} \times X^\alpha$ into X. Assume $f(t+p,x) = f(t,x)$ for all (t,x), some $p > 0$, and also assume $x_0(t)$ is a p-periodic solution:

$$\frac{dx_0}{dt} + Ax_0 = f(t,x_0), \text{ all } t$$

$$x_0(t+p) = x_0(t).$$

Let $x = x_0(t) + z$, z measuring the distance to the periodic solution; if $x(t)$ is a solution of $\frac{dx}{dt} + Ax = f(t,x)$, then $z(t) = x(t)-x_0(t)$ satisfies

$$\frac{dz}{dt} + Az = f(t,x_0(t)+z) - f(t,x_0(t)),$$

and we study a neighborhood of the periodic solution by examining the *linear variational equation*

$$\frac{dy}{dt} + Ay = \frac{\partial f}{\partial x}(t,x_0(t))y.$$

Observe that this equation has p-periodic coefficients.

Theorem 8.1.1. Let A be a sectorial operator in X, $0 \le \alpha < 1$, and $f(t,x)$ maps a neighborhood $U \subset \mathbb{R} \times X^\alpha$ of $\{(t,x_0(t)), t \in \mathbb{R}\}$ into X, where $f(t,x)$ and $\frac{\partial f}{\partial x}(t,x)$ are continuous in this neighborhood, $f(t,x)$ is locally Hölder continuous in t, and $\frac{\partial f}{\partial x}(t,x_0(t))$ is Hölder continuous. We assume $f(t+p,x) = f(t,x)$ for $(t,x) \in U$, some $p > 0$, and $x_0(\cdot)$ is a p-periodic solution of

(NL) $$\frac{dx}{dt} + Ax = f(t,x)$$

If the period map for the linear variational equation

(L) $$\frac{dy}{dt} + Ay = \frac{\partial f}{\partial x}(t,x_0(t))y$$

has its spectrum strictly inside the unit circle, i.e. the zero solution of (L) is exponentially asymptotically stable, then the solution $x = x_0(t)$ of the nonlinear equation (NL) is exponentially asymptotically stable. Specifically, there exist positive constants ρ, β, M such that, if $x_1(t)$ is any solution of (NL) with $\|x_1(t_0)-x_0(t_0)\|_\alpha \le \rho/2M$, then $x_1(t)$ exists on $t_0 \le t < \infty$ and has

$$\|x_1(t) - x_0(t)\|_\alpha \le 2M\|x_1(t_0) - x_0(t_0)\|_\alpha e^{-\beta(t-t_0)}.$$

<u>Proof</u>. Let $z = x-x_0(t)$, and

$$g(t,z) = f(t,x_0(t) + z) - f(t,x_0(t)) - \frac{\partial f}{\partial x}(t,x_0(t))z.$$

Since $\{t,x_0(t)), 0 \le t \le p\}$ is compact, there exist $\rho_0 > 0$ and a nondecreasing function $k(\rho)$, $0 \le \rho \le \rho_0$, $k(\rho) \to 0$ as $\rho \to 0+$, such that $\|g(t,z)\| \le k(\rho)\,\|z\|_\alpha$ for $\|z\|_\alpha \le \rho$.

Let $T(t,s)$ be the evolution operator for the linear variational equation. By hypothesis the spectral radius $r(T(t_0+p,t_0)) < 1$, so there exists $M > 0$, $\beta' > 0$ such that, for $t > s$, $x \in X^\alpha$

$$\|T(t,s)x\|_\alpha \le Me^{-\beta'(t-s)}\|x\|_\alpha$$
$$\|T(t,s)x\|_\alpha \le M(t-s)^{-\alpha}e^{-\beta'(t-s)}\|x\|.$$

Choose $0 < \beta < \beta'$ and choose $\rho > 0$ so small that $\{(t,x_0(t)+z), \|z\|_\alpha \le \rho\} \subset U$ and

$$k(\rho)M\int_0^\infty u^{-\alpha}e^{-(\beta'-\beta)u}du < \frac{1}{2}.$$

Let $z_1(t) = x_1(t) - x_0(t)$, $t \ge t_0$; if $\|z_1(t_0)\|_\alpha < \rho/2M$ then, as long as $\|z_1(t)\|_\alpha \le \rho$ we have

$$z_1(t) = T(t,t_0)z_1(t_0) + \int_{t_0}^t T(t,s)g(s,z_1(s))ds$$

so

$$\|z_1(t)\|_\alpha e^{\beta(t-t_0)} \le M\|z_1(t_0)\|_\alpha$$
$$+ Mk(\rho)\int_{t_0}^t (t-s)^{-\alpha}e^{-(\beta'-\beta)(t-s)}e^{\beta(s-t_0)}\|z_1(s)\|_\alpha$$

$$\leq M\|z_1(t_0)\|_\alpha + \tfrac{1}{2}\max_{[t_0,t]} \{e^{\beta(s-t_0)}\|z_1(s)\|_\alpha\}$$

so

$$\|z_1(t)\|_\alpha \leq 2M\,\|z_1(t_0)\|_\alpha e^{-\beta(t-t_0)} < \rho.$$

It follows that the solution exists for all $t \geq t_0$ and the result is proved.

Theorem 8.1.2. Suppose the assumptions on A, f, x_0 of Th. 8.1 hold, but the period map $U(t) = T(t+p,t)$ of the linear variational equation has

$$\sigma(U(t)) \cap \{\mu \mid |\mu| > 1\}$$

a nonempty spectral set. (If A has compact resolvent this says that there is a characteristic multiplier with modulus greater than one.)

Then the periodic solution $x_0(t)$ is unstable.

Remark. Ex. 2 below assumes only $r(U(t)) > 1$, with a bit more smoothness.

Proof. We prove there exists a function $x^*(t,t_0,a)$ which satisfies the equation for all $t < t_0$, with $x^*(t_0,t_0,a) \neq x_0(t_0)$ and $\|x^*(t,t_0,a) - x_0(t)\|_\alpha \to 0$ as $t \to -\infty$. If we then choose ($n = 1,2,3,\cdots$) $x_n = x^*(t_0-np,t_0,a)$, then the solution $x(t;t_0,x_n) = x^*(t-np,t_0,a)$ for $t_0 \leq t \leq t_0+np$ $x(t_0;t_0,x_n) = x_n$, has $\|x_n-x_0(t_0)\|_\alpha \to 0$ but, for all n,

$$\sup_{t\geq t_0} \|x(t;t_0,x_n)-x_0(t)\|_\alpha \geq \|x(t_0+np;t_0,x_n) - x_0(t_0)\|_\alpha$$

$$= \|x^*(t_0;t_0,a) - x_0(t_0)\|_\alpha > 0$$

The proof of existence of x^* is essentially the same as the corresponding proof in Th. 5.1.3, solving an integral equation by the contraction mapping principle. We will merely set up the integral equation, leaving the rest as an exercise.

Let $\sigma_1 = \sigma(U(t)) \cap \{|\mu| > 1\}$, and let $X = X_1(t) \oplus X_2(t)$ be the corresponding decomposition into $U(t)$-invariant subspace (see Th. 7.2.3). Let $E_j(t)$ be the projection of X onto $X_j(t)$ ($j = 1,2$). There exist $M > 0$, $\beta > 0$ such that, for $t > s$, $x_2 \in X_2^\alpha(s)$,

$$\|T(t,s)x_2\|_\alpha \leq Me^{\beta(t-s)}\|x_2\|_\alpha, M(t-s)^{-\alpha}e^{\beta(t-s)}\|x_2\| ,$$

and for $x_1 \in X_1(s)$, $T(t,s)x_1$ may be defined consistently for $t < s$ with

$$\|T(t,s)x_1\|_\alpha \leq Me^{3\beta(t-s)}\|x_1\|_\alpha , \; Me^{3\beta(t-s)}\|x_1\| \quad \text{for} \quad t < s.$$

Suppose $g:(-\infty,t_0) \to X$ has $\|g(t)\| = O(e^{2\beta(t-t_0)})$.

The unique solution of the linear equation

$$\frac{dz}{dt} + (A - \frac{\partial f}{\partial x}(t,x_0(t)))z = g(t) \quad \text{for} \quad t < t_0$$

with $E_1(t_0)z(t_0) = a$ and $\|z(t)\|_\alpha = O(e^{2\beta(t-t_0)})$ as $t \to -\infty$ is

$$z(t) = T(t,t_0)a + \int_{t_0}^{t} T(t,s)E_1(s)g(s)ds + \int_{-\infty}^{t} T(t,s)E_2(s)g(s)ds.$$

Substituting $g(s,z(s)) = g(s)$ give the integral equation which determines the function $x^* = z^* + x_0$.

Exercise 1. Apply ex. 1 and 4, sec. 5.1, with $T(\zeta) = x(t_0+p;t_0, x_0(t_0)+\zeta) - x_0(t_0)$, $\zeta \in X$, to give alternate proofs of Th. 8.1.1 and 8.1.2.

Exercise 2. Suppose A, f, $x_0(t)$ as in Th. 8.1.1 and also, for some $q > 0$, $\|f(t,x_0(t)+z) - f(t,x_0(t)) - \frac{\partial f}{\partial x}(t,x_0(t))z\| = O(\|z\|_\alpha^{1+q})$ as $z \to 0$ uniformly in t. If the spectral radius of the period map of the linearization is greater than one, prove the solution $x_0(t)$ is unstable. (See Cor. 5.1.6.)

8.2. Orbital stability and instability for autonomous systems

An important class of problems not covered by Theorem 8.1.1 is the autonomous case, when the equation is not explicitly time-dependent. In this case, there is always a multiplier on the unit circle (lemma 8.2.2) and even though the instability theorem Th. 8.1.2 applies in the autonomous case, the conclusion is too weak. For the ordinary differential equation $dx_1/dt = x_2$, $dx_2/dt = -\sin x_1$ (the pendulum equation) every nontrivial periodic solution is unstable, since the period depends on the amplitude and initially close solutions eventually get out of phase. The proper notion for autonomous equations is orbital stability and instability.

Definition 8.2.1. A periodic solution $x_0(t) = x_0(t+p)$ is orbitally stable if the set $\Gamma = \{x_0(t),\ 0 \le t \le p\}$ is stable, i.e. if for any neighborhood U of Γ there exists a neighborhood V of Γ such that if $x_1 \in V$ then the solution $x(t;x_1) \in U$ for all $t \ge 0$. Similarly orbital asymptotic stability, orbital instability, etc. all refer to the corresponding stability property of the set Γ.

Lemma 8.2.2. Let A be sectorial and $f(x)$ be continuously differentiable from X^α to X in a neighborhood of a nonconstant periodic solution $x_0(t)$ of

$$\frac{dx}{dt} + Ax = f(x).$$

Also assume $t \to f_x(x_0(t)) \in \mathscr{L}(X^\alpha, X)$ is Hölder continuous.

Then the linear variational equation

$$\frac{dy}{dt} + Ay = f_x(x_0(t))y$$

has 1 as a characteristic multiplier.

Proof. We prove the derivative $\dot{x}_0(t)$ is a nontrivial periodic solution of the linear equation, so $\dot{x}_0(t_0) = U(t_0)\dot{x}_0(t_0) \ne 0$, and 1 is a multiplier.

Both $x_0(t)$ and $x_0(t+h)$ are solutions, and $\dot{x}_0(t) \in X^\alpha$ is Hölder continuous (Th. 3.5.2), so differentiating $x_0(t+h) = e^{-At}x_0(h) + \int_0^t e^{-A(t-s)} f(x_0(s+h))$ with respect to h, it follows that $y(t) = \dot{x}_0(t)$ solves $dy/dt + Ay = f'(x_0(t))y$. Also, by uniqueness, $\dot{x}_0(t) \ne 0$.

Theorem 8.2.3. Suppose $x = x_0(t)$ is a nonconstant periodic solution of the autonomous equation with period p, where A is sectorial and $f(x)$ is continuously differentiable from a neighborhood of the orbit $\Gamma = \{x_0(t),\ 0 \le t \le p\} \subset X^\alpha$ into X, and that $t \to f_x(x_0(t))$ is Hölder continuous into $\mathscr{L}(X^\alpha, X)$.

Assume the multiplier 1 is an isolated simple eigenvalue of the period map, and the remainder of the spectrum lies in $\{|\mu| < e^{-\beta p}\}$ for some $\beta > 0$.

Then Γ is orbitally asymptotically stable with asymptotic phase. I.e. there exist $\rho > 0$, $M > 0$ such that if

$$\operatorname{dist}_{X^\alpha}\{x(0),\Gamma\} = \min_t \|x(0) - x_0(t)\|_\alpha < \rho/2M$$

then the solution $x(t)$ of $dx/dt + Ax = f(x)$ through $x(0)$ exists on $0 \leq t < \infty$ and there exists real $\theta = \theta(x(0))$ such that

$$\|x(t) - x_0(t-\theta)\|_\alpha \leq 2\rho e^{-\beta t} \quad \text{for} \quad t \geq 0.$$

Proof. Let $x = x_0(t) + z$,

$$\frac{dz}{dt} + [A - f_x(x_0(t))]z = g(t,z)$$

$$g(t,0) = 0, \|g(t,z_1)-g(t,z_2)\| \leq k(\rho)\|z_1-z_2\|_\alpha \text{ if } \|z_1\|_\alpha, \|z_2\|_\alpha \leq \rho$$

with $k(\rho) \to 0$ as $\rho \to 0+$.

Decomposing the space according to the spectral set $\{1\}$, $X(t) = X_1(t) \oplus X_2(t)$ where $X_1(t) = \text{span}\{\dot{x}_0(t)\}$ for all t (recall $\dot{x}_0(t) \neq 0$), and $\|T(t,s)x_1\|_\alpha \leq M\|x_1\|$ for all t, $x_1 \in X_1(s)$. For $x_2 \in X_2^\alpha(s)$, $t > s$,

$$\|T(t,s)x_2\|_\alpha \leq Me^{-\beta'(t-s)}\|x_2\|_\alpha, M(t-s)^{-\alpha}e^{-\beta'(t-s)}\|x_2\|$$

for some $\beta' > \beta > 0$. Choose $\rho > 0$ so small

$$Mk(\rho)M_0(\frac{1}{\beta} + \int_0^\infty e^{-(\beta'-\beta)u}u^{-\alpha}du) \leq \frac{1}{2}$$

with $M_0 = \sup\{\|E_1(s)\|, \|E_2(s)\|\}$, $E_j(s)$ projection onto $X_j(s)$.

In the norm $|z|_{\alpha,\beta} = \sup_{t\geq t_0}\{\|z(t)\|_\alpha e^{\beta(t-t_0)}\}$, the map $z \to F(z)$:

$$F(z)(t) = T(t,t_0)a + \int_{t_0}^t T(t,s)E_1(s)g(s,z(s)ds$$
$$- \int_t^\infty T(t,s)E_2(s)g(s,z(s))ds, \quad t \geq t_0,$$

is a contraction map of the continuous $z: [t_0,\infty) \to X^\alpha$ with $|z|_{\alpha,\beta} \leq \rho$, provided $a \in X_2^\alpha(t_0)$, $\|a\|_\alpha \leq \rho/2M$.

Let $z = z^*(\cdot,a)$ be the unique fixed point of F. For $a,b \in X_2^\alpha(t_0)$ with $\|a\|_\alpha, \|b\|_\alpha \leq \rho/2M$, $|z^*(\cdot,a) - z^*(\cdot,b)|_{\alpha,\beta} \leq 2M\|a-b\|_\alpha$. Define $H(a) = z^*(t_0,a) - a$; then

$$\|H(a)-H(b)\|_\alpha \leq 2M^2M_0\beta^{-1}k(2M \max(\|a\|_\alpha,\|b\|_\alpha))\|a-b\|_\alpha .$$

It may be shown $t \to z^*(t,a)$ is locally Hölder continuous (for $t>t_0$) into X^α, and it follows $x^*(t,a) = x_0(t) + z^*(t,a)$ is a solution of the differential equation for $t > t_0$, whenever $a \in X_2^\alpha(t_0)$,

$\|a\|_\alpha \le \rho/2M$.

Consider $x(t,\xi)$, the solution of $\dot{x}+Ax = f(x)$, $t > t_0-p$, with $x(t_0-p,\xi) = \xi$. If $\|\xi-x_0(t_0)\|_\alpha$ is sufficiently small, $x(t,\xi)$ exists on $t_0-p \le t \le t_0+p$ and we prove there exist θ real and $a \in X_2^\alpha(t_0)$ such that $|\theta| + \|a\|_\alpha$ is small and

$$x(t_0+\theta,\xi) = x^*(t_0,a).$$

This implies, by uniqueness, $x(t,\xi) = x^*(t-\theta,a)$ for $t \ge t_0+\theta$, so

$$\|x(t,\xi) - x_0(t-\theta)\|_\alpha \le \rho e^{-\beta(t-t_0-\ )} \le 2\rho e^{-\beta(t-t_0)}.$$

By periodicity, it suffices that $\|x(t_0+np) - x_0(t_0)\|_\alpha$ is sufficiently small for some integer n, and by continuous dependence, that $\|x(t_1)-x_0(t_1)\|_\alpha$ is small for some t_1. Thus the result is proved once we show solvability of $-x(t_0+\theta,\xi) + x^*(t_0,a) = 0$, or equivalently

$$a - \theta\dot{x}_0(t_0) = G(\theta,a;\xi), \text{ where}$$

$$G(\theta,a;\xi) = x(t_0+\theta,\xi) - x_0(t_0+\theta) - H(a) + x_0(t_0+\theta) - x_0(t_0)-\theta\dot{x}_0(t_0).$$

Now $G(0,0;x_0(t_0)) = 0$ and

$$E_1(t_0)(a - \theta\dot{x}_0(t_0)) = -\theta\dot{x}_0(t_0)$$

$$E_2(t_0)(a - \theta\dot{x}_0(t_0)) = a$$

so the "linear part" is invertible. We need to show $(\theta,a) \to G(\theta,a;\xi)$ has an arbitrarily small Lipschitz constant near $(0,0)$, provided ξ is near $x_0(t_0)$: this implies we have a contraction and the equation is solvable when $\|\xi-x_0(t_0)\|_\alpha$ is small enough.

Given $\varepsilon > 0$, there exists $0 < \delta < p$ so that $\|H(a)-H(\bar{a})\|_\alpha \le \varepsilon\|a-\bar{a}\|_\alpha, \|\dot{x}_0(t_0+\theta)-\dot{x}_0(t_0)\|_\alpha \le \varepsilon/2$ for $|\theta| \le \delta$, and $\|a\|_\alpha \le \delta$. Also for $t_0-p < t \le t_0+p$, $\alpha < \gamma < 1$, $\|\dot{x}(t,\xi)-\dot{x}_0(t)\|_\alpha \le C(t-t_0+p)^{-1}\|\xi-x_0(t_0)\|_\alpha^{1-\gamma}$ by Th. 3.5.3, so for $|\theta|$, $|\bar{\theta}|$, $\|a\|_\alpha$, $\|\bar{a}\|_\alpha \le \delta$

$$\|G(\theta,a;\xi) - G(\bar{\theta},\bar{a};\xi)\|_\alpha \le C_1|\theta-\bar{\theta}|\ \|\xi-x_0(t_0)\|_\alpha^{1-\gamma}$$

$$+ \varepsilon\|a-\bar{a}\|_\alpha + \varepsilon/2|\theta-\bar{\theta}| \le \varepsilon(\|a-\bar{a}\|_\alpha + |\theta-\bar{\theta}|),$$

if $\|\xi-x_0(t_0)\|_\alpha$ is sufficiently small.

Exercise 1. Use ex. 1, sec. 5.1, to give an alternate proof of Th. 8.2.3 with

$$T(\xi) = x(t_0+p(\xi);t_0,x_0(t_0)+\xi) - x_0(t_0)$$

for $\xi \in X_2^\alpha(t_0)$, and with $p(\xi)$ chosen $(p(\xi) = p + O(\|\xi\|_\alpha)$ so that $T(\xi) \in X_2^\alpha(t_0)$ also. First show this map is well-defined and C^1 near $\xi = 0$, by the implicit function theorem. (Cf. the proof of Th. 8.3.2. below).

Theorem 8.2.4. Assume the same regularity conditions as in Th. 8.2.3, but suppose the period map $U(t)$ for the linear variational equation has $\sigma(U(t)) \cap \{|\mu| > 1\}$ a nonempty spectral set. Then the solution $x = x_0(t)$ is orbitally unstable: there exists a neighborhood W of $\Gamma = \{x_0(t),\ 0 \le t \le p\}$ and a sequence $\{x_n\}$ such that $\mathrm{dist}_{X^\alpha}(x_n,\Gamma) \to 0$ as $n \to \infty$, but for each n, $x(t;t_0,x_n)$ eventually leaves W.

Proof. Theorem 8.1.2 applies and proves $x_0(t)$ is unstable, a weaker conclusion. We use the function $x^*(t;t_0,a) = x_0(t) + z^*(t;t_0,a)$ from the proof of Th. 8.1.2, with $a \in X_1(t_0)$ (the "unstable" subspace), $\|a\|_\alpha \le \rho/2M$, $\|z^*(t_0;t_0,a)-a\|_\alpha = o(\|a\|_\alpha)$ as $\|a\|_\alpha \to 0$, $x_0(t) + z^*(t;t_0,a)$ is a solution of the differential equation for $t < t_0$ and $\|z^*(t;t_0,a)\|_\alpha \le 2M\,\|a\|_\alpha e^{2\beta(t-t_0)}$.

Now it is possible to choose a $X_1^\alpha(t_0)$, and $\varepsilon > 0$, such that $0 < \|a\|_\alpha \le \rho/2M$ and $\|x_0(t_0) + z^*(t_0;t_0,a) - x_0(t)\|_\alpha \ge \varepsilon > 0$ for all t. For if $p > 0$ is the least period of x_0, then $\|x_0(t) - x_0(t_0)\|_\alpha > 0$ for $t_0 < t < t_0+p$; and for t near t_0 (or t_0+p) with $\|a\|_\alpha$ small but nonzero,

$$\|x_0(t_0)+z^*(t_0;t_0,a) - x_0(t)\|_\alpha \ge \|a-\dot{x}_0(t_0)(t-t_0)\|_\alpha$$

$$- \|z^*(t_0;t_0,a)-a\|_\alpha - \|x_0(t)-x_0(t_0)-\dot{x}_0(t_0)(t-t_0)\|_\alpha$$

and this is bounded from zero since $a \in X_1(t_0)$ and $\dot{x}_0(t_0) \in X_2(t_0)$, and $\dot{x}_0(t)$ is Hölder ocontinuous into X^α.

Choosing such a and $\varepsilon > 0$, let

$$W = \{x: \mathrm{dist}_{X^\alpha}(x,\Gamma) < \varepsilon\}$$

and $x_n = x_0(t_0)=z^*(t_0-np;t_0,a)$. Then $\|x_n-x_0(t_0)\|_\alpha \to 0$ as $n \to +\infty$, and

$$x(t;t_0,x_n) = x_0(t)+z^*(t-np;t_0,a) \text{ for } t_0 \le t \le t_0+np,$$

so $x(t_0+np;t_0,x_n) = x_0(t_0) + z^*(t_0;t_0,a) \notin W$, by choice of a and ε.

Exercise 2. Use ex. 4, sec. 5.1, to give an alternate proof of Th. 8.2.3. (cf. ex. 1 above.)

Exercise 3. Assume $\|f(x_0(t)+z) - f(x_0(t)) - f_x(x_0(t))z\| = O(\|z\|_\alpha^{1+q})$ as $z \to 0$, uniformly in t, for some $q > 0$. If the period map of the linearization has spectral radius greater than 1, prove $x_0(t)$ is orbitally unstable. (cf. ex. 2 in sec. 8.1.)

8.3 Perturbation of periodic solutions

Suppose $x_0(t)$ is a p-periodic solution $(p > 0)$ of $\frac{dx}{dt} + Ax = f(t,x,0)$, with A sectorial and $f\colon \mathbb{R} \times X^\alpha \times \mathbb{R} \to X$ is continuously differentiable near $\{(t,x,\varepsilon)\colon x = x_0(t),\ \varepsilon = 0\}$ and p-periodic in t. Let $x = x_0(t) + z$ in

$$\frac{dx}{dt} + Ax = f(t,x,\varepsilon):$$

then

$$\frac{dz}{dt} + A(t)z = g(t,z,\varepsilon)$$

where

$$A(t) = A - f_x(t,x_0(t),0).$$

$g(t,z,\varepsilon) = f(t,x_0(t)+z,\varepsilon) - f(t,x_0(t),0) - f_x(t,x_0(t),0)z$, $\|g(t,z,\varepsilon)\| \le B(|\varepsilon| + \eta(\|z\|_\alpha)\|z\|_\alpha)$ for $\|z\|_\alpha$, $|\varepsilon|$ small, where $\eta(\rho) \to 0$ as $\rho \to 0+$. Assume also $t \to f_x(t,x_0(t),0) \in \mathscr{L}(X^\alpha,X)$ is Hölder continuous.

Theorem 8.3.1. Assume A, f, x_0 are as above, and assume that 1 is not in the spectrum of the period map for the linear variational equation for $x_0(\cdot)$, $\varepsilon = 0$. Then for each sufficiently small ε, there exists a unique p-periodic solution $x_\varepsilon(\cdot)$ in a small neighborhood of $x_0(\cdot)$ and $\|x_\varepsilon(t)-x_0(t)\|_\alpha \to 0$ as $\varepsilon \to 0$.

Proof. Let $T(t,s)$ be the evolution operator for $dz/dt + A(t)z = 0$. If $\|z_0\|_\alpha$, $|\varepsilon|$ are sufficiently small, the solution $z(t;t_0,z_0,\varepsilon)$ of the differential equation exists on $t_0 \le t \le t_0+p$ and

$$z(t_0+p;t_0,z_0,\varepsilon) - z_0 = (T(t_0+p,t_0)-1)z_0 + \int_{t_0}^{t_0+p} T(t_0+p,s)g(s,z(s;t_0,z_0,\varepsilon),\varepsilon)ds.$$

Now $(1-T(t_0+p,t_0))^{-1} \equiv L$ exists, by hypothesis, and the condition that this be a p-periodic solution is

$$z_0 = L \int_{t_0}^{t_0+p} T(t_0+p,s)g(s,z(s;t_0,z_0,\varepsilon),\varepsilon)ds.$$

The implicit function theorem applies, and proves, for small $|\varepsilon|$ there exists a unique $z_0(\varepsilon)$ in a neighborhood of the origin $\{\|z\|_\alpha < r\}$ solving this equation.

Exercise 1. If $x_0(\cdot)$ is asymptotically stable (or unstable) by the linear approximation, in the sense of Th. 8.1.1 (or 8.1.2) show the same is true for $x_\varepsilon(\cdot)$.

Now we know the hypothesis that $1 \notin \sigma(T(t+p,t))$ is never verified for autonomous equations, so we treat this case separately.

Theorem 8.3.2. Assume A, f, x_0 as above, but suppose f does not depend on the time variable t. Then 1 is an eigenvalue of the period map, but we shall assume it is an isolated simple eigenvalue. Then there is a neighborhood U of $\{x_0(t),\ 0 \le t \le p\}$ in X^α and $\varepsilon_0 > 0$, $\delta_0 > 0$, such that, for each ε in $|\varepsilon| < \varepsilon_0$, there exists a unique periodic solution $x_\varepsilon(t)$ which remains in U and has period $p(\varepsilon)$ $|p(\varepsilon)-p| < \delta_0$ and $p(\varepsilon) \to p$, $x_\varepsilon(t) \to x_0(t)$ as $\varepsilon \to 0$.

Proof. We use Poincare's "method of sections" to eliminate the troublesome eigenvalue. Let $U(0) = T(p,0)$ be the period map (for $\varepsilon = 0$) and $U(0)\dot{x}_0(0) = \dot{x}_0(0)$, $U^*(0)y_0 = y_0 \in X^*$, $\langle y_0,\dot{x}_0(0)\rangle = 1$. Such y_0 exists, by our assumption,

Define $W = \{x \in X^\alpha: \langle y_0, x-x_0(0)\rangle = 0\}$. If $x_1 \in W$, $\|x_1-x_0\|_\alpha + |\varepsilon|$ is sufficiently small, then $x(t;x_1,\varepsilon)$ is defined on $0 \le t \le 2p$.

Define

$$G(x_1,\varepsilon) \in W \text{ by}$$
$$G(x_1,\varepsilon) = x(\tau(x_1,\varepsilon) + p;x_1,\varepsilon),$$

where $\tau = \tau(x_1,\varepsilon)$ is the unique time near 0 such that

$$g(\tau;x_1,\varepsilon) = \langle y_0, x(p+\tau;x_1,\varepsilon) - x_0(0)\rangle = 0.$$

By the implicit function theorem, $\tau(x_1,\varepsilon)$ is a well-defined C^1 function:

$$g(0;x_0(0),0) = 0$$

$$\frac{\partial g}{\partial \tau}(0;x_0(0),0) = \langle y_0,\dot{x}_0(0)\rangle = 1$$

Now we prove the existence of a fixed point $x_1 = G(x_1,\varepsilon) \in W$, for each small ε, hence the existence of a periodic solution of period $p + \tau(x_1(\varepsilon),\varepsilon)$.

We know $G(x_0(0),0) = x_0(0)$, and

$$G_x(x_0(0),0)z = \dot{x}_0(0)\tau_x(x_0(0),0)z + T(p,0)z; \quad \text{if } \langle y_0 z\rangle = 0,$$

then

$$\langle y_0, G_x(x_0(0),0)z\rangle = 0 = \tau_x(x_0(0),0)z + \langle y_0, T(p,0)z\rangle$$

so

$$\tau_x(x_0(0),0)z = 0.$$

Thus in the space $Z_1 = \{z: \langle y_0,z\rangle = 0\} = R(T(p,0))$, $G_x(x_0(0),0) = T(p,0)|_{Z_1}$. But $\sigma(T(p,0)|_{Z_1}) = \sigma(T(p,0))\setminus\{1\}$ so $G_x(x_0(0),0)-I$ has a bounded inverse on Z_1. By the implicit function theorem, there exists a unique $x_1(\varepsilon)$ with $x_1(\varepsilon) - x_0(0)$ in Z_1, $\|x_1(\varepsilon) - x_0(0)\|_\alpha < r$, such that

$$G(x_1(\varepsilon),\varepsilon) = x_1(\varepsilon)$$

Corollary 8.3.3. Under the hypotheses of Th. 8.3.2, if we assume p is the least period of x_0 and also that $\sigma(U(t))\setminus\{1\}$ contains no n^{th} root of unity for $1 \le n \le N$, then the periodic solution $x_1(t,\varepsilon)$ constructed above is the only periodic solution near $\{x_0(t),\ 0 \le t \le p\}$ with period less than $(N+\frac{1}{2})p$, for sufficiently small $|\varepsilon|$.

Exercise 2. Show that $x(\cdot,\varepsilon)$ shares the orbital stability or instability of x_0, when x_0 satisfies the hypotheses of Th. 8.2.3 or 8.2.4.

Exercise 3. Use ex. 3, sec. 4.2, to give a criterion for the existence of a small neighborhood U of $\{x_0(t),\ 0 \le t \le p\}$ which is positively invariant under the period map of $dx/dt + Ax = f(t,x,\varepsilon)$ for all sufficiently small ε, when x_0 is a p-periodic asymptotically stable solution of the equation with $\varepsilon = 0$.

Examine also the autonomous case, assuming x_0 is orbitally asymptotically stable.

8.4. The Poincaré map

The Poincaré (or period) map is a powerful tool for studying the neighborhood of a periodic solution. We used it in Th. 8.3.1 and 8.3.2, and it may also be used (exercise in sec. 8.1 and 8.2) to prove stability and instability results. It will be used extensively in the next section, so it deserves careful study. For the ODE case, see Lefschetz [68, p. 160].

First consider a periodic equation (period $p > 0$)

$$dx/dt + Ax = f(t,x), \quad f(t+p,x) = f(t,x)$$

where A is sectorial and f is C^1 from $\mathbb{R} \times X^\alpha$ to X, and suppose $x_0(\cdot)$ is a p-periodic solution. For $\xi \in X^\alpha$ near $x_0(t_0)$, define

$$\Phi(\xi) = x(t_0+p;t_0,\xi);$$

then Φ is $C^1(X^\alpha,X^\alpha)$ near $\xi = x_0(t_0)$, $\Phi(x_0(t_0)) = x_0(t_0)$, and

$$\Phi'(x_0(t_0)) = T_0(t_0+p,t_0)$$

where $T_0(t,s)$ is the evolution operator for

$$dy/dt + Ay = f_x(t,x_0(t))y.$$

The map Φ is the Poincaré map for this (nonautonomous) case; a different form is used in the autonomous case (see below). Observe that any fixed point of Φ yields a p-periodic solution, while a fixed point of Φ^N (the N-fold composition) yields an Np-periodic solution ($N = 2,3,\cdots$); a solution having *least* period Np is called an N^{th} order subharmonic, if $N > 1$. In view of the following exercise, we expect *in general* that a periodic solution of a nonautonomous periodic equation will have the same period, or be a subharmonic.

Exercise 1. [79] Suppose A, f as above with $f(t+p,x) = f(t,x)$. Suppose $x_1(\cdot)$ is a periodic solution of $dx/dt + Ax = f(t,x)$ with least period $q > 0$. If q/p is irrational, prove

$f(t,x_1(s)) = f(s,x_1(s))$ for all t,s. If q/p is rational, let $q/p = m/n$ where m,n are relatively prime positive integers, and show $f(t,x_1(s)) = f(s,x_1(s))$ whenever $t-s = \frac{k}{n}p$ for some integer k.

Exercise 2. With A, f, Φ, x_0 as above, prove x_0 is stable (or unstable, or asymptotically stable) as a solution of $dx/dt + Ax = f(t,x)$, if and only if $x_0(t_0)$ is stable (or ...) as a fixed point of Φ. We say $x_0(t_0)$ is stable, for example, if for any $\varepsilon > 0$, there exists $\delta > 0$ so that whenever $\|\xi - x_0(t_0)\|_\alpha < \delta$, $\Phi^n(\xi)$ is defined for all $n \geq 1$ and $\|\Phi^n(\xi) - x_0(t_0)\|_\alpha < \varepsilon$.

Exercise 3. = Exercise 1 of sec. 8.1.

Now suppose the equation is autonomous: $f(t,x) = f(x)$, independent of t. Assume $x_0(\cdot)$ is a nonconstant p-periodic solution of

$$dx/dt + Ax = f(x), \qquad f: X^\alpha \to X \text{ is } C^1.$$

Let S be a C^1 manifold (in X^α) of codimension one such that $x_0(t_0) \in S$ and $\dot{x}_0(t_0)$ is not tangent to S at $x_0(t_0)$. For $\xi \in S$ near $x_0(t_0)$, define $\Phi(\xi) \in S$ by

$$\Phi(\xi) = x(t_0 + p(\xi); t_0, \xi),$$

where $p(\xi) = p + O(\|\xi - x_0(t_0)\|_\alpha$ is chosen to ensure $\Phi(\xi) \in S$. We first show Φ is a well-defined C^1 function on S near $x_0(t_0)$, the Poincaré map on the "surface of section" S.

Suppose without loss $x_0(t_0) = 0$ and S is represented near 0 in the form $x = \sigma(y)$ for $y \in Y$, Y = tangent space to S at 0, so $\|\sigma(y) - y\|_\alpha = o(\|y\|_\alpha)$ and as $y \to 0$ in Y. Then define $F(z,\theta,y) = x(t_0+\theta; t_0, \sigma(y)) - \sigma(z)$ for small $y,z \in Y$ and real θ near p; we use the implicit function theorem to prove there exist $(z(y),\theta(y))$ near $(0,p)$ in $Y \times \mathbb{R}$ so that $F(z(y),\theta(y),y) = 0$ for y near 0 in Y. First observe that F is continuously differentiable, so it suffices to prove the linearization has a continuous inverse at $(0,p,0)$. But

$$F_z \delta z + F_\theta \delta\theta = -\delta z + \delta\theta \dot{x}_0(t_0) = \delta x$$

can be solved uniquely for $(\delta z, \delta\theta) \in Y \times \mathbb{R}$ as continuous functions of $\delta x \in X^\alpha$, since $X^\alpha = \operatorname{span}\{\dot{x}_0(t_0), Y\}$ so the result is proved.

Exercise 4. If $\Phi(x_1) = x_1 \in S$ (near $x_0(t_0)$), then the solution $x_1(t)$ of $dx/dt + Ax = f(x)$ through x_1 is periodic. It is orbitally stable (or orbitally unstable, or orbitally asymptotically stable) if x_1 is stable (or unstable or asymptotically stable) as a fixed point of Φ.

Exercise 5. If A has compact resolvent, then Φ is a compact map of S. (That is, for any bounded sequence $\{x_n\}_{n=1}^{\infty}$ in $S \subset X^\alpha$, in a neighborhood of $x_0(t_0)$, $\{\Phi(x_n)\}_{n=1}^{\infty}$ has an X^α-convergent subsequence).

Now we examine the spectrum of the derivative $\Phi'(x_0(t_0))$, comparing this with the spectrum of $T_0(t_0+p,t_0)$. First suppose, as above, $x_0(t_0) = 0$

$$S = \{\sigma(y): y \in Y, \|y\|_\alpha < r\}$$

with $\sigma(y) = y + o(\|y\|_\alpha)$; then

$$\Phi'(0)y = \dot{x}_0(t_0)\theta'(0)y + T_0(t_0+p,t_0)y$$

so

$$\Phi'(0) = P_y T_0(t_0+p,t_0)|_Y,$$

where P_y is the projection onto Y, along $\dot{x}_0(t_0)$. If $Y = \{y | \langle y,\ell\rangle = 0\}$ for some nonzero $\ell \in (X^\alpha)^*$, then $\langle \dot{x}_0(t_0),\ell\rangle \neq 0$; we may assume $\langle \dot{x}_0(t_0),\ell\rangle = 1$, and then

$$P_y x = x - \dot{x}_0(t_0)\langle x,\ell\rangle, \quad x \in X^\alpha.$$

Exercise 6. Let X be a Banach space, Y a closed subspace of codimension one of X, and L a continuous linear map of X to itself. Assume $a \in X$, $a \neq 0$, $a \notin Y$, and $La = 0$. Finally, let $M = P_y L|_Y : Y \to Y$, where P_y is the projection onto Y, along a ($P_y a = 0$; $P_y y = y$ for $y \in Y$).

If X_1 is an L-invariant subspace, $L(X_1) \subset X_1$, and if $Y_1 = P_y X_1$, then Y_1 is M-invariant. If X_1 is finite dimensional, then $\dim Y_1 = \dim X_1$ (if $a \notin X_1$), or $\dim Y_1 = \dim X_1 - 1$ (if $a \in X_1$).

If X_1 is an L-invariant subspace, $Y_1 = P_y X_1$, then $\sigma(L|_{X_1}) \setminus \{0\} \subset \sigma(M|_{Y_1}) \subset \sigma(L|_{X_1})$. (Hint: if $\lambda \notin \sigma(L|_{X_1})$, $y_2 \in Y_1$, then $(\lambda - M)y_1 = y_2$ is uniquely solvable for $y_1 \in Y_1$, namely

$y_1 = P_y(\lambda - L_1)^{-1}x_2$, $L_1 = L|_{X_1}$, where $x_2 \in X_1$ has $P_y x_2 = y_2$.)

If $Lx = \lambda x$, $x \neq 0$, and $y = P_y x$, then $My = \lambda y$: if $\lambda \neq 0$; then $y \neq 0$; if $\lambda = 0$, then $y \neq 0$ unless x is a multiple of a. More generally, if $\lambda \neq 0$, and $k = 1,2,3,\ldots$,

$$N(\lambda - M)^k = P_y N(\lambda - L)^k.$$

Applying the results of ex. 6 with $L = T_0(t_0+p,t_0)-1$, $a = \dot{x}_0(t_0)$, and $M = \Phi'(0)$ we see in particular that, if $\{1\}$ is an isolated eigenvalue of $U(t_0) = T_0(t_0+p,t_0)$ and $X^\alpha = X_1^\alpha \oplus X_2^\alpha$ is the corresponding decomposition with

$$\sigma(U_0(t_0)|_{X_1{}^\alpha}) = \{1\},$$

$$\sigma(U_0(t_0)|_{X_2{}^\alpha}) = \sigma(U_0(t_0)) \setminus \{1\},$$

then $Y = Y_1 \oplus Y_2$ where the $Y_j = P_y X_j^\alpha$ are $\Phi'(0)$-invariant subspaces and

$$\sigma(\Phi'(0)|_{Y_1}) \subset \{1\}$$

$$\sigma(\Phi'(0)|_{Y_2}) = \sigma(U_0(t_0)) \setminus \{1\}.$$

In particular, if $\{1\}$ is an isolated simple eigenvalue of $T_0(t_0+p,t_0) = U_0(t_0)$, then $Y_1 = \{0\}$ and $1 \notin \sigma(\Phi'(0))$. If $\{1\}$ is an eigenvalue of multiplicity m, $\dim Y_1 = m-1$.

<u>Exercise 7</u>. = Exercises 1 and 2 of section 8.2.

<u>Exercise 8</u>. Give an alternate proof of Th. 8.3.2, applying the implicit function theorem directly to the Poincaré map on any surface of section.

8.5. Bifurcation and transfer of stability for periodic solutions

If $x = x_0(t)$ is a p-periodic solution of $dx/dt + Ax = f(t,x)$ where $f(t+p,x) = f(t,x)$ (or perhaps, f is independent of t) we can study periodic solutions near $x_0(\cdot)$ by looking for fixed points of the Poincaré map (or period map). Thus the problem becomes that of existence, bifurcation, and stability of fixed points of a (nonlinear) smooth map of a Banach space or manifold into itself. This approach has been developed by several authors, from Poincaré and G. D. Birkhoff to Smale and his school, especially in the finite dimensional case. We refer to Lanford's excellent review [66] (of some results of Ruelle and Takens [85]) for several of the proofs below. Our exposition would not likely be an improvement, and some of the results are similar to those in sections 5.2 and 6.1.

<u>Theorem 8.5.1</u>. (Center-unstable manifold theorem [66]) Let Z be a real Banach space and suppose $\Phi \in C^k_{Lip}(Z)$ (i.e. Φ is a C^k map of Z to itself with Lipschitz continuous k^{th} derivatives) for some $1 \leq k < \infty$, on a neighborhood of the origin. Assume $\Phi(0) = 0$ and the derivative $\Phi'(0)$ has $\sigma(\Phi'(0)) \cap \{|\lambda| \geq 1\}$ a spectral set. Let $Z = X \oplus Y$ be the corresponding decomposition so the spectral radius $r(\Phi'(0)|_X) < 1$ while $\sigma(\Phi'(0)|_Y)$ is in $\{|\lambda| \geq 1\}$.

Assume also (what is trivial if Y is finite dimensional or has a smooth norm) that there exists r, $0 < r < 1$, and $\phi \in C^k_{Lip}(Y,\mathbb{R})$ with $0 \leq \phi(y) \leq 1$, $\phi(y) = 1$ if $\|y\| \leq r$ and $\phi(y) = 0$ if $\|y\| \geq 1$.

Then there exist $\varepsilon > 0$ and a C^k_{Lip} map u from $\{y \in Y \mid \|y\| < \varepsilon\}$ into X, $\|u(y)\| = O(\|y\|^2)$ as $y \to 0$, such that the manifold

$$\Sigma = \{y + u(y) \mid \|y\| < \varepsilon\} \subset X \oplus Y$$

satisfies:

(a) (local invariance) If $z = u(y) + y \in \Sigma$ and $\Phi(z) = z_1 = x_1 + y_1 \in X \oplus Y$ has $\|y_1\| < \varepsilon$, then $z_1 \in \Sigma$ (i.e. $x_1 = u(y_1)$).

(b) (locally attracting) If $z_{n+1} = \Phi(z_n)$ for $n = 1,2,\cdots,m$, $z_n = x_n + y_n \in X \oplus Y$ with $\|x_n\| < \varepsilon$, $\|y_n\| < \varepsilon$ for $n = 1,2,\cdots,m$, then $\|x_n - u(y_n)\| \leq C\alpha^n \|x_1 - u(y_1)\|$ for $n = 1,\cdots,m$. Here C, α are constants with $0 < \alpha < 1$.

<u>Remark</u>. In place of C^k_{Lip}, we may use $C^{k+\theta}$ smoothness throughout, $0 < \theta < 1$; compare with Thm. 6.1.7.

<u>Exercise 1</u>. Suppose Φ is continuous and $\Sigma = \{u(y)+y\colon \|y\| < \varepsilon_0\} \subset X \oplus Y$ is locally invariant and locally attracting under Φ. Assume $\bar{z} = \Phi(\bar{z}) \in \Sigma$ is asymptotically stable with respect to $\Phi|_\Sigma$, i.e. if $\|z - \bar{z}\| < \delta_0$ and $z \in \Sigma$, $\|\Phi^n(z) - \bar{z}\| \leq \theta_n \phi(\|z - \bar{z}\|)$ for all $n \geq 0$, where $\theta_n \to 0$ as $n \to \infty$, $\phi(\cdot)$ is continuous with $\phi(0) = 0$.

Then $\bar{z}$ is asymptotically stable under Φ. If $\bar{z}$ is unstable under $\Phi|_\Sigma$, then $\bar{z}$ is unstable. (Hint: Construct a Liapunov function, starting with theone given on Σ by ex. 1, sec. 4.2.)

To apply Th. 8.5.1 to perturbation and bifurcation problems, consider a one-parameter family of maps $\Phi_\mu : Z \to Z$ for each μ in $-\mu_0 \le \mu \le \mu_0$, assuming $(z,\mu) \to \Phi_\mu(z)$ is C^k_{Lip} near $(0,0) \in Z \times \mathbb{R}$ and $\Phi_0(0) = 0$. Suppose $\sigma(\partial\Phi_0(0)/\partial z) \cap \{|\lambda| \ge 1\}$ is a spectral set, and apply the theorem to the map $(z,\mu) \to (\Phi_\mu(z),\mu)$ on the Banach space $Z \times \mathbb{R}$. The center manifold, rather the μ-section of the center manifold is

$$\Sigma_\mu = \{u(y,\mu) + y,\ \|y\| < \varepsilon\}$$

which is locally invariant and locally attracting under Φ_μ, for each μ in $(-\varepsilon,\varepsilon)$. The stability properties of fixed points of Φ_μ (or of Φ_μ-invariant sets) in Σ_μ may be determined by studying $\Phi_\mu|\Sigma_\mu$, a smooth map of a manifold whose dimension is the same as the dimension of Y. The cases $\dim Y = \dim \Sigma_\mu = 1$ or 2 are most important, and are discussed below.

<u>Exercise 2</u>. Suppose Φ_μ as above, $\Phi_0(0) = 0$ and $\Phi_0'(0)$ has its spectrum strictly inside the unit circle (i.e., $\dim Y = 0$). Then for small $|\mu|$, Φ_μ has a unique fixed point $z(\mu)$ near the origin $z(\mu) \to 0$ as $\mu \to 0$, and $z(\mu)$ is asymptotically stable under Φ_μ.

<u>The case $\dim Y = 1$</u>.

Suppose that, for $\mu = 0$, $\Phi_0'(0)$ has 1 as a simple eigenvalue, with the remainder of the spectrum strictly inside the unit circle. (If -1 is simple in $\sigma(\Phi_0'(0)$, consider instead $\Phi_\mu^2 = \Phi_\mu \circ \Phi_\mu$). In this case $\Phi_{\mu|\Sigma_\mu}$ has the representation $y \to f(y,\mu)$, a real valued function of the real variables, y, μ, with $\frac{\partial f}{\partial y}(0,\mu) = \lambda(\mu)$, $\lambda(\mu) =$ the (simple, real) eigenvalue of $\Phi_\mu'(0)$ with $\lambda(\mu) \to 1$ as $\mu \to 0$. This case has been discussed in sec. 6.3 (and, more generally, by Crandall and Rabinowitz [19]). If $d\lambda/d\mu(0) > 0$ and

$$f(y,\mu) = f(0,\mu) + \lambda(\mu)y + by^m + o(|\mu y| + |y|^m)$$

$$\partial f/\partial y(y,\mu) = \lambda(\mu) + mby^{m-1} + o(|\mu| + |y|^{m-1})$$

as $(y,\mu) \to (0,0)$, for some $b \ne 0$ and integer $m \ge 2$ we have the following possibilities:

(i) $f(0,\mu) = a\mu + o(|\mu|)$, $a \ne 0$

(ii) $f(0,\mu) = a\mu^2 + o(\mu^2)$, $a \neq 0$, with $m = 2$

(α) $\Delta = (\lambda'(0))^2 - 4ab > 0$, (β) $\Delta < 0$, (γ) $\Delta = 0$

(iii) $f(0,\mu) = o(\mu^2)$ or $m > 2$ with

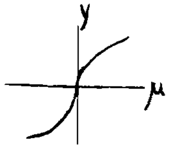
(α) m even

(β) m odd, $b > 0$

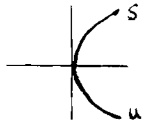
(γ) m odd, $b < 0$

The case (ii) (γ) with $\Delta = 0$ requires further information about f and will not be discussed.

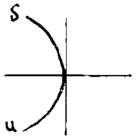
In case (i), there is a curve for fixed points $y(\mu) = c\mu^{1/m} + o(\mu^{1/m})$, $a + bc^m = 0$, of one of the forms



m odd



m even, ab < 0

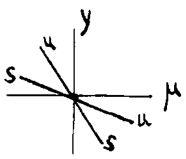


m even, ab > 0

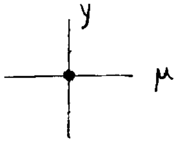
Here we suppose $a > 0$; if $a < 0$, interchange "s" and "u". In case m is odd, both branches are "s" if $ab < 0$, both "u" if $ab > 0$.

In case (ii) (α) there are two curves of fixed points, and (ii) (β) there are no fixed points

(α)

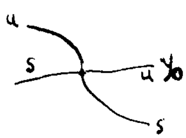


(β)

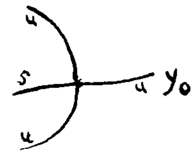


In (ii) (α), interchange "s" and "u" if $b < 0$.

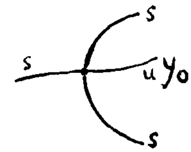
In cases (iii) there is a curve of fixed points $y_0(\mu) \simeq f(0,\mu)/\lambda(\mu) = O(|\mu|)$ which plays the role of the trivial solution in lemma 6.3.1 so we have



(α)



(β)



(γ)

Here the branches have been marked "stable" or "unstable" ("s" or "u") according to whether $\partial f/\partial y < 1$ or > 1 at that point, i.e. according to their stability or instability under iteration of $f(\cdot,\mu)$ (of $\Phi_\mu|\Sigma_\mu$).

Remark. If $\frac{d\lambda(0)}{d\mu} = 0$, bifurcation and stability questions may still generally be settled by use of Newton's polygon, but the branching may be more complicated than these diagrams indicate.

Remark. P. Brunovsky [9] studied some generic cases of bifurcation of diffeomorphisms; it may be reassuring to know we will cover all the generic possibilities listed there.

Exercise.3. Suppose $f(t+2\pi,x) = f(t,x)$, $a > 0$, and $|\varepsilon| << 1$; consider

$$u_t = u_{xx} + u - au^3 + \varepsilon f(t,x)u, \quad (0 < x < \pi,\ t > 0)$$
$$u(0,t) = 0, \quad u(\pi,t) = 0.$$

Assume $\beta \equiv \int_0^{2\pi} dt \int_0^{\pi} dx f(t,x)\sin^2 x \neq 0$; compute the Poincaré map in the critical manifold and conclude that when $\beta > 0$, the zero solution loses stability for small positive ε and a pair of asymptotically stable 2π-periodic solutions branch off from the zero solution. (Actually, one solution is the negative of the other.) If $\beta < 0$, simply reverse the sign of ε.
(Hint: In view of ex. 3, sec. 7.2, the critical manifold has the form $u = s \sin x + O(|s|^3+|\varepsilon s|)$ and the restriction of the Poincaré map is $s \to h(s,\varepsilon)$ where

$$h(0,\varepsilon) = 0,\ \frac{\partial h}{\partial s}(0,\varepsilon) = 1 + \frac{2\beta\varepsilon}{\pi} + O(\varepsilon^2)$$

$$h(s,0) = s - \frac{3\pi}{2} as^3 + O(s^5)$$

<u>Exercise 4</u>. (<u>Entrainment of frequency</u>) Suppose $x_0(\cdot)$ is a nonconstant p-periodic solution of $dx/dt + Ax = f(x)$, which has a simple multiplier {1} and the remaining eigenvalues strictly inside the unit circle.

Consider the equation

$$dx/dt + Ax = f(t) + \varepsilon g(t,x,\varepsilon)$$

where f,g are smooth on $\mathbb{R} \times X^{\alpha} \times (-\varepsilon_0,\varepsilon_0) \to X$, $g(t+p(\varepsilon),x,\varepsilon) = g(t,x,\varepsilon)$, with $p(\varepsilon)$ smooth, $p(\varepsilon) = p + O(\varepsilon)$. Also define $y^*(t)$ by $(T_0(t,s))^*y^*(t) = y^*(s)$, $y^*(s+p) = y^*(s)$, $\langle y^*(s),\dot{x}_0(s)\rangle = 1$ for all s. Assume $\int_0^p \langle y^*(s),g(s,x_0(s),0)\rangle ds \neq 0$ and $\int_0^p \langle y^*(s),f'''(x_0(s))(\dot{x}_0(s))^3\rangle ds \neq 0$.

Then for small ε, there exists a unique $p(\varepsilon)$-periodic solution near $x_0(\cdot)$ depending continuously on ε, which is asymptotically stable for small $\varepsilon \neq 0$. (Note that $\int_0^p \langle y^*(s),f''(x_0(s))(\dot{x}_0(s))^2\rangle ds=0$.)

<u>The case n = 2</u>.

Now suppose that, for $\mu = 0$, $\Phi_0'(0)$ has two simple eigenvalues $\{e^{\pm i\theta_0}\}$ on the unit circle, with the remainder of the spectrum strictly inside the unit circle. If $e^{i\theta_0 m} = 1$ for some integer $m \geq 3$, then the operator Φ_μ^m has, for $\mu = 0$, a double eigenvalue at 1; this case is discussed briefly below, as it is important in the study of subharmonic resonance. However, for the present we shall suppose $e^{i\theta_0}$ is not a (small) root of unity, and state a result of Ruelle and Takens [85] which give a useful canonical form for 2-dimensional maps such as $\Phi_{\mu|\Sigma_\mu}$. Lanford [66] gives an elementary proof which easily extends to prove our slight generalization.

<u>Theorem 8.5.2</u>. Suppose $F_\mu: \mathbb{R}^2 \to \mathbb{R}^2$ has $F_\mu(0) = 0$ and the spectrum of $F_\mu'(0)$ is $\{\lambda(\mu),\overline{\lambda}(\mu)\}$ where $\lambda(0) = e^{i\theta_0}$, θ_0 real, and $e^{im\theta_0} \neq 1$ for $m = 1,2,\ldots,N+1$ $(N \geq 4)$. Assume $(\mu,x,y) \to F_\mu(x,y)$ is N times continuously differentiable in a neighborhood of the origin. There is a C^N μ-dependent change of variables in $\mathbb{R}^2$ which bring F_μ into the form (in polar coordinates)

$$F_\mu: (r,\phi) \to (|\lambda(\mu)|r + \sum_1^{\nu} f_m(\mu)^{2m+1} + o(r^N),\ \phi+\theta(\mu)+\sum_1^{\nu} g_m(\mu)r^{2m} + o(r^{N-1}))$$

where $\lambda(\mu) = |\lambda(\mu)|e^{i\theta(\mu)}$, $\nu = [\frac{N-1}{2}]$, and $f_m(\cdot)$, $g_m(\cdot)$ are real C^N functions of μ for μ near 0.

Using this canonical form, it is not difficult to prove the existence of invariant circle: we refer again to Lanford [66] who proves the case $N = 4$, but the proof again generalizes.

Theorem 8.5.3. Assume F_μ as in Th. 8.5.2 above, $N \geq 4$, and assume also that $\frac{d}{d\mu}|\lambda(\mu)| > 0$ at $\mu = 0$ (so the origin loses stability) and that, in the canonical form, some $f_k(0) \neq 0$ with $1 \leq k < \frac{N+1}{4}$, but $f_m(0) = 0$ for $1 \leq m < k$.

If $f_k(0) < 0$, then for small $\mu > 0$ there exists a unique closed invariant curve Γ_μ encircling the origin, $F_\mu(\Gamma_\mu) = \Gamma_\mu$, $\Gamma_\mu \to 0$ as $\mu \to 0+$, and there is no invariant set near the origin for small $\mu \leq 0$, except the origin itself. Also, Γ_μ is attracting: in a neighborhood of the origin independent of μ, if $z \neq 0$ then $F_\mu^n(z) \to \Gamma_\mu$ as $n \to \infty$, provided $\mu > 0$, but $F_\mu^n(z) \to 0$ if $\mu \leq 0$.

If $f_k(0) > 0$, there exists a small invariant repelling curve for small $\mu < 0$ - in fact, the first case applies to F_μ^{-1}.

Remark. The condition that $f_k(0) \neq 0$ for some $k < (N+1)/4$ may be equivalently stated: the stability properties of the origin under F_0 are independent of the terms of order $(|x| + |y|)^q$ for some $q < (N+3)/2$.

The "flow" on Γ_μ is a rotation $\phi \mapsto \phi + \theta(0) + o(1)$ as $\mu \to 0$, and may be considered as a smooth diffeomorphism of the circle S^1 to itself, without fixed points (when μ is small). Generic diffeomorphisms of the circle have been studied in the theory of structural stability and the basic result (Peixoto, 1962; see [76, p. 51]) is that a dense, open subset of $\mathrm{Diff}(S^1)$ consists of those diffeomorphisms such that all (finitely many) non wandering points are periodic, and all periodic points are hyperbolic. Thus one might expect that, for most maps F_μ (or Φ_μ) encountered in practical situations, the invariant curve Γ_μ would be replaced, in effect, by a finite number of periodic points $\{p_1(\mu),\ldots,p_k(\mu)\}$ on Γ_μ; this was conjectured by Ruelle and Takens [85]. However, since Γ_μ is so to speak adapted to the particular F_μ, it is not clear that a generic choice of F_μ gives a generic restriction $F_\mu|_{\Gamma_\mu}$, and the question remains open. That it is not without interest is shown by the formal computations of D. Joseph [54] which suggest that quasi-periodic solutions with two basic frequencies may bifurcate from a periodic solution of the (periodically forced) Navier-Stokes equation, when the solution loses stability by a pair of multipliers moving through the unit circle at points which are not roots of unity.

Since the rotation number $\rho(\mu)$ depends continuously on μ for μ near zero (as follows from the uniform convergence proved in, for example, [37, p. 67]) if $\rho(0) = \frac{\theta(0)}{2\pi}$ is irrational, then $\rho(\mu)$ must be irrational for nonzero μ arbitrarily close to zero - thus almost periodic solutions must exist. It remains to be shown whether the series obtained by Joseph do have any validity; the difficulty is the classical one of "small divisors", and may perhaps be resolved in the same way.

In any case, if (notation of Th. 8.5.2) $e^{im\theta_0} \neq 1$ for $m = 1,2,\cdots,M$, then $F_\mu^m|_{\Gamma_\mu}$ can have no fixed points for $m = 1,\ldots,M$ if μ is sufficiently small. Thus if $p(\mu)$ is a nonzero periodic point on Γ_μ, its period must be greater than M and $\{F_\mu^m(p(\mu)), 0 \leq m \leq M\}$ are all distinct periodic points, distributed fairly uniformly about Γ_μ. Thus if M is large, the invariant curve retains its identity in any practical sense, and the periodic motions with long period would be practically indistinguishable from almost periodic motions. As μ gets larger, however, these arguments lose their force and one might expect the curve Γ_μ to degenerate; A. Stokes and others [7] studied such a problem for two coupled van der Pol equations.

Exercise 5. Examine the problem of bifurcation of a periodic orbit from an equilibrium point (see sec. 6.4) using the results of this section. (See [84].)

Now we examine the case when $F_0(0) = 0$ and $F_\mu'(0)$ has simple complex eigenvalues $\lambda(\mu),\overline{\lambda}(\mu)$ with $\lambda(0) = e^{i\theta_0}$, θ_0 real, $e^{im\theta_0} = 1$ for some integer $m \geq 3$, and $\frac{d}{d\mu}|\lambda(\mu)| > 0$ at $\mu = 0$.

First observe, that, by the implicit function theorem, there exists a unique fixed point $z(\mu) = F_\mu(z(\mu))$ for small μ, $z(\mu) = O(|\mu|)$ as $\mu \to 0$.

Exercise 6. Prove the eigenvalues $\lambda_1(\mu)$, $\overline{\lambda}_1(\mu)$ of $F_\mu'(z(\mu))$ have $\lambda_1(\mu) = \lambda(\mu) + O(\mu^2)$ as $\mu \to 0$. Thus we may, without loss of generality, assume $F_\mu(0) = 0$ for all small μ.

Now $\lambda(\mu) = e^{i\theta_0} + \ell\mu + O(\mu^2)$ and $\mathrm{Re}(e^{-i\theta_0}\ell) > 0$, so the eigenvalues $\lambda^m(\mu),\overline{\lambda}^m(\mu)$ of $(F_\mu'(0))^m$ have

$$\lambda^m(\mu) = 1 + me^{-i\theta_0}\ell\mu + O(\mu^2).$$

It follows that $G_\mu(x) \equiv F_\mu^m(z)$ has the form

$$G_\mu(z) = z + \mu Mz + C_n(z) + o(|\mu|\ |z| + |z|^n)$$

where M is a 2×2 matrix whose eigenvalues have real part positive, and $C_n(\cdot)$ is an n^{th} order homogeneous polynomial, the first nontrivial term in the Taylor series for $G_0(z)-z$.

Let $z = \mu^p\zeta$, $p = 1/(n-1)$, and consider the equation

$$0 = [G_\mu(z)-z]\mu^{-n/n-1} = M\zeta + C_n(\zeta) + o(1) \quad \text{as} \quad \mu \to 0.$$

Suppose there exists a nonzero real ζ_0 such that $M\zeta_0 + C_n(\zeta_0) = 0$, and suppose further that this is a simple root: $\det\{M+C_n'(\zeta_0,\cdot)\} \neq 0$. Then by the implicit function theorem, there exists a unique fixed point $z(\mu) = G_\mu(z(\mu)) = \zeta(\mu)\mu^{1/n-1}$ such that $\zeta(\mu) \to \zeta_0$ as $\mu \to 0$.

Exercise 7. Assume $v \in \mathbb{R}^2$, $C_n(v) = 0$ only when $v = 0$. Prove that any fixed point $z(\mu) = G_\mu(z(\mu))$ in $\mathbb{R}^2$ with $z(\mu) \to 0$ as $\mu \to 0$ must satisfy $z(\mu)\mu^{-1/(n-1)} \to \{\zeta \in \mathbb{R}^2 \mid M\zeta + C_n(\zeta) = 0\}$. If $z(\mu)\mu^{-1/(n-1)} \to 0$ as $\mu \to 0$, show $z(\mu) = 0$ for small $\mu \neq 0$. What is the conclusion when $\zeta \in \mathbb{R}^2$, $M\zeta + C_n(\zeta) = 0$ only for $\zeta = 0$?

Exercise 8. Suppose $M\zeta_0 + C_n(\zeta_0) = 0$, $\zeta_0 \neq 0$, but $L_1 \equiv M + C_n'(\zeta_0,\cdot)$ has rank 1 ($\det L_1 = 0$, $L_1 \neq 0$). Substitute $z = \mu^{1/(n-1)}(\zeta_0+\eta)$ and obtain

$$0 = L_1\eta + H_1(\eta,\mu^{1/(n-1)})$$

where $H_1(\eta,0) = o(|\mu|^2)$ and H_1 is a smooth function of its arguments near 0. Applying elementary row operations, we may assume one row of L_1 is zero; the other row may be solved for one of the components, say η_1, as a smooth function of η_2 and $\mu^{1/(n-1)}$. Substituting this function in the remaining equation, we have one equation in two variables, η_2 and $\mu^{1/(n-1)}$. Then the usual Newton polygon may be applied [21, 48].

Observe that, after scaling variables in the manner of Newton's polygon, such bifurcation problems reduce generally to the determination of all real solution of polynomial systems such as

$M\zeta_0 + C_n(\zeta_0) = 0$ (two n^{th} order polynomials in the two components of ζ_0). This is a purely algebraic problem whose general solution (Kronecker elimination) is described by van der Waerden [101]. For the simple cases ordinarily encountered, it is generally easier to determine the highest common factor, considering these equations as polynomials in (say) the first component ζ_0^1, whose coefficients are polynomials in ζ_0^2. The requirement that this highest common factor depends on ζ_0^1 (i.e., that a solution exists) gives a polynomial equation in the single variable ζ_0^2.

An introduction to bifurcation problems is presented in Stakgold's article [95], and references given there; illustrations of some of the difficulties of bifurcation at multiple eigenvalues are given in [75] and [84].

We summarize our results in two theorems, formulating the problem in terms of the differential equation.

<u>Theorem 8.5.4</u>. (The nonautonomous case). Let A be sectorial in X, $\alpha < 1$, and let $f: \mathbb{R} \times X^\alpha \times (-\varepsilon_0, \varepsilon_0) \to X$ be C^2 and N times continuously differentiable in x, with $f(t+p,x,\varepsilon) = f(t,x,\varepsilon)$ for all t,x,ε and some fixed $p > 0$. Assume $x_0(\cdot)$ is a p-periodic solution of $dx/dt + Ax = f(t,x,0)$ and assume its linear variational equation $dy/dt + Ay = f_x(t,x_0(t),0)y$ has all multipliers strictly inside the unit circle, with the exception of a simple multiplier at 1 or -1, or a pair of simple multipliers $\{e^{\pm i\theta_0}\}$. Assume the stability or instability of $x_0(\cdot)$ is independent of terms of order $O(\|x-x_0(t)\|_\alpha^q)$ for some $q \le N$.

(A) Suppose 1 is a simple multiplier. Let $\phi(t)$, $\psi(t)$ be p-periodic solutions of $\dot{\phi} + A(t)\phi = 0$, $-\dot{\psi} + A(t)^*\psi = 0$, $(A(t) = A - f_x(t,x_0(t),0))$ with $\langle\psi(t),\phi(t)\rangle = 1$.

(i) If $a_1 = \int_0^p \langle\psi(s), f_\varepsilon(s,x_0(s),0)\rangle ds \neq 0$, where $f_\varepsilon = \frac{\partial f}{\partial \varepsilon}$, then we have either one-sided bifurcation of a stable-unstable pair of p-periodic solutions for small $\varepsilon > 0$ (or < 0) with no p-periodic solutons near x_0 for small $\varepsilon < 0$ (or > 0, respectively); or else there is a unique p-periodic solution $x_\varepsilon(t)$ near $x_0(t)$ for small $|\varepsilon|$, having the same stability properties as x_0. (When x_0 is asymptotically stable, the last case occurs.)

(ii) If $a_1 = 0$,

$$a_2 = \frac{1}{2}\int_0^p <\psi(s), f^0_{\varepsilon\varepsilon}(s) + 2f^0_{x\varepsilon}(s)x_1(s) + f^0_{xx}(s)x_1(s)^2> ds$$

where

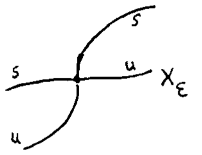
$$g^0(s) \equiv g(s, x_0(s), 0) \quad \text{and} \quad \dot{x}_1 + A(t)x_1 = f^0_\varepsilon(t),\ x_1(0) = 0$$

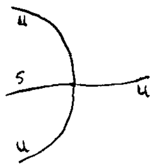
$$b = \int_0^p <\psi(s), f^0_{x\varepsilon}(s)\phi(s) + f^0_{xx}(s)\phi(s)x_1(s)> ds$$

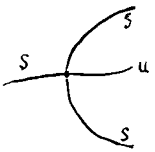
$$c = \frac{1}{2}\int_0^p <\psi(s), f^0_{xx}(s)\phi(s)^2> ds$$

then if $b^2 < 4a_2c$, there are no p-periodic solutions near x_0 for small $\varepsilon \neq 0$, but if $b^2 > 4a_2c$ with $a_2c \neq 0$, there is a stable-unstable pair of p-periodic solutions for each small $\varepsilon \neq 0$. (If x_0 is asymptotically stable, $c = 0$ and this case does not occur.)

(iii) If $b \neq 0$, $a_1 = 0$ and $a_2c = 0$, there is a p-periodic solution $x_\varepsilon(t)$ which is a differentiable function of ε near 0 and which changes stability as ε passes through 0, and there is also a bifurcating branch of p-periodic solutions according to one of the schemes (with ε the horizontal coordinate):

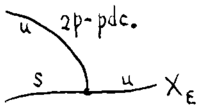


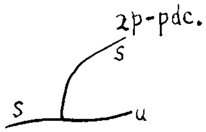




If x_0 is asymptotically stable, the last case occurs. If $\lambda(\varepsilon)$ is the multiplier of x_ε near 1, then $b \neq 0$ is equivalent to $\frac{d\lambda}{d\varepsilon}(0) \neq 0$.

(B) Suppose {-1} is a simple multiplier. Then there is a smooth curve x_ε of p-periodic solutions, $x_\varepsilon \to x_0$ as $\varepsilon \to 0$, and we suppose this changes stability as ε crosses 0. In fact, assume the critical multiplier $\lambda(\varepsilon)$, $\lambda(0) = -1$, has $d\lambda(\varepsilon)/d\varepsilon \neq 0$ at $\varepsilon = 0$. Then in addition to x_ε there is one-sided bifurcation of a 2p-periodic solution according to one of the schemes





(If $x(t)$ is a 2p-periodic solution, so is $x(t+p)$, but we don't distinguish between these.) The second case occurs when x_0 is asymptotically stable.

(C) If $\{e^{\pm i\theta_0}\}$ are simple multipliers with $e^{im\theta_0} \neq 1$ for $1 \leq m \leq N+1$, $N \geq 4$, and $q < (N+3)/2$, there is a unique p-periodic solution $x_\varepsilon(t)$, depending smoothly on ε. We suppose this solution changes stability as ε passes through 0 and the critical multipliers $\lambda(\varepsilon)$, $\overline{\lambda(\varepsilon)}$ (with $\lambda(0) = e^{i\theta_0}$) satisfy $\frac{d}{d\varepsilon}|\lambda(\varepsilon)| > 0$ when $\varepsilon = 0$.

If x_0 is asymptotically stable, there exists for small $\varepsilon > 0$ an attracting invariant tube $T_\varepsilon \subset \mathbb{R} \times X^\alpha$, a p-periodic manifold whose cross-section is a closed curve, $T_\varepsilon \to \{(t,x_0(t))\}$ as $\varepsilon \to 0+$. If $\theta_0/2\pi$ is irrational, there exist almost periodic solutions for arbitrary small $\varepsilon > 0$. In any case, there may be subharmonics of order greater than $N+1$ for small $\varepsilon > 0$. For small $\varepsilon < 0$, the only solution which remains near $\{x_0(t), -\infty < t < \infty\}$ for all t is $x_\varepsilon(t)$.

If x_0 is unstable then T_ε exists for small $\varepsilon < 0$ and is repelling, but otherwise the case is similar to that above.

<u>Theorem 8.5.5</u>. (The autonomous case). Suppose A is sectorial, $\alpha < 1$, $f: X^\alpha \times (-\varepsilon_0,\varepsilon_0) \to X$ is N times continuously differentiable ($N \geq 2$) and x_0 is a nonconstant p-periodic solution of $dx/dt + Ax = f(x,0)$ whose stability or instability is independent of terms which vanish to order q on the orbit $\{x_0(t), 0 \leq t \leq p\}$ for some $q \leq N$. Assume that, in addition to the obvious multiplier $\{1\}$, corresponding to $\dot{x}_0(t)$, there are one real or two complex other multipliers on the unit circle with the remainder of the spectrum strictly inside the unit circle. We study a neighborhood of $\{x_0(t), 0 \leq t \leq p\}$ for $dx/dt + Ax = f(x,\varepsilon)$ with $|\varepsilon|$ small.

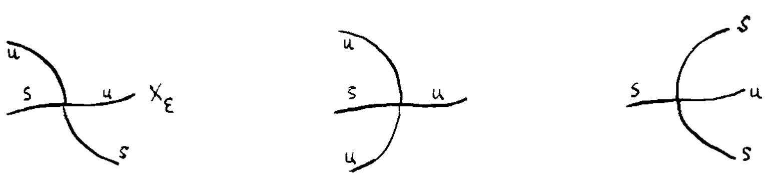
(A) If $\{1\}$ is a double eigenvalue of the period map with two independent eigenvectors, there is a p-periodic solution $\psi(t) \neq 0$ of

$-\dot{\psi} + A(t)^*\psi = 0$, $A(t) = A - f_x(x_0(t),0)$, with $\langle\psi(t),\dot{x}_0(t)\rangle = 0$. If

$$(*) \qquad \int_0^p \langle\psi(s), \frac{\partial f}{\partial\varepsilon}(x_0(s),0)\rangle ds \neq 0$$

then as ε varies we have either one-sided bifurcation of a stable-unstable pair of periodic solutions with period close to p, or a unique periodic solution x_ε near x_0 with period near p for each small $|\varepsilon|$, depending continuously on ε, with the same stability properties as x_0. If x_0 is orbitally asymptotically stable, the second case occurs.

When the integral (*) vanishes, there are many possibilities and we consider only the case when there is a curve of periodic solutions $x_\varepsilon(t)$ which is differentiable in ε and has period close to p for each small $|\varepsilon|$. Suppose the multipliers $\{1,\lambda(\varepsilon)\}$ near the unit circle, $\lambda(0) = 1$, satisfy $d\lambda/d\varepsilon \neq 0$ when $\varepsilon = 0$. Then we have bifurcation of periodic solutions according to one of the schemes



where ε is the horizontal coordinate, and all the solutions have period close to p. When x_0 is asymptotically stable, the last case occurs.

(B) If $\{1,-1\}$ are both simple eigenvalues, there is a differentiable curve $\varepsilon \to x_\varepsilon$ of periodic solution (period near p) with a multiplier $\lambda(\varepsilon)$, $\lambda(0) = -1$; assume $d\lambda/d\varepsilon \neq 0$ when $\varepsilon = 0$. Then there is bifurcation of periodic solutions with period approximately $2p$ according to one of the following

When x_0 is asymptotically stable, the first case occurs.

(C) Suppose $\{1, e^{i\theta_0}, e^{-i\theta_0}\}$ are simple multipliers with $e^{im\theta_0} \neq 1$ for $m = 1,2,\dots,N+1$ and $N \geq 4$, $q < (N+3)/2$. There is a unique periodic solution x_ε with period near p depending smoothly on ε whose multipliers near the unit circle are $\{1, \lambda(\varepsilon), \overline{\lambda(\varepsilon)}\}$, $\lambda(0) = e^{i\theta_0}$; assume $\frac{d}{d\varepsilon}|\lambda(\varepsilon)| > 0$ when $\varepsilon = 0$.

(i) If x_0 is orbitally asymptotically stable, then for small $\varepsilon > 0$ an attracting invariant two-dimensional torus bifurcates from the orbit $\{x_0(t),\ 0 \leq t \leq p\}$; in a neighborhood of this orbit, the maximal invariant set is the orbit of x_ε if $\varepsilon \leq 0$, or this together with the torus and the orbits joining these when $\varepsilon > 0$.

(ii) If x_0 is unstable, a repelling invariant torus bifurcates from the orbit for small $\varepsilon < 0$.

Chapter 9

The Neighborhood of an Invariant Manifold

9.1 Existence, stability and smoothness of invariant manifolds

We will prove a rather general (and lengthy) invariant manifold theorem. The invariant manifold might be considered a perturbation of the manifold $x = 0$ for $\dot{x} + A(t,y(t))x = 0$, $\dot{y} = g_0(t,y)$, $-\infty < t < \infty$. We assume, for each solution y, that the x-equation has an exponential dichotomy on $\mathbb{R}$ with exponent $\beta > 0$ and bound independent of y, and β is also greater than the exponential rate at which solutions of the y-equation separate as $t \to \pm\infty$. With certain smoothness and boundedness assumptions, we show

$$\dot{x} + A(t,y)x = f(t,x,y), \quad \dot{y} = g(t,x,y), \quad -\infty < t < \infty$$

has an invariant manifold $x = \sigma(t,y)$ near $x = 0$ provided f, $\partial f/\partial x$ and $g-g_0$ are integrally-small at $x = 0$, uniformly in y. This invariant manifold has the same stability properties as $x = 0$ in the unperturbed equation. In light of the many applications of the simpler results of Ch. 6, and also the examples below, I may hope to be forgiven a long theorem and longer proof.

Several parts of the argument follow Coppel and Palmer [ ]; the connection would be much closer if A were independent of y, but this is not true in some of our examples. The smoothness argument also differs from [ ] and is simpler even for finite dimensions.

In section 9.2 we introduce a coordinate system near a given invariant manifold and in 9.3 apply the results of sec. 9.1 to the resulting system and its perturbations.

Theorem 9.1.1. Let X, Y be Banach spaces, $0 \le \alpha < 1$, and suppose A_0 is sectorial in X, $(t,y) \to A(t,y)-A_0: \mathbb{R} \times Y \to \mathscr{L}(X^\alpha,X)$ is bounded, locally Hölder continuous in t and differentiable in y. U is a neighborhood of 0 in X^α and $(f,g,g_0): \mathbb{R} \times U \times Y \to X \times Y \times Y$ are bounded, locally Hölder continuous in t, and differentiable in $(x,y) \in U \times Y$, with $g_0(t,y)$ independent of x. Assume the following:

(i) There exist $\mu \geq 0$, $N \geq 1$ such that, if $y_1(t)$, $y_2(t)$ are both solutions of $\dot{y} = g_0(t,y)$ on $-\infty < t < \infty$, then $\|y_1(t)-y_2(t)\| \leq Ne^{\mu|t-\tau|} \|y_1(\tau)-y_2(\tau)\|$ for all t,τ.

(ii) There exists $\beta > \mu$ such that, whenever $\dot{y} = g_0(t,y)$, $-\infty < t < \infty$, the equation

$$\dot{x} + A(t,y(t))x = 0$$

has an exponential dichotomy on $\mathbb{R}$ with exponent β and bound N.

(iii) With $f_x = \partial f/\partial x$, etc., f, g, g_x, g_y, g_{0y}, f_x, f_y, $A-A_0$ and A_y (in $\mathscr{L}(Y,\mathscr{L}(X^\alpha,X))$) are uniformly bounded by N on $\mathbb{R} \times U \times Y$, and for some θ with $0 < \theta \leq 1$, $\mu(1+\theta)<\beta$,

$$(x,y) \to f_x,\ f_y,\ g_x,\ g_y,\ g_{0y} \text{ and } A_y$$

satisfy a uniform Hölder condition with exponent θ and bound N.

(iv) If h is f or f_x or $g-g_0$, then

$$\left\|\int_{t_1}^{t_2} h(t,0,y)\,dt\right\| \leq q$$

whenever $|t_1-t_2| \leq 1$ and y is in Y.

Then there exist positive constants q_0, r_0 depending only on A_0, α, θ, β, $\varepsilon \equiv \beta-\mu(1+\theta) > 0$ and N, such that when $q \leq q_0$ there is an invariant manifold

$$S = \{(t,x,y) \mid x = \sigma(t,y),\ (t,y) \in \mathbb{R} \times Y\}$$

for $\dot{x} + A(t,y)x = f(t,x,y)$, $\dot{y} = g(t,x,y)$ which is the maximal invariant subset of

$$\{(t,x,y) \mid \|x\|_\alpha \leq r_0,\ (t,y) \in \mathbb{R} \times Y\}.$$

Uniformly in t,y and $0 \leq q \leq q_0$, $y \to \sigma(t,y)$ is differentiable from Y to X^α and its derivative satisfies a uniform Hölder continuous with exponent θ, and as $q \to 0$

$$\sigma(t,y) \to 0, \qquad \frac{\partial\sigma}{\partial y}(t,y) \to 0$$

uniformly on $\mathbb{R} \times Y$.

If $t \to A(t,y) - A_0$, $f(t,x,y)$, $g(t,x,y)$ are periodic with period $p > 0$ (or constant, or uniformly almost periodic) then $t \to \sigma(t,y)$ is also p-periodic (or constant, or uniformly almost periodic with frequency module contained in the joint module of A, f, g). If A, f, g are unchanged whenever y is replaced by $y + \omega$ (for some fixed $\omega \in Y$) then $\sigma(t,y+\omega) = \sigma(t,y)$.

If the zero solution of $\dot{x} + A(t,y(t))x = 0$ is stable for some (hence, every) solution y of $\dot{y} = g_0(t,y)$, the invariant manifold is asymptotically stable with asymptotic phase. Specifically if $\beta_0 = \beta - \frac{\varepsilon}{4}$, any solution $x(t)$, $y(t)$ for $t \geq \tau$ with $\|x(\tau)\|_\alpha \leq r_0/8N$ exists with $\|x(t)\|_\alpha \leq r_0$ for all $t \geq \tau$, and there is a unique solution $(\overline{x}(t),\overline{y}(t))$ in S $(\overline{x}(t) = \sigma(t,\overline{y}(t))$ with

$$\|x(t)-\overline{x}(t)\|_\alpha + \|y(t)-\overline{y}(t)\| \leq Ke^{-\beta_0(t-\tau)}\|x(\tau)-\sigma(\tau,y(\tau))\|_\alpha$$

for all $t \geq \tau$. The constant K depends only on A_0, α, β, θ, ε and N.

If $\dot{x} + A(t,y(t))x = 0$ is unstable for some solution $y(t)$ of $\dot{y} = g_0(t,y)$, then S is unstable. In fact any solution $x(t),y(t)$ $(t \geq \tau)$ with $x(\tau)$ in an open dense set of $\{\|x\|_\alpha \leq r_0/16N^2\}$, depending on $(\tau,y(\tau))$, must eventually have $\|x(t)\|_\alpha > r_0$.

<u>Remark</u>. We shall assume without loss of generality that the dichotomy in (ii) is with respect to the space X^α (see ex. 5, sec. 7.6) and the bounds in (iv) also apply when $h = f_y$ or $\frac{\partial}{\partial y}(g-g_0)$. Indeed (iii) and (iv) already imply this with q replaced by $(N+2)q^{\theta/(1+\theta)}$; for example, $|t_1-t_2| \leq 1$, $\eta \in Y$, imply

$$\left\|\int_{t_1}^{t_2} f_y(t,0,y)\eta dt\right\| \leq N\|\eta\|^{1+\theta} + 2q = (N+2)q^{\theta/(1+\theta)}\|\eta\|$$

if $\|\eta\|^{1+\theta} = q$.

If $Y_0 \subset Y$ is convex and invariant under $\dot{y} = g_0(t,y)$ and $\dot{y} = g(t,x,y)$ for each $x \in U$, and the assumptions (i)-(iv) hold only for $y \in Y_0$, the same arguments give an invariant manifold $x = \sigma(t,y)$ as a graph over $\mathbb{R} \times Y_0$. If Y_0 is not convex, but is a $C^{1+\theta}$ submanifold of Y such that any points y_1, y_2 of Y_0 can be joined by a curve in Y_0 of length $\leq N\|y_1-y_2\|$, only slight changes in the arguments (and estimates) are required. In either case, there is a slight weakening of the smoothness condition (unless $\theta = 1$) near the boundary of Y_0, if any (see remark following proof of Lemma 9.1.8).

A simple example of the unpleasant consequences when $\beta \leq \mu$ is in Hale [37]. For $\beta < \mu$, it can be even worse: see Jarnik and Kurzweil [122].

<u>Theorem 9.1.2</u>. With the hypotheses and notation of Thm. 9.1.1, if $q \leq q_0$, then for every $(\tau,\eta) \in \mathbb{R} \times Y$ there is a unique solution of

$$\dot{x} + A(t,y)x = f(t,x,y), \quad \dot{y} = g(t,x,y), \quad -\infty < t < \infty,$$

with $y(\tau) = \eta$ and $\|x(t)\|_\alpha \leq r_0$ for all t. Denoting this solution $x(t;\tau,\eta)$, $y(t;\tau,\eta)$, we have $\eta \to x(t;\tau,\eta)$, $y(t;\tau,\eta)$ differentiable,

$$\|x(t;\tau,\eta)\|_\alpha \leq Kq^{\delta/2}, \quad \left\|\frac{\partial y}{\partial \eta}(t;\tau,\eta)\right\|_{\mathscr{L}(Y)} \leq Ke^{\gamma|t-\tau|},$$

$$\left\|\frac{\partial x}{\partial \eta}(t;\tau,\eta)\right\|_{\mathscr{L}(Y,X^\alpha)} \leq Kq^{\delta^2\theta/2}e^{\gamma|t-\tau|},$$

$$\left\|\frac{\partial x}{\partial \eta}(t;\tau,\eta_2) - \frac{\partial x}{\partial \eta}(t;\tau,\eta_1)\right\|_{\mathscr{L}(Y,X^\alpha)}$$

$$+ \left\|\frac{\partial y}{\partial \eta}(t;\tau,\eta_2) - \frac{\partial y}{\partial \eta}(t;\tau,\eta_1)\right\|_{\mathscr{L}(Y)} \leq K\|\eta_2-\eta_1\|^\theta e^{\gamma(1+\theta)|t-\tau|}$$

where $\gamma = \mu + \frac{\varepsilon}{4}$, $\delta = (1-\alpha)/2$. The constant K depends only on A_0, α, β, θ, ε and N.

This result implies the existence and smoothness of the invariant manifold of Thm. 9.1.1; the other properties are relegated to exercises (ex. 2, 3, 4, below). Define $\sigma(\tau,\eta) = x(\tau;\tau,\eta)$ for $(\tau,\eta) \in \mathbb{R} \times Y$, the claims regarding smoothness follow immediately. If $\dot{x} + A(t,y) = f(t,x,y)$, $\dot{y} = g(t,x,y)$ for all t and $\|x(t)\|_\alpha \leq r_0$ for all t, the uniqueness assertion of Thm. 9.1.2 implies

$$x(t) = x(t;\tau,y(\tau)), \quad y(t) = y(t;\tau,y(\tau)) \quad \text{for all } t,\tau$$

and in particular $x(\tau) = \sigma(\tau,y(\tau))$ for all τ. Conversely if $x(\tau) = \sigma(\tau,y(\tau))$ for some τ, the solution with this initial value exists for all time and has $x(t) = \sigma(t,y(t))$ for all t. Thus $S = \{(t,x,y) \mid x = \sigma(t,y)\}$ is invariant and the maximal invariant subset of $\mathbb{R} \times \{\|x\|_\alpha \leq r_0\} \times Y$.

The proof of Thm. 9.1.2 is broken into a series of lemmas (9.1.3 - 9.1.8) designated (a) - (f) for brevity. We first state the lemmas, show they prove Thm. 9.1.2, then prove the lemmas. In the following, $C_1, C_2, \ldots$ are positive constants (independent of q,r)

depending only on A_0, α, β, θ, ε and N, and "sufficiently small" means being less than such a constant.

(a) <u>Lemma 9.1.3</u>. Suppose $\tilde{\mu} > \mu$, $y: \mathbb{R} \to Y$ is differentiable with $\|\dot{y}(t)\| \le N$ and $\|\dot{y}(t) - g(t,0,y(t))\| \le \Delta$ for all t, and $z: \mathbb{R} \to \mathscr{L}(Y)$ is differentiable with $\|\dot{z}(t) - \frac{\partial g}{\partial y}(t,0,y(t))z(t)\| \le \Delta \|z(t)\|$ for all t. If q and Δ are sufficiently small (depending on $\tilde{\mu}-\mu$) then $\|z(t)\| \le 2Ne^{\tilde{\mu}|t-\tau|}\|z(\tau)\|$ for all t,τ.

(b) <u>Lemma 9.1.4</u>. Let $\tilde{\mu} = \mu + \frac{\varepsilon}{8}$ and suppose $x_1, x_2: \mathbb{R} \to X^\alpha$ are continuous with $\|x_j(t)\|_\alpha \le r$ for all t $(j = 1,2)$. If r, q are sufficiently small and $\tilde{y}(t;\tau,\eta_j,x_j)$ is the solution of $\dot{y} = g(t,x_j(t),y)$, $y(\tau) = \eta_j$ $(j = 1,2)$ for any $\tau \in \mathbb{R}$ and $\eta_1,\eta_2 \in Y$ then for all t,

$$\|\tilde{y}(t;\tau,\eta_1,x_1) - \tilde{y}(t;\tau,\eta_2,x_2)\|$$
$$\le 2Ne^{\tilde{\mu}|t-\tau|}\|\eta_1-\eta_2\| + 2N^2 \left|\int_\tau^t e^{\tilde{\mu}|t-s|}\|x_1(s)-x_2(s)\|_\alpha \, ds\right|.$$

(c) is a technical lemma we needn't describe here.

(d) <u>Lemma 9.1.6</u>. Suppose $\tilde{\beta} = \beta - \frac{\varepsilon}{4}$ and q is sufficiently small. There is a constant $C_2 > 0$ such that whenever $y: \mathbb{R} \to Y$ has $\|\dot{y}(t)\| \le N$, $\|\dot{y}(t) - g(t,0,y(t))\| \le C_2$ for all t, and $B: \mathbb{R} \to \mathscr{L}(X^\alpha, X)$ is locally Hölder continuous with $\|B(t)\| \le N$ and $\|\int_{t_1}^{t_2} B(t)dt\| \le C_2$ for $|t_1 - t_2| \le 1$, the equation

$$\dot{x} + A(t,y(t))x = B(t)x$$

has an exponential dichotomy on $\mathbb{R}$ with exponent $\tilde{\beta}$ and bound $2N$.

(e) <u>Lemma 9.1.7</u>. If r, q are sufficiently small and $2C_3 q^{\delta/2} \le r$, $\delta = (1-\alpha)/2$, then for any $y: \mathbb{R} \to Y$ with $\|\dot{y}(t)\| \le N$ and $\|\dot{y}(t) - g(t,0,y(t))\| \le C_2$, there is a unique solution $x(t) = \tilde{x}(t;y)$ of

$$\dot{x} + A(t,y(t))x = f(t,x,y(t)), \quad -\infty < t < \infty,$$

with $\sup\|x(t)\|_\alpha \le r$; it has $\|\tilde{x}(t;y)\|_\alpha \le 2C_3 q^{\delta/2}$.

If y_1, y_2 both satisfy the conditions for y above, $\tau \in \mathbb{R}$, $0 \le \gamma \le \beta - \frac{\varepsilon}{2}$, and for all t

$$\|y_1(t)-y_2(t)\| \ , \ \|\dot{y}_1(t)-\dot{y}_2(t)\| \le \zeta e^{\gamma|t-\tau|},$$

then

$$\|\tilde{x}(t;y_1) - \tilde{x}(t;y_2)\|_\alpha \le C_5 q^{\delta^2\theta/2} e^{\gamma|t-\tau|}\zeta.$$

These results suffice to prove the existence of the functions $x(t;\tau,\eta)$, $y(t;\tau,\eta)$ of Thm. 9.1.2, and smoothness follows from the next result.

(f) Lemma 9.1.8. Let $\gamma = \mu + \frac{\varepsilon}{4}$ so $\gamma-\tilde{\mu} = \frac{\varepsilon}{8} > 0$ and $\tilde{\beta}-\gamma(1+\theta) \gg \frac{\varepsilon}{4} > 0$. There are constants C_6, C_7 such that, for small q and all $t,\tau \in \mathbb{R}$,

$$\eta \to x(t;\tau,\eta): Y \to X^\alpha \text{ is differentiable}$$

$$\left\|\frac{\partial x}{\partial \eta}(t;\tau,\eta)\right\| \le 2C_5C_6\, q^{\delta^2\theta/2} e^{\gamma|t-\tau|}$$

and

$$\left\|\frac{\partial x}{\partial \eta}(t;\tau,\eta_1) - \frac{\partial x}{\partial \eta}(t;\tau,\eta_2)\right\| \le C_7\|\eta_1-\eta_2\|^\theta\, e^{\gamma(1+\theta)|t-\tau|}$$

for all η, η_1, η_2 in Y.

Using the lemmas, we now prove Thm. 9.1.2. Suppose q,r small enough that (b) and (e) hold $Nr \le C_2$ and $\{x: \|x\|_\alpha \le r\} \subset U$, and choose $(\tau,\eta) \in \mathbb{R} \times Y$. If $x: \mathbb{R} \to X^\alpha$ is continuous with $\|x(t)\|_\alpha \le r$ for all t, define $y(t) = \tilde{y}(t;\tau,\eta,x)$ as in (b). Observe $\|\dot{y}(t)\| = \|g\| \le N$ and $\|\dot{y}(t)-g(t,0,y(t))\| \le Nr \le C_2$. Using (e) define

$$\Phi(x;\tau,\eta)(t) = \tilde{x}(t;\tilde{y}(\cdot,\tau,\eta,x))$$

and note $\|\Phi(x;\tau,\eta)(t)\|_\alpha \le r$ for all t. For $\gamma = \mu + \frac{\varepsilon}{4}$, fixed τ in $\mathbb{R}$, any η_1,η_2 in Y and $x_1,x_2: \mathbb{R} \to X^\alpha$ bounded by r, let

$$z(t) = \tilde{y}(t;\tau,\eta_1,x_1) - \tilde{y}(t;\tau,\eta_2,x_2),$$

and $|||x_1-x_2|||_\tau = \sup_t\{\|x_1(t)-x_2(t)\|\, e^{\gamma|t-\tau|}\}$. According to (b),

$$\|z(t)\| \text{ and } \|\dot{z}(t)\| \le C_6 e^{\gamma|t-\tau|}(\|\eta_1-\eta_2\| + |||x_1-x_2|||_\tau).$$

Thus by (e)

$$\|\Phi(x_1;\tau,\eta_1)(t) - \Phi(x_2;\tau,\eta_2)(t)\|_\alpha$$
$$\le C_5C_6 q^{\delta^2\theta/2} e^{\gamma|t-\tau|}(\|\eta_1-\eta_2\| + |||x_1-x_2|||_\tau).$$

If $C_5C_6q^{\delta^2\theta/2} \le 1/2$, then for fixed τ,η

$$x \to \Phi(x;\tau,\eta)$$

is a contraction in the norm $|||\cdot|||_\tau$ on the r-ball. Let $x(t;\tau,\eta)$ be the fixed point, and let

$$y(t;\tau,\eta) = \tilde{y}(t;\tau,\eta,x(\cdot,\tau,\eta)).$$

Then $x(t;\tau,\eta)$, $y(t;\tau,\eta)$ is the unique solution of

$$\dot{x} + A(t,y)x = f(t,x,y), \quad \dot{y} = g(t,x,y), \quad -\infty < t < \infty,$$

with $y(\tau) = \eta$ and $\|x(t)\|_\alpha \le r$ for all t.

<u>Preliminary calculation</u>. Suppose $\|h(t,y_1)-h(t,y_2)\| \le N\|y_1-y_2\|^p$

$$\left\|\int_{t_1}^{t_2} h(t,y)dt\right\| \le q \quad \text{for} \quad |t_1-t_2| \le 1,$$

and $\|y(t_1)-y(t_2)\| \le N|t_1-t_2|$. Then if $q \le N^{1+p}$ and $|t_1-t_2| \le 1$,

$$\left\|\int_{t_1}^{t_2} h(t,y(t))dt\right\| \le 3Nq^{p/(1+p)}.$$

For $|\Delta t| \le 1$ implies $\left\|\int_t^{t+\Delta t} h(s,y(s))ds\right\| \le q + N^{1+p}|\Delta t|^{1+p}$; if we divide the interval $[t_1,t_2]$ into m equal subintervals where $m-1 \le Nq^{-1/(1+p)} \le m$, then

$$\left\|\int_{t_1}^{t_2} h(t,y(t))\right\| \le q + (m-1)q + N^{1+p}m^{-p} \le 3Nq^{p/(1+p)}.$$

(a) <u>Proof of Lemma 9.1.3</u>. Choose integer $n \ge 1$ so $2Ne^{\mu n} \le e^{\tilde{\mu}n}$. We will prove $\|z(t)\| \le 2Ne^{\mu|t-\tau|}\|z(\tau)\|$ when $|t-\tau| \le n$, provided q, Δ are small, which implies $\|z(t)\| \le 2Ne^{\tilde{\mu}|t-\tau|}\|z(\tau)\|$ for all t,τ. To do this we first estimate $\|y(t)-\bar{y}(t)\|$ for $|t-\tau| \le n$, where $\bar{y}$ satisfies $\dot{\bar{y}} = g_0(t,\bar{y})$, $\bar{y}(\tau) = y(\tau)$. Let $Z(t,s)$ be the $\mathcal{L}(Y)$-valued function defined by $\frac{\partial}{\partial t} Z(t,s) = \frac{\partial g_0}{\partial y}(t,\bar{y}(t))Z(t,s)$, $Z(s,s) = I$; by (i) $\|Z(t,s)\| \le Ne^{\mu|t-s|}$ and so

$$\left\|\frac{\partial}{\partial s} Z(t,s)\right\| = \|-Z(t,s)g_{0y}(s,\bar{y}(s))\| \le N^2e^{\mu|t-s|}.$$

If $g_1 = g-g_0$, the preliminary calculation above shows

$$\left\|\int_{t_1}^{t_2} g_1(s,0,y(s))ds\right\| \le 3Nq^{1/2} \quad \text{for} \quad |t_1-t_2| \le 1,$$

provided $q \le N^2$, so the integral is bounded by $3nNq^{\frac{1}{2}}$ for $|t_1-t_2| \le n$. For $|t-\tau| \le n$ we have

$$\frac{d}{dt}(y-\bar{y}) = \dot{y} - g(t,0,y) + g_1(t,0,y) + g_0(t,y) - g_0(t,\bar{y})$$

so $\|y(t)-\bar{y}(t)\| \le n(\Delta + 3Nq^{\frac{1}{2}})e^{Nn}$ by Gronwall's inequality.

Also

$$\begin{aligned}\dot{z}(t) - g_{0y}(t,\bar{y}(t))z(t) = \dot{z} &- g_y(t,0,y)z \\ &+ g_{1y}(t,0,y)z \\ &+ (g_{0y}(t,y) - g_{0y}(t,\bar{y}))z\end{aligned}$$

so the variation of constants formula gives

$$\begin{aligned}\|z(t)\| \le Ne^{\mu|t-\tau|}\|z(\tau)\| &+ \left\|\int_\tau^t Z(t,s)g_{1y}(s,0,y(s))z(s)ds\right\| \\ &+ N\left|\int_\tau^t e^{\mu|t-s|}(\Delta\|z(s)\| + N\|y(s)-\bar{y}(s)\|^\theta\|Z(s)\|)ds\right|.\end{aligned}$$

As above, $\left\|\int_{t_1}^{t_2} g_{1y}(s,0,y(s))ds\right\| \le 3nNq^{\theta/(1+\theta)}$ for $|t_1-t_2| \le n$, so integration by parts shows

$$\begin{aligned}&\left\|\int_\tau^t Z(t,s)\frac{d}{ds}\left(-\int_s^t g_{1y}\right)z(s)ds\right\| \\ &\le 3nN^2q^{\theta/(1+\theta)}e^{\mu|t-\tau|}\|z(\tau)\| \\ &\quad + \left|\int_\tau^t e^{\mu|t-s|}\|z(s)\|ds\right| 3nN^2(2N+\Delta)q^{\theta/(1+\theta)}\end{aligned}$$

for $|t-\tau| \le n$. Substituting this and the estimate for $\|y-\bar{y}\|$, Gronwall's inequality (for some q,Δ) proves

$$\|z(t)\| \le 2Ne^{\mu|t-\tau|}\|z(\tau)\| \quad \text{for} \quad |t-\tau| \le n.$$

(b) Proof of Lemma 9.1.4. With $\tilde{\mu} = \mu + \frac{\varepsilon}{8}$, let q,Δ be small enough that (a) holds. Choose $r > 0$ so small that $Nr \le \Delta$, $Nr^\theta \le \Delta$ and $\{x: \|x\|_\alpha \le r\} \subset U$. For $0 \le \sigma \le 1$, let $x_\sigma = (1-\sigma)x_1 + \sigma x_2$, $\eta_\sigma = (1-\sigma)\eta_1 + \sigma\eta_2$ and $y_\sigma(t) = \tilde{y}(t;\tau,\eta_\sigma,x_\sigma)$. By (a) and the variation of constants formula

$$\|\frac{\partial}{\partial\sigma}y_\sigma(t)\| \leq 2Ne^{\tilde{\mu}|t-\tau|}\|\eta_2-\eta_1\|$$
$$+ 2N\left|\int_\tau^t e^{\tilde{\mu}|t-s|}\|g_x(s,x_\sigma(s),y_\sigma(s))\| \, \|x_2(s)-x_1(s))\|_\alpha ds\right|.$$

Since $\|g_x\| \leq N$, integration over $0 \leq \sigma \leq 1$ completes the proof.

(c) Lemma 9.1.5. With X, A_0, α as in Thm. 9.1.1, suppose $t \to A(t)-A_0\colon \mathbb{R} \to \mathscr{L}(X^\alpha,X)$ is locally Hölder continuous and bounded by K for all t, and $\dot{x} + A(t)x = 0$ has an exponential dichotomy on $\mathbb{R}$ (with respect to X^α) with exponent $\beta_1 > 0$ and bound K. If $0 < \beta_2 < \beta_1$ there is a constant Q depending only on A_0, α, β_1, β_2 and K such that the following holds.

If $G(t,s)$ is the Green's function for the dichotomy, $\|G(t,s)\|_{\mathscr{L}(X,X^\alpha)} \leq Q\psi_\alpha(t-s)e^{-\beta_1|t-s|}$, where $\psi_\alpha(s) = s^{-\alpha}$ on $(0,1)$, $\phi_\alpha(s) = 1$ elsewhere. If Z is a Banach space, $B\colon \mathbb{R} \to \mathscr{L}(Z,X)$ and $z\colon \mathbb{R} \to Z$ satisfy

$$\|B(t)\| \leq N, \ \left\|\int_{t_1}^{t_2} B(t)dt\right\| \leq \eta \leq N \quad \text{when} \quad |t_1-t_2| \leq 1$$
$$\|z(t)\| \leq \zeta e^{\gamma|t-\tau|}, \quad \|\dot{z}(t)\| \leq \zeta e^{\gamma|t-\tau|}$$

for some fixed real τ and $|\gamma| \leq \beta_2 < \beta_1$, then

$$\left\|\int_{-\infty}^{\infty} G(t,s)B(s)z(s)ds\right\|_\alpha \leq Q\eta^\delta N^{1-\delta}\zeta e^{\gamma|t-\tau|}$$

for all t, where $\delta = (1-\alpha)/2$.

Remark. If we identify $\mathscr{L}(\mathbb{R},X)$ with X, we may take $Z = \mathbb{R}$ and $B\colon \mathbb{R} \to X$.

Proof. By Lemma 7.5.1, if $t_1 = t_0+1$,

$$\left\|\int_{t_0}^{t_1} T(t_1,s)B(s)z(s)ds\right\|_\alpha \leq 3C\eta^\delta N^{1-\delta}\zeta e^{\gamma|t^*-\tau|}\int_0^1\{(1-s)^{-\alpha-\delta}+(1-s)^{-\alpha}s^{-\delta}\}$$

where $\gamma|t^*-\tau| = \max(\gamma|t_1-\tau|, \gamma|t_0-\tau|)$, so there is a constant C^1 depending on A_0, α, K and β_2) with

$$\left\|\int_{t_0}^{t_1} T(t_1,s)B(s)z(s)ds\right\|_\alpha \leq C^1\eta^\delta N^{1-\delta}\zeta e^{\gamma|t_1-\tau|}.$$

Then

$$\left\|\int_{-\infty}^{t} G(t,s)B(s)z(s)ds\right\|_\alpha \le \sum_{n=0}^{\infty} \|G(t,t-n)\int_{t-n-1}^{t-n} T(t-n,s)B(s)z(s)\|_\alpha$$

$$\le \sum_{n=0}^{\infty} Ke^{-\beta_1 n} C'\eta^\delta N^{1-\delta}\zeta e^{\gamma|t-\tau-n|}$$

$$\le \{KC'/(1-e^{-(\beta_2-\beta_1)})\}\eta^\delta N^{1-\delta}\zeta e^{\gamma|t-\tau|}.$$

A similar estimate for $\|\int_t^\infty G(t,s)B(s)z(s)\|_\alpha$ completes the proof.

(d) Proof of Lemma 9.1.6. Choose integer $n \ge 1$ so $Ne^{-\beta n} < e^{-\tilde{\beta}n}$. Suppose $\|\dot{y}(t)\| \le N$, $\|\dot{y}(t) - g(t,0,y(t))\| \le \Delta$ for all t and $\|\int_{t_1}^{t_2} B(t)dt\| \le \Delta$ for $|t_1-t_2| \le 1$. If $(\tau,\eta) \in \mathbb{R} \times Y$ let $\overline{y}(t;\tau,\eta)$ be the solution of $\dot{\overline{y}} = g_0(t,\overline{y})$, $\overline{y}(\tau) = \eta$. We apply Thm. 7.6.12 with $\Lambda = \mathbb{R} \times Y$, $\lambda(\tau) = (\tau,y(\tau))$ and $A(t;\lambda(\tau)) = A(t,\overline{y}(t;\tau_1,y(\tau))$ to prove the result

We showed in the proof of (a) that

$$\|\overline{y}(t;\tau,y(\tau))-y(t)\| \le ne^{Nn}(\Delta+3Nq^{1/2}) \quad \text{for} \quad |t-\tau| \le n,$$

so

$$\left\|\int_\tau^t (A(s,y(s))-B(s)-A(s;\lambda(\tau)))ds\right\|_{\mathscr{L}(X^\alpha,X)} \le n\Delta + n^2Ne^{Nn}(\Delta+3Nq^{1/2})$$

for $0 \le t-\tau \le n$.

If $\dot{y}_j = g_0(t,y_j)$ for all t $(j = 1,2)$ and G_j, P_j are the corresponding Green's function and projection, then (see ex. 17, sec. 7.6)

$$P_2(\tau)-P_1(\tau) = \int_{-\infty}^{\infty} G_2(\tau,s)(A(s,y_2(s))-A(s,y_1(s))G_1(s,\tau)ds$$

and $\|y_1(s)-y_2(s)\| \le Ne^{\mu|s-\tau|}\|y_1(\tau)-y_2(\tau)\|$, so

$$\|P_2(\tau)-P_1(\tau)\|_{\mathscr{L}(X^\alpha)} \le QN^3\|y_1(\tau)-y_2(\tau)\|\int_{-\infty}^{\infty} \psi_\alpha(\tau-s)e^{-(2\beta-\mu)|\tau-s|}ds$$

$$\le C_1\,\|y_1(\tau)-y_2(\tau)\|\ .$$

If $y_1(t) = \overline{y}(t;\tau,y(\tau))$, $y_2(t) = \overline{y}(t;\tau+n,y(\tau+n))$, then

$$\|P_{\lambda(\tau+n)}(\tau)-P_{\lambda(\tau)}(\tau)\|_{\mathscr{L}(X^\alpha)} \le C_1\,\|\overline{y}(\tau;\tau+n,y(\tau+n))-y(\tau)\|$$

$$\le C_1 ne^{Nn}(\Delta + 3Nq^{1/2}).$$

If Δ, q are sufficiently small, we have an exponential dichotomy on $\mathbb{R}$ with exponent $\tilde{\beta} = \beta - \frac{\varepsilon}{4}$ and bound $2N$ (Thm. 7.6.12). The condition on Δ is expressed as $\Delta \le C_2$.

(e) Proof of Lemma 9.1.7. Suppose q is small enough that (d) holds. If $\|\dot{y}(t)\| \le N$, $\|\dot{y}(t)-g(t,0,y(t))\| \le C_2$ for all t, then

$$\left\|\int_{t_1}^{t_2} f_x(t,0,y(t))dt\right\| \le 3Nq^{\theta/(1+\theta)} \quad \text{for} \quad |t_1-t_2| \le 1$$

and for small q (so $3Nq^{\theta/(1+\theta)} \le C_2$),

$$\dot{x} + A(t,y(t))x = f_x(t,0,y(t))x$$

has an exponential dichotomy on $\mathbb{R}$ with exponent $\tilde{\beta}$ and bound $2N$. Let $G(t,s)$ be the corresponding Green's function and Q the constant from (c) with $\beta_1 = \tilde{\beta}$, $\beta_2 = \beta - \frac{\varepsilon}{2} = \tilde{\beta} - \frac{\varepsilon}{4}$.

If $x(t)$ is a bounded solution of

$$(*) \qquad \dot{x} + A(t,y(t))x = f(t,x,y(t)), \quad -\infty < t < \infty,$$

then $x = \tilde{x}$ where

$$(**) \qquad \tilde{x}(t) = \int_{-\infty}^{\infty} G(t,s)\{f(s,0,y(s)) + F(s,x(s),y(s))\}ds,$$

$$F(t,x,y) = f(t,x,y) - f(t,0,y) - f_x(t,0,y)x.$$

For small q, r with $2C_3q^{\delta/2} \le r$, we show the map $x \to \tilde{x}$ defined by (**) is a contraction in the uniform norm on the space of continuous $x\colon \mathbb{R} \to X^\alpha$ with $\sup \|x(t)\|_\alpha \le r$. The fixed point of this map is the unique solution of (*) with $\sup \|x(t)\|_\alpha \le r$.

First suppose $\|x\|_\alpha \le r$ implies $x \in U$ and

$$Nr^\theta \int_{-\infty}^{\infty} \|G(t,s)\|_{\mathscr{L}(X^\alpha,X)} ds \le QNr^\theta \int_{-\infty}^{\infty} \max(|s|^{-\alpha},1)e^{-\tilde{\beta}|s|}ds \le \frac{1}{2}.$$

Now $\left\|\int_{t_1}^{t_2} f(s,0,y(s))ds\right\| \le 3Nq^{1/2}$ for $|t_1-t_2| \le 1$, so by (c) (with $Z = \mathbb{R}$, $\gamma = 0$, $z(s) = 1$)

$$\left\|\int_{-\infty}^{\infty} G(t,s)f(s,0,y(s))ds\right\|_\alpha \le QN^{1-\delta}(3Nq^{1/2})^\delta \equiv C_3q^{\delta/2}.$$

If $C_3q^{\delta/2} \le r/2$ then $\sup \|x(t)\|_\alpha \le r$ implies

$$\|\tilde{x}(t)\|_\alpha \le C_3 q^{\delta/2} + \frac{1}{2}\sup \|x(t)\|_\alpha \le r.$$

If $x_1, x_2 : \mathbb{R} \to X^\alpha$ are both bounded by r,

$$\|\tilde{x}_1(t) - \tilde{x}_2(t)\|_\alpha \le \frac{1}{2} \cdot \sup_s \|x_1(s) - x_2(s)\|_\alpha$$

and $x \to \tilde{x}$ is a contraction. If $\tilde{x}(t;y)$ is the fixed point, $\|\tilde{x}(t;y)\|_\alpha \le 2C_3 q^{\delta/2}$ for all t.

Suppose $y_1, y_2 : \mathbb{R} \to Y$ both satisfy the conditions on y mentioned above, and $x_j(t) = \tilde{x}(t;y_j)$ $(j = 1,2)$, $u = x_2 - x_1$. Then u is a bounded solution of

$$\dot{u} + A(t,y_2(t))u = B_2(t)u + M_2(t)(y_2 - y_1) + (A(t,y_1(t)) - A(t,y_2(t)))x_1(t)$$

where

$$B_2(t) = \int_0^1 f_x(t, x_1(t) + \sigma(x_2(t) - x_1(t)),\ y_2(t))\,d\sigma$$

$$M_2(t) = \int_0^1 f_y(t, x_1(t),\ y_1(t) + \sigma(y_2(t) - y_1(t)))\,d\sigma.$$

Since $\|x_1(t) + \sigma u(t)\|_\alpha \le 2C_3 q^{\delta/2}$

$$\left\|\int_{t_1}^{t_2} B_2(t)\,dt\right\| \le N(2C_3 q^{\delta/2})^\theta + 3Nq^{\theta/(1+\theta)} \le C_4 q^{\delta\theta/2}$$

for $|t_1 - t_2| \le 1$, so for small q

$$\dot{u} + A(t,y_2(t))u = B_2(t)u$$

has an exponential dichotomy on $\mathbb{R}$ with exponent $\tilde{\beta}$ and bound $2N$. Similarly $\|\int_{t_1}^{t_2} M_2(t)\,dt\| \le C_4 q^{\delta\theta/2}$ for $|t_1 - t_2| \le 1$.

If $z = y_2 - y_1$ satisfies

$$\|z(t)\|,\ \|\dot{z}(t)\| \le \zeta e^{\gamma|t-\tau|} \quad \text{for all } t$$

for some fixed τ and $0 \le \gamma \le \tilde{\beta} - \frac{\varepsilon}{4}$, then by (c),

$$\|u(t)\|_\alpha \le QN^{1-\delta}(C_4 q^{\delta\theta/2})\,\zeta e^{\gamma|t-\tau|} + Q(2C_3 q^{\delta/2})N \int_{-\infty}^{\infty} e^{-\tilde{\beta}|t-s|}\max(|t-s|^{-\alpha}, 1)\zeta e^{\gamma|s-\tau|}\,ds$$

and so

$$\|\tilde{x}(t;y_1)-\tilde{x}(t;y_2)\|_\alpha \le C_5 q^{\delta^2\theta/2}\zeta e^{\gamma|t-\tau|}.$$

(f) Proof of Lemma 9.1.8. Fix $\tau \in \mathbb{R}$ and let $\gamma = \mu + \frac{\varepsilon}{4}$ (so $\tilde{\mu} < \gamma$, $\gamma(1+\theta) < \tilde{\beta}$) and let $|||x|||_{\tau,\gamma} = \sup\{\|x(t)\|_\alpha e^{-\gamma|t-\tau|}\}$

$$|||y|||'_{\tau,\gamma} = \sup\{\|y(t)\|e^{-\gamma|t-\tau|}, \|\dot{y}(t)\|e^{-\gamma|t-\tau|}\}$$

and similarly with γ replaced by $\gamma(1+\theta)$.

Suppose y_1, y_2 both satisfy the conditions of (e), $\tilde{x}_j(t) = \tilde{x}(t;y_j)$ and define $\xi(t;y_2-y_1)$ as the solution of

$$\dot{\xi} + (A(t,y_1(t))-f_x(t,\tilde{x}_1(t),y_1(t)))\xi$$

$$= (f_y(t,\tilde{x}_1,y_1) - A_y(t,y_1)\tilde{x}_1)(y_2(t)-y_1(t))$$

with $\|\xi(t)\|_\alpha = O(e^{\gamma|t-\tau|})$ as $|t| \to \infty$; ξ is well defined (Thm. 7.6.3).

Let $u(t) = \tilde{x}(t;y_2)-\tilde{x}(t;y_1)-\xi(t;y_2-y_1)$; then

$$\|\dot{u} + (A(t,y_1) - f_x(t,\tilde{x}_1,y_1))u\|$$

$$= \|(A(t,y_1)-A(t,y_2))(\tilde{x}_2-\tilde{x}_1) - (A(t,y_2)-A(t,y_1)-A_y(t,y_1)(y_2-y_1))\tilde{x}_1$$

$$+ f(t,\tilde{x}_2,y_2)-f(t,\tilde{x}_1,y_1)-f_x(t,\tilde{x}_1,y_1)(\tilde{x}_2-\tilde{x}_1)-f_y(t,\tilde{x}_1,\tilde{y}_1)(y_2-y_1)\|$$

$$\le C(\|\tilde{x}(t;y_2)-\tilde{x}(t;y_1)\|_\alpha^{1+\theta} + \|y_2(t)-y_1(t)\|^{1+\theta})$$

$$\le C'(|||y_2-y_1|||'_{\gamma,\tau})^{1+\theta}\, e^{\gamma(1+\theta)|t-\tau|}$$

and so

$$\|\tilde{x}(t;y_2)-\tilde{x}(t;y_1) - \xi(t;y_2-y_1)\|_\alpha$$

$$\le C'' e^{\gamma(1+\theta)|t-\tau|}(|||y_2-y_1|||'_{\gamma,\tau})^{1+\theta}.$$

We write $\xi(t;y_2-y_1) = \frac{\partial}{\partial y}\tilde{x}(t;y_1)(y_2-y_1)$ and recall from the proof of (e)

$$\|\frac{\partial\tilde{x}}{\partial y}(t;y_1)(y_2-y_1)\|_\alpha \le Cq^{\delta^2\theta/(1+\theta)}|||y_2-y_1|||'_{\gamma,\tau}e^{\gamma|t-\tau|},$$

which holds also with γ replaced by $\gamma(1+\theta)$.

Now for $\eta_1, \eta_2 \in Y$ and continuous $x_1, x_2 : \mathbb{R} \to X^\alpha$ with $\sup \|x_j(t)\|_\alpha \le r$, let $y_j(t) = \tilde{y}(t;\tau,\eta_j,x_j)$ $(j = 1,2)$ and let z be the solution of

$$\dot{z} - g_y(t,x_1(t),y_1(t))z = g_x(t,x_1,y_1)(x_2(t)-x_1(t)),$$
$$z(\tau) = \eta_2-\eta_1.$$

Then

$$\|(\tfrac{d}{dt} - g_y(t,x_1,y_1))(y_2-y_1-z)\| \le N(\|x_2-x_1\|_\alpha^{1+\theta}+\|x_1-x_2\|_\alpha\|y_2-y_1\|^\theta + \|y_2-y_1\|^{1+\theta})$$
$$\le C(|||x_2-x_1|||_{\tau,\gamma}^{1+\theta} + \|\eta_2-\eta_1\|^{1+\theta})e^{\gamma(1+\theta)|t-\tau|}$$

so

$$\|y_2(t)-y_1(t)-z(t)\|$$
$$\le C'(|||x_2-x_1|||_{\tau,\gamma}^{1+\theta} + \|\eta_2-\eta_1\|^{1+\theta})e^{\gamma(1+\theta)|t-\tau|}$$

for all t, and

$$z(t) = \frac{\partial}{\partial x}\tilde{y}(t;\tau,\eta_1,x_1)(x_2-x_1) + \frac{\partial}{\partial\eta}\tilde{y}(t;\tau,\eta_1,x_1)(\eta_2-\eta_1).$$

Replacing C' by $NC'+C$ if necessary, the same estimate holds for $\|\frac{d}{dt}(y_2-y_1-z)\|$, and then if $x_j(t) = x(t;\tau,\eta_j)$ (so $|||x_2-x_1|||_{\gamma,\tau} \le \text{Const.}\|\eta_2-\eta_1\|$) we have

$$|||y_2-y_1 - \frac{\partial\tilde{y}}{\partial x}(x_2-x_1) - \frac{\partial\tilde{y}}{\partial\eta}(\eta_2-\eta_1)|||'_{\gamma(1+\theta),\tau} \le C''\|\eta_2-\eta_1\|^{1+\theta}.$$

Also

$$|||x_2-x_1 - \frac{\partial\tilde{x}}{\partial y}(y_2-y_1)|||_{\gamma(1+\theta),\tau} \le C'''(|||y_2-y_1|||'_{\gamma,\tau})^{1+\theta}$$
$$\le C^{iv}\|\eta_2-\eta_1\|^{1+\theta},$$

$$|||x_2-x_1 - \frac{\partial\tilde{x}}{\partial y}\frac{\partial\tilde{y}}{\partial x}(x_2-x_1) - \frac{\partial\tilde{x}}{\partial y}\frac{\partial\tilde{y}}{\partial\eta}(\eta_2-\eta_1)|||_{\gamma(1+\theta),\tau}$$
$$\le C^{v}\|\eta_2-\eta_1\|^{1+\theta}$$

Since

$$|||\frac{\partial\tilde{x}}{\partial y}\frac{\partial\tilde{y}}{\partial x}(x_2-x_1)|||_{\gamma(1+\theta),\tau} \le Cq^{\gamma^2\theta/(1+\theta)}|||x_2-x_1|||_{\gamma(1+\theta),\tau}$$

it follows (for small q) that there is a constant C_7 so

$$\|\tilde{x}(t;\tau,\eta_2)-\tilde{x}(t;\tau,\eta_1) - L(t,\tau,\eta_1)(\eta_2-\eta_1)\|_\alpha$$
$$\le \frac{1}{6}C_7\|\eta_2-\eta_1\|^{1+\theta}e^{\gamma(1+\theta)|t-\tau|}$$

for some linear L. It follows from the exercise (1) below that $L(t,\tau,\eta_1) = \frac{\partial\tilde{x}}{\partial\eta}(t;\tau,\eta_1)$ and

$$\|\frac{\partial\tilde{x}}{\partial\eta}(t;\tau,\eta_2) - \frac{\partial\tilde{x}}{\partial\eta}(t,\tau,\eta_1)\| \le C_7\|\eta_2-\eta_1\|^{\theta}e^{\gamma(1+\theta)|t-\tau|}.$$

Exercise 1*. If $\|\phi(y_2)-\phi(y_1)-\phi'(y_1)(y_2-y_1)\| \le K\|y_2-y_1\|^{1+\theta}$ for all y_2,y_1, prove

$$\|\phi'(y_2)-\phi'(y_1)\| \le 6K\|y_2-y_1\|^{\theta}.$$

Hint: choose any u, $\|u\| = 1$, let $d = \|y_2-y_1\|$ and evaluate the inequality at the pairs of points

$$(y_2,y_1),\quad (y_2+ud,y_1),\quad (y_2+ud,y_2).$$

Remark. If we work on $Y_0 \subset Y$, a convex set or submanifold of Y, and $\theta < 1$, the above argument may run into trouble near the boundary of Y_0 (if any), and we obtain smoothness only over a slightly smaller set. For example, if the condition of ex. 1 holds for y_1, y_2 in Y_0 (say Y_0 is convex)

$$\|\phi'(y_1)-\phi'(y_2)\| \le 6K\|y_1-y_2\|^{\theta} \text{ when } \|y_1-y_2\| \le \max\{\text{dist}(y_j,\partial Y_0);\ j=1,2\}.$$

Exercise 2*. If $t \to A(t,y)-A_0$, $f(t,x,y)$, $g(t,x,y)$ are uniformly almost periodic prove $t \to \sigma(t,y)$ is uniformly a.p. and its frequency module is contained in the joint module of A, f, g.

Hint: Suppose $\{t_n\}$ is a sequence in $\mathbb{R}$ and $\|A_n-A_m\|$, $\|f_n-f_m\|$, $\|g_n-g_m\| \le \Delta_m \to 0$ as $m \to \infty$, $n \ge m$, uniformly on $\mathbb{R} \times U \times Y$ ($A_n(t,y) = A(t+t_n,y)$, etc.) It suffices to prove $\|\sigma(t+t_n,y) - \sigma(t+t_m,y)\|_\alpha \to 0$ uniformly on $\mathbb{R} \times Y$.

For some constant B, let $x\colon \mathbb{R} \times \mathbb{R} \to X^\alpha$ be any continuous function with $\|x(t,s)\|_\alpha \le r_0$ and for $n \ge m$,

$$\|x_n(t,s)-x_m(t,s)\|_\alpha \le B\Delta_m e^{\gamma|t-\tau|}, \quad x_n(t,\tau) = x(t+t_n,\tau+t_n).$$

If $\tilde{x}(t,\tau) = \Phi(x(\cdot,\tau);\tau,\eta)(t)$ for fixed $\eta \in Y$, show

$$\|\tilde{x}_n(t,\tau)-\tilde{x}_m(t,\tau)\|_\alpha \le C_8(1 + Bq^{\delta^2\theta/2})\Delta_m e^{\gamma|t-\tau|},$$

and choose $B = 2C_8$.

<u>Exercise 3*</u>. Suppose $\dot{x} + A(t,y)x = 0$ is stable when $\dot{y} = g_0(t,y)$. Prove there exists $r_0 > 0$ (for small q and $\beta_0 = \beta - \frac{\varepsilon}{2} < \tilde{\beta}$) so any solution of $\dot{x} + A(t,y)x = f(t,x,y)$, $\dot{y} = g(t,x,y)$ for $t > \tau$ with $\|x(\tau)\|_\alpha \le r_0/8N$ has $\|x(t)\|_\alpha \le r_0$ and

$$\|x(t)-\sigma(t,y(t))\|_\alpha \le 4Ne^{-\beta_0(t-\tau)}\|x(\tau)-\sigma(\tau,y(\tau))\|_\alpha \text{ for all } t \ge \tau.$$

Also there is a unique solution $\bar{y}$ of $\dot{\bar{y}} = g(t,\sigma(t,\bar{y}),\bar{y})$ such that $\|y(t)-\bar{y}(t)\| \le C_8 e^{-\beta_0(t-\tau)}\|x(\tau)-\sigma(\tau,y(\tau))\|_\alpha$ for $t \ge \tau$. We may let $C_8 = 16N^3/(\beta_0-\tilde{\mu})$.

<u>Hint</u>: Let $Z(t,s)$ be the evolution operator for $\dot{z} = g_y(t,x,y)z$ and find $\bar{y}$ as the solution of

$$\bar{y}(t) = y(t) + \int_t^\infty Z(t,s)[g(s,x,y)-g(s,\sigma(s,\bar{y}),\bar{y})-g_y(s,x,y)(y-\bar{y})]$$

with

$$\|\bar{y}(t)-y(t)\| \le \eta e^{-\beta_0(t-\tau)}, \quad \eta = \frac{16N^3}{\beta_0-\tilde{\mu}}\|x(\tau)-\sigma(\tau,y(\tau))\|_\alpha.$$

<u>Exercise 4*</u>. Suppose $\dot{x} + A(t,y(t))x = 0$ is unstable for some solution of $\dot{y} = g_0(t,y)$. If $(\tau,\eta) \in \mathbb{R}$ let $\dot{\bar{y}} = g(t,0,\bar{y})$, $\bar{y}(\tau) = \eta$, and let $P_{\tau,\eta}(t)$ be the projection corresponding to the dichotomy for $\dot{x} + A(t,\bar{y}(t))x = f_x(t,0,\bar{y}(t))x$. Prove (for small r_0, q) there is a function $w(t,\cdot,\cdot): U \times Y \to X^\alpha$, Lipschitzian near $x = 0$, so the "stable manifold"

$$W^s(S) = \{(t,x,y) \mid x = w(t,(1-P_{t,y}(t))x,y),\ \|(1-P_{t,y}(t)x\|_\alpha \le \frac{r_0}{8N}\}$$

contains the initial value $(\tau,x(\tau),y(\tau))$ of any solution of $\dot{x} + A(t,y)x = f(t,x,y)$, $\dot{y} = g(t,x,y)$ for $t > \tau$ with $\|(1-P_{\tau,\eta}(\tau))x(\tau)\|_\alpha \le r_0/8N$ and $\|x(t)\|_\alpha \le r_0$ for $t \ge \tau$. Any such solution has $x(t)-\sigma(t,y(t)) \to 0$ exponentially as $t \to \infty$. For $\xi \in N(P_{\tau,\eta}(\tau))$, $\|\xi\|_\alpha \le r_0/8N$, the map $\xi \to (1-P_{\xi,\eta}(\tau))w(\tau,\xi,\eta)-\xi$ satisfies a Lipschitz condition with bound $\le 1/2$. If

$\|x(\tau)\|_\alpha \leq r_0/16N^2$ and $(\tau,x(\tau),y(\tau)) \notin W^s(S)$, then eventually $\|x(t)\|_\alpha > r_0$. (We suppose q is so small that $\|\sigma(t,y)\|_\alpha < r_0/16N^2$.)

Exercise 5. Prove there is a constant C_{10} such that, for $0 \leq h \leq 1$, $\delta = (1-\alpha)/2$,

$$\|\sigma(t+h,y)-\sigma(t,y)\|_\alpha \leq C_{10}h^\delta$$

$$\|\frac{\partial\sigma}{\partial y}(t+h,y) - \frac{\partial\sigma}{\partial y}(t,y)\|_{\mathscr{L}(Y,X^\alpha)} \leq C_{10}h^\delta$$

Hint: in (e) show $\|\tilde{x}(t+h;\tau,y)-\tilde{x}(t;\tau,y)\|_\alpha \leq \text{Const}.h^\delta$ and in (f), $\|\frac{\partial x}{\partial\eta}(t+h,\eta) - \frac{\partial x}{\partial\eta}(t,\eta)\| \leq \text{Const}.h^\delta e^{\gamma|t-\tau|}$.

Exercise 6. Suppose, in addition to the assumptions of Thm. 9.1.1, that $t \to A(t,y)-A_0$, $f(t,x,y)$, $g(t,x,y)$ are uniformly Hölder continuous. Prove $t \to \sigma(t,y): \mathbb{R} \to X$ is differentiable, and $\|\frac{\partial\sigma}{\partial t}(t,y)\|$ and $\|A_0\sigma(t,y)\|$ are uniformly bounded. (Hint: first estimate the derivative of $t \to x(t;\tau,\eta)$, using ex. 5.)

Conclude from this that

$$\frac{\partial\sigma}{\partial t}(t,y) + A(t,y)\sigma(t,y) = f(t,\sigma(t,y),y) - \frac{\partial\sigma}{\partial y}(t,y)g(t,\sigma(t,y),y),$$

for all t,y and obtain further smoothness in t,y - in particular, $\partial\sigma/\partial t$ and $A_0\sigma$ are uniformly Hölder continuous from $\mathbb{R} \times Y$ to X.

Remark. Without assuming uniform Holder continuity in time, we still have $\partial\sigma/\partial t$ and $A_0\sigma$ defined and the equation for σ holds; this follows by invariance and the differentiability of $y \to \sigma(t,y)$.

Corollary 9.1.9. Suppose Λ is a convex set in some Banach space and A,f,g,g_0 of Thm. 9.1.1 depend also on $\lambda \in \Lambda$. Suppose the hypotheses of Thm. 9.1.1 hold uniformly for $\lambda \in \Lambda$, that $\lambda \to A-A_0$, f, g, g_0 are differentiable with derivative bounded by N, and A_λ, f_λ, g_λ, $g_{0\lambda}$ (as well as A_y, f_x, f_y, g_x, g_y, g_{0y}) are Hölder continuous in $(x,y,\lambda) \in U \times Y \times \Lambda$ with exponent θ and bound N.

If q is sufficiently small, for each $\lambda \in \Lambda$ there is an invariant manifold

$$S_\lambda = \{(t,x,y) \mid x = \sigma(t,y,\lambda),\ (t,y) \in \mathbb{R} \times Y\}$$

and $(\lambda,y) \to \sigma(t,y,\lambda)$ is differentiable with derivatives bounded and uniformly Hölder continuous with exponent θ, and

$$\sigma(t,y,\lambda), \quad \frac{\partial\sigma}{\partial y}(t,y,\lambda), \quad \frac{\partial\sigma}{\partial\lambda}(t,y,\lambda) \to 0$$

as $q \to 0$ uniformly on $\mathbb{R} \times Y \times \Lambda$.

Proof: If we replace Y by $Y \times \Lambda$ and add the equation $\dot{\lambda} = 0$, this follows immediately from Thm. 9.1.1 and the remarks following that theorem.

Example 1. Suppose A, f, g satisfy the smoothness and boundedness conditions (iii) of Thm. 9.1.1. Consider, for small $\varepsilon > 0$, the system

$$\dot{x} + \varepsilon A(t,y)x = \varepsilon f(t,x,y), \quad \dot{y} = \omega + \varepsilon g(t,x,y)$$

where $\omega \in Y$ is constant (possibly zero). Change variables to

$$\tau = \varepsilon t \quad \text{and} \quad z = y - \omega t = y - \omega\tau/\varepsilon;$$

with $x' = dx/d\tau$, we have

$$x' + A(\tau/\varepsilon, z+\omega\tau/\varepsilon)x = f(\tau/\varepsilon, x, z+\omega\tau/\varepsilon)$$

$$z' = g(\tau/\varepsilon, x, z+\omega\tau/\varepsilon).$$

Suppose the limit

$$\overline{f}(x,z) = \lim_{T\to\infty} \frac{1}{T}\int_{t_0}^{t_0+T} f(s,x,z+s\omega)ds$$

exists (at least when $x = 0$) uniformly in t_0, z, and is independent of t_0, and the same is true when f is replaced by f_x, A or g.

Assume $\overline{f}(0,z) = 0$, $\overline{f}_x(0,z) = 0$, $\overline{g}(0,z) = 0$ and $|\text{Re}\,\sigma(\overline{A}(z))| \geq \beta > 0$, and $\{\overline{A}(z) : z \in Y\}$ is in a compact set of $\mathscr{L}(X^1,X)$. If $\overline{A}(z)$ is not independent of z, assume the essential spectrum of any $\tilde{A} = \lim_{\nu\to\infty} \overline{A}(z_\nu)$ is disjoint from the strip $-\beta < \text{Re}\,\lambda < \beta$. Then the family of evolution operators $\{e^{-\overline{A}(z)(t-s)}, t \geq s\}$, $z \in Y$, has a uniform exponential dichotomy (as in Thm. 7.6.13) and so, by Thm. 7.6.11, the equation $x' + A(\tau/\varepsilon, z+\omega\tau/\varepsilon)x = 0$ has an exponential dichotomy on $\mathbb{R}$ with exponent $\beta/2$ and bound M (for $0 < \varepsilon \leq \varepsilon_0$ and each $z \in Y$). Hypotheses (i) and (iv) of Thm. 9.1.1 are easily verified (with $g_0 = 0$), so for small $\varepsilon > 0$ the (x,z)-system has an invariant manifold of the form $x = \sigma_\varepsilon(\tau/\varepsilon, z)$, or in original variables

$$\{(t,x,y) \mid x = \sigma_\varepsilon(t,y-\omega t),\quad (t,y) \in \mathbb{R} \times Y\}.$$

This is asymptotically stable with asymptotic phase if $\operatorname{Re} \sigma(\overline{A}(z)) > 0$, and is unstable otherwise. If $t \to (A,f,g)(t,x,z+\omega t)$ is uniformly almost periodic, $t \to \sigma_\varepsilon(t,z)$ is uniformly almost periodic as well. If A, f, g are multiply periodic in certain components of y, so they are unchanged when y is replaced by $y+\omega_k$ $(k = 1,\dots,m)$, the same is true for σ.

In particular, if $Y = 0$, $x = \sigma_\varepsilon(t)$ is the unique solution of $\dot{x} + \varepsilon A(t)x = \varepsilon f(t,x)$ on $-\infty < t < \infty$ with $\sup \|x(t)\|_\alpha \le r_0$, it is almost periodic when A, f are uniformly almost periodic, and it has the same stability properties as the solution $x = 0$ of the averaged equation $x' + \overline{A}x = \overline{f}(x)$. For ordinary differential equations, this result is due to Bogoliubov.

Example 2. Suppose A is sectorial in X, $\operatorname{Re} \sigma(A) \ge 0$, and $\sigma(A)$ meets the imaginary axis in a finite set of eigenvalues of finite multiplicity. Then $X = X_0 \oplus X_1$, $A = A_0 \oplus A_1$, $\operatorname{Re} \sigma(A_0) = 0$, $\operatorname{Re} \sigma(A_1) > 0$, $\dim X_0 < \infty$. We assume also $e^{A_0 t}$ is uniformly bounded on $-\infty < t < \infty$ (hence, almost periodic).

Assume $0 \le \alpha < 1$, $f\colon \mathbb{R} \times X^\alpha \to X$ is locally Hölder continuous in t, and in a neighborhood of each $x \in X^\alpha$, $t \to f(t,x)$ is uniformly almost periodic, and $x \to f(t,x)$ is uniformly $C^{1+\theta}$ (or C^1_{Lip}, if $\theta = 1$). Corresponding to $X = X_0 \oplus X_1$ write $f = f_0 + f_1$ and let

$$\overline{f}_0(\xi) = \lim_{T\to\infty} \frac{1}{T} \int_{t_0}^{t_0+T} e^{A_0 s} f_0(s, e^{-A_0 s}\xi)$$

for ξ in X_0. The limit exists by almost-periodicity, and is uniform in t_0.

If $\overline{f}_0(\xi_0) = 0$ and the derivative $\overline{f}_0'(\xi_0)$ has no eigenvalue on the imaginary axis, we show there is an almost-periodic solution $x_\varepsilon(t)$ of

$$\dot{x} + Ax = \varepsilon f(t,x),\quad -\infty < t < \infty,$$

with $\|x_\varepsilon(t) - e^{-A_0 t}\xi_0\|_\alpha$ uniformly small for small $\varepsilon > 0$. If $\operatorname{Re} \sigma(\overline{f}'(\xi_0)) < 0$, x_ε is asymptotically stable; otherwise it is unstable.

To see this let $x = e^{-A_0 t}\xi + \eta$ $(\xi \in X_0,\ \eta \in X_1^\alpha)$ so the equation becomes

$$\dot{\xi} = \varepsilon e^{A_0 t} f_0(t, e^{-A_0 t}\xi + \eta)$$

$$\dot{\eta} + A_1\eta = \varepsilon f_1(t, e^{-A_0 t}\xi + \eta).$$

Choose R so $\|e^{A_0 t}\xi_0\|_\alpha \leq R$ for all t; modify f outside $\{\xi + \eta \mid \xi \in X_0, \eta \in X_1, \|\xi\|_\alpha \leq R+1, \|\eta\|_\alpha \leq r\}$ for some $r > 0$ so $x \to f(t,x)$ is uniformly $C^{1+\theta}$ on $X_0 + \{\|\eta\|_\alpha \leq r, \eta \in X_1\}$; this is certainly possible since X_0 is finite dimensional. For small ε, Thm. 9.1.1 applies and there is an asymptotically stable invariant manifold $\eta = \sigma_\varepsilon(t,\xi)$ with σ_ε, $\partial\sigma_\varepsilon/\partial\xi \to 0$ as $\varepsilon \to 0$ uniformly in (t,ξ), and $t \to \sigma_\varepsilon(t,\xi)$ is uniformly almost periodic.

The flow in the invariant manifold is

$$\dot{\xi} = \varepsilon e^{A_0 t} f(t, e^{-A_0 t}\xi + \sigma_\varepsilon(t,\xi))$$

which is the standard form for the method of averaging [37]. In a neighborhood of ξ_0, this involves only the original (unmodified) function f. By Bogoliubov's theorem (or example 1 with X replaced by X_0) there is an almost periodic solution $\xi_\varepsilon(t)$ uniformly close to ξ_0 (for small ε), sharing the stability properties of the solution ξ_0 of $\xi' = \bar{f}_0(\xi)$. Since the invariant manifold is asymptotically stable with asymptotic phase, the solution $x_\varepsilon(t) = e^{-A_0 t}\xi_\varepsilon(t) + \sigma_\varepsilon(t,\xi_\varepsilon(t))$ also shares the stability properties of the equilibrium ξ_0.

<u>Exercise 7</u>. Suppose $t \to a(t,x)$ is uniformly a.p. for $0 \leq x \leq \pi$, with mean value $a_0(x)$ and

$$\bar{a}_0 = \frac{2}{\pi}\int_0^\pi a_0(x)\sin x\,dx \neq 0.$$

Prove the zero solution of

$$u_t = u_{xx} + u + \varepsilon a(t,x)u \qquad (0 < x < \pi)$$
$$u(0,t) = 0, \quad u(\pi,t) = 0$$

is asymptotically stable for small $\varepsilon > 0$ if $\bar{a}_0 < 0$, and is unstable if $\bar{a}_0 > 0$.

<u>Exercise 8</u>. Suppose D_1, D_2, ω are positive constants

$t \to (a,b,c,d)(x,t)$ are uniformly almost periodic for $0 \le x \le 1$ with mean value $(a_0,b_0,c_0,d_0)(x)$, and the mean value of $(a,b,c,d)(x,t)e^{\pm 2i\omega t}$ is $(a_{\mp 2},\dots,d_{\mp 2})(x)$. Assume $\int_0^1 (a_{\pm 2},b_{\pm 2},c_{\pm 2},d_{\pm 2})(x)dx = 0$ and

$$\int_0^1 (a_0,b_0,c_0,d_0)(x)dx = (\overline{a}_0,\overline{b}_0,\overline{c}_0,\overline{d}_0)$$

with $\overline{a}_0 + \overline{d}_0 \neq 0$. For small $\varepsilon > 0$, the zero solution of

$$\begin{aligned} u_t &= D_1 u_{xx} - \omega v + \varepsilon(au+bv) \\ v_t &= D_2 v_{xx} + \omega u + \varepsilon(cu+dv) \qquad (0 < x < 1) \\ u_x, v_x &= 0 \quad \text{when} \quad x = 0,1 \end{aligned}$$

is asymptotically stable if $\overline{a}_0+\overline{d}_0 < 0$ and unstable if $\overline{a}_0+\overline{d}_0 > 0$. What happens if $\int_0^1 (a_{\pm 2},\dots,d_{\pm 2})dx \neq 0$? (In obvious notation, if $\overline{a}_0+\overline{d}_0 \neq 0$, we have stability for small $\varepsilon > 0$ if

$$\overline{a}_0+\overline{d}_0 < 0 \quad \text{and} \quad (\overline{a}_0+\overline{d}_0)^2 + (\overline{b}_0-\overline{c}_0)^2 > |\overline{b}_2+\overline{c}_2+i\overline{a}_2-i\overline{d}_2|^2,$$

and instability if one or both inequalities are reversed.)

Example 3. Assume A, f, g satisfy (iii) of Thm. 9.1.1 uniformly for $0 \le \varepsilon \le \varepsilon_0$, and are uniformly continuous as $\varepsilon \to 0+$. Consider

$$\begin{aligned} \varepsilon \frac{dx}{dt} + A(t,y)x &= f(t,x,y,\varepsilon) \\ \frac{dy}{dt} &= g(t,x,y,\varepsilon) \end{aligned}$$

with

$$f(t,0,y,0) = 0, \quad f_x(t,0,y,0) = 0$$

and

$$|\mathrm{Re}\ \sigma(A(t,y))| \ge \beta > 0 \quad \text{for all} \quad t,y$$

$t \to A(t,y)-A_0$ uniformly continuous and $\{A(t,y)-A_0 \mid (t,y) \in \mathbb{R} \times Y\}$ in a compact set in $\mathscr{L}(X^\alpha,X)$. Also A_0 has compact resolvent.

Let $\tau = t/\varepsilon$ be the new time-variable so

$$dx/d\tau + A(\varepsilon\tau,y)x = f(\varepsilon\tau,x,y,\varepsilon)$$

$$dy/d\tau = \varepsilon g(\varepsilon\tau,x,y,\varepsilon).$$

For sufficiently small $\varepsilon > 0$ and any fixed $y \in Y$,

$$dx/d\tau + A(\varepsilon\tau, y)x = 0$$

has an exponential dichotomy on $\mathbb{R}$ with exponent $\beta/2$ and bound M independent of y and ε (Thm. 7.6.13). The Other hypotheses of Thm. 9.1.1 are easily verified so there is an invariant manifold of the form

$$x = \sigma_\varepsilon(\varepsilon\tau, y) = \sigma_\varepsilon(t,y)$$

with

$$\sigma_\varepsilon(t,y) \to 0, \ \frac{\partial\sigma_\varepsilon}{\partial y}(t,y) \to 0$$

uniformly as $\varepsilon \to 0+$.

In case $\mathrm{Re}\,\sigma(A(t,y)) > 0$, any solution (x,y) with $\|x(t_0)\|_\alpha$ sufficiently small has

$$\|x(t)-\sigma_\varepsilon(t,y(t))\|_\alpha \le Ce^{-\beta(t-t_0)/2\varepsilon}\|x(t_0)-\sigma_\varepsilon(t_0,y(t_0))\|_\alpha$$

for $t \ge t_0$.

Exercise 9. Suppose A is sectorial in X with compact resolvent, $0 \le \alpha < 1$, U is an open set in X^α, $f\colon \mathbb{R} \times U \times Y \times [0,\varepsilon_0] \to X$ is smooth and $Ax = f(t,x,y,0)$ has a smooth solution $x = \phi(t,y)$ with values in U, such that $\sigma(A-f_x(t,\phi(t,y),y,0))$ is uniformly bounded away from the imaginary axis.

Show, for small $\varepsilon > 0$, the system

$$\varepsilon\dot{x} + Ax = f(t,x,y,\varepsilon), \quad \dot{y} = g(t,x,y,\varepsilon)$$

has an invariant manifold (a graph over $\mathbb{R} \times Y$) near $x = \phi(t,y)$. (Use example 3 above; this case is similar to one treated by Zadiraka [ ] for ODEs.)

Thm. 9.1.1 and the lemmas may be combined to treat more complicated systems. As an example, we consider (briefly) the system

$$(*) \qquad \begin{aligned} &\dot{x} + A(t,y,z)x = f(t,x,y,z) \\ &\dot{y} = g(t,x,y,z) \\ &\varepsilon\dot{z} + B(t,x,y)z = h(t,x,y,z), \end{aligned}$$

for small $\varepsilon > 0$.

Theorem 9.1.10. Suppose Z is a Banach space, B_0 is sectorial in Z, W is a neighborhood of 0 in $Z^\alpha = D(B_0^\alpha)$. Assume X, A_0, α, f, g satisfy the requirements (i), (ii), (iv) of Thm. 9.1.1 when $z = 0$, and the smoothness and boundedness hypotheses of (iii) hold uniformly for $z \in W$, with Hölder continuity in z, and they apply as well to A_z, f_z, g_z.

Assume $(t,x,y) \to B(t,x,y)-B_0: \mathbb{R} \times U \times Y \to \mathscr{L}(Z^\alpha,Z)$ is bounded, locally Hölder continuous in t, differentiable in (x,y), and for each $(\lambda,\eta) \in \mathbb{R} \times Y$

$$\dot{z} + B(\lambda,0,\eta)z = 0$$

has an exponential dichotomy on $\mathbb{R}$ with exponent β' and bound N. (See Thm. 7.6.13 for some sufficient conditions.) Finally, $h: \mathbb{R} \times U \times Y \times W \to Z$ is locally Hölder continuous in t, differentiable in (x,y,z),

$$B-B_0,\ B_x,\ B_y,\ h,\ h_x,\ h_y,\ h_z$$

are bounded by N,

$$(x,y,z) \to B_x,\ B_y,\ h_x,\ h_y,\ h_z$$

are Hölder continuous with exponent θ and bound N and for all t,y

$$\|(h,h_z)(t,0,y,0)\| \le q$$

$$\|h_x(t,0,y,0)\| \le M_1, \quad \|f_z(t,0,y,0)\| \le M_2.$$

There exist positive constants r_0, ε_0, q_0, m_0 such that if $\varepsilon \le \varepsilon_0$, $q \le q_0$ and the product $M_1M_2 \le m_0$, there is an invariant manifold

$$S_\varepsilon = \{(t,x,y,z) \mid x = \sigma_\varepsilon(t,y),\ z = \rho_\varepsilon(t,y),\ (t,y) \in \mathbb{R} \times Y\}$$

for (*) which is the maximal invariant subset of $R \times \{\|x\|_\alpha \le r_0\} \times Y \times \{\|z\|_\alpha \le r_0\}$. The functions $y \to (\sigma_\varepsilon(t,y),\rho_\varepsilon(t,y)): Y \to X^\alpha \times Z^\alpha$ are differentiable, the derivatives are Holder continuous with exponent θ uniformly in t,y,ε,q and as $q \to 0$

$$\sigma_\varepsilon,\rho_\varepsilon,\ \frac{\partial\sigma_\varepsilon}{\partial y},\ \frac{\partial\rho_\varepsilon}{\partial y} \to 0$$

uniformly in t,y,ε.

If $t \to A, f, g, B, h$ are uniformly almost periodic, then $t \to (\sigma_\varepsilon(t,y), \rho_\varepsilon(t,y))$ is also uniformly almost periodic with frequency module in the joint module of A, f, g, B, h. If $x = 0$ is stable for $\dot{x} + A(t,y(t),0)x = 0$ when $\dot{y} = g_0(t,y)$, and $\mathrm{Re}\,\sigma(B(t,0,y)) > 0$, then S_ε is asymptotically stable with asymptotic phase. Otherwise S_ε is unstable.

<u>Sketch of proof of Thm. 9.1.10</u>. If $z: \mathbb{R} \to Z^\alpha$ is continuous with $\|z(t)\|_\alpha \le r_2$ (r_2 small) then, for example,

$$\left\| \int_{t_1}^{t_2} f_x(t,0,y,z(t))dt \right\| \le q + Nr_2^\theta \quad \text{when} \quad |t_1 - t_2| \le 1$$

and by Thm. 9.1.2 with q replaced by $q+Nr_2^\theta$ there is a unique solution of

$$\dot{x} + A(t,y,z(t))x = f(t,x,y,z(t)), \quad \dot{y} = g(t,x,y,z(t))$$

with $y(\tau) = \eta$ and $\sup \|x(t)\|_\alpha \le r_1 = C(q^{\delta/2} + M_2 r_2)$, for small q, r_2 and a constant C. Denoting this solution $\tilde{x}(t;\tau,\eta,z)$, it follows that

$$(\eta, z) \to \tilde{x}(\cdot;\tau,\eta,z)$$

is $C^{1+\theta}$ in the norms $\|\eta\|$, $|||z|||_{\gamma(1+\theta),\tau}$ and $|||x|||_{\gamma(1+\theta),\tau}$ (see Lemma 9.1.8) and following the proof of that lemma we find

$$\|\tilde{x}(t;\tau,\eta,z_1) - \tilde{x}(t;\tau,\eta,z_2)\|_\alpha$$

$$\le C'[M_2 + O((r_1+r_2)^{\delta\theta} + q^{\delta\theta/2})]e^{\gamma|t-\tau|}\,|||z_1 - z_2|||_{\gamma,\tau},$$

with a constant C' depending only on A_0, α, β, θ, μ and N.

Also if $\|\dot{y}(t)\| \le N$, $\varepsilon > 0$ is sufficiently small,

$$dz/ds + B(\varepsilon s,0,y(\varepsilon s))z = 0$$

has an exponential dichotomy on $\mathbb{R}$ with exponent $\beta'/2$ and bound $2N$ (Thn. 7.6.12) and we may suppose the same holds for $dz/ds + B(\varepsilon s,x(\varepsilon s),y(\varepsilon s))z = 0$ whenever $\sup \|x(t)\|_\alpha \le r_1$, with r_1 small enough. Applying Thm. 9.1.2 to

$$dz/ds + B(\varepsilon s,x(\varepsilon s),y(\varepsilon s))z = h(\varepsilon s,x(\varepsilon s),y,z)$$

$$dy/ds = \varepsilon g(\varepsilon s,x(\varepsilon s),y,z)$$

given $y = \eta$ for $s = \varepsilon\tau$, there is a unique solution with $\sup \|z\|_\alpha \le r_2$ provided ε, q, r_1, r_2 are small enough and $r_2 \ge C''(q + M_1 r_1)$. Writing $\tilde{z}(t;\tau,\eta,x,\varepsilon)$ for this solution, in the original time variables, let $\hat{x}\colon \mathbb{R} \to X^\alpha$ be continuous and $O(e^{\gamma|t-\tau|})$ and $\zeta(t) = \frac{\partial \tilde{z}}{\partial x}(t;\tau,\eta,x,\varepsilon)\hat{x}$. Then $\varepsilon\, d\zeta/dt + (B-h_z)\zeta = (h_x - B_x\tilde{z})\hat{x} + (h_y - B_y\tilde{z})\frac{\partial\tilde{y}}{\partial x}\,\hat{x}$ and note $\|h_x\| \le M_1 + N(r_1^\theta + r_2^\theta)$,

$$\|h_y\| \le (N+2)\, q + M_1 r_1 + q r_2 + N(r_1^{1+\theta} + r_2^{1+\theta})^{\theta/(1+\theta)}$$

(cf. remark following statement of Thm. 9.1.1). Also $|||\frac{\partial\tilde{y}}{\partial x}\,\hat{x}|||'_{\gamma,\tau} \le C'''|||\hat{x}|||_{\gamma,\tau}$ for a constant C''', so

$$|||\frac{\partial\tilde{z}}{\partial x}\,\hat{x}|||_{\gamma,\tau} \le C'''[M_1 + O(r_1+r_2)^\theta + q^{\theta/(1+\theta)})]\,|||\hat{x}|||_{\gamma,\tau}.$$

If the product $M_1 M_2$ is sufficiently small, we may choose $r_1, r_2 = O(q^{\delta/2})$ so all the estimates above hold (for small q,ε) and so that

$$x \to \tilde{x}(\cdot,\tau,\eta,\tilde{z}(\cdot,\tau,\eta,x,\varepsilon))$$

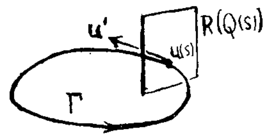
is a contraction in the norm $|||\cdot|||_{\gamma,\tau}$ on the ball $\sup\|x(t)\|_\alpha \le r_1$, uniformly for (τ,η) in $\mathbb{R} \times Y$. The fixed point x, with the corresponding y and z, is the unique solution of (*) on $-\infty < t < \infty$ with $\sup\|x(t)\|_\alpha \le r_1$, $\sup\|z(t)\|_\alpha \le r_2$ and $y(\tau) = \eta$.

9.2. A coordinate system near an invariant manifold

We introduce a coordinate system generalizing that used by Urabe and Hale to study the neighborhood of a periodic orbit of an autonomous ODE. If $u\colon \mathbb{R} \to \mathbb{R}^n$ is a C^{r+1} function $(r \ge 0)$ with derivative $u'(s) \ne 0$ and $u(s+p) = u(s)$ for all s and some (least) $p > 0$, then $\Gamma = \{u(s) \mid 0 \le s \le p\}$ is a C^{r+1} simple closed curve in $\mathbb{R}^n$. It may be shown that there is a C^r function $Q\colon \mathbb{R} \to \mathscr{L}(\mathbb{R}^{n-1}, \mathbb{R}^n)$ with $Q(s+p) = Q(s)$, $Q(s)$ is injective and $\operatorname{span}\{u'(s), Q(s)\mathbb{R}^{n-1}\} = \mathbb{R}^n$ for each s. We may approximate Q by a C^{r+1} function with the same properties, so assume Q is C^{r+1}. For small $\delta > 0$, let $B_\delta(\mathbb{R}^{n-1}) = \delta$-ball about 0 in $\mathbb{R}^{n-1}$; then

$$(\exp(2\pi i s/p), v) \mapsto (s,v) \mapsto u(s) + Q(s)v$$

defines a C^{r+1} diffeomorphism from $S^1 \times B_\delta(\mathbb{R}^{n-1})$ onto a neighborhood of Γ.



Now suppose M is a compact C^{r+1} m-dimensional submanifold of $\mathbb{R}^n$ $(m < n)$. See Abraham and Robbins [104], Spivak [133] or Golubitsky and Guillemin [129] for definitions and basic theorems about manifolds and vector bundles. Roughly, a vector bundle over M is a disjoint union $E = \bigcup_{p \in M} \{p\} \times E_p$ of vector spaces E_p, one associated with each point p of M, tied together with continuity conditions to make a manifold; and a map between two such bundles E and F consists of a map h of the "base space" M and a linear map of the vector spaces $h\#(p): E_p \to F_{h(p)}$ for each $p \in M$, both depending continuously on p. If E,F are vector bundles over M, their sum $E \oplus F$ is a vector bundle over M such that the vector space $(E \oplus F)_p$ associated with $p \in M$ is $E_p \oplus F_p$. Important examples of vector bundles are the tangent bundle TM and the normal bundle N of M, T_pM = tangent space to M at $p \in M$, N_p = orthogonal complement of T_pM in $\mathbb{R}^n$; then $TM \oplus N = M \times \mathbb{R}^n$. A vector bundle E over M is called <u>trivial</u> if it is a product $M \times V$, for some vector space V, or equivalent to a product, i.e., there is a linear isomorphism $Q(p): V \to E_p$ for each $p \in M$, depending continuously on p, thus $(p,v) \mapsto (p,Q(p)v): M \times V \to E$ is a vector bundle map, as is its inverse.

Suppose there is a C^{r+1} map $Q: M \to \mathscr{L}(\mathbb{R}^{n-m},\mathbb{R}^n)$ with $Q(p)$ injective for each p and $T_pM \oplus R(Q(p)) = \mathbb{R}^n$ for each $p \in M$. By the implicit function theorem, $(p,v) \mapsto p + Q(p)v$ is a diffeomorphism of $M \times B_\delta(\mathbb{R}^{n-m})$ onto a neighborhood of M (some $\delta > 0$), generalizing the coordinate system described above for the case $m = 1$. If $R(Q(p)) = N_p$ for each p, Q defines an equivalence from $M \times \mathbb{R}^{n-m}$ to the normal bundle N; in any case, we may follow $Q(p)$ by the orthogonal projection onto N_p. Thus the existence of such a map Q is equivalent to triviality of the normal bundle, a property studied with more care below.

Suppose M is invariant under the flow $\dot{z} = h(z)$, h defined and C^r on a neighborhood of $M \subset \mathbb{R}^n$. In the new coordinates (x,y) with $y \in M$, $x \in \mathbb{R}^{n-m}$ $(|x| < \delta)$ and $z = y + Q(y)x$, the flow becomes

$$\dot{y} = g(x,y) = (I+P(y)Q'(y)x)^{-1}P(y)h(y+Q(y)x)$$

$$\dot{x} = A(y)x + f(x,y)$$

where $P(y)$ is the projection of $\mathbb{R}^n$ onto T_yM, along $R(Q(y))$, so $g(x,y) \in T_yM$, $g(0,y) = h(y)$, and

$$A(y)x = Q(y)^{-1}(1-P(y))(h'(y)(Q(y)x) - (Q'(y)h(y))x)$$

$$f(x,y) = o(|x|) \quad \text{as} \quad x \to 0.$$

Since $P(y)$ and $Q'(y)$ are C^r, $(x,y) \to g(x,y)$, $A(y)x + f(x,y)$ are C^r functions, although $y \to A(y)$ is merely C^{r-1}.

To apply our invariant manifold theorem to the (x,y)-system, we need information about

$$\dot{\xi} = A(y(t))\xi \quad \text{where} \quad \dot{y} = h(y), \quad y(t) \in M. \qquad (*)$$

Such information may often be derived from information about solutions ζ of

$$\dot{\zeta} = h'(y(t))\zeta. \qquad (**)$$

Indeed if ζ is any solution of (**) and $\xi(t)$ is defined by $(1-P(y(t)))\zeta(t) = Q(y(t))\xi(t)$, then ξ solves (*); and conversely, for any solution ξ of (*) there is a solution ζ of (**) with $\zeta(t) - Q(y(t))\xi(t) \in T_{y(t)}M$ for all t. To see this, let $z_\varepsilon(t)$ be the solution of $\dot{z} = h(z)$ with $z_\varepsilon(0) = y(0) + \varepsilon\zeta(0)$, where ζ solves (**). Then $z_\varepsilon(t) = y(t) + \varepsilon\zeta(t) + o(\varepsilon) = y_\varepsilon(t) + Q(y_\varepsilon(t))x_\varepsilon(t)$ (uniformly on compact t-invervals) so $y_\varepsilon(t) - y(t) = O(\varepsilon)$, $x_\varepsilon(t) = O(\varepsilon)$, and $(1-P(y(t)))(y_\varepsilon(t)-y(t)) = o(\varepsilon)$ since $(1-P(y))T_yM = 0$. Define $\xi(t) \in \mathbb{R}^{n-m}$, $\eta(t) \in T_{y(t)}M$ by $\zeta(t) = \eta(t) + Q(y(t))\xi(t)$; it follows that $y_\varepsilon(t) = y(t) + \varepsilon\eta(t) + o(\varepsilon)$, $x_\varepsilon(t) = \varepsilon\xi(t) + o(\varepsilon)$, and $\dot{x}_\varepsilon(t) = A(y_\varepsilon(t))x_\varepsilon(t) + o(\varepsilon) = \varepsilon A(y(t))\xi(t) + o(\varepsilon)$ so $\dot{\xi}(t) = A(y(t))\xi(t)$. Conversely, if $\xi(t)$ solves (*) and $\eta(t) \in T_{y(t)}M$ is defined by

$$\dot{\eta} = g_x(0,y(t))\xi + g_y(0,y(t))\eta, \ \eta(0) = 0,$$

we find $\zeta = \eta + Q(y)\xi$ solves (**).

Thus, for example, if every solution ζ of (**) satisfies

$$\zeta(t) - \eta(t) = O(e^{-\beta t})$$

for some solution $\eta(t) \in T_{y(t)}M$ of $\dot{\eta} = h'(y(t))\eta$ and some $\beta > 0$, then every solution ξ of (*) is $O(e^{-\beta t})$. For more detailed results, see Ex. 2 below.

The essential property needed to construct this coordinate system is triviality of the normal bundle, so we will study this property in some detail. If the normal bundle is not trivial, we can cover a neighborhood of M with open sets, in each of which we have a coordinate system as above. But fortunately the most important applications involve a trivial normal bundle and a single coordinate system is sufficient.

First observe that a trivial vector bundle is orientable. If the normal bundle N over M is trivial, then $TM \oplus N = M \times \mathbb{R}^n$ and N are both orientable and it follows that TM and M are orientable. A basis of T_pM may be oriented by requiring that, combined with a positively oriented basis for N_p, it gives a positive basis for $\mathbb{R}^n$. Thus orientability of M is a necessary condition for triviality of N. Another necessary condition is that TM is stably trivial, i.e., that there exists a trivial bundle (in this case, N) whose sum with TM is trivial. Observe that stable triviality of TM is an intrinsic property of M, unchanged by diffeomorphism. Now suppose TM is stably trivial; then there exists a trivial bundle E such that $TM \oplus E$ is trivial, so $N \oplus (TM \oplus E) = (M \times \mathbb{R}^n) \oplus E$, a sum of trivial bundles, is trivial and N is stably trivial. Then for some integer p, $N \oplus M \times \mathbb{R}^p$ is trivial, and if we consider M as a submanifold of $\mathbb{R}^{n+p}$ $(M \subset \mathbb{R}^n \times 0 \subset \mathbb{R}^{n+p})$, the normal bundle expands to $N \oplus M \times \mathbb{R}^p$. Thus, if we are able to expand the containing space to $\mathbb{R}^{n+p}$, stable triviality of TM is a necessary and sufficient condition for triviality of N. If $n > 2m$ (m = dimension M, $M \subset \mathbb{R}^n$), we need not expand the space: according to exercise 1 below, if N is stably trivial and $n > 2m$, then N is trivial.

Examples.

(i) If U is an open set in $\mathbb{R}^n$, U has trivial normal bundle (namely $U \times \{0\}$).

(ii) If U is an open set in $\mathbb{R}^n$ and $M = \partial U$, and U lies on one side of M, M has trivial normal bundle. The unit outward normal vector defines a basis for the normal space at each point of M.

(iii) The normal bundle of the sphere $S^2 \subset \mathbb{R}^3$ is trivial. The tangent bundle $T(S^2)$ is stably trivial, but not trivial. For a trivial bundle has nonzero continuous sections, so if $T(S^2)$ were trivial there would exist a continuous tangent vector field on S^2 which never vanishes, contrary to the "hairy ball" theorem. (A marvelously simple proof of this theorem is given by Milnor, Am. Math. Mo., Aug.-Sept. 1978, p. 521-524.)

(iv) The normal bundle of $S^1 \subset \mathbb{R}^2$ is trivial - as is the tangent bundle. Thus any C^2 simple closed curve in $\mathbb{R}^n$ $(n \geq 2)$ has a trivial normal bundle.

(v) If $M_1 \subset \mathbb{R}^{n_1}$ and $M_2 \subset \mathbb{R}^{n_2}$ both have trivial normal bundles, then $M_1 \times M_2 \subset \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ has a trivial normal bundle.

(vi) The Möbius strip in $\mathbb{R}^3$ is not orientable so both normal and tangent bundles are nontrivial.

Exercise 1. Suppose M is a compact C^2 m-dimensional submanifold of $\mathbb{R}^n$, $n > 2m$, with stably trivial normal bundle N. Specifically $N \oplus M \times \mathbb{R}^p$ is trivial. Prove N is trivial. (It is sufficient to prove this for $p = 1$.)

To do this consider $M \subset \mathbb{R}^n \times 0 \subset \mathbb{R}^{n+1}$ $(p = 1)$ and let $w = (0,\dots,0,1) \in \mathbb{R}^{n+1}$ so $w \perp N_x, T_x(M)$ for each $x \in M$. If $k = n-m$, the assumed triviality says there is a C^1 $Q: M \to \mathscr{L}(\mathbb{R}^{k+1},\mathbb{R}^{n+1})$ with $Q(x)$ an isomorphism of $\mathbb{R}^{k+1}$ onto $N_x \oplus \text{span}\{w\}$ for each x, and we may suppose $Q(x)$ is orthogonal for each x. Define $v(x)$ by $Q(x)v(x) = w$, $v(x) \in S^k \subset \mathbb{R}^{k+1}$, and choose an orthonormal basis for $\mathbb{R}^{k+1}\{e_1,\dots,e_{k+1}\}$ with $e_{k+1} \neq \pm v(x)$ for all $x \in M$, so $Q(x)e_{k+1} \neq \pm w$ for all x. There is a rotation $S(x)$ of $\mathbb{R}^{n+1}$ which takes $Q(x)e_{k+1}$ to w and leaves fixed the orthogonal complement of $\text{span}\{Q(x)e_{k+1},w\}$. It follows that $\{S(x)Q(x)e_1,\dots,S(x)Q(x)e_k\}$ is a basis for N_x, a C^1 function of $x \in M$, and so N is trivial.

Remark. I am indebted to Prof. Paul Baum of Brown University for suggesting the use of stable triviality.

The result of exercise 1 can be generalized: any stably trivial vector bundle whose fiber dimension exceeds its base dimension is trivial. The proof outlined above can be modified to prove this result, using the fact that a stably trivial bundle is orientable. This generalization was shown to me by Paul Baum with a different proof.

The next result is a version of Whitney's embedding theorem, proved using the transversality theorem [104].

Lemma 9.2.1. Suppose X is a Banach space of dimension $\geq 2m+1$, and M is a C^r $(r \geq 2)$ m-dimensional compact submanifold of X. There exists a projection P in $\mathscr{L}(X)$ whose image is $(2m+1)$-dimensional such that the restriction $P|M$ is a C^r diffeomorphism of M to its image in $R(P)$, and $T_x(M) \cap N(P) = \{0\}$ for each $x \in M$. If $\tilde{P}$ is any projection in $\mathscr{L}(X)$ with $\|\tilde{P}-P\|$ sufficiently small, then $\tilde{P}$ satisfies the same conditions.

Proof. Let $M^{(2)} = \{(x,y) \in M \times M \mid x \neq y\}$ and let $(X^*)^n$ be the n-fold Cartesian product $(n = 2m+1)$, and $(TM)_o = \{(x,v) \in TM \mid x \in M,\ v \neq 0\}$. Define

$$F\colon M^{(2)} \times (X^*)^n \to \mathbb{R}^n\colon\ (x,y,\zeta_1,\dots,\zeta_n) \mapsto (\zeta_1(x-y),\dots,\zeta_n(x-y))$$

$$G\colon (TM)_o \times (X^*)^n \to \mathbb{R}^n\colon\ (x,v,\zeta_1,\dots,\zeta_n) \mapsto (\zeta_1(v),\dots,\zeta_n(v)).$$

Suppose for some $z = (\zeta_1,\dots,\zeta_n)$ that $0 \in \mathbb{R}^n$ is a regular value of $F(\cdot,z)$ and of $G(\cdot,z)$. Then $F(\cdot,z)^{-1}(0)$ and $G(\cdot,z)^{-1}(0)$ are either empty or submanifolds of $M^{(2)}$ or $(TM)_o$ or codimension n. Since $n > 2m$, they are empty. Since $F(x,y,z) \neq 0$ on $M^{(2)}$, $x \mapsto z(x)\colon M \to \mathbb{R}^n$ is injective; and $z(v) \neq 0$ if $v \neq 0$, $(x,v) \in TM$. Choose $w_1,\dots,w_n$ in X so $\zeta_k(w_j) = \delta_{jk}$, $z = (\zeta_1,\dots,\zeta_n)$. Then $P = \sum_1^n w_k \otimes \zeta_k$, $Px = \sum_1^n w_k\zeta_k(x)$, satisfies the conditions of the lemma.

It remains to prove such z exists. If X^* is separable, we apply the transversality theorem directly and show an open dense set in $(X^*)^n$ will serve, by proving zero is a regular value of F and G [104]. Suppose $F(x,y,z) = 0$ and $u \in \mathbb{R}^n$; $(x,y) \in M^{(2)}$ means $x-y \neq 0$ so there exists $\zeta \in X^*$ with $\zeta(x-y) = 1$ and $u\zeta = (u_1\zeta,\dots,u_n\zeta) \in (X^*)^n$ has $D_zF(x,y,z)(u\zeta) = u$. Thus $D_zF(x,y,z)$ is surjective and 0 is a regular value of F; the argument for G is similar.

Suppose X^* is not separable. Choose a dense sequence $\{(x_k,v_k)\}_{k=1}^\infty$ in TM and let

$$X_n = \operatorname{span}\{x_i, x_j - x_k, v_\ell \mid 1 \leq i,j,k,\ell \leq n\}.$$

Define $X_n^* \subset X^*$ of dimension $= \dim X_n$ for each n with $X_n^* \supset X_{n-1}^*$, $X_0^* = 0$; and define Z^* as the closure of $\bigcup_{n=1}^\infty X_n^*$. Then Z^* is separable, and the restrictions of F (or G) to $M^{(2)}$ (or $(TM)_o$) $\times (Z^*)^n$ may be studied as above, with the same conclusion.

Remark. The argument we present is essentially that in Guillemin and Pollack, Differential Topology (Prentice-Hall, 1974, p. 51), modified for infinite dimensions.

Theorem 9.2.2. Suppose X is a Banach space, A is sectorial in X, $D(A^*)$ is dense in X^* (which is true if X is reflexive, but see ex. 3), $0 \leq \alpha < 1$, U is an open set in X^α, $f\colon U \to X$ is a C^r map, $r \geq 1$, and M is a compact C^{r+1} submanifold of X of dimension m, invariant under $dx/dt + Ax = f(x)$. Also assume that the tangent bundle TM is stably trivial and $\dim X > 2m$, or that M has trivial normal bundle.

There is a closed subspace $Z \subset X$ of codimension m and a C^{r+1} map $Q: M \to \mathscr{L}(X)$ such that $Q(y)|Z$ is an isomorphism of Z to its image with $Q(y)Z \oplus T_yM = X$ for each $y \in M$, $AQ(y) - Q(y)A$ extends uniquely to a bounded linear operator on X, and a neighborhood of M is represented as

$$\{y + Q(y)z: y \in M,\ z \in Z,\ \|z\| < \delta\}$$

with $(y,z) \to y + Q(y)z$ a C^{r+1} diffeomorphism onto this neighborhood.

The flow $\dot{x} + Ax = f(x)$ in a neighborhood of M becomes

$$\dot{y} = g(y,z),\quad \dot{z} + Bz = h(y,z)$$

where B is sectorial in Z, $D(B) = D(A) \cap Z$, $Z^\alpha = Z \cap X^\alpha$, $h(y,0) = 0$, $g(y,0) = -Ay + f(y) \in T_yM$. The functions g,h are C^r functions of $y \in M$, $z \in Z^\alpha$, $\|z\|_\alpha < \delta$. If $\dot{y} + Ay = f(y)$ in M and $\dot{x} + Ax = f'(y(t))x$, there is a corresponding solution z of $\dot{z} + Bz = \frac{\partial h}{\partial z}(y(t),0)z$ with $x(t) - Q(y(t))z(t) \in T_{y(t)}M$ for each t, and conversely.

Proof. The case $\dim X < \infty$ was discussed above so assume X is infinite dimensional. Choose a projection P as in Lemma 9.2.1; we may assume that $R(P) \subset D(A)$ and $R(P^*) \subset D(A^*)$, since $D(A)$, $D(A^*)$ are dense, and that $P(M)$ has trivial normal bundle in $R(P)$. Note AP and A^*P^* are bounded, so PA extends uniquely to a bounded linear operator on X. There is a subspace $W \subset R(P)$ and a C^{r+1} map $Q_0: P(M) \to \mathscr{L}(W,R(P))$ such that $Q_0(y)$ is injective and $R(Q_0(y)) \oplus T_y(PM) = R(P)$ for each $y \in PM$. Since $N(P) \cap T_yM = 0$ for $y \in M$, P maps T_yM isomorphically onto $T_{Py}(PM)$ and we may take $Z = N(P) \oplus W$ and define $Q(y)(v+w) = v + Q_0(Py)w$ for $v \in N(P)$, $w \in W$, $y \in M$. Also $X = Z \oplus \overline{Z}$, $\dim \overline{Z} = m$, $\overline{Z} \subset R(P)$, and we let $Q(y)\overline{Z} = 0$. If $y \in M$, $z \in Z$, $\overline{z} \in \overline{Z}$,

$$(AQ(y)-Q(y)A)(z+\overline{z}) = APQ_0(Py)Pz + PA(1-P)z - (Q(y)PA+(1-P)AP)(z+\overline{z})$$

so $AQ(y) - Q(y)A$ may be extended to a bounded linear map on X. It is easy to see $Q(y)|Z$ is an isomorphism of Z onto $R(Q(y))$.

If $B = (1-P)A|Z$, it is easily seen that B is sectorial in Z with domain $D(A) \cap Z$ dense in Z. If $P(y)$ is the projection of X onto T_yM along $R(Q(y))$, the flow $\dot{x} + Ax = f(x)$ becomes

$$(1+P(y)Q'(y)z)\dot{y} + Ay - f(y) = P(y)[f(y+Q(y)z-f(y)+(Q(y)A-AQ(y))z]$$

or $\dot{y} = g(y,z)$, and

$$Q(y)(\dot{z}+Bz) = -Q(y)[PA + (1-P)AP]z + (1-P(y))[f(y+Q(y)z)-f(y) + (Q(y)A-AQ(y))z] - Q'(y)g(y,z)z$$

with $x = y + Q(y)z$. Since the right-side of the z-equation has values in $R(Q(y)) = N(P(y))$, we may apply $(Q(y)|Z)^{-1}$ and solve for

$$\dot{z} + Bz = h(y,z).$$

The other claims of the theorem are easily verified.

Remarks. If we apply the same change of variables to $\dot{x} + Ax = F(t,x)$, the resulting y and z equations are obtained by replacing the term $f(y+Q(y)z)$ by $F(t,y+Q(y)z)$.

Observe that, if A has compact resolvent, then B also has compact resolvent.

In place of C^r smoothness we may equally consider C^{r-1}_{Lip} when r is an integer ≥ 2.

Exercise 2. Assume the hypotheses of Thm. 9.2.2 hold, and in addition, for every solution $y(t) \in M$ of $\dot{y} + Ay = f(y)$, every solution x of $\dot{x} + Ax = f'(y(t))x$ is a sum $x(t) = x_+(t) + x_0(t) + x_-(t)$ of solutions with $x_0(t) \in T_{y(t)}M$, $\|x_+(t)\| \leq Me^{-\beta(t-\tau)}\|x_+(\tau)\|$ for $t \geq \tau$, $\|x_-(t)\| \leq Me^{-\beta(\tau-t)}\|x_-(\tau)\|$ for $t \leq \tau$, (for some positive constants β, M) and there are projections $P_{+,-,0}(t)$, $x_\sigma(t) = P_\sigma(t)x(t)$ for $\sigma = +,-,0$, $\|P_\sigma(t)\| \leq M$. Finally suppose $(1-P_0(t))|R(Q(y(t)))$ is an isomorphism of $R(Q(y(t)))$ onto $N(P_0(t))$ with

$$\frac{1}{K}\|z\| \leq \|(1-P_0(t))Q(y(t))z\| \leq K\|z\|, \quad z \in Z.$$

Prove $\dot{z} + Bz = h_z(y(t),0)z$ has an exponential dichotomy on $\mathbb{R}$ with exponent β and bound K^2M. The projections may be taken to be $\tilde{Q}(t)^{-1}P_\pm(t)\tilde{Q}(t)$ where $\tilde{Q}(t) = (1-P_0(t))Q(y(t))$.

It may be useful to consider first the case when M is a periodic orbit with 1 as a simple multiplier and the remainder of the spectrum of the period map off the unit circle. See also example 1 of the next section.

Exercise 3. In Thm. 9.2.2, $D(A^*)$ was assumed to be dense merely to ensure we could take $R(P^*) \subset D(A^*)$. In fact $D(A^*)$ is always dense in the weak* topology. Using compactness of M prove we may choose the projection P with $R(P) \subset D(A)$ and $R(P^*) \subset D(A^*)$ even if $D(A^*)$ is not dense.

9.3 Examples.

Example 1. Suppose A is sectorial in X with compact resolvent, $0 \le \alpha < 1$, U is open in X^α, and $f: U \to X$ is C^r $(r \ge 1)$, or C^{r-1}_{Lip}. Assume there is a nonconstant periodic solution $u(t)$ in U of

$$\dot{x} + Ax = f(x),$$

$u(t+p) = u(t)$, and the period map of the linearization,

$$\dot{x} + Ax = f'(u(t))x$$

has 1 as a simple multiplier, with the remainder of the spectrum off the unit circle. Assume $t \to u(t) \in X$ is C^{r+1} (or C^r_{Lip}, respectively); it is certainly $C^{r+\mu}$ for any $\mu < 1$.

Introduce coordinates (s,z) in a neighborhood of the periodic orbit Γ

$$x = u(s) + Q(s)z \quad (s \text{ real}, z \in Z, \|z\| < \delta)$$

where $Z \subset X$ has codimension one and $s \to Q(s)$ is periodic with period p and is of class C^{r+1}. The flow near the periodic orbit has the form

$$\dot{s} = 1 + S(s,z)$$
$$\dot{z} + \tilde{A}z = h(s,z)$$

where S,h are C^r, p-periodic in s and vanish when $z = 0$, and $\tilde{A}$ is sectorial in Z with compact resolvent.

Linearizing about $z = 0$ we obtain

$$\dot{s} = 1, \quad \dot{z} + \tilde{A}z = \frac{\partial h}{\partial z}(s,0)z.$$

Since $s(t) = t + s(0)$, the z-equation has p-periodic coefficients, and its multipliers are the numbers $e^{\lambda p}$ such that there is a nontrivial solution $z(t)$ with $e^{-\lambda t}z(t)$ p-periodic. Suppose there is such a solution $z(t)$ bounded on $-\infty < t < \infty$, i.e. $|e^{\lambda p}| = 1$, and $s(0) = 0$ (for simplicity); then there is a solution $x(t)$ of $\dot{x} + Ax = f'(u(t))x$ with $x(t) - Q(t)z(t) \in T_{u(t)}\Gamma$, i.e., for some scalar $\eta(t)$, $x(t) = Q(t)z(t) + \eta(t)u'(t)$. Now there is a constant c such that $x(t) - cu'(t)$ is identically zero or is unbounded as $t \to +\infty$ or as $t \to -\infty$. It follows that $\eta(t) = c$ and $z(t) = 0$ for all t. Thus the linearized z-equation has no multipliers on the unit circle and so, has an exponential dichotomy on $\mathbb{R}$. (In fact, a similar argument shows the multipliers of $\dot{z} + \tilde{A}z = h_z(t,0)z$ are the multipliers $\neq 1$ of $\dot{x} + Ax = f'(u(t))x$.) If A does not have compact resolvent, we still have an exponential dichotomy (see ex. 2, sec. 9.2).

If the multipliers $\neq 1$ of the linearized x-equation are all inside the unit circle, it follows from Thm. 9.1.1 that the solution $x = u(t)$ is orbitally asymptotically stable with asymptotic phase, but it is orbitally unstable if some multiplier has modulus greater than 1. Of course, this was proved before with weaker assumptions, but introduction of the coordinate system enables us to treat perturbations of the system in a unified way.

Now suppose f is C^1_{Lip}, u is C^2_{Lip}, $F: \mathbb{R} \times U \to X$ is bounded, locally Hölder continuous in t and C^1_{Lip} in x. The equation $\dot{x} + Ax = f(x) + F(t,x)$ becomes

$$\dot{s} = 1 + g(s,z,t), \quad \dot{z} + \tilde{A}(s)z = h_1(s,z,t)$$

where $\tilde{A}(s) = \tilde{A} - \frac{\partial h}{\partial z}(s,0)$ and $s \to \tilde{A}(s)$, g, h_1 are p-periodic $g = 0$ if $z = 0$ and $F = 0$, and $h_1(s,z,t) = O(\|z\|^2_\alpha)$ if $F = 0$. When $t \to F(t,x)$ is uniformly almost periodic then $t \to g, h_1$ are also uniformly a.p. with the same frequency module. If $t \to F(t,u(s))$, $F_x(t,u(s))$ are integrally small uniformly in s, there is an invariant manifold $z = \sigma(s,t)$, or in the original variables

$$x = u(s) + Q(s)\sigma(s,t).$$

This is p-periodic in s, and when $t \to F(t,x)$ is uniformly almost periodic, it is u.a.p. in t with frequency module contained in the module of F, and σ, $\frac{\partial\sigma}{\partial s} \to 0$ as $q \to 0$ uniformly in s,t $(\|\int_{t_1}^{t_2}(F(t,u(s)),F_x(t,u(s)))dt\| \le q$ for $|t_1-t_2| \le 1)$. If F is independent of t (frequency module $= \{0\}$) σ is also independent of t and $x = u(s) + Q(s)\sigma(s)$ is the unique periodic orbit near Γ of the perturbed equation, and it shares the stability properties of Γ. The period is the solution T_F of $s(T_F) = p$, $\dot{s} = 1 + g(s,\sigma(s))$, $s(0) = 0$. If F is T-periodic in t $(T > 0)$, the frequency module is $\{nT: n = 0,\pm1,\pm2,\ldots\}$ and σ is also T-periodic in t. We may identify the time-slices of the invariant manifold at $t = 0$ and $t = T$, thus obtaining a two-dimensional torus. The flow on this torus is given by

$$\dot{s} = 1 + g(s,\sigma(s,t),t)$$

and the function on the right is p-periodic in s, T-periodic in t, and C^1_{Lip} as a function of s. This degree of smoothness is sufficient to apply Denjoy's theorem and other results on the flow on a torus (see [37 , 89]).

Example 2. Suppose X, A, f satisfy the conditions of Example 1, but there is a (k+1)-parameter family of periodic solutions

$$x(t) = u(\omega(b)t+\phi,b)$$

$(-\infty < \phi < \infty,\ b \in \mathbb{R}^k$ with $|b| \le 1)$ of $\dot{x} + Ax = f(x)$, where $u(s+1,b) = u(s,b)$, $\omega(b) > 0$, $(s,b) \to u(s,b),\omega(b)$ are C^{r+1} functions and

$$\frac{\partial u}{\partial s}, \frac{\partial u}{\partial b_1}, \ldots, \frac{\partial u}{\partial b_k}$$

are linearly independent for each (s,b) in $\mathbb{R} \times B$, $B = \{b \in \mathbb{R}^k: |b| \le 1\}$. Then

$$M = \{u(s,b) \mid 0 \le s \le 1,\ \ b \in B\}$$

is a compact C^{r+1} (k+1)-dimensional invariant manifold with boundary, essentially $S^1 \times B$. Since $S^1 \times \overset{\circ}{B}$ ($\overset{\circ}{B}$ = interior of B) has trivial normal bundle, we may construct a coordinate system $x = u(s,b) + Q(s,b)z$ where $z \in Z$, a subspace of X of codimension (k+1), and the flow near $\overset{\circ}{M}$ (= M, less its boundary), becomes

$$\dot{s} = \omega(b) + S(s,b,z)$$

$$\dot{b} = B(s,b,z)$$

$$\dot{z} + \tilde{A}z = h(s,b,z)$$

where S,B,h have period 1 in s and vanish when $z = 0$ (since the flow in $\overset{\circ}{M}$ is $\dot{s} = \omega(b)$, $\dot{b} = 0$, $z = 0$). To avoid complications near the boundary of M, we may choose a smooth function $\psi(b)$ such that $\psi(b) = 1$ for $|b| \le 1-\delta < 1$ and $\psi(b) = 0$ for $|b| \ge 1-\delta/2$, and replace B in the equations above by $\psi(b)B(s,b,z)$. This has no effect on solutions which remain in $|b| \le 1-\delta$.

In the linearization

$$\dot{x} + Ax = f'(u(\omega(b)t+\phi,b))x$$

with ϕ and b constant, 1 is a multiplier of multiplicity at least (k+1), since $\partial u/\partial s$ and $\partial u/\partial b_j + t\, \partial u/\partial s\ \partial\omega/\partial b_j$ $(1 \le j \le k)$ satisfy the equation. We assume the multiplicity of 1 is exactly (k+1) and there are no other multipliers on the unit circle. As before, it follows that

$$\dot{z} + \tilde{A}z = \frac{\partial h}{\partial z}(s,b,0)z \quad (\dot{b} = 0,\ \ \dot{s} = \omega(b))$$

has an exponential dichotomy on $\mathbb{R}$. By Thm. 7.6.11, we may assume the exponent and bound of the dichotomy are independent of $s(0)$ and b, at least for $|b| \leq 1-\delta/2$.

If the multipliers $\neq 1$ are all inside the unit circle, $\mathring{M}$ is asymptotically stable with asymptotic phase. Specifically if $x(t)$ is a solution with $\|x(0) - u(\phi,b)\|_\alpha$ sufficiently small for some $\phi \in \mathbb{R}$, $|b| \leq 1-2\delta$, there exists ϕ^* near ϕ and b^* near b ($|b^*| < 1-\delta$) so that

$$\|x(t) - u(\omega(b^*)t+\phi^*,b^*)\|_\alpha \to 0$$

exponentially as $t \to +\infty$. The solution remains in $|b| < 1-\delta$ for all $t > 0$ so $\psi(b) = 1$ and our modified equation is equivalent to the original equation. This result (generalizing one of Hale and Stokes [44]) was originally proved by N. Alikakos [107] in a more direct way, and was applied to the Volterra-Lotka system with diffusion for predator-prey population dynamics. (See ex. 1 below.)

Example 3. Suppose X_1,X_2 are Banach spaces, A_j is sectorial in X_j with compact resolvent ($j = 1,2$), $0 \leq \alpha < 1$, and $f_j: X_j^\alpha \to X_j$ are C^r functions. Assume $\dot{x}_j + A_jx_j = f_j(x_j)$ has a C^{r+1} periodic orbit with period p_j, $x_j = u_j(t)$, which has 1 as a simple multiplier and no other multipliers on the unit circle.

Assume $F: \mathbb{R} \times X_1^\alpha \times X_2^\alpha \times X_1 \times X_2$ is locally Hölder continuous in t and C^r in (x_1,x_2) in a neighborhood in $X_1^\alpha \times X_2^\alpha$ of

$$M = \{(u_1(s_1),u_2(s_2) \mid -\infty < s_1,s_2 < \infty\}.$$

We suppose $F = \mathrm{col}(F_1,F_2)$ has

$$\left\|\int_{t_1}^{t_2}(F,F_{x_1},F_{x_2})(t,u_1(s_1),u_2(s_2))dt\right\| \leq q$$

when $|t_1-t_2| \leq 1$ and s_1,s_2 are arbitrary. For small q, consider the perturbed equation

$$\dot{x}_1 + A_1x_1 = f_1(x_1) + F_1(t,x_1,x_2)$$
$$\dot{x}_2 + A_2x_2 = f_2(x_2) + F_2(t,x_1,x_2)$$

near M. Since M is a two dimensional torus, we may introduce new variables in the form

$$x_j = u_j(s_j) + Q_j(s_j)z_j \quad (j = 1,2)$$

so the perturbed equations take the form

$$\dot{s}_j = 1 + S_j(s_1,s_2,z_1,z_2,t)$$
$$\dot{z}_j + \tilde{A}_j z_j = h_j(s_1,s_2,z_1,z_2,t), \quad j = 1,2.$$

Here Q_j, S_j and h_j have period p_1 in s_1, period p_2 in s_2, and $S_j = 0$, $h_j = 0$ when $z_1 = 0$, $z_2 = 0$ and $F = 0$. If $F = 0$, the equations are uncoupled and independent of t (for each j, S_j and h_j depend only on s_j and z_j) and the linearization

$$\dot{z}_j + \tilde{A}_j z_j = \frac{\partial h_j}{\partial z_j}(s_1,s_2,0,0,t_o)z_j$$

with the coefficient dependent only on $s_j = t +$ constant) has an exponential dichotomy on $\mathbb{R}$.

Suppose, for example, F is independent of t and is C^1_{Lip} near M. If q is sufficiently small, by Thm. 9.1.1 there is a unique invariant manifold M_F near M for the perturbed system. If each of the solutions u_1,u_2 is orbitally stable, then M_F is asymptotically stable with asymptotic phase; otherwise M_F is unstable. In any case, M_F is a C^1_{Lip} manifold and the flow in M_F is given by a C^1_{Lip} vector field without equilibrium points on this 2-torus.

Example 4. Suppose A is sectorial in X with compact resolvent, $0 \le \alpha < 1$, U is open in X^α, $f\colon \mathbb{R} \times U \to X$ is locally Hölder continuous and uniformly almost periodic in t and uniformly C^r in x (or C^{r-1}_{Lip}), $r \ge 2$. We consider the equation

$$\dot{x} + \varepsilon Ax = \varepsilon f(t,x)$$

for small $\varepsilon > 0$. Define

$$\overline{f}(x) = \lim_{T\to\infty} \frac{1}{T}\int_0^T f(t,x)dt, \quad \overline{f}'(x) = \lim_{T\to\infty} \frac{1}{T}\int_0^T f_x(t,x)dt$$

and suppose $u(\tau)$ is a C^{r+1} nonconstant p-periodic solution of the averaged equation

$$dx/d\tau + Ax = \overline{f}(x),$$

1 is a simple multiplier of the linearization and there are no other multipliers on the unit circle. We introduce new variables

$$x = u(s) + Q(s)z$$

with $s \to Q(s)$ p-periodic, and the flow becomes

$$\dot{s} = \varepsilon(1 + S(t,s,z))$$
$$\dot{z} + \varepsilon\tilde{A}z = \varepsilon h(t,s,z)$$

where S,h vanish when $z = 0$, are p-periodic in s and uniformly almost periodic in t. Also, if

$$h_0'(s) = \lim_{T\to\infty} \frac{1}{T}\int_0^T h_z(t,s,0)dt$$

then $dz/d\tau + \tilde{A}z = h_0'(s)z$ $(s = \tau + \text{constant})$ has an exponential dichotomy on $\mathbb{R}$.

Let $\tau = \varepsilon t$, $\tilde{A}(s) = \tilde{A} - h_0'(s)$; the equations take the form

$$ds/d\tau = 1 + S(\tau/\varepsilon,s,z)$$
$$dz/d\tau + \tilde{A}(s)z = h(\tau/\varepsilon,s,z) - h_0'(s)z.$$

When $z = 0$, we have $S = 0$ and $h = 0$, and $h_z(\tau/\varepsilon,s,0) - h_0'(s)$ is integrally small for small ε, uniformly in s. Thus by Thm. 9.1.1, for small $\varepsilon > 0$, there is an invariant manifold of the form $z = \sigma_\varepsilon(\tau/\varepsilon,s)$ or, in the original variables

$$x = u(s) + Q(s)\sigma_\varepsilon(t,s) \equiv v_\varepsilon(t,s)$$

where $s \to v_\varepsilon(t,s)$ is p-periodic. As $\varepsilon \to 0+$, $\sigma_\varepsilon(t,s) \to 0$ and $(\partial\sigma_\varepsilon/\partial s)(t,s) \to 0$ uniformly in t,s. The flow in the invariant manifold is

$$\dot{s} = \varepsilon(1 + S(t,s,\sigma_\varepsilon(t,s)))$$

so the resulting solutions $x(t)$ are small perturbations of the slowly-moving function $u(s(t))$, $\dot{s} = \varepsilon + o(\varepsilon)$. The invariant manifold shares the stability properties of the solution $u(\tau)$ of the averaged equation. If $t \to f(t,x)$ is periodic, the same is true for S,h and σ and we are again faced with a flow on a torus; we have the necessary smoothness if $(t,x) \to f(t,x)$ is C^1_{Lip} and $u(\tau): \mathbb{R} \to X$ is C^2_{Lip}.

<u>Exercise 1</u>. Suppose Ω is a bounded smooth connected open set in $\mathbb{R}^n$, a, b, ν_1, ν_2 are positive constants and consider the Volterra-Lotka system with diffusion:

$$\dot{u}_1 = \nu_1\Delta u_1 + u_1(a-u_2) \text{ in } \Omega, \quad \dot{u}_2 = \nu_2\Delta u_2 + u_2(u_1-b) \text{ in } \Omega,$$
$$\partial u_1/\partial N = 0, \quad \partial u_2/\partial N = 0 \text{ on } \partial\Omega.$$

The space of nonnegative functions in $W^{1,p}(\Omega, \mathbb{R}^2)$ is positively invariant. This system has as an invariant manifold -the 2-dimensional space of constant functions (independent of $x \in \Omega$), with the flow in this manifold given by (*) $\dot{u}_1 = u_1(a-u_2)$, $\dot{u}_2 = u_2(u_1-b)$. N. Alikakos [107] proved that any solution with smooth positive initial values must have a positive solution of the ODE (*) as its ω-limit set. He proved that the solution approaches with asymptotic phase provided the limiting solution is not the equilibrium $u_1 = b$, $u_2 = a$. Introduce coordinates near this two dimensional invariant manifold in a neighborhood of this equilibrium point (example 2 does <u>not</u> apply) and prove asymptotic stability with asymptotic phase uniformly in such a neighborhood.

CHAPTER 10

TWO EXAMPLES

10.1 A selection-migration model in population genetics

A model for the change of the gene frequency u in a population under the combined influence of selection and migration is

$$\partial u/\partial t = \Delta u + \lambda s(x) f(u) \quad \text{for} \quad x \in \Omega,\ t > 0 \qquad (*)_\lambda$$
$$\partial u/\partial \nu = 0 \quad \text{on} \quad \partial\Omega$$

with initial value given in $0 \leq u(x,0) \leq 1$, $x \in \Omega$. Here Ω is a bounded open connected set in $\mathbb{R}^n$ with C^2 boundary (assume $n \leq 3$ for simplicity); λ is a positive constant, essentially the ratio of the intensity of selection to the migration rate; $s(x)$ is the local relative selective advantage (or disadvantage, if $s(x) < 0$) of the gene at position $x \in \Omega$, with $s \in L^\infty(\Omega)$; and $f\colon [0,1] \to \mathbb{R}$ is a C^2 function with $f(0) = 0 = f(1)$, $f(u) > 0$ in $0 < u < 1$, $f'(0) > 0$, $f'(1) < 0$. In the genetics problem, $f(u) = u(1-u)[hu+(1-h)(1-u)]$ for a constant h in $0 < h < 1$. Below we shall assume $f''(u) < 0$ on $0 < u < 1$, i.e., $1/3 \leq h \leq 2/3$. This model was introduced by Fisher, and studied more recently by Fleming [118], whose paper inspired this investigation.

It is easily shown by maximum principle arguments that $0 \leq u(x,t) \leq 1$ for $t \geq 0$, $x \in \Omega$, if this holds at the initial time, and the equation defines a dynamical system in X

$$X = \{u \in H^1(\Omega) \mid 0 \leq u(x) \leq 1 \ \text{ a.e. in } \Omega\}$$

and in fact we have a gradient flow (see sec. 5.3). Thus the stability properties of the solutions will be determined by the equilibrium solutions in X. In case $f''(u) < 0$, $0 < u < 1$, we obtain an essentially complete description of the stability properties for every $\lambda > 0$.

An essential role in the argument is played by $\bar{s} = \int_\Omega s(x)dx$, the average selective advantage. If $\bar{s} < 0$ then for small λ (rapid migration) every solution u in X (except $u \equiv 1$) tends to 0 as $t \to +\infty$; but if $s(x) > 0$ on a set of positive measure, then for large λ (slow migration) every solution u in X (except $u \equiv 0$ or

$u \equiv 1$) tends to the unique equilibrium ϕ_λ in $0 < \phi_\lambda(x) < 1$ as $t \to +\infty$. With mild assumptions, ϕ_λ converges to the characteristic function of $\{x \in \Omega: s(x) > 0\}$ as $\lambda \to +\infty$ (ex. 3). If $\overline{s} > 0$, we obtain analogous results by replacing u by $1-u$. The case $\overline{s} = 0$ is considered in ex. 2.

We say $u = 0$, $u = 1$ are <u>trivial equilibria</u> and set

$$E_\lambda = \{\phi \in X \cap W^{2,p}(\Omega) \mid \Delta\phi + \lambda s f(\phi) = 0 \text{ in } \Omega, \frac{\partial\phi}{\partial\nu} = 0 \text{ on } \partial\Omega\}.$$

<u>Lemma 10.1.1</u>. If $\overline{s} \neq 0$, $E_\lambda = \{0,1\}$ for sufficiently small $\lambda > 0$, i.e., there are no nontrivial equilibria.

<u>Proof</u>: Suppose ϕ is a nontrivial equilibrium and $\overline{\phi}$ is the space-average value of ϕ; then $0 < \overline{\phi} < 1$, and $\phi = \overline{\phi} + \psi$ where $\int_\Omega \psi dx = 0$, $\Delta\psi = -\lambda s f(\overline{\phi}+\psi)$ and $\partial\psi/\partial\nu = 0$ on $\partial\Omega$. Provided $\int_\Omega s f(\overline{\phi}+\psi) dx = 0$, this Neumann problem is uniquely solvable for ψ with $\int_\Omega \psi dx = 0$, and for a constant $C = C(\Omega) < \infty$

$$\|\psi\|_{L_2} \leq C\lambda \|s f(\overline{\phi}+\psi)\|_{L_2}$$

For small λ this implies $\|\psi\|_{L_2} \leq C_1 \lambda f(\overline{\phi})$ for a constant C_1, so

$$\left|\int_\Omega s(x)dx\right| f(\overline{\phi}) = \left|\int_\Omega s(x)(f(\overline{\phi}) - f(\overline{\phi}+\psi))dx\right| \leq C_2 \lambda f(\overline{\phi})$$

which gives a positive lower bound for λ. The following eigenvalue computation is due to Fleming [118].

<u>Lemma 10.1.2</u>. If $\overline{s} < 0$, $s(x) > 0$ on a set of positive measure, let $\lambda_o = \inf\{\int_\Omega |\nabla\phi|^2 dx: f'(0)\int_\Omega s(x)\phi^2 = 1\}$; then $0 < \lambda_o < \infty$ and

(i) For $0 < \lambda < \lambda_o$, the zero solution is asymptotically stable

(ii) For $\lambda > \lambda_o$, the zero solution is unstable by the linear approximation, and there exists $\varepsilon(\lambda) > 0$ so any solution u in X of $(*)_\lambda$ with $u|_{t=0} \not\equiv 0$ has $\sup_\Omega u(x,t) \geq \varepsilon(\lambda)$ for all large t.

(iii) For any $\lambda > 0$, the solution $u \equiv 1$ is unstable by the linear approximation, and any solution u in X of $(*)_\lambda$ with $u|_{t=0} \not\equiv 1$ has $\inf_\Omega u(x,t) \leq 1-\varepsilon(\lambda)$ for all large t.

<u>Proof</u>: There exists $\phi \in H^1(\Omega)$ with $f'(0)\int_\Omega s(x)\phi^2 = 1$, $\int_\Omega |\nabla\phi|^2 dx = \lambda_o$ (an exercise in weak convergence, using the fact $\overline{s} \neq 0$ to control the average value of the approximants) so $\Delta\phi + \lambda_o s f'(0)\phi = 0$ in Ω, $\partial\phi/\partial\nu = 0$ on $\partial\Omega$. Thus when $\lambda > \lambda_o$

$$\int_\Omega (\Delta\phi+\lambda s f'(0)\phi)\phi dx = (\lambda-\lambda_o)f'(0) > 0$$

But $\max\{\int_\Omega(-|\nabla\theta|^2 + \lambda f'(0)s\theta^2) \mid \theta \in H^1(\Omega), \int_\Omega\theta^2 = 1\}$ is the largest eigenvalue of $\Delta_N + \lambda f'(0)s$, so for $\lambda > \lambda_o$ this eigenvalue is positive and the zero solution is unstable by the linear approximation.

Suppose $\zeta = \zeta(\lambda)$ is the largest eigenvalue; then (see ex. 1) there exists $\psi > 0$ in $H^1(\Omega)$ with $\Delta\psi + \lambda s f'(0)\psi = \zeta\psi$ in $\partial\Omega$, $\partial\psi/\partial\nu = 0$ on $\partial\Omega$, $\int_\Omega\psi^2 = 1$. If $\int_\Omega s\psi^2 \leq 0$ then $\zeta \leq \int_\Omega \psi\Delta\psi = -\int_\Omega |\nabla\psi|^2 < 0$. If $\int_\Omega s\psi^2 > 0$ then, by definition of λ_o, $\int_\Omega |\nabla\psi|^2 \geq \lambda_o f'(0)\int_\Omega s\psi^2$ so $\zeta \leq (\lambda-\lambda_o)f'(0)\int_\Omega s\psi^2$. Thus when $0 < \lambda < \lambda_o$, we have $\zeta < 0$ and the zero solution is asymptotically stable, while for $\lambda > \lambda_o$ it is unstable. The solution $u \equiv 1$ is unstable for all $\lambda > 0$ since $\int_\Omega(\Delta+\lambda s f'(0))1\, dx = \lambda\bar{s}f'(1) > 0$.

Now suppose $\lambda > \lambda_o$, so $\zeta > 0$, and ψ is the positive eigenfunction mentioned above. Define

$$v(t) = \int_\Omega \psi(x)u(x,t)dx$$

where u solves $(*)_\lambda$, $0 \leq u \leq 1$, $u \not\equiv 0$, so $v(0) > 0$. If $\varepsilon = \varepsilon(\lambda) > 0$ is small enough and for some $t \geq 0$ $\sup_x u(x,t) \leq 2\varepsilon$, then $dv/dt \geq \frac{1}{2}\zeta v > 0$ so any nonzero equilibrium ϕ has $\sup \phi \geq 2\varepsilon$. And since the ω-limit set of the solution u consists of a connected set of nonzero equilibria, $\sup u(x,t) \geq \varepsilon$ for all sufficiently large t. The argument near $u = 1^x$ is similar.

Remark. If $s \leq 0$ a.e. in Ω the above argument shows $u = 0$ is asymptotically stable for every $\lambda > 0$. In effect, we set $\lambda_o = +\infty$.

Lemma 10.1.3. If $\bar{s} < 0$, $s(x) > 0$ on a set of positive measure and $f''(0) \neq 0$, the only nontrivial equilibria (ψ,λ) near $(0,\lambda_o)$ lie on a C^1 curve of the form

$$\psi = \varepsilon\phi + O(\varepsilon^2), \quad \lambda = \lambda_o + \varepsilon\lambda_1 + o(\varepsilon)$$

for small $\varepsilon > 0$, where $\phi > 0$, $\Delta\phi + \lambda_o s f'(0)\phi = 0$, $\partial\phi/\partial\nu = 0$ on $\partial\Omega$, and $\lambda_1 f''(0) < 0$. If $f''(0) < 0$ then $\lambda_1 > 0$ and we have branching to the right. If $f''(u) = bu^m + o(u^m)$ as $u \to 0+$ $(b < 0, m \geq 0)$ we again have branching to the right but the curve has $\lambda = \lambda_o + \varepsilon^{m+1}\lambda_1 + o(\varepsilon^{m+1})$, $\lambda_1 > 0$.

Proof: This is a standard bifurcation argument, once it is realized that 0 is a simple eigenvalue of $\Delta_N + \lambda_o s f'(0)$, with eigenfunction

ϕ (see ex. 1), for lemma 10.1.2 says 0 is the largest eigenvalue.

Lemma 10.1.4. Suppose $q \in L^{\infty}(\Omega)$, $\Delta u + qu = 0$ in Ω, $\partial u/\partial \nu = 0$ on $\partial\Omega$, $u \geq 0$ a.e. in Ω. If $u \not\equiv 0$ then $u(x) \geq \varepsilon > 0$ on $\bar{\Omega}$ for some $\varepsilon > 0$.

Proof: Choose $\lambda > 0$ so $\lambda + q(x) \geq 1$ a.e., so $(\lambda-\Delta)u = (\lambda+q)u \geq u \geq 0$ in Ω. If N_λ is the Neumann function for $\lambda-\Delta$ in Ω then $N_\lambda(x,y) > 0$ for $x \neq y$ in $\bar{\Omega}$, by the maximum principle, and if $u(x) \not\equiv 0$

$$u(x) = \int_\Omega N_\lambda(x,y)(\lambda+q(y))u(y)dy > 0.$$

Lemma 10.1.5. Suppose $s \in L^{\infty}(\Omega)$, $s \not\equiv 0$, $\lambda > 0$, $f\colon [0,1] \to \mathbb{R}$ is C^2, has $f(0) = f(1) = 0$ and $u \mapsto f'(u)$ is strictly decreasing on $0 < u < 1$. Then any nontrivial equilibrium of $(*)_\lambda$ is simple, i.e., the linearization about the equilibrium does not have zero as an eigenvalue.

Proof: Suppose ϕ is a nontrivial equilibrium of $(*)_\lambda$. By lemma 10.1.4 with $q = \lambda s f(\phi)/\phi$, $\phi(x) > 0$ on $\bar{\Omega}$, and similarly with $q = -\lambda s f(\phi)/(1-\phi)$, $1-\phi(s) > 0$ on $\bar{\Omega}$. Thus $f(\phi) > 0$ on $\bar{\Omega}$.

Suppose $\Delta\psi + \lambda s f'(\phi)\psi = 0$ in Ω, $\partial\psi/\partial\nu = 0$ on $\partial\Omega$; we must prove $\psi = 0$. Let $\theta = \psi/f(\phi)$; then

$$\Delta\theta + 2\nabla\theta\cdot\nabla\phi f'(\phi)/f(\phi) + \theta f''(\phi)|\nabla\phi|^2/f(\phi) = 0$$

$$\partial\theta/\partial\nu = 0 \quad \text{on} \quad \partial\Omega.$$

The coefficient of θ is ≤ 0 so the maximum principle says the maximum of θ occurs at the boundary, and if $\theta \not\equiv$ constant, it occurs at a point where $\partial\theta/\partial\nu > 0$. Thus θ is constant. If $\theta \not\equiv 0$ (i.e. $\psi \not\equiv 0$) then $f''(\phi)\nabla\phi = 0$ in Ω so $f'(\phi) =$ constant in Ω, so $\phi =$ constant and $\lambda s f(\phi) \equiv 0$, a contradiction.

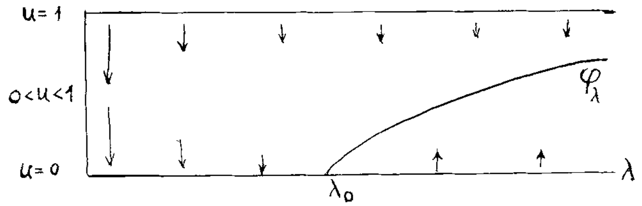
Theorem 10.1.6. Suppose Ω, s, f satisfy the conditions stated with the problem $(*)_\lambda$; $\int_\Omega s(x)dx < 0$; for some $m \geq 0$, $u^{-m}f''(u)$ has a finite nonzero limit as $u \to 0+$; and $u \mapsto f'(u)$ is strictly decreasing (for example, $f''(u) < 0$ on $0 \leq u < 1$). Define λ_0 as in Lemma 10.1.2, but $\lambda_0 = +\infty$ if $s(x) \leq 0$ a.e.

For any $\lambda > \lambda_0$, there is a unique nontrivial equilbrium ϕ_λ for $(*)_\lambda$, $0 < \phi_\lambda(x) < 1$ on $\bar{\Omega}$, and it is asymptotically stable. There are no nontrivial equilibria for $0 < \lambda \leq \lambda_0$.

If u is any solution of $(*)_\lambda$, $0 \leq u \leq 1$, with $u \not\equiv 0$, $u \not\equiv 1$, then

$$u(x,t) \to \begin{cases} 0 & \text{if } 0 < \lambda \le \lambda_0 \\ \phi_\lambda(x) & \text{if } \lambda > \lambda_0 \end{cases}$$

as $t \to +\infty$, uniformly in x.



Proof: Let N_λ be the number of nontrivial equilibria of $(*)_\lambda$; we prove $N_\lambda = 0$ for $0 < \lambda \le \lambda_0$, $N_\lambda = 1$ for $\lambda > \lambda_0$. Since this is a gradient flow, and by lemma 10.1.2, the results about asymptotic behavior follow.

By Lemma 10.1.5 and the implicit function theorem, N_λ is finite and locally constant for $\lambda \ne \lambda_0$, $\lambda > 0$, so it is constant on $0 < \lambda < \lambda_0$ and on $\lambda_0 < \lambda < \infty$. By Lemma 10.1.1, $N_\lambda = 0$ for small $\lambda > 0$, hence for all λ in $(0,\lambda_0)$. By Lemma 10.1.5 and compactness (in $H^1(\Omega)$) of the set of possible equilibria, $N_{\lambda_0} = 0$ as well, and by Lemma 10.1.3, $N_\lambda = 1$ for small $\lambda-\lambda_0 > 0$, hence for all $\lambda > \lambda_0$.

Remark. If $f(u) = u(1-u)[hu + (1-h)(1-u)]$ with h slightly greater than $2/3$, then $f''(0) > 0$ and the curve of equilibria branches to the left initially, before turning to the right.

Exercise 1. Suppose $q \in L^\infty(\Omega)$

$$\mu = \inf\{\int_\Omega (|\nabla\phi|^2+q\phi^2)dx: \int_\Omega \phi^2 dx = 1\}.$$

Then μ is the first eigenvalue of $-\Delta_N + q$ and it is a simple eigenvalue with an eigenfunction $\phi(x) > 0$ in $\bar{\Omega}$ ($\Delta\phi + (\mu-q(x))\phi = 0$ in Ω, $\partial\phi/\partial\nu = 0$ on $\partial\Omega$).

Hint: If ϕ minimizes the quadratic form, so does $|\phi|$, so we may assume $\phi \ge 0$ and so $\phi > 0$ in $\bar{\Omega}$ by 10.1.4 above. Then if also $\Delta_N\psi + (\mu-q)\psi = 0$ show

$$0 = \int_\Omega |\nabla\psi|^2 + (q-\mu)\psi^2 = \int_\Omega \phi^2|\nabla(\psi/\phi)|^2.$$

Exercise 2. Suppose $s \not\equiv 0$, $\bar{s} = \int_\Omega s(x)dx = 0$, and $f''(u) < 0$ on $0 < u < 1$. When $\lambda > 0$, the trivial solutions are unstable by the linear approximation [118] so the number of nontrivial equilibria is constant for all $\lambda > 0$. Prove there is exactly one nontrivial equilibrium for all $\lambda > 0$, by proving there is exactly one equilibrium

ϕ_λ, $0 < \phi_\lambda < 1$, for small $\lambda > 0$, and it has $\phi_\lambda(x) \to c$ as $\lambda \to 0+$, where $f'(c) = 0$, $0 < c < 1$.

Hint: Write the equilibrium as the sum of its average value and the deviation from the average, and solve for this deviation in terms of λ and the average value.

Exercise 3. Suppose $s \in L^\infty(\Omega)$, $s(x) \neq 0$ a.e. and the set $P = \{x: s(x) > 0\}$ has finite capacity, i.e., there exists $\psi \in H^1(\mathbb{R}^n)$ with $\psi = 1$ on P and $0 \leq \psi < 1$ a.e. on $\mathbb{R}^n \setminus P$. Suppose $f(0) = f(1) = 0$, $f(u) > 0$ on $0 < u < 1$ and ϕ_λ minimizes

$$Q_\lambda(\phi) = \int_\Omega (\tfrac{1}{2}|\nabla\phi|^2 - \lambda s(x)F(\phi))dx$$

$F(\phi) = \int_0^\phi f(s)ds$, $0 \leq \phi \leq 1$. Let $\chi_P = 1$ on P, $\chi_P = 0$ on $\Omega \setminus P$.

Prove $\phi_\lambda \to \chi_P$ in measure as $\lambda \to +\infty$, hence $\phi_\lambda \to \chi_P$ in $L_q(\Omega)$ for any $q < \infty$.

Hint: Let $m = \sqrt{\lambda}$ and show

$$\frac{1}{\lambda} Q_\lambda(\psi^m) \to -F(1)\int_P s(x)dx \quad \text{as} \quad \lambda \to \infty$$

$$\frac{1}{\lambda} Q_\lambda(\psi^m) \geq \frac{1}{\lambda} Q_\lambda(\phi_\lambda) \geq -F(1)\int_P s(x)dx$$

so $\int_\Omega |s(x)(F(\chi_P)-F(\phi_\lambda))|dx \to 0$ as $\lambda \to \infty$. Note $\nabla\psi = 0$ a.e. in P.

10.2 A problem in the theory of combustion

In sec. 5.1 and 6.1 above, we studied the system

$$\partial n/\partial t = D\Delta n - \varepsilon n f(T) \quad \text{in } \Omega, \quad \partial n/\partial\nu = 0 \text{ on } \partial\Omega$$

$$\partial T/\partial t = \Delta T + qnf(T) \quad \text{in } \Omega, \quad T = 1 \text{ on } \partial\Omega$$

where Ω is a bounded smooth domain in $\mathbb{R}^3$, $f(T) = \exp(-H/T)$ and D, q, H, ε are positive constants with ε small. If $n \geq 0$, $T \geq 0$ at the initial time, these remain true for all later time and the system defines a dynamical system in the space of nonnegative functions n, T in $H^1(\Omega)$ with $T = 1$ on $\partial\Omega$. Every solution of this kind has $(n,T) \to (0,1)$ as $t \to \infty$ but the convergence is slow, $O(\exp(-\varepsilon f(1)t))$. In sec. 6.1, invariant manifolds were used to discuss the solutions on $0 \leq t \leq O(1/\varepsilon)$, but here we take a different approach and justify the "pseudo-steady-state hypothesis" under fairly mild assumptions.

Let $\langle\phi\rangle = |\Omega|^{-1}\int_\Omega \phi(x)dx$ be the spatial average and set $n = n_0 + n_1$ with $n_0(t) = \langle n(\cdot,t)\rangle$ so $\int_\Omega n_1(x,t)dx = 0$. If λ_0, λ_1

are the first positive eigenvalues of the Dirichlet and Neumann problems respectively in Ω, then (see sec. 6.1)

$$\lambda_1\|n_1(t)\|_{L_2}^2 \le \|\nabla n_1(t)\|_{L_2}^2 \le e^{-\lambda_0 Dt}\|\nabla n(0)\|_{L_2}^2 + \frac{\varepsilon^2}{\lambda_0^2 D^2}\|n(0)\|^2.$$

After a time interval of order $\log(\varepsilon^{-1}\|\nabla n(0)\|_{L_2})$ we have $\|n_1(t)\|_{H^1} = O(\varepsilon)$ for all later time. For simplicity, assume $\|n_1(t)\|_{L_2} = O(\varepsilon)$ uniformly for $t \ge 0$. Then

$$\dot{n}_0(t) = -\varepsilon\langle nf(T)\rangle < 0$$

and

$$\dot{n}_0(t) = -\varepsilon n_0(t)\langle f(T)\rangle + O(\varepsilon^2).$$

Let $\lambda(x,t) = qn(x,t)$; then for some constant k,

$$\|\lambda(\cdot,t) - \langle\lambda(\cdot,\tau)\rangle\|_{L_2(\Omega)} \le k\varepsilon(1 + |t-\tau|)$$

for all $t,\tau \ge 0$. In fact we will study the T-equation with qn replaced by any such slowly-varying, nearly constant function λ, also supposing (on occasion) that $t \mapsto \langle\lambda(\cdot,t)\rangle$ is strictly decreasing. Solutions T of

$$\partial T/\partial t = \Delta T + \lambda(x,t)f(T),\ T = 1 \text{ on } \partial\Omega$$

will be compared with solutions S of

$$\partial S/\partial t = \Delta S + \mu f(S),\quad S = 1 \text{ on } \partial\Omega \qquad (*)_\mu$$

for appropriate constants μ. Observe $(*)_\mu$ is a gradient flow, so its equilibria will play a crucial role.

<u>Theorem 10.2.1</u>. Assume for $\mu_0 = \langle\lambda(\cdot,0)\rangle$, there is a compact interval $J \subset \mathbb{R}$ containing μ_0 such that for each $\mu \in J$, $(*)_\mu$ has an equilibrium ϕ_μ with $\Delta_D + \mu f'(\phi_\mu) < 0$ and $\mu \mapsto \phi_\mu \in H^1(\Omega)$ is continuous. Assume the solution S^o of $(*)_{\mu_o}$ with $S^o = T$ at $t = 0$ has $S^o(t) \to \phi_{\mu_o}$ as $t \to +\infty$. Then for sufficiently small $\varepsilon > 0$

$$\|S^o(t) - T(t)\|_{H^1} = O(\varepsilon^p) \quad \text{on} \quad 0 \le t \le t_\varepsilon,$$

and $\|T(t) - \phi_{\langle\lambda(\cdot,t)\rangle}\|_{H^1} = O(\varepsilon \ln 1/\varepsilon)$ for $t \ge t_\varepsilon$ uniformly in t as long as $\langle\lambda(\cdot,t)\rangle$ remains in J. Here p is a positive constant and $t_\varepsilon = O(\ln 1/\varepsilon)$, and we may expect $\langle\lambda(\cdot,t)\rangle$ to remain in J on a time interval $O(1/\varepsilon)$, long compared to t_ε.

Remark. This is the so-called pseudo-steady-state hypothesis, that T remains close to the stable, slowly moving equilibrum $\phi_{<\lambda(\cdot,t)>}$, following an initial transient. See exercise 1 for a more general version.

Another remark. If the equilibria of $(*)_{\mu_0}$ are all hyperbolic (see sec. 5.3), then an open dense set of initial values $S(0)$ give solutions $S(t)$ converging to a stable equilibrium as $t \to +\infty$. Given $\mu > 0$, it can be shown that, for most regions Ω (in a Baire category sense), all equilibria of $(*)_\mu$ are hyperbolic. Thus the hypotheses, while difficult to verify in a particular case, are generally very plausible.

Since the time-one map is compact, for a closed bounded set of initial values $S(0)$ in the region of attraction of ϕ_{μ_0}, we may find $\varepsilon > 0$ "sufficiently small" uniformly on this set so the conclusions of the theorem hold.

Proof of Thm. 10.2.1. Since $\Delta_D + \mu_0 f'(\phi_{\mu_0}) < 0$ there are positive constants C, β so

$$\|S^0(t) - \phi_{\mu_0}\|_{H^1} \le Ce^{-\beta t}.$$

Let $\theta = T - S^0$, so $\theta = 0$ on $\partial\Omega$ and

$$\partial\theta/\partial t = \Delta_D\theta + (\lambda(\cdot,t)-\mu_0)f(T) + \mu_0(f(S^0+\theta)-f(S^0)),\ \theta = 0 \text{ when } t = 0.$$

There are constants C_1, M (depending only on Ω, H and μ_0) so $\|\theta(t)\|_{H^1} \le C_1 e^{Mt} \sup_{[0,t]} \|\lambda(\cdot,s)-\mu_0\|_{L_2} \le \varepsilon k C_1 e^{Mt}(1+t)$. Define t'_ε by $\varepsilon \exp((\beta+M)t'_\varepsilon) = 1$; then for $0 \le t \le t'_\varepsilon$

$$\|T(t) - S^0(t)\|_{H^1} = O(\varepsilon^p \ell n\ 1/\varepsilon)$$

and

$$\|T(t'_\varepsilon) - \phi_{\mu_0}\|_{H^1} = O(\varepsilon^p \ell n\ 1/\varepsilon)$$

where $p = \beta/(M+\beta) > 0$.

Now by the implicit function theorem and compactness of J, $\mu \mapsto \phi_\mu$ is differentiable on J and for some constant $\gamma > 0$

$$-A_\mu \equiv \Delta_D + \mu f'(\phi_\mu) < -\gamma$$

on J. There is a constant C_2 so

$$\|e^{-A_\mu t}\nu\|_{H^1} \le C_2 e^{-\gamma t}\|\nu\|_{H^1},$$
$$\le C_2 t^{-\frac{1}{2}} e^{-\gamma t}\|\nu\|_{L_2}$$

for $t > 0$ and μ in J, $v \in H_0^1(\Omega)$. Also

$$\mu\| f(S+\theta) - f(S) - f'(S)\theta\|_{L_2} \leq C_2\|\theta\|^2_{H^1}$$

for all S,θ in $H^1(\Omega)$ and μ in J; here we have used the Sobolev inclusion $H^1(\Omega) \subset L_4(\Omega)$.

If $\mu = \langle\lambda(\cdot,\tau)\rangle$ is in J and $\theta = T-\phi_\mu$ has $\|\theta(\tau)\|_{H^1} \leq r_o$ (r_o chosen below), then

$$\partial\theta/\partial t + A_\mu\theta = (\lambda(\cdot,t)-\mu_o)f(T) + \mu(f(T) - f(\phi_\mu)-f'(\phi_\mu)\theta)$$

for $t > \tau$ so

$$\|\theta(t)\|_{H^1} \leq C_2 e^{-\gamma(t-\tau)}\|\theta(\tau)\|_{H^1} + C_2\int_\tau^t e^{-\gamma(t-s)}(t-s)^{-\frac{1}{2}}\{k\varepsilon(1+s-\tau)+C_2\|\theta(s)\|^2_{H^1}\}ds.$$

Suppose $8r_o C_2^3\cdot\max\limits_{p\geq 0}\{\int_0^p e^{-\gamma s}(p-s)^{-\frac{1}{2}}ds\} = 1$, and let $C_3 = 4C_2k\int_0^\infty s^{-\frac{1}{2}}e^{-\gamma s}ds$ and $\ell = k/2C_2C_3^2$. Then for $t \geq \tau$, as long as $\varepsilon(1+t-\tau) \leq \ell$, we have

$$\|\theta(t)\|_{H^1} \leq 2C_2 e^{-\gamma(t-\tau)}\|\theta(\tau)\|_{H^1} + \varepsilon C_3(1+t-\tau).$$

The inequality certainly holds for small $t-\tau > 0$ and if $[\tau,t]$ is the largest interval where it holds, the choice of the constants ensures the inequality is strict at time t, so we must have $t-\tau+1 = \ell/\varepsilon$. (The estimates are rather crude, and use $(a+b)^2 \leq 2(a^2+b^2)$.)

Choose t_ε so $\gamma(t_\varepsilon - t_\varepsilon') = \ell n\ 1/\varepsilon$, so again $t_\varepsilon = O(\ell n\ 1/\varepsilon)$ and (supposing ε small)

$$\|\phi_{\langle\lambda(t_\varepsilon)\rangle} - \phi_{\langle\lambda(t_\varepsilon')\rangle}\|_{H^1} = O(\varepsilon\ell n\ 1/\varepsilon)$$

and

$$\|\theta(t_\varepsilon)\|_{H^1} = O(\varepsilon\ \ell n\ 1/\varepsilon).$$

Suppose $\|T(\tau) - \phi_{\langle\lambda(\tau)\rangle}\|_{H^1} \leq B\ \varepsilon\ \ell n\ 1/\varepsilon$ and choose L so $2C_2\exp(-\gamma L) = 1/2$; then

$$\|T(\tau+L) - \phi_{\langle\lambda(\tau+L)\rangle}\|_{H^1} \leq \frac{1}{2}B\varepsilon\ \ell n\ 1/\varepsilon + \varepsilon C_3(1+L)$$
$$\leq B\varepsilon\ \ell n\ 1/\varepsilon$$

for small $\varepsilon > 0$. Therefore

$$\|T(t) - \phi_{<\lambda(t)>}\|_{H^1} = O(\varepsilon \, \ln 1/\varepsilon)$$

uniformly for $t \geq t_\varepsilon$ as long as $<\lambda(\cdot,t)>$ remains in J.

Remark. With a bit more care one can improve the estimate to $O(\varepsilon)$.

Exercise 1. Suppose A is sectorial in X, $0 \leq \alpha < 1$, $U \subset \mathbb{R} \times X^\alpha$ is open, (Λ,d) is a metric space, $f\colon U \times \Lambda \to X$ is bounded with $f(t,x,\lambda)$ locally Hölder continuous in t, uniformly continuously differentiable in x and uniformly Hölder continuous (exponent θ, $0 < \theta \leq 1$) in λ. Assume for each $\lambda \in \Lambda$ there is a solution $\xi_\lambda(t)$ of

$$\dot{\xi}_\lambda + A\xi_\lambda = f(t,\xi_\lambda,\lambda), \quad t \geq 0$$

with $(t,\xi_\lambda(t))$ in U, $\lambda \mapsto \xi_\lambda(t)$ is locally Hölder continuous (some uniformity implied below) and the evolution operators $T_\lambda(t,s)$ for

$$\dot{\zeta} + A\zeta = f_x(t,\xi_\lambda(t),\lambda)\zeta$$

satisfy $\|T_\lambda(t,s)\|_{\mathscr{L}(X^\alpha)} \leq Me^{-\beta(t-s)}$, $t \geq s \geq 0$. Define $\ell > 0$ by $2\,M\exp(-\beta\ell/2) = 1$ and prove, for each $C > 0$, there are positive constants r_0, K_0, ε_0 such that whenever $\lambda\colon \mathbb{R}^+ \to \Lambda$ satisfies

$$\|\xi_{\lambda(t)}(t) - \xi_{\lambda(\tau)}(t)\|_\alpha \leq Cd(\lambda(t),\lambda(\tau))\theta \quad \text{for} \quad 0 \leq t-\tau \leq \ell,$$

and

$$d(\lambda(t),\lambda(\tau)) \leq \varepsilon(1+|t-\tau|) \quad \text{for all} \quad t,\tau \geq 0$$

with some $0 < \varepsilon \leq \varepsilon_0$, and x is a solution of

$$\dot{x} + Ax = f(t,x,\lambda(t)), \quad t \geq 0$$

with $\|x(0) - \xi_{\lambda(0)}(0)\|_\alpha \leq r_0$, we have

$$\|x(t) - \xi_{\lambda(t)}(t)\|_\alpha \leq 2Me^{-\beta t/2}\|x(0) - \xi_{\lambda(0)}(0)\|_\alpha + K_0\varepsilon^\theta$$

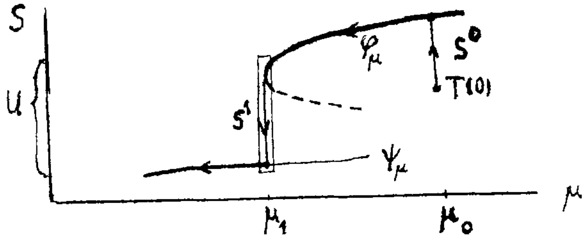
for all $t \geq 0$, as long as $(t,x(t))$ remains in U. A similar conclusion holds if x satisfies

$$\|\dot{x} + Ax - f(t,x,\lambda(t))\| \leq K_1\varepsilon^\theta$$

as long as $(t,x(t))$ remains in U.

Next we aexamine the behavior of the solutions when the equilibrium ϕ_μ loses stability.

Theorem 10.2.2. Suppose $t \to \langle\lambda(\cdot,t)\rangle$ is strictly decreasing. Assume ϕ_μ is an equilibrium of $(*)_\mu$, depending continuously on μ for $\mu_1 \le \mu$ (μ near μ_1) with $\Delta_D + \mu f'(\phi_\mu) < 0$ when $\mu > \mu_1$. Suppose ϕ_{μ_1} is an isolated equilibrium of $(*)_{\mu_1}$ and, for small $\mu - \mu_1 < 0$, there are no equilibria of $(*)_\mu$ near ϕ_{μ_1}. Then there is a nonconstant solution $S^1(t)$ of $(*)_{\mu_1}$ on $-\infty < t < \infty$ with $S^1(t) \to \phi_{\mu_1}$ as $t \to -\infty$. S^1 is unique to within a phase-shift. We suppose $S^1(t) \to \psi_{\mu_1}$ as $t \to +\infty$ where ψ_{μ_1} is an equilibrium of $(*)_{\mu_1}$ with $\Delta_D + \mu_1 f'(\psi_{\mu_1}) < 0$ For μ near μ_1, there are stable equilibria ψ_μ of $(*)_\mu$ near ψ_{μ_1}, by the implicit function theorem. If $\delta > 0$ is small and U is a small H^1-neighborhood of the closure of $\{S^1(t): -\infty < t < \infty\}$, then, for sufficiently small $\varepsilon > 0$, if $(\langle\lambda(\cdot,t)\rangle, T(t))$ enters $(\mu_1-\delta, \mu_1+\delta) \times U$ within $O(\varepsilon \ln 1/\varepsilon)$ of $(\mu_1+\delta, \phi_{\mu_1+\delta})$, it remains in $(\mu_1-\delta, \mu_1+\delta) \times U$ until it exists within $O(\varepsilon \ln 1/\varepsilon)$ of $(\mu_1-\delta, \psi_{\mu_1-\delta})$, then $\|T(t) - \psi_{\langle\lambda(\cdot,t)\rangle}\|_{H^1} = O(\varepsilon \ln 1/\varepsilon)$ for later time as long as $\psi_{\langle\lambda(t)\rangle}$ remains an asymptotically stable equilibrium. (The previous theorem applies.)

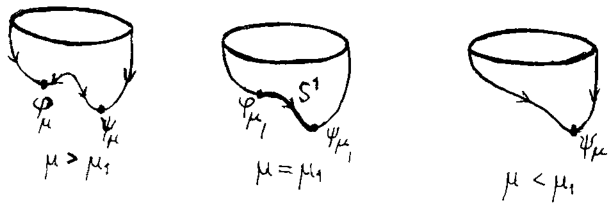


Remark. The assumed behavior of ϕ_μ would ordinarily arise by a saddle point coalescing with ϕ_μ as $\mu \to \mu_1+$, and this apparently does happen when Ω is a ball (see below). Equilibria of $(*)_\mu$ are critical points of

$$Q_\mu(S) = \int_\Omega \{\tfrac{1}{2} |\nabla S|^2 - \mu F(S)\} dx$$

$F(S) = \int_1^S f(\sigma)d\sigma$, for $S \ge 0$ in $H^1(\Omega)$ with $S = 1$ on $\partial\Omega$. The assumed behavior near ϕ_{μ_1} might occur as follows:

graph Q_μ



(There may be other critical points not indicated here.) Of course, these pictures preceded and inspired the theorem.

Proof of Thm. 10.2.2. Let $-A_\mu = \Delta_D + \mu f'(\phi_\mu)$, $\mu \geq \mu_1$. For $\mu > \mu_1$, $A_\mu > 0$, and by the implicit function theorem, 0 must be an eigenvalue of A_{μ_1}, hence the first eigenvalue of A_{μ_1}, hence simple with an eigenfunction v:

$$A_{\mu_1} v = 0, \quad v \neq 0 \quad \text{in } \Omega \quad (v = 0 \text{ on } \partial\Omega), \int_\Omega v^2 = 1,$$

Applying Thm. 6.2.1 to the system $\partial S/\partial t = \Delta S + \mu f(S)$, $S|_{\partial\Omega} = 1$, $d\mu/dt = 0$, at the equilibrium $S = \phi_{\mu_1}$, $\mu = \mu_1$, gives the local center manifold in the form

$$S = h(\sigma,\mu) = \phi_{\mu_1} + \sigma v + O(\sigma^2 + |\mu-\mu_1|)$$

for $|\sigma| \leq r_1$, $|\mu-\mu_1| \leq \delta_o$ with

$$\sigma = \int_\Omega v(h(\sigma,\mu) - \phi_{\mu_1})dx.$$

The flow in the center manifold is $\dot\sigma = g(\sigma,\mu)$ where $g(\sigma,\mu_1) = O(\sigma^2)$, $g(\sigma,\mu_1) \neq 0$ for small $\sigma \neq 0$ (since ϕ_{μ_1} is isolated) and $g(\sigma,\mu) \neq 0$ for $|\sigma| < r_o$, and $-\delta_o < \mu-\mu_1 < 0$. We may assume $g(\sigma,\mu) > 0$ for $\mu < \mu_1$; otherwise replace v,σ by $-v,-\sigma$ to change sign of g. Then $g(\sigma,\mu_1) > 0$ for small $\sigma \neq 0$.

If the equation is modified outside a small neighborhood of ϕ_{μ_1}, as in the proof of Thm. 6.2.1, we obtain an invariant manifold for the modified equation to which Cor. 6.1.5 applies. Restricting attention to a neighborhood of ϕ_{μ_1}, so we deal with the original equation $(*)_{\mu_1}$, we conclude:

If S is a solution of $(*)_{\mu_1}$ with $\|S(0)-\phi_{\mu_1}\|_{H^1} \leq r_o$ then there is a solution $\sigma(t)$ of $\dot\sigma = g(\sigma,\mu_1)$ such that, as long as

$\|S(t)-\phi_{\mu_1}\|_{H^1}$ remains $\leq r_o$

$$\|S(t)-h(\sigma(t),\mu_1)\|_{H^1} \leq C_5 e^{-\gamma t}\|S(0)-h(\sigma^*(0),\mu_1)\|_{H^1}$$

where $\sigma^*(0) = \int_\Omega v(S(0)-\phi_{\mu_1})\,dx$ and $C_5 \geq 1$, γ, r_o are positive constants. We may assume r_o chosen small enough that

$$\frac{1}{2}\|v\|_{H^1}|\sigma-\sigma'| \leq \|h(\sigma,\mu_1)-h(\sigma',\mu_1)\|_{H^1} \leq 2\|v\|_{H^1}|\sigma-\sigma'|$$

for $-r_o \leq \sigma,\sigma' \leq r_o$.

Let $\sigma_1(t)$ be the solution $\dot{\sigma} = g(\sigma,\mu_1)$ on $(-\infty,0]$ with $\sigma_1(0) = r_o/4\|v\|_{H^1}$ and let $S^1(t)$ be the solution of $(*)_{\mu_1}$ on $-\infty < t < \infty$ with

$$S^1(t) = h(\sigma_1(t),\mu_1) \quad \text{for} \quad t \leq 0.$$

Note $\|S^1(t)-\phi_{\mu_1}\|_{H^1} \leq 2\|v\|_{H^1}$, $\sigma_1(t) \leq \frac{1}{2}r_o$ for $t \leq 0$, $S^1(t) \to \phi_{\mu_1}$ as $t \to -\infty$.

It is assumed that $S^1(t) \to \psi_{\mu_1}$ as $t \to +\infty$ where ψ_{μ_1} is asymptotically stable by the linear approximation. The technical heart of the argument is in the following lemma, whose proof is deferred.

<u>Lemma 10.2.3</u>. If $\Gamma = H^1(\Omega)$ - closure $\{S^1(t) \mid -\infty < t < \infty\}$ then Γ is an asymptotically stable compact set for $(*)_{\mu_1}$.

The converse theorem of Liapunov stability (Thm. 4.2.1) applies to asymptotically stable compact sets as well as to points, with virtually the same proof. Thus there is a real-valued function V defined in an H^1-neighborhood of Γ with

$$|V(S) - V(\bar{S})| \leq C_7 \|S-\bar{S}\|_{H^1(\Omega)}$$

$$V = 0 \quad \text{on} \quad \Gamma$$

$$V(S) \geq a(\text{dist}(S,\Gamma))$$

for $S,\bar{S}$ in this neighborhood with value 1 on $\partial\Omega$, and $a(\cdot)$ is a continuous strictly increasing function with $a(0) = 0$, and finally

$$\dot{V}(S) \leq -V(S)$$

when the derivative is computed along a solution of $(*)_{\mu_1}$.

Suppose $\ell > 0$ is so small that the above properties of V hold on

$$\{S \in H^1(\Omega) \mid S = 1 \text{ on } \partial\Omega,\ V(S) < 2\ell\}$$

and the only equilibria of $(*)_{\mu_1}$ in this neighborhood are $\phi_{\mu_1}, \psi_{\mu_1}$. If S solves $(*)_{\mu_1}$ for $t \geq \tau$, $S = T$ when $t = \tau$ and $\|\lambda(\cdot,t)-\mu_1\|_{L_2} \leq \delta$ for $0 \leq t-\tau \leq \ell n\ 2$ then $\|S(t) - T(t)\|_{H^1} \leq C_8\delta$ on this interval. Suppose $\delta \leq \delta_\ell \equiv \ell/(2C_7C_8)$ and $V(T(\tau)) \leq \ell$; then

$$\begin{aligned} V(T(t)) &\leq V(S(t)) + C_7\|S(t) - T(t)\|_{H^1} \\ &\leq e^{-(t-\tau)}V(T(\tau)) + C_7C_8\delta \leq \tfrac{3}{2}\ell \end{aligned}$$

for $0 \leq t-\tau \leq \ell n\ 2$ and $V(T(\tau+ \ell n\ 2)) \leq \ell$. Thus $V(T(t)) \leq 3\ell/2$ for $t \geq \tau$, and $V(T(t)) \leq \ell$ somewhere in each interval of length $\ell n\ 2$, as long as $\|\lambda(\cdot,t)-\mu_1\|_{L_2}$ remains less than δ_ℓ. If $|<\lambda(\cdot,\tau)>-\mu_1| \leq \delta_\ell/2|\Omega|^{\frac{1}{2}}$, this will be true on a time interval of length $O(\delta_\ell/\varepsilon)$.

Let $\mu_2 = \mu_1 - \delta_\ell/2|\Omega|^{\frac{1}{2}}$. If S is any solution of $(*)_{\mu_2}$ with initial value $V(S) \leq \ell$ then by the above argument for T (with λ replaced by μ_2), $V(S(t)) \leq 3\ell/2$ for all $t \geq 0$. There exists $b > 0$ which plays the role of r_0 in the estimate of the previous theorem (p. 322), roughly the radius of a region of attraction for the equilibrium ψ_μ. We may assume ℓ chosen less than $b/4$ so $3\ell/2 < b/2$ and $S(t)$ eventually enters and remains in the $b/2$-ball about ψ_{μ_2}. (For small $\ell > 0$, ψ_{μ_2} is the only equilibrium of $(*)_{\mu_2}$ in $\{V \leq \frac{3}{2}\ell\}$). Further, by compactness of the tine-one map for $(*)_{\mu_1}$, $S(t)$ enters the $b/2$-ball at a time no greater than some $\tau = \tau(b,\ell)$ independent of S.

If $<\lambda(\cdot,t_2)> = \mu_2$ and $V(T(t_2)) \leq \ell$ and S is the solution of $(*)_{\mu_2}$ for $t \geq t_2$ with $S = T$ when $t = t_2$, then $\|T(t_2+\tau(b,\ell)) - \psi_{\mu_2}\|_{H^1} \leq 3b/4$ (if ε is small) and also $\|T(t_3) - \psi_{<\lambda(\cdot,t_3)>}\|_{H^1} \leq b$ for $t_3 = t_2 + \tau(b,\ell)$. This is the situation considered in Thm. 10.2.1, and since we remain in $<\lambda(\cdot,t)> > \mu_1-\delta$ for a further time interval of length $O(\ell/\varepsilon)$, the solution exists within $O(\varepsilon\ \ell n\ 1/\varepsilon)$ of $(\mu_1-\delta,\ \psi_{\mu_1-\delta})$. It remains only to prove asymptotic stability of Γ.

<u>Proof of Lemma 10.2.3</u>. We must show that any solution S of $(*)_{\mu_1}$ with initial value in some neighborhood of Γ tends to Γ as $t \to +\infty$, and the convergence is uniform on this neighborhood. If $\|S(0)-\phi_{\mu_1}\|_{H^1} \leq r_o$ there is associated a solution $\sigma(t)$ of $\dot{\sigma} = g(\sigma,\mu_1)$. We consider first the case when $\sigma(0) \leq 0$, then the case when $\|S(0) - \phi_{\mu_1}\|_{H^1} \geq r_o$, and then the case $\sigma(0) > 0$ (which leads the solution into the second case).

Let $\rho = r_o/9C_5(1 + 2\|v\|_{H^1})$ and suppose $\|S(0) - \phi_{\mu_1}\|_{H^1} \leq \rho$ while the associated $\sigma(t)$ has $\sigma(0) \leq 0$. Since $\dot{\sigma} = g(\sigma,\mu_1) > 0$ for $\sigma \neq 0$, $\sigma(t)$ increases to zero as $t \to +\infty$ and we have

$$|\sigma^*(0)| = \left|\int_\Omega v(S(0) - \phi_{\mu_1})\right| \leq \rho$$

$$\|S(0)-h(\sigma^*(0),\mu_1)\|_{H^1} \leq \rho + 2\|v\|\rho$$

$$\tfrac{1}{2}\|v\|\,|\sigma(0)| \leq \|h(\sigma(0),\mu_1) - \phi_{\mu_1}\|_{H^1} \leq \rho + \rho(1+2\|v\|)C_5$$

so

$$\|h(\sigma(t),\mu_1) - \phi_{\mu_1}\|_{H^1} \leq 2\|v\|\,|\sigma(t)| \leq 8\rho C_5(1 + 2\|v\|)$$

for all $t \geq 0$ and so

$$\|S(t) - \phi_{\mu_1}\|_{H^1} \leq 9\rho C_5(1 + 2\|v\|) = r_o$$

for all $t \geq 0$.

Let $\tilde{\sigma}$ be the solution of $\dot{\sigma} = g(\sigma,\mu_1)$ with $\tilde{\sigma}(0) = -r_o/2$ so $\tilde{\sigma}(t) \leq \sigma(t) \leq 0$. Then

$$\operatorname{dist}(S(t),\Gamma) \leq \|S(t) - \phi_{\mu_1}\|_{H^1} \leq C_5(1+2\|v\|)\rho e^{-\gamma t} + 2\|v\|\,|\tilde{\sigma}(t)|$$

which tends to zero as $t \to +\infty$, uniformly for these initial values ($\|S(0) - \phi_{\mu_1}\|_{H^1} \leq \rho$, $\sigma(0) \leq 0$).

Now if $r_2 > 0$ is sufficiently small (and $r_2 \leq r_o/2$) and S is any solution of $(*)_{\mu_1}$ with

$$\|S(0) - S^1(\tilde{t})\|_{H^1} \leq r_2 \quad \text{for some } \tilde{t} \geq 0,$$

then

$$\|S(t) - S^1(t+\tilde{t})\|_{H^1} \leq C_9 e^{-\gamma t}\|S(0) - S^1(\tilde{t})\|_{H^1}$$

for all $t \geq 0$. To see this, note such an estimate is easily proved when $\tilde{t}$ is large enough to bring $S^1(\tilde{t})$ near ψ_{μ_1}, so both S and S^1 approach ψ_{μ_1} exponentially. By Lipschitz-continuous dependence on the initial values, we may choose $r_2 > 0$ so it holds for all $\tilde{t} \geq 0$. Thus dist $(S(0),\Gamma) = \delta \leq r_2$ with $\|S(0) - \phi_{\mu_1}\|_{H^1} \geq r_o$ implies $\text{dist}(S(t),\Gamma) \leq \delta C_9 e^{-\gamma t}$ for all $t \geq 0$. We take the same exponent γ as in the estimate for the center manifold, or rather replace both by the smaller exponent.

Finally suppose $\|S(0) - S^1(\tilde{t})\|_{H^1} = \delta \leq r_3$, $r_3 = r_2/C_5(1 + 2\|v\|_{H^1})$, for some $\tilde{t} < 0$, with the corresponding $\sigma(0) > 0$. Then

$$|\sigma^*(0) - \sigma_1(\tilde{t})| = |\int_\Omega v(S(0) - S^1(\tilde{t}))| \leq \delta$$

$$\|S(0) - h(\sigma^*(0),\mu_1)\|_{H^1} \leq \delta(1+2\|v\|)$$

as long as $\|S(t) - \phi_{\mu_1}\|_{H^1}$ remains less than r_o. There is some first $t_1 > 0$ when $\|S(t_1) - \phi_{\mu_1}\|_{H^1} = r_o$, and

$$\|S(t_1) - h(\sigma(t_1),\mu_1)\| \leq \delta C_5 e^{-\gamma t_1}(1 + 2\|v\|) \leq r_2,$$

so for $t \geq t_1$ (by the previous case)

$$\text{dist}(S(t),\Gamma) \leq C_9 e^{-\gamma(t-t_1)} \cdot \delta C_5 e^{-\gamma t_1}(1 + 2\|v\|)$$

$$\leq \delta C_5 C_9 (1 + 2\|v\|) e^{-\gamma t}$$

and $\delta = \|S(0) - S^1(\tilde{t})\|_{H^1} \leq r_3$.

Putting the three cases together proves asymptotic stability of Γ.

The equilibrium problem for $(*)_\mu$ has been studied extensively in case Ω is a ball. If $\mu \geq 0$ then every equilibrium solution S has $S \geq 1$ in Ω (maximum principle), and a recent result of Nirenberg shows every such solution is radial when Ω is a ball. S. Parter (SIAM J. Appl. Math. 26(1974), p. 687-716) and Parter, Stein and Stein (Stud. Appl. Math. 54(1975), 293-314) have studied the equilibrium problem for radial solutions, and proved that in certain parameter ranges there are at least three equilibria, and in other ranges there is exactly one. The extensive computations suggest there are no more than three equilibria - if that is so, the pictures of Q_μ given above would describe the behavior for a ball.

Exercise 2. For each $\mu \geq 0$, there is an equilibrium solution of (*) which minimizes Q_μ. If μ is sufficiently small show there is exactly one equilibrium. ($\mu < \frac{1}{4}\lambda_o He^2$, λ_o = first e.v. of $-\Delta_D$ in Ω.)

Exercise 3. Suppose $F(x,u) > 0$ for $u \geq 0$, $u \mapsto F(x,u)$ is analytic for $u \geq 0$, and consider the equilibrium problem

$$\Delta u + \mu F(x,u) = 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega.$$

Suppose there exist $\mu_m \to \mu_o > 0$ $(\mu_m \neq \mu_o)$ and corresponding solutions $u_m \geq 0$ in Ω with $\Delta_D + \mu_m \frac{\partial F}{\partial u}(\cdot, u_m(\cdot)) < 0$ and $u_m \to u_o$ uniformly in Ω. Show all equilibria (μ, u) near (μ_o, u_o) lie on an analytic curve

$$\mu = \mu_o + \mu_1 \varepsilon^p + O(\varepsilon^{p+1}), \quad u = u_o + \varepsilon v + O(\varepsilon^2), \qquad |\varepsilon| \text{ small},$$

for some $p \geq 1$, $\mu_1 \neq 0$ and if $p > 1$, $v > 0$ in Ω. What does this say about $(*)_\mu$? Examine stability in the cases p odd, p even.

NOTES

1.3 The term "sectorial operator" generalizes the usage of Kato [56]. H. B. Stewart (Trans. Am. Math. Soc. 199(1974), 141-162) studies analytic semigroups in the uniform norm.

1.4 Fractional powers of operators are studied extensively in [63] and [103]; the operators need not be sectorial. The connection with interpolation of Hilbert spaces is developed in [71] - see in particular the results of Grisvard described in [71, p. 107].

3.2 More general results are available [57, 63, 93].

3.3 More general results are available [93].

3.4 Analyticity has been proved for some more general linear equations; see in particular [57].

4.1, 4.3 Many of these notions apply to more general dynamical systems; see, for example, [38]. Sell [89] provides an introduction to recent work for nonautonomous ODE's.

4.2 The argument follows Yoshizawa [135].

5.1 Stability of equilibrium for the Navier-Stokes equation was proved by Prodi [80]. Sattinger later proved stability and instability results for weak (Hopf) solutions of the Navier-Stokes equations, and some stability and instability results (of strong solutions) are proved by Kirchgässner and Kielhöfer [59]. The instability theorems 5.1.5, 5.1.6 were inspired by Th. 2.3 (Ch. 7) of Daleckii and Krein [114]; I am grateful to E. Poulson for bringing this result to my attention.

5.2 The stable manifold was also studied by Crandall and Rabinowitz, and these manifolds have been studied in more general context by Hirsch, Pugh and Shub.

5.3 The work of Chafee and Infante [14] was done in 1971; our arguments differ in detail. Some of the results on gradiant flows were reported in Nonlinear Diffusion, Pitman (1976).

6.1 The proof of existence is modeled on [37, Ch. 7]. Another approach to stability of the manifold is presented by Lykova [72], but her differential equations involve only bounded operators. Smoothness of the invariant manifold may be proved along the lines of [58], [66], [104], under plausible assumptions. Note the example in [66], proving the manifold cannot be analytic, in general, without some global bounds.

6.2 An example of van Strein (Math. Z. 166(1979), 143-145) shows the critical or center manifold need not be C^∞:

$$\dot{x} = -x + y^2, \quad \dot{y} = -yz - y^3, \quad \dot{z} = 0.$$

A center manifold which is C^{2N} has the form

$$x = \sigma(y,z) = \sum_{m=1}^{N} 2^{m-1}(m-1)!y^{2m}/\prod_{j=1}^{m}(1-2jz) + o(|y|^{2N})$$

as $y \to 0$, for $z < 1/2N$. If $z < 0$, $y \to \sigma(y,z)$ is analytic near $y = 0$. When $z = 0$, this function is C^∞ but not analytic, and when $z \geq 1/2N$, it is not $2N$-times differentiable. My thanks to Jack Carr for pointing out this example.

6.3 Results of this kind date back to Poincaré, in the case of ordinary differential equations. The fundamental reason for the relation between stability and the geometry of the bifurcation seems to be the homotopy invariance of topological degree: this was discovered by Gavalas [33] and Sattinger [86], although the paper of Sattinger involves a (correctable) error, confusing two notions of the simplicity of an eigenvalue. Some results of section 8.5 may be proved in this manner (assuming the Poincaré map is compact, i.e. A has compact resolvent.) Our work is modeled, in part, on [41].

6.4 E. Hopf proved this result for analytic ODE's, and his work was generalized by many people in particular Chafee [12] who also studied this problem for retarded functional differential equation. Hale [41] extended the treatment to neutral FDE's, and our work is roughly modeled on [41]. Marsden and McCracken present many approaches in The Hopf Bifurcation and its Applications (Springer, Appl. Math. Sciences 19, 1976).

7.2 The argument is modeled on the work of Stokes [98] for retarded functional differential equations. These must also be some connections with the work of Krein [63] on evolution of subspaces.

7.3 More general backward uniqueness results are available; see [29, 30, 64].

7.5 Thm. 3.4.8 can be used to prove "averaging" results, but the direct proof of Thm. 7.5.2 gives sharper results.

7.6 Coppel's recent book [113] is a good source for dichotomies in ODE theory. Many of the results were extended to infinite dimensions (especially Hilbert space) for bounded operators by Daleckii and Krein [115]. C. V. Coffman and J. J. Schäffer (Math. Ann. 172(1967), 139-166) discuss dichotomies for difference equations (in a Banach space) on a half-line, emphasizing the connections between dichotomies and admissibility, a connection studied by Massera and Schäffer [128] for ODE's. A proof of the continuous-time version of Thm. 7.6.5 might allow simplification of this section. Theorems 7.6.12 and 7.6.14 seem to be new even for ODE's.

8.2 Orbital stability was proved by Iooss [52]. Orbital instability is apparently rarely proved even for ordinary differential equations.

8.4, 8.5 Use of the Poincaré map for ODE's is described in [68]. Krasnoselski [61] studies some PDE problems by applying topological degree theory to the Poincaré map.

8.5 Extensive recent work on "generic" properties of maps should have applications in this field: see [9, 66, 76, 85]. Applicability of this work to parabolic equations, in particular the Navier-Stokes equations, has been widely suspected, but was apparently not proved before [66, 85].

9.1 Thm. 9.1.1 was inspried by Coppel and Palmer [114], but is more general even for ODE's, the generality arising from the use of Thm. 7.6.12.

9.2 The coordinate system generalizes the construction of Urabe and Hale [37]. Changing variables in an equation with unbounded operators is complicated, but the complications here seem excessive. There must be an easier way. We could avoid the assumption of "stable triviality" by developing the theory of parabolic equations on infinite dimensional manifolds rather than Banach spaces. The cost seems high for this application, but it would also allow a natural treatment of nonlinear boundary conditions.

10.1 The problem is formulated following Fleming [118].

10.2 Our "small parameter" ε is not quite that of Sattinger [132], so comparison of the results needs more care than is displayed in the text. Parter, Stein and Stein (Studies in Appl. Math. 54(1975), 293-314) examined the equilibrium problem in more detail. Their results do not prove there are no more than three equilibria, but the bounds are such that this seems very likely.

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