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Topological chaos and statistical triviality

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ABSTRACT

While transferring one flow ϕ with fixed points homeomorphically to its equivalent flow ψ, we show that positive topological entropy degenerates to zero (such degeneracy happens for equivalent continuous flows [3] and equivalent differential flows [6]) if and only if all ergodic invariant measures with positive measure-theoretic entropy degenerate to fixed points. Whenever ϕ is assumed to be topological transitive, the measure degeneracy implies that the resulted equivalent flow ψ is topologically chaotic but statistically trivial, meaning that all ergodic invariant measures are supported on fixed points. Using different approaches in different areas people constructed examples of topological chaotic but statistical trivial systems, see [3] for C^0 flows, see [6] for C^r, $r \geq 1$, flows, see [1] [11] for C^0 homeomorphisms, see time one map in [6] for C^r, $r \geq 1$ diffeomorphisms. We point out it is non-hyperbolic singularity causes the degeneracy while changing one flow equivalently to another.

1. Preliminaries

Two flows defined on a compact metric space are *equivalent* if there exists a homeomorphism of the space that sends each orbit of one flow onto an orbit of the other flow while preserving the time orientation. The *topological entropy (measure-theoretic entropy)* of a flow is defined as the entropy of its time-1 map. While topological entropy is an invariant for equivalent homeomorphisms (see Theorem 7.2 in [10]), finite non-zero topological entropy for a flow cannot be an invariant because its value is affected by time reparameterization. However, 0 and ∞ topological entropy are invariants for equivalent flows *without* fixed points (see [3][7][8][9]).

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In equivalent *continuous* flows *with* fixed points, Ohno [3] constructed a counterexample, showing that neither 0 nor ∞ entropy is preserved by equivalence. In equivalent *differential* flows *with* singularities Sun, Young and Zhou constructed a counterexample in [6], showing that 0 topological entropy is not preserved by equivalence. Note that a *differentiable* flow on a compact manifold cannot have ∞ entropy (see Theorem 7.15 in [10]).

In the examples in [3] [6], the phenomena that positive topological entropy degenerates to 0 (entropy degeneracy) and that all ergodic invariant measures with positive measure-theoretic entropy degenerate to the atomic measures supposed on fixed points or singularities (measure degeneracy) happen simultaneously, while shifting homeomorphically one flow to its equivalent flow on a compact metric space or compact smooth manifold. We then ask naturally: is the entropy degeneracy characterized by measure degeneracy for any given equivalent continuous (or differential) flows with fixed points (or singularities)? In the present paper we give a positive answer. Now we introduce this result.

For a flow $\phi : M \times \mathbb{R} \to M$ denote by $\phi_t : M \to M$ the homeomorphism given by $\phi_t(x) = \phi(x,t)$. A Borel probability measure (measure for short) μ is called ϕ_t-invariant if for any Borel set B it holds that $\mu(\phi_t(B)) = \mu(B)$. It is called ϕ-invariant if it is ϕ_t-invariant for all t. A ϕ-invariant measure is called ergodic with respect to ϕ if any Borel set ϕ_t-invariant for any t has measure 0 and 1. The set of all ergodic ϕ-invariant measures is denoted by $\mathcal{M}_{erg,\phi}$. We denote by $\mathcal{M}_{inv,\psi}$ the set of ϕ-invariant measures. We use $h(\phi) := h(\phi_1)$ and $h_\mu(\phi) := h_\mu(\phi_1)$ to denote the topological entropy and measure-theoretic entropy, respectively. Set

$$\mathcal{M}^+_{erg,\phi} = \{\mu \in \mathcal{M}_{erg,\phi}\,;\ h_\mu(\phi) > 0\}, \quad \text{and} \quad \mathcal{M}^0_{erg,\phi} = \{\mu \in \mathcal{M}_{erg,\phi}\,;\ h_\mu(\phi) = 0\}.$$

We use $\{fixed\ points\}$ to denote both the set of all fixed points for a continuous flow and the set of all atomic measures supported on fixed points. And we use the terminology $\{singularities\}$ similarly for a differential flow.

Definition 1.1. Let two continuous flows $\phi,\ \psi : M \times \mathbb{R} \to M$ on a compact metric space be equivalent and let $\pi : M \to M$ denote a homeomorphism preserving the time orientation such that

$$\pi(Orb(x,\phi)) = Orb(\pi(x),\psi), \quad \forall x \in M.$$

If $h(\phi) > 0$ and $h(\psi) = 0$ hold simultaneously, we say that the positive entropy of ϕ degenerates to 0, or that the phenomenon of entropy degeneracy happens, while transferring ϕ to ψ by π. If $\mathcal{M}^+_{erg,\phi} \neq \emptyset$, $\mathcal{M}^+_{erg,\psi} = \emptyset$ and $\pi_*\mathcal{M}^+_{erg,\phi} \cap \mathcal{M}_{erg,\psi} = \emptyset$ hold simultaneously, we say that all measures in $\mathcal{M}^+_{erg,\phi}$ degenerate to fixed points, or that the phenomenon of measure degeneracy happens, while transferring ϕ to ψ by π.

The follow lemma is from [4].

Lemma 1.2. *Let two continuous flows $\phi,\ \psi : M \times \mathbb{R} \to M$ on a compact metric space be equivalent and let $\pi : M \to M$ denote a homeomorphism preserving the time orientation such that*

$$\pi(Orb(x,\phi)) = Orb(\pi(x),\psi), \quad \forall x \in M.$$

Set $M_0 = \{fixed\ points\ of\ \phi\}$. There exists a continuous function $\theta(x,t),\ x \in M \setminus M_0,\ t \in \mathbb{R}$ such that

1) $\theta(x,0) = 0$ and $\theta_x = \theta(x,.) : \mathbb{R} \to \mathbb{R}$ is strictly increasing; and

2) $\theta_x(s+t) = \theta_x(s) + \theta_{\phi_{s(x)}}(t)$; and

3) $\pi \circ \phi_t(x) = \psi_{\theta(x,t)} \circ \pi(x)$.

We call $\theta(x,t) : M \setminus M_0 \times \mathbb{R} \to \mathbb{R}$ a reparameterization while transferring ϕ to ψ by π.

Theorem 1.3. (Main Theorem) *Consider two equivalent continuous flows $\phi,\ \psi : M \times \mathbb{R} \to M$ on a compact metric space M with transferring homeomorphism $\pi : M \to M$, that is,*

$$\pi(Orb(x,\phi)) = Orb(\pi(x),\psi), \quad \forall x \in M.$$

Then the following are equivalent.

(1). The positive entropy of ϕ degenerates to 0, while transferring ϕ to ψ by π;

(2). The ϕ invariant ergodic measures with positive measure-theoretic entropy degenerate to fixed points, while transferring ϕ to ψ by π;

Corollary 1.4. *If the phenomenon of measure degeneracy or the entropy degeneracy happens while transferring ϕ to ψ by π, then $\limsup_{t\to+\infty} \frac{\theta(x,t)}{t} \to \infty$ for $\mu - a.a.x \in M$, $\forall \mu \in \mathcal{M}^+_{erg,\phi}$, where $\theta(x,t)$ is the reparameterization while transferring ϕ to ψ by π.*

In Section 2 we will show explicitly how the phenomenon of measure degeneracy happens. We will prove in Section 3 Main Theorem, by using which we will classify all the probability systems in Section 4.

2. Time reparameterization and measure degeneracy

While transferring one flow ϕ to its equivalent flow ψ, time reparameterization $\theta(x,t)$ (see Lemma 1.2) may increase very quickly as $t \to +\infty$. In this section we show explicitly how an invariant ergodic measure $\mu \in \mathcal{M}_{erg,\phi}$ degenerates, provided

$$\lim_{t\to+\infty} \frac{\theta(x,t)}{t} = \infty, \quad \mu - a.a.\ x \in M.$$

Proposition 2.1. *Denote by M_0 the set of fixed points of a given continuous flow ϕ on a compact metric space M. Suppose*

$$\theta(x,t), \quad x \in M \setminus M_0, \quad t \in \mathbb{R}$$

is a continuous function satisfying the following properties:

1) $\theta(x,0) = 0$ and $\theta_x = \theta(x,.) : \mathbb{R} \to \mathbb{R}$ is strictly increasing; and

2) $\theta_x(s+t) = \theta_x(s) + \theta_{\phi_{s(x)}}(t)$.

Then it holds for a given $\mu \in \mathcal{M}_{erg,\phi}$ that either

$$\lim_{t\to+\infty} \frac{\theta(x,t)}{t} = \lim_{t\to+\infty} \frac{\int_0^t \theta(\phi_s x,1)\,ds}{t} = \int \theta(x,1)\,d\mu, \quad \text{for} \ \ \mu - a.a.\ x \in M,$$

or

$$\lim_{t\to+\infty} \frac{\theta(x,t)}{t} = \infty, \quad \mu - a.a.\ x \in M.$$

Proof. For $t > 0$ take integer n such that $n \le t \le n+1$, then

$$\frac{n}{n+1}\frac{\theta(x,n)}{n} \le \frac{\theta(x,t)}{t} \le \frac{n+1}{n}\frac{\theta(x,n+1)}{n+1}.$$

Since $\theta(x,n) = \theta(x,1) + \theta(\phi_1 x,1) + \cdots + \theta(\phi_{n-1}x,1)$, applying Birkhoff ergodic theorem for ϕ_1 the following limits exist

$$\lim_{n\to+\infty}\frac{\theta(x,n)}{n}=\lim_{t\to+\infty}\frac{\theta(x,t)}{t}=\lim_{n\to+\infty}\frac{\theta(x,n+1)}{n+1}$$

for $\mu-a.a.x\in M$. Moreover, this limit is a constant b for $\mu-a.a.x\in M$, this is because $\lim_{t\to+\infty}\frac{\theta(x,t)}{t}$ is ϕ invariant and μ is ergodic for ϕ. For any ϕ_1 ergodic measure τ we have

$$b=\lim_{n\to+\infty}\frac{\theta(x,n)}{n}=\lim_{n\to+\infty}\frac{\Sigma_{i=0}^{n-1}\theta(\phi_i x,1)}{n}=\int\theta(x,1)\,d\tau.$$

By the ergodic decomposition theorem it holds that

$$\int\theta(x,1)\,d\mu=\int(\int\theta(x,1)\,d\tau)\,d\mu=\int b\,d\mu=b.$$

So we have

$$\lim_{t\to\pm\infty}\frac{\theta(x,t)}{t}=\int\theta(x,1)\,d\mu,\quad\text{for}\quad \mu-a.a.\ x\in M.$$

Applying Birkhoff ergodic theorem to (ϕ,μ) it holds that

$$\lim_{t\to\pm\infty}\frac{\theta(x,t)}{t}=\int\theta(x,1)\,d\mu=\lim_{t\to\pm\infty}\frac{\int\theta(\phi_s x,1)ds}{t},\ \mu-a.a.\ x\in M. \tag{2.1}$$

Next we consider the case that $\theta(.,1)$ is not integrable with μ, meaning $\int\theta(x,1)\,d\mu=\infty$. We will show that $\lim_{t\to+\infty}\frac{\theta(x,t)}{t}=+\infty,\ \mu-a.a.\ x\in M$, by showing that $\liminf_{t\to+\infty}\frac{\theta(x,t)}{t}=+\infty,\ \mu-a.a.\ x\in M$.

Let us introduce a new function

$$\theta^-(x)=\liminf_{t\to+\infty}\frac{\int_0^t\theta(\phi_s x,1)\,ds}{t},\quad x\in M.$$

Since

$$\frac{\int_0^t\theta(\phi_s\phi_\tau x,1)\,ds}{t}=\frac{t+\tau}{t}\frac{\int_0^{t+\tau}\theta(\phi_s x,1)\,ds}{t+\tau}-\frac{\int_0^\tau\theta(\phi_s x,1)\,ds}{t},$$

we have $\theta^-(\phi_\tau x)=\theta^-(x),\ \tau\in\mathbb{R},\ x\in M$. Note μ is both ϕ-invariant and ϕ-ergodic, $\theta^-(x)=c$ a constant $\mu-a.a.\ x\in M$. It suffices from (2.1) to complete the proposition by showing the following

Assertion. $c=+\infty$.

Let $N>1$ be a given big integer and let

$$A_N=\{x\in M\mid\theta(x,1)>N\}.$$

A_N is measurable since $\theta(x,1)$ is continuous. Define

$$\theta_N(x,1)=\begin{cases}\theta(x,1), & x\ni M\setminus A_N\\ N, & x\in A_N.\end{cases}$$

Then $\{\theta_N(x,1)\}_{N=1}^{+\infty}$ is an increasing sequence of integrable functions and

$$\lim_{N\to+\infty}\theta_N(x,1)=\theta(x,1).$$

Since $\int \theta(x,1)\,d\mu = +\infty$, $\int \theta_N(x,1)\,d\mu \to +\infty$ by using the Monotone Convergence Theorem. For given positive real R take N_R such that $\int \theta_{N_R}(x,1)\,d\mu - 1 > R$. Since by the Birkhoff ergodic theorem $\lim_{t\to+\infty} \frac{\int_0^t \theta_{N_R}(\phi_s x,1)\,ds}{t} = \int \theta_{N_R}(y,1)\,d\mu$, there exists $T_R > 0$ such that for all $t \geq T_R$ it holds that

$$\frac{\int_0^t \theta_{N_R}(\phi_s x,1)\,ds}{t} > \int \theta_{N_R}(y,1)\,d\mu - 1 > R.$$

So,

$$\frac{\int_0^t \theta(\phi_s x,1)\,ds}{t} \geq \frac{\int_0^t \theta_{N_R}(\phi_s x,1)\,ds}{t} > R.$$

This implies that assertion, $c = \infty$. □

Proposition 2.2. *Consider a continuous flow ϕ on a compact metric space M. Suppose*

$$\theta(x,t), \quad x \in M, \quad t \in \mathbb{R}$$

is a continuous function satisfying the following properties:

1) $\theta(x,0) = 0$ and $\theta_x = \theta(x,.) : \mathbb{R} \to \mathbb{R}$ is strictly increasing; and

2) $\theta_x(s+t) = \theta_x(s) + \theta_{\phi_{s(x)}}(t)$.

Let $B = \{x \in M \mid \frac{\theta(x,t)}{t} \to \infty (t \to +\infty)\}$. Then B is ϕ_s-invariant, for all $s \in \mathbb{R}$.

Proof. Note $\theta(x,s+t) = \theta(x,s) + \theta(\phi_s(x),t)$, $\frac{\theta(x,s+t)}{t} \to \infty (t \to +\infty)$ if and only if $\frac{\theta(\phi_s x,t)}{t} \to \infty (t \to +\infty)$. This implies that $\phi_s(x) \in B$, $\forall x \in B$. □

Proposition 2.3. *Consider two equivalent flows ϕ, $\psi : M \times \mathbb{R} \to M$ on a compact metric space M with transferring homeomorphism $\pi : M \to M$, that is,*

$$\pi(Orb(x,\phi)) = Orb(\pi(x),\psi), \quad \forall x \in M.$$

We suppose that $\lim_{t\to+\infty} \frac{\theta(x,t)}{t} = \infty$ for μ-a.e. $x \in M$ for a given non-atomic measure $\mu \in \mathcal{M}_{erg,\phi}$ and suppose that $Supp(\mu) \cap \{fixed\ points\} = \{p\}$ consisting of exact one fixed point of ϕ, where $\theta : M \setminus M_0 \times \mathbb{R} \to \mathbb{R}$ is the reparameterization while transferring ϕ to ψ by π. Let $\nu := \pi_*\mu$. Then ν is not a ψ-invariant measure, $\nu \notin \mathcal{M}_{inv,\psi}$ (remark: ν is ψ-ergodic but not ψ-invariant), or in other words, the ψ-ergodic invariant measure ν degenerates to the fixed point $q := \pi(p)$ in the sense that $\nu(B(q,\eta)) \to 1$ as $\eta \to 0$, where $B(q,\eta) = \{y \in M \mid d(q,y) < \eta\}$.

Proof. When $\psi_t(A) = A$, $\forall\, t \in \mathbb{R}$ for some Borel set A, it follows that $\phi_t(\pi^{-1}A) = \pi^{-1}(A)$, $\forall t \in \mathbb{R}$. Since μ is ϕ ergodic, $\mu(\pi^{-1}(A)) = 0,\ 1$ and thus $\nu(A) = 0, 1$. So ν is ψ ergodic. ν is not atomic on fixed point because μ is not. Now we show that ν is not ψ-invariant. In fact, otherwise, we could deduce as follows that ν is atomic on the fixed point q, a contradiction.

Since $p \in Supp(\mu)$, $q \in Supp(\nu)$. Take arbitrarily two reals $\eta_1 > \eta > 0$ and consider two balls $B(q,\eta) = \{y \in M \mid d(q,y) < \eta\}$ and $B(q,\eta_1) = \{y \in M \mid d(q,y) < \eta_1\}$ centered at q. Their boundaries are denoted by $\partial B(q,\eta)$ and $\partial B(q,\eta_1)$, respectively. Clearly, $\partial B(q,\eta)$ and $\partial B(q,\eta_1)$ are compact. We take η_1 small such that $\mu(\pi^{-1}B(q,\eta_1)) < 1$. We take η small enough such that the time used to go from $\pi^{-1}\partial B(q,\eta)$ to $\pi^{-1}\partial B(q,\eta_1)$ or vice versa along any orbit of ϕ is bigger than 2. This can be down because the fixed point p is inside $\pi^{-1}B(q,\eta)$ and $\pi^{-1}B(q,\eta_1)$, and near p every orbit goes slowly.

Denote by M_0 the set of fixed points of ϕ. By Lemma 1.2 the reparameterization $\theta(x,t)$ is continuous and satisfies $\pi\phi_t(x) = \psi_{\theta(x,t)}(\pi(x))$, $\forall x \in M \setminus M_0$. Take $C > 0$ such that $\theta(x,t) \leq C$, $\forall 1 \leq t \leq 2$, $\forall x \in M \setminus \pi^{-1}B(q,\eta)$. For $\tau \geq 2$ we write $\tau = r + \ell$, where $r \in \mathbb{N}$, $1 \leq \ell < 2$. Then

$$\begin{aligned} \theta(x,\tau) &= \theta(x,1) + \theta(\phi_1(x),1) + \cdots + \theta(\phi_{r-1}x,1) + \theta(\phi_r(x),\ell) \\ &\leq C(r+1) \\ &\leq C\tau, \end{aligned} \tag{2.2}$$

provided $\phi(x,[0,\tau)) \subset M \setminus \pi^{-1}B(q,\eta)$.

Since μ is both ϕ invariant and ϕ ergodic, there exists a Borel set D of μ full measure such that

$$\mu(A) = \lim_{t\to+\infty} \frac{\int_0^t \chi_A(\phi_s(x))ds}{t}, \ \forall x \in D$$

holds for any Borel set A, where χ_A denotes the characteristic function of A. Similarly, there exists a Borel set E of ν full measure such that

$$\nu(A) = \lim_{t\to+\infty} \frac{\int_0^t \chi_A(\psi_s(x))ds}{t}, \ \forall x \in E$$

holds for any Borel set A. Since $\nu = \pi_*\mu$, $\pi(D) \cap E$ is of ν full measure. Take $x \in Supp(\mu) \cap D$ and $y \in Supp(\nu) \cap E$ with $y = \pi(x)$. Since $\nu(B(q,\eta)) > 0$, the orbit $Obt(y,\psi)$ starting at y will pass through $B(q,\eta)$ infinite times. Without loss of generality we assume $y \in M \setminus B(q,\eta)$ and thus $x \in M \setminus \pi^{-1}B(q,\eta)$.

Take a_1 such that $\phi(x,[0,a_1)) \cap \pi^{-1}B(p,\eta_1) = \emptyset$ and $\phi_{a_1}(x) \in \partial\pi^{-1}B(q,\eta_1)$. Take $b_1 > a_1$ such that $\phi(x,[a_1,b_1)) \cap \pi^{-1}B(p,\eta) = \emptyset$ and $\phi_{b_1}(x) \in \partial\pi^{-1}B(q,\eta)$. Because the orbit starting at $\phi_{a_1}(x)$ enters $\pi^{-1}B(q,\eta)$ with positive μ measure and thus enters infinite times, the above b_1 exists. Take $b_2 > b_1$ such that $\phi(x,[b_1,b_2]) \subset \pi^{-1}\bar{B}(p,\eta)$ and $\phi(x,(b_2,b_2+r)) \cap \pi^{-1}B(p,\eta) = \emptyset$ for some small $r > 0$. Because the orbit starting at $\phi_{b_1}(x)$ enters $\pi^{-1}(B(q,\eta_1) \setminus B(q,\eta))$ with positive μ measure and thus enters infinite times, the above b_2 exists. If it happens that the orbit starting at $\phi_{b_2}(x)$ stays $\pi^{-1}B(q,\eta_1)$ forever, then $\mu(\pi^{-1}B(q,\eta_1)) = 1$, which contradicts to the choice of η_1. So we can take $a_2 > b_2$ such that $\phi(x,[b_2,a_2]) \subset \pi^{-1}\bar{B}(p,\eta_1)$ and $\phi(x,(a_2,a_2+r)) \cap \pi^{-1}B(p,\eta_1) = \emptyset$ for some small $r > 0$. Take $a_3 > a_2$ such that $\phi(x,(a_2,a_3)) \cap \pi^{-1}B(p,\eta_1) = \emptyset$ and $\phi_{a_3}(x) \in \partial\pi^{-1}B(q,\eta_1)$. Take $b_3 > a_3$ such that $\phi(x,(a_3,b_3)) \cap \pi^{-1}B(p,\eta) = \emptyset$ and $\phi_{b_3}(x) \in \partial\pi^{-1}B(q,\eta)$. Take $b_4 > b_3$ such that $\phi(x,[b_3,b_4]) \subset \pi^{-1}\bar{B}(p,\eta)$ and $\phi(x,(b_4,b_4+r)) \cap \pi^{-1}B(p,\eta) = \emptyset$ for some small $r > 0$. Take $a_4 > b_4$ such that $\phi(x,[b_4,a_4]) \subset \pi^{-1}\bar{B}(p,\eta_1)$ and $\phi(x,(a_4,a_4+r)) \cap \pi^{-1}B(p,\eta_1) = \emptyset$ for some small $r > 0$. One can show the existence of $a_3,\ b_3,\ b_4,\ a_4$ by similar argument as showing the existence of $a_1,\ b_1,\ a_2,\ b_2$. By repeating this procedure we get two sequences

$$a_1, a_2, a_3, a_4, a_5, \cdots, \quad \text{and} \quad b_1, b_2, b_3, b_4, b_5, \cdots$$

such that the segments $Orb(x,[a_{2k-1},a_{2k}])$ are inside the closed set $\pi^{-1}\bar{B}(p,\eta_1)$ and $Orb(x,(a_{2k},a_{2k+1}))$ are outside the open set $\pi^{-1}B(p,\eta_1)$, and the segments $Orb(x,[b_{2k-1},b_{2k}])$ are inside the closed set $\pi^{-1}\bar{B}(p,\eta)$ and $Orb(x,(b_{2k},b_{2k+1}))$ are outside the open set $\pi^{-1}B(p,\eta)$, for all $k \in \mathbb{N}$. We note that

$$\pi\phi(x,a_i) = \psi(y,\theta(x,a_i)), \quad \text{and} \quad \pi\phi(x,b_i) = \psi(y,\theta(x,b_i)), \quad i = 1,2,\cdots.$$

We also note that $a_{2k} \geq 2k,\ k = 1,2,3,\cdots$.

Note that $\theta(x,a_{2k-1}) - \theta(x,a_{2k-2}) = \theta(\phi_{a_{2k-2}}(x), a_{2k-1} - a_{2k-2})$ and $\phi_{a_{2k-2}}(x) \in M \setminus \pi^{-1}B(q,\eta)$. When $a_{2k-1} - a_{2k-2} < 2$, then $\theta(\phi_{a_{2k-2}}(x), a_{2k-1} - a_{2k-2}) \leq \theta(\phi_{2k-2}(x),2) \leq C$. It follows when $n \to +\infty$

$$0 \leq \frac{\Sigma_{a_{2k-1}-a_{2k-2}\leq 2}[\theta(x,a_{2k-1})-\theta(x,a_{2k-2})]}{\theta(x,a_{2n})} \leq \frac{nC}{\theta(x,a_{2n})} = \frac{a_{2n}}{\theta(x,a_{2n})}\frac{nC}{a_{2n}} \to 0.$$

Recall by our assumption $\nu \in \mathcal{M}_{erg,\psi}$. Then

$$\begin{aligned}
\nu(\bar{B}(q,\eta_1)) &\geq \lim_{n\to+\infty} \frac{[\theta(x,a_2)-\theta(x,a_1)]+[\theta(x,a_4)-\theta(x,a_3)]+\cdots+[\theta(x,a_{2n})-\theta(x,a_{2n-1})]}{\theta(x,a_{2n})} \\
&= 1-\lim_{n\to+\infty} \frac{[\theta(x,a_3)-\theta(x,a_2)]+[\theta(x,a_5)-\theta(x,a_4)]+\cdots+[\theta(x,a_{2n-1})-\theta(x,a_{2n-2})]}{\theta(x,a_{2n})} \\
&= 1-\lim_{n\to+\infty} \frac{\Sigma_{a_{2k-1}-a_{2k-2}>2}[\theta(x,a_{2k-1})-\theta(x,a_{2k-2})]}{\theta(x,a_{2n})} \\
&\quad -\lim_{n\to+\infty} \frac{\Sigma_{a_{2k-1}-a_{2k-2}\leq 2}[\theta(x,a_{2k-1})-\theta(x,a_{2k-2})]}{\theta(x,a_{2n})} \\
&= 1-\lim_{n\to+\infty} \frac{\Sigma_{a_{2k-1}-a_{2k-2}>2}[\theta(x,a_{2k-1})-\theta(x,a_{2k-2})]}{\theta(x,a_{2n})} \\
&\geq 1-\lim_{n\to+\infty}\frac{a_{2n}}{\theta(x,a_{2n})}\lim_{n\to+\infty}\frac{C[(a_3-a_2)+\cdots(a_{2n-1}-a_{2n-2})]}{[(a_3-a_2)+\cdots(a_{2n-1}-a_{2n-2})]+[(a_2-a_1)+\cdots(a_{2n}-a_{2n-1})]} \\
&\quad \text{(by (2.2))} \\
&\geq 1-\lim_{n\to+\infty}\frac{a_{2n}}{\theta(x,a_{2n})}C,
\end{aligned}$$

which tends to 1 as $n \to +\infty$. By taking $\eta_1 \to 0$ arbitrarily small, this implies that $\nu = \delta_q$, the atomic measure supported on the singularity q, a contradiction. □

3. Entropy for equivalent probability flows

In this section we prove the Main theorem. We adapt Katok's definition of measure theoretic entropy [2] in the proof.

Given a flow ϕ_t on a compact metric space M, $q \in M$, $t \in \mathbb{R}$ and $\varepsilon > 0$, we set a (t,ε,ϕ)-ball

$$D(q,t,\varepsilon,\phi) = \{w \in M \mid d(\phi_s w, \phi_s q) < \varepsilon,\ 0 \leq s \leq t\}.$$

Definition 3.1. Given a ϕ invariant and ϕ ergodic measure μ and given $\delta \in (0,1)$, let $R(\delta,t,\varepsilon,\phi)$ denote the smallest number of (t,ε,ϕ)-balls needed to cover a set whose μ-measure is greater than $1-\delta$. Then the measure theoretic entropy of ϕ, denoted by $h_\mu(\phi)$, is defined by

$$h_\mu(\phi) := \lim_{\varepsilon\to 0}\limsup_{t\to\infty}\frac{1}{t}\ln R(\delta,t,\varepsilon,\phi).$$

This definition is a flow version of what Katok defined for homeomorphism in [2], where he pointed out by the Shannon theorem that the limit in definition is independent of the choice of δ. The following lemma talks about the relation between the entropy in Definition 3.1 and the entropy defined by time-1 map.

Lemma 3.2.

$$(1).\ \textit{For } \mu \in \mathcal{M}_{erg,\phi} \textit{ it follows that } h_\mu(\phi) = h_\mu(\phi_1);$$
$$(2).\ h(\phi) = h(\phi_1) = \sup\{h_\mu(\phi_1) \mid \mu \in \mathcal{M}_{erg,\phi}\}.$$

Proof. This is Theorem A in [5]. □

Observe by definition that a ϕ_1-invariant measure is not necessarily ϕ-invariant and a ϕ-ergodic measure is not necessarily ϕ_1-ergodic, the variational principle given in the second term in the above lemma thus differs from the usual one for homeomorphisms.

Definition 3.3. Let $\phi,\ \psi : M \times \mathbb{R} \to M$ be two flows on a compact metric space and let $\mu \in \mathcal{M}_{erg,\phi}$ and $\nu \in \mathcal{M}_{erg,\psi}$. We say that two probability flows (ϕ, μ) and (ψ, ν) are equivalent, if there exist a μ-full measure ϕ-invariant set $A \subset M$ and a homeomorphism $\pi : A \to \pi(A)$ and a continuous map $\theta : A \times \mathbb{R} \to \mathbb{R}$ such that the following holds:

1. $\pi(A)$ is with ν-full measure; and
2. $0 < \lim_{t\to+\infty} \frac{\theta(x,t)}{t} < +\infty,\ \forall\ x \in A$; and
3. $\theta_x : \mathbb{R} \to \mathbb{R}$ is strictly increasing, $\forall\ x \in A$; and
4. $\theta_x(s + t) = \theta_x(s) + \theta_{\phi_{s(x)}}(t),\ \forall\ x \in A,\ \forall\ s, t \in \mathbb{R}$; and
5. $\pi \circ \phi_t(x) = \psi_{\theta(x,t)} \circ \pi(x),\ \forall\ x \in A,\ \forall\ t \in \mathbb{R}$.

As in Lemma 1.2, we call $\theta(x, t)$ a reparameterization while shifting (ϕ, μ) to (ψ, ν) by π.

One may compare the equivalent probability flows in Definition 3.3 with the measure-theoretic equivalent flows defined in [7], where the reparameterization $\theta(x, t)$ was assumed to be extended continuously to the whole support of μ. Observe $\theta(x, t)$ can not be extended continuously to $Supp(\mu)$ in general provided $Supp(\mu)$ contains a fixed point, the measure-theoretic equivalence defined in [7] contains only parts of cases in Definition 3.3. The following Theorem 3.6 points out that zero entropy and infinity entropy are preserved for equivalent probability flows defined in Definition 3.3, which generalizes the same result for equivalent flows defined in [7]. Instate of $Supp(\mu)$ in [7] we need here to deal with a μ-full measure set not compact. We have to argue in a new approach due to the lack of compactness. Now we start by a lemma.

Lemma 3.4. *Suppose there exist $\mu \in \mathcal{M}_{erg,\phi}$ and $\nu \in \mathcal{M}_{erg,\psi}$ such that the two probability flows (ϕ, μ) and (ψ, ν) are equivalent. Then*

$$\int f d\nu = \frac{1}{\int \theta(x,1) d\mu} \int \Big(\int_0^{\theta(x,1)} f(\psi_t(\pi x))\, dt \Big) d\mu, \quad \forall f \in C^0(M, \mathbb{R}),$$

where $\theta(x, t)$ denotes the reparameterization while transferring (ϕ, μ) to (ψ, ν) by π.

Proof. Set

$$Q_\mu(\phi) = \left\{ x \in M \mid \lim_{t\to\pm\infty} \frac{1}{t} \int_0^t f(\phi_s x) ds = \int_M f\, d\mu(x), \quad \forall f \in C^0(M, \mathbb{R}) \right\}.$$

By the Birkhoff ergodic theorem, $Q_\mu(\phi)$ is a ϕ-invariant and μ-full measure set. One can define similarly $Q_\nu(\psi)$. Take $x \in Q_\mu(\phi)$ such that $\pi(x) \in Q_\nu(\psi)$. Since $x \in Q_\mu(\phi)$ it holds by Proposition 2.1 and its proof that

$$\lim_{n\to+\infty} \frac{1}{n} \Sigma_{i=0}^{n-1} \theta(\phi_i(x), 1) = \int \theta(y, 1)\, d\mu(y) = \lim_{t\to+\infty} \frac{\theta(x, t)}{t}.$$

Note $\theta(x, t) \to \pm\infty$ as $t \to \pm\infty$ and $\pi\phi_t(x) = \psi_{\theta(x,t)}\pi(x)$, it follows for a given $f \in C^0(M, \mathbb{R})$ that

$$\int f d\nu = \lim_{t\to\infty} \frac{1}{t} \int_0^t f(\psi_s(\pi x))\, ds$$

$$= \lim_{n\to\infty} \frac{1}{\frac{1}{n}\Sigma_{i=0}^{n-1}\theta(\phi_i(x),1)} \frac{1}{n} \int_0^{\Sigma_{i=0}^{n-1}\theta(\phi_i(x),1)} f(\psi_t(\pi x))\, dt$$

$$= \frac{1}{\int \theta(y,1)\, d\mu(y)} \lim_{n\to\infty} \frac{1}{n}\Sigma_{i=0}^{n-1} \int_0^{\theta(\phi_i(x),1)} f(\psi_s \psi_{\Sigma_{\tau=0}^{i-1}\theta(\phi_\tau(x),1)}(\pi x))\, ds$$

$$= \frac{1}{\int \theta(y,1)\, d\mu(y)} \lim_{n\to\infty} \frac{1}{n}\Sigma_{i=0}^{n-1} \int_0^{\theta(\phi_i(x),1)} f(\psi_s \psi_{\theta(x,i)}(\pi x))\, ds$$

$$= \frac{1}{\int \theta(y,1)\, d\mu(y)} \lim_{n\to\infty} \frac{1}{n}\Sigma_{i=0}^{n-1} \int_0^{\theta(\phi_i(x),1)} f(\psi_s \pi\phi_i(x))\, ds, \quad \mu - a.a.\ x \in X.$$

Denote

$$F(x) = \lim_{n\to\infty} \frac{1}{n}\Sigma_{i=0}^{n-1} \int_0^{\theta(\phi_i(x),1)} f(\psi_s \pi\phi_i(x))\, ds.$$

Since μ is ϕ_1-invariant, by the Birkhoff ergodic theorem $F(x)$ is defined for $\mu - a.a.\ x$ and

$$\int F\, d\mu = \int (\int_0^{\theta(y,1)} f(\psi_s \pi y)\, ds)\, d\mu(y).$$

Observe that both $\int f\, d\mu$ and $\int \theta(y,1)\, d\mu(y)$ are constants and remain unchanged when taking integral, so

$$\int f d\nu = \frac{1}{\int \theta(y,1)\, d\mu(y)} \int (\int_0^{\theta(y,1)} f(\psi_s \pi y)\, ds)\, d\mu(y). \quad \square$$

Corollary 3.5. *Let two ergodic and invariant probability flows (ϕ, μ) and (ψ, ν) are equivalent, see Definition 3.1, with corresponding reparameterization $\theta(x,t)$ and homeomorphism π. Then*

$$\mu(A) = 0 \Leftrightarrow \nu(\pi(A)) = 0 \qquad \textit{for any Borel set } A.$$

Proof. If the sufficient part does not hold, there would exist a Borel set A with $\mu(A) > 0$ and $\nu(\pi(A)) = 0$. We denote by $\chi_{\pi(A)}$ the characteristic function of $\pi(A)$ and take a sequence $\{f_n\}$ of positive and bounded continuous function with $f_n \to \chi_{\pi(A)}$. So $\int f_n\, d\nu \to \int \chi_{\pi(A)}\, d\nu = \nu(\pi(A)) = 0$. We have by (4) in Definition 3.3 and Lemma 3.4

$$\int f_n d\nu = \frac{1}{\int \theta(x,1) d\mu} \int (\int_0^{\theta(x,1)} f_n(\psi_t(\pi x))\, dt) d\mu$$

$$= \frac{1}{\int \theta(x,1) d\mu} \int (\int_0^1 f_n(\pi\phi_t(x))\, dt) d\mu$$

$$= \frac{1}{\int \theta(x,1) d\mu} \int (\int_0^1 (f_n \circ \pi)(\phi_t(x))\, dt) d\mu.$$

Observe that $f_n \circ \pi \to \chi_A,\ n \to \infty$. Since $\int f_n d\nu \to 0$, then $\int (\int_0^1 (f_n \circ \pi)(\phi_t(x))\, dt) d\mu \to 0$ as $n \to \infty$. So $\int (\int_0^1 \chi_A(\phi_t(x))\, dt) d\mu \to 0$, which implies that $\mu(A) = \int \chi_A d\mu = 0$, a contradiction. So the sufficient part holds. One can show the necessary part similarly. □

Theorem 3.6. *Let $\phi,\ \psi : M \times \mathbb{R} \to M$ be two continuous flows on a compact metric space and let $\mu \in \mathcal{M}_{erg,\phi}$ and $\nu \in \mathcal{M}_{erg,\psi}$. If two probability flows (ϕ, μ) and (ψ, ν) are equivalent with reparameterization $\theta(x, t)$ for $\mu - a.a.\ x \in M$ while transferring ϕ to ψ, then*

$$h_\mu(\phi) = 0 \ \Leftrightarrow\ h_\nu(\psi) = 0, \quad \textit{and} \quad h_\mu(\phi) = \infty \ \Leftrightarrow\ h_\nu(\psi) = \infty.$$

Proof. Let $\eta > 0$ and take $\epsilon > 0$ such that

$$d(y_1, y_2) < \epsilon \quad \Rightarrow \quad d(\pi^{-1}y_1, \pi^{-1}y_2) < \eta, \quad \forall y_1, y_2 \in M.$$

By Proposition 2.1 and Lemma 3.4 $\lim_{t\to+\infty} \frac{\theta(x,t)}{t} = \int \theta(x, 1)\, d\mu = a < \infty,\ \mu - a.a.\ x \in M$. Note that $\theta(x, t) \geq 0,\ a \geq 0$. Define for big T

$$A(T) = \{x \in M \mid \frac{\theta(x, t)}{t} \leq a + 1,\ \forall t \geq T\}.$$

Then $\mu(A(T))$ tends to 1 as T tends to ∞. For given $0 < \delta << 1$ we take $T_0 > 0$ large such that $\mu(A(T_0)) > 1 - \delta$ and thus $\nu(\pi(A(T_0))) > 0$ by Corollary 3.5. Without loss generality (taking T_0 large when necessary) we assume that $\nu(\pi(A(T_0))) > 1 - \delta$. Denote $\beta = \sup_{x\in A(T_0)} \theta(x, 1)$. Then $0 < \beta < \infty$.

Set $N := R(\delta, t, \epsilon, \psi)$ for $t \geq T_0$ and take

$$D(y_1, t, \epsilon, \psi), D(y_2, t, \epsilon, \psi), \cdots, D(y_N, t, \epsilon, \psi)$$

to cover a subset of M of ν measure large than $1 - \delta$. Then

$$\nu((\cup_{i=1}^N D(y_i, t, \epsilon, \psi)) \cap (\pi A(T_0))) > 1 - 2\delta.$$

Assertion.

$$\nu(\cup_{i=1}^N D(y_i, t, \epsilon, \psi) \cap \pi A(T_0)) \leq \frac{\beta}{\int \theta(x, 1) d\mu} \mu(\cup_{i=1}^N \pi^{-1} D(y_i, t, \epsilon, \psi)) \cap A(T_0)).$$

Set $P = \cup_{i=1}^N D(y_i, t, \epsilon, \psi) \cap \pi A(T_0)$. Take a sequence of closed sets C_k and a sequence of open sets U_k such that $C_k \subset P \subset U_k$ and $\mu(U_k \setminus C_k) < \frac{1}{k}$, and take a sequence of continuous functions f_k such that $f_k(x) = 1$ on C_k, and $f_k(x) = 0$ on $M \setminus U_k$, and $0 \leq f_k \leq 1$. It is clear that $\lim_{k\to+\infty} f_k = \chi_P$, $\lim_{k\to+\infty} \int f_k d\nu = \int \chi_P d\nu$, and $\lim_{k\to+\infty} \int_{U_k \setminus P} f_k d\nu = 0$. By using Lemma 3.4 it follows that

$$\begin{aligned} \nu(P) &= \int \chi_P d\nu \\ &= \lim_{k\to+\infty} \int_P f_k d\nu + \lim_{k\to+\infty} \int_{U_k\setminus P} f_k d\nu \\ &= \lim_{k\to+\infty} \int_P f_k d\nu \\ &= \lim_{k\to+\infty} \frac{1}{\int \theta(x, 1) d\mu} \int_{\pi^{-1}P} (\int_0^{\theta(x,1)} f_k(\psi_t(\pi x))\, dt) d\mu \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\int \theta(x,1)d\mu} \int\limits_{\pi^{-1}P} \lim_{k\to+\infty} (\int\limits_{0}^{\theta(x,1)} f_k(\psi_t(\pi x))\, dt) d\mu \\
&= \frac{1}{\int \theta(x,1)d\mu} \int\limits_{\pi^{-1}P} (\int\limits_{0}^{\theta(x,1)} \lim_{k\to+\infty} f_k(\psi_t(\pi x))\, dt) d\mu \\
&\le \frac{\sup_{x\in A_0(T_0)} \theta(x,1)}{\int \theta(x,1)d\mu} \mu(\pi^{-1}P) \\
&= \frac{\beta}{\theta(x,1)} \mu(\pi^{-1}P).
\end{aligned}
$$

This proves the Assertion.

Arranging order when necessary we may assume the existence of $\tilde{N}$, $\tilde{N} \le N$, such that the first $\tilde{N}$ $(t, \epsilon, \psi)-$ balls

$$
D(y_1, t, \epsilon, \psi), D(y_2, t, \epsilon, \psi), \cdots, D(y_{\tilde{N}}, t, \epsilon, \psi)
$$

are exactly the ones that intersect with $\pi(A)$ with positive ν measure. By moving the centers slightly we may assume that all the centers $y_1, \cdots, y_{\tilde{N}}$ are in $\pi A(T_0)$, which makes sense for $\theta(\pi^{-1}y_i, t)$, $i = 1, \cdots, \tilde{N}$. Observe that

$$
D(\pi^{-1}y_i, \frac{t}{a+1}, \eta, \phi) \cap A(T_0) \supset \pi^{-1}(D(y_i, t, \epsilon, \psi) \cap \pi A(T_0)),\ i = 1, 2, \cdots, \tilde{N},\ \forall t \ge T_0.
$$

We thus have by the Assertion that

$$
\begin{aligned}
&\mu(\cup_{i=1}^{\tilde{N}} D(\pi^{-1}y_i, \frac{t}{a+1}, \eta, \phi) \cap A(T_0)) \\
\ge &\mu(\pi^{-1} \cup_{i=1}^{\tilde{N}} D(y_i, t, \epsilon, \psi) \cap \pi A(T_0)) \\
\ge &\frac{\beta}{a} \nu(\cup_{i=1}^{\tilde{N}} D(y_i, t, \epsilon, \psi) \cap \pi A(T_0)) \\
> &(1 - 2\delta) =: \delta'.
\end{aligned}
$$

Thus we have

$$
R(\delta', \frac{t}{a+1}, \eta, \phi) \le \tilde{N} \le N = R(\delta, t, \epsilon, \psi), \quad \forall t \ge T_0.
$$

By Definition 3.1 we have

$$
\begin{aligned}
h_\nu(\psi) &= \lim_{\epsilon\to 0} \lim_{t\to\infty} \frac{1}{t} \log R(\delta, t, \epsilon, \psi) \\
&\ge \lim_{\eta\to 0} \lim_{t\to\infty} \frac{1}{t} \log R(\delta', \frac{t}{a+1}, \eta, \phi) \\
&= \frac{1}{a+1} \lim_{\eta\to 0} \lim_{t\to\infty} \frac{1}{\frac{t}{a+1}} \log R(\delta', \frac{t}{a+1}, \eta, \phi) \\
&= \frac{1}{a+1} h_\mu(\phi).
\end{aligned}
$$

This shows that $h_\nu(\psi) \ge C h_\mu(\phi)$, where $C = \frac{1}{a+1} > 0$. One can show similarly that $h_\nu(\psi) \le C' h_\mu(\phi)$ for some positive constant C'. Therefore,

$$h_\mu(\phi) = 0 \Leftrightarrow h_\nu(\phi) = 0, \quad h_\mu(\phi) = \infty \Leftrightarrow h_\nu(\phi) = \infty. \quad \square$$

Proof of Main Theorem. (1) → (2). Assume (1), that is by Definition 1.1 we assume that $h(\phi) > 0$ and $h(\psi) = 0$. By Lemma 3.2(2), these deduce $\mathcal{M}^+_{erg,\phi} \neq \emptyset$ and $\mathcal{M}^+_{erg,\psi} = \emptyset$. By Theorem 3.6 and Proposition 2.3 $\pi_*\mathcal{M}^+_{erg,\psi} \cap \mathcal{M}_{erg,\psi} = \emptyset$. So the phenomenon of measure degeneracy happens.

(2) → (1). Since $\mathcal{M}^+_{erg,\phi} \neq \emptyset$ and $\mathcal{M}^+_{erg,\psi} = \emptyset$ by assumption, it holds by Lemma 3.2(2) that $h(\phi) > 0$ and $h(\psi) = 0$. So the phenomenon of entropy degeneracy happens. □

Proof of Corollary of Main Theorem. Now by Main Theorem 1.3 $h(\phi) > 0$ and $h(\psi) = 0$. To get the corollary let us suppose, on contradictory, that

$$\lim_{t\to+\infty} \frac{\theta(x,t)}{t} < \infty, \ \mu - a.a.x \in M$$

for some $\mu \in \mathcal{M}^+_{erg,\phi}$. Define a ψ ergodic invariant measure ν by

$$\int f d\nu = \frac{1}{\int \theta(x,1)d\mu} \int (\int_0^{\theta(x,1)} f(\psi_t(\pi x))\, dt) d\mu, \quad \forall f \in C^0(M,\mathbb{R}).$$

Clearly (ϕ,μ) and (ψ,ν) are measure-theoretic equivalent. By Theorem 3.6 $h_\nu(\psi) > 0$. And by Lemma 3.2(2), $h(\psi) > 0$, a contradiction. □

4. Probability systems in natural time-changed flows

Based on Theorem 1.3, we make an analysis in this section for probability systems in natural time-changed flows described as follows.

Let $\phi : M \times \mathbb{R} \to M$ be a C^r flow induced by a C^r vector field X, $r \geq 1$ and let M_0 denote the set of singularities of X. Let $\theta(x,t),\ x \in M \setminus M_0,\ t \in \mathbb{R}$ be a continuous function satisfying the following:

1) $\theta(x,0) = 0$ and $\theta_x = \theta(x,.) : \mathbb{R} \to \mathbb{R}$ is strictly increasing $\forall x \in M \setminus M_0$; and

2) $\theta_x(s+t) = \theta_x(s) + \theta_{\phi_{s(x)}}(t)$, $\forall s,\ t \in \mathbb{R},\ \forall x \in M \setminus M_0$.

Define a flow $\psi : M \times \mathbb{R} \to M$,

$$\psi_t(x) = \phi_{\theta(x,t)}(x), \quad \forall x \in M \setminus M_0, \quad t \in \mathbb{R},$$
$$\psi_t(x) = x, \ \forall x \in M_0, \ \forall t \in \mathbb{R}.$$

We call ψ to be a time-changed flow from ϕ by $\theta(x,t)$. (M,ϕ) and (M,ψ) are clearly equivalent, with transferring homeomorphism $\pi := id : M \to M$. If further the function $\theta(x,t),\ x \in M \setminus M_0,\ t \in \mathbb{R}$ is C^1 differentiable in both variables, then we call ψ a natural time-changed flow from ϕ by $\theta(x,t)$. In this case there is a C^1 vector field Y on M such that ψ is induced by Y.

Remark. If both ϕ and ψ are C^r and x is not a fixed point, then the Implicit Function Theorem immediately implies that $\theta(x,t)$ is C^r in both variables.

Proposition 4.1. *For a natural time-changed flow from ϕ to ψ by $\theta(x,t)$ it holds*

$$\theta(x,t) = \int_0^t \frac{\|Y(\psi_s x)\|}{\|X(\psi_s x)\|} ds, \quad \forall t \in \mathbb{R}, \quad x \in M \setminus M_0.$$

Proof. From

$$\psi_t(\psi_s x) = \psi_{s+t}(x) = \phi_{\theta(x,s+t)}(x) = \phi_{\theta(\phi_s x,t)}(\phi_{\theta(x,s)}x),$$

we get

$$Y(\psi_s x) = \frac{d\psi_t(\psi_s x)}{dt}|_{t=0} = \frac{d\phi_{\theta(\phi_s x,t)}(\phi_{\theta(x,s)}x)}{d\theta(\phi_s x,t)}|_{\theta=0}\,\frac{d\theta(\phi_s x,t)}{dt}|_{t=0} = X(\phi_{\theta(x,s)}x)\dot{\theta}(\phi_s x,0),$$

where $\dot{\theta}(\phi_s x,0) = \frac{d\theta(\phi_s x,t)}{dt}|_{t=0}$, and $x \in M \setminus M_0$. Note $\theta(x,t)$ is increasing with t, $\dot{\theta}(\phi_s x,0) \geq 0$. So

$$\dot{\theta}(\phi_s x,0) = |\dot{\theta}(\phi_s x,0)| = \frac{\|Y(\psi_s x)\|}{\|X(\phi_{\theta(x,s)}x)\|} = \frac{\|Y(\psi_s x)\|}{\|X(\psi_s x)\|}.$$

Observe that

$$\dot{\theta}(\phi_s x,0) = \frac{d\theta(\phi_s x,t)}{dt}|_{t=0} = \frac{d\theta(x,s+t)}{d(s+t)}|_{t=0} = \frac{d\theta(x,s)}{ds},$$

and $\theta(x,0) = 0$, we have

$$\theta(x,t) = \int_0^t \frac{\|Y(\psi_s x)\|}{\|X(\psi_s x)\|}\,ds, \quad \forall t \in \mathbb{R}, \quad x \in M \setminus M_0. \quad \square$$

As showed in 1.3 and [6] it is singularity that causes measure degeneracy. And one singularity is enough to cause such degeneracy, see [1] [11] for discrete case. We show in the next theorem that the singularity that causes measure degeneracy is **not** hyperbolic.

Theorem 4.2. *Let M be a compact manifold and let $\phi : M \times \mathbb{R} \to M$ be a C^r flow induced by a C^r vector field X ($r \geq 1$) with exact one singularity p. Let ψ be a natural time-changed flow from ϕ by $\theta(x,t)$ and denote Y the vector field of ψ. If the entropy degeneracy happens while changing ψ to ϕ, that is, $h(\phi) > 0$ and $h(\psi) = 0$, then p is not hyperbolic.*

Proof. p is the unique singularity for both ϕ and ψ. We denote $f(x) = \frac{\|Y(x)\|}{\|X(x)\|}$, $x \in M \setminus \{p\}$. Denote by $\lambda_1, \cdots \lambda_n$ the eigenvalues of $D\phi|_p$ and by $\gamma_1, \cdots, \gamma_n$ the eigenvalues of $D\psi|_p$, where $n = dimM$. Since p is a hyperbolic singularity for both X and Y, non eigenvalue is zero. There exist a neighborhood N of p, four positive numbers a, b, c, d such that for $x \in N$ it holds that

$$a\sqrt{|\lambda_1|^2 + \cdots |\lambda_n|^2} \leq \|X(x)\| \leq b\sqrt{|\lambda_1|^2 + \cdots |\lambda_n|^2}$$
$$c\sqrt{|\gamma_1|^2 + \cdots |\gamma_n|^2} \leq \|Y(x)\| \leq d\sqrt{|\gamma_1|^2 + \cdots |\gamma_n|^2}$$

Denote $A = \frac{c\sqrt{|\gamma_1|^2+\cdots|\gamma_n|^2}}{b\sqrt{|\lambda_1|^2+\cdots|\lambda_n|^2}}$, $B = \frac{d\sqrt{|\gamma_1|^2+\cdots|\gamma_n|^2}}{a\sqrt{|\lambda_1|^2+\cdots|\lambda_n|^2}}$. Then $A \leq f(x) \leq B$, $x \in N$. By Proposition 4.1

$$\theta(x,t) = \int_0^t \frac{\|Y(\psi_s x)\|}{\|X(\psi_s x)\|}\,ds, \quad \forall t \in \mathbb{R}, \quad x \in M \setminus \{p\}.$$

So $At \leq \theta(x,t) \leq Bt$, $x \in N$, $t \geq 0$. Observe there is no singularity in $M \setminus N$, $Ct \leq \theta(x,t) \leq Dt$, $x \in M \setminus N$, $t \geq 0$. Set $E = \min\{A, C\}$ and $F = \max\{B, D\}$, then

$$Et \leq \theta(x,t) \leq Ft, \quad x \in M, \ t \geq 0.$$

This implies by a standard argument that entropy degeneracy does not happen, a contradiction. So the singularity p is not hyperbolic. □

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