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Nonlinear dispersive equations: classical and new frameworks

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Abstract

The purpose of this manuscript associated to the Golden Jubilee of the IME-USP is to present selected material from the author's scientific contribution dealing with nonlinear phenomena of dispersive type. That study has been a source for modern research in the dynamic of traveling wave solutions of different type and it has been disseminated through publication of scientific articles and/or books. Here, we provide the reader with information about current research in stability theory for nonlinear dispersive equations and possible developments. Also, I hope it inspires future developments in this important and fascinating subject. In this manuscript we consider the following topics: stability theory of solitary waves and the applicability of the concentration–compactness principle, the existence and orbital (in)stability of periodic traveling wave solutions for nonlinear dispersive models, nonlinear Schrödinger and Korteweg–de Vries models on star-shaped metric graphs. The use of tools of the theory of spaces of Hilbert, the spectral theory for unbounded self-adjoint operators, Sturm–Liouville's theory, variational methods, analytic perturbation theory of operators, and the extension theory of symmetric operators are pieces fundamental in our study. The methods presented in this manuscript have prospect for the study of the dynamic of solutions for nonlinear evolution equations around of different traveling waves profiles which may appear in non-standard environments such as star-shaped metric graphs.

Keywords Dispersive equations · Variational methods · Traveling waves · Orbital stability · Linear instability · Korteweg–de Vries models · BBM models · Schrödinger models · Point interactions · Metric graphs · Analytic perturbation · Sturm–Liouville theory · Extension theory of symmetric operators

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1 Introduction

During the last 40 years the theory of stability for traveling waves solutions of nonlinear evolutions equations has grown into a large field that attracts the attention of both mathematicians and physicists in view of its applications to real-world nonlinear models and the novelty of the problems. These models emerge in various areas of applications such as natural science (physics, biology, etc.), engineering and fluid mechanics (optics, laser beams, water waves, Bose–Einstein condensates), as well as, the propagation through junctions in networks (so called, metric star graphs), among others.

Nonlinear dispersive evolutions equations for modelling waves take into account both nonlinearity and dispersion effects. The discovery of the "*great wave of translation or solitary wave*" in the Edinburgh–Glasgow canal in 1834 by Russell [123], was the birth of one of the most relevant and fascinating topics of study for dispersive equations in the last 40 years. Fascinated with this long water wave without a change in shape, Russell made some laboratory experiments on this phenomenon, generating solitary waves by dropping a weight at one end of a water channel (see, for instance, the YouTube video "Shallow water wave generation (quasi solitary wave with breaking), by Nick Pizzo). It was not until the 1870s that Russell's prediction was finally and independently confirmed by both Boussinesq [51] and Rayleigh [121]. Assuming that a solitary wave has a length much greater than the depth of the water, they derived from the equations of motion for an inviscid incompressible liquid that the wave height above the mean level h, $z = \psi(x, t)$, is given by

$$\psi(x,t) = a\text{sech}^2[\beta(x-ct)], \tag{1.1}$$

where $\beta^2 = 3a/[4h^2(h+a)]$ for any positive amplitude $a > 0$. Although these authors found the sech^2 profile, they did not write any evolution equation that produces (1.1) as a solution. However, Boussinesq did much more [52] and discovered that if a water wave propagates along a flat-bottomed channel of undisturbed depth h, and has large wavelength and small amplitude relative to h, then the elevation $u = u(x, t)$ of the water surface, considered as a function of the coordinate x along the channel and the time t, will approximately satisfy the nonlinear evolution equation

$$u_{tt} - ghu_{xx} - gh\left[\frac{3}{2h}u^2 + \frac{h^2}{3}u_{xx}\right]_{xx} = 0, \tag{1.2}$$

where g is the gravitational acceleration and $\sqrt{gh}$ is the speed of the shallow water waves. This equation is currently known as the *Boussinesq (bidirectional) equation*. Using this equation, he obtained an explicit representation of solitary waves traveling in both positive and negative x-directions, $u(x, t) = \phi_{Bou}(x \pm ct)$, where the profile

ϕ_{Bou} is given by

$$\phi_{Bou}(\xi) = a\ \mathrm{sech}^2\left[\sqrt{\frac{3a}{h^3}}\xi\right]. \tag{1.3}$$

In 1895, Korteweg and de Vries [90] formulated a mathematical model which provided an explanation of the phenomenon observed by Scott Russell (who were apparently unaware of the work of Boussinesq). They derived the now-famous equation for the propagation of waves in one direction on the surface of water of density ρ, which in dimensional variables can be written in the form

$$v_t = \frac{\sqrt{gh}}{h}\left[\left(\varepsilon + \frac{3}{2}v\right)v_x + \frac{1}{2}\sigma v_{xxx}\right] \tag{1.4}$$

where $v = v(x,t)$, x is a coordinate chosen to be moving with the wave, ε is a small parameter and

$$\sigma = h\left(\frac{h^2}{3} - \frac{T}{g\rho}\right) \sim \frac{1}{3}h^3,$$

when the surface tension $T(\ll \frac{1}{3}g\rho h^2)$ is negligible. This is essentially the original form of the *Korteweg–de Vries equation*.

Now, the Galilean invariance is one of the fundamental properties of any mathematical model arising in classical mechanics. Nowadays, it is common to speak about this principle in terms of a *symmetry* of the governing equations. Hereinbelow, the Galilean invariance property is referred to the fact that the governing equations are invariant under a Galilean boost transformation, in the sense that this kind of transformations preserves the space of the solutions of the respective problem. For example, model (1.4) posses the Galilean invariance of translation and scale, and they are giving by the following transformation of variables:

$$x \to \gamma x, \quad x \to x + x_0, \quad t \to \theta t, \quad x \to x - \lambda t, \quad v \to \eta v,$$

with real constants $\gamma, x_0, \theta, \lambda$ and η. Thus, if v is a solution of (1.4), so is u defined by $u(x,t) = \eta v(\gamma(x - \lambda t), \theta t)$, with

$$3\gamma^3 = \sigma, \quad \gamma\eta = 1, \quad 3\sqrt{gh}\theta = -2h, \quad 3\lambda = -2\epsilon/\gamma.$$

In this moving frame of reference (1.4) becomes to

$$u_t + uu_x + u_{xxx} = 0. \tag{1.5}$$

We shall call it the KdV equation. We note that in the approximation used to derive Eq. (1.4) one considers long wave propagating in the direction of increasing x. Equation (1.4) or (1.5) is one of the simplest and most useful nonlinear dispersive model equation for solitary waves or periodic traveling waves, and it represents the long-time evolution of wave phenomena in which *the steepening effect of the nonlinear term* uu_x

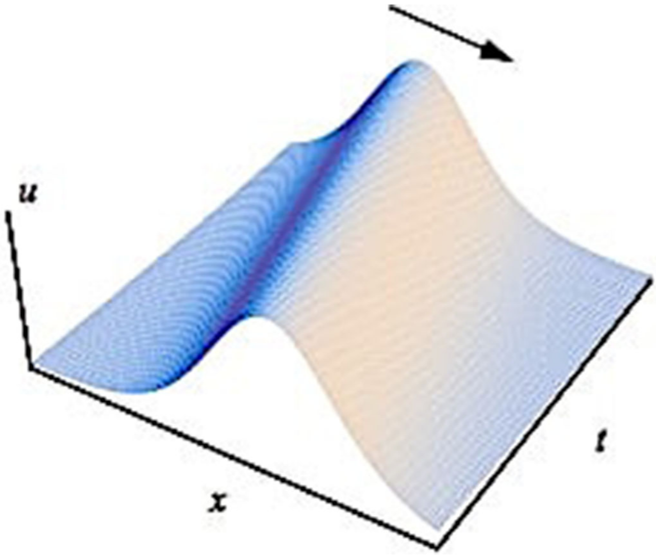

Fig. 1 Time evolution for the solitary wave solution of the KdV model (1.5)

is counterbalanced by the dispersion term u_{xxx}. From Galilean invariance above, we obtain that the KdV model (1.5) has the following two-parameter family of solitary waves solutions (see Fig. 1)

$$u(x,t) = \phi_{KdV}(x - ct + x_0), \quad \phi_{KdV}(\xi) = 3c \operatorname{sech}^2\left(\frac{1}{2}\sqrt{c}\xi\right), \quad c > 0, x_0 \in \mathbb{R}. \tag{1.6}$$

Some consequences of the presence of symmetries in the models are well known. They include, for example, the Hamiltonian formulation and the generation of conserved quantities. In the case of Galilean invariance, it is worth mentioning its influence in the existence and stability of solitary and periodic traveling wave solutions (see e.g., [19]). However, the practical implications of losing the Galilean symmetry are not sufficiently known. In the present study we will give recent results on this issue and its influence on the stability and existence of traveling waves for dispersive wave models on metric graph, i.e., a network-shaped structure of vertices connected by edges (see Fig. 2), with a nonlinear dispersive model (such as a KdV model or a Schrödinger model) suitably defined on functions that are supported on the edges (see Sects. 5 and 6 below). These models arise as simplified models for wave propagation, for instance, in a quasi one-dimensional (e.g. meso- or nanoscale) system that looks like a thin neighborhood of a graph.

There are few analytic studies of soliton propagation through networks by nonlinear flows. Results on the stability or instability mechanism of these profiles are still unclear. One of the objectives of this work is to shed light on these themes. A central point that makes this analysis a delicate problem is the presence of a vertex where the underlying one-dimensional metric graph should bifurcate (or multi-bifurcate in a general metric graph with several vertex) and so some specific Galilean invariance can break down, by instance, *the translation symmetry*. We note that not branching angles but the topology of bifurcation is essentially the most delicate point in the analysis. Indeed, a soliton-profile coming into the vertex along one of the bonds shows a complicated motion around the vertex such as reflection and emergence of the radiation there, moreover, in particular one can not see easily how energy travels across the network.

By returning to our solitary waves or periodic traveling waves study for a specific nonlinear dispersive model, in general, it is not obvious that these profiles exist. Not even when a particular theory shows only the existence of such solutions, it is not easy to find an explicit formula, specially for models in high dimensions. For real applications in several fields of the nature science, it is not only sufficient the existence of specific profiles, but also its behavior when small disturbances of these profiles are made, namely, its stability or instability by the nonlinear flow of the associated model. In the papers [9,14–17,21,33,42] and in the book [19], the author has established methods and several results in the study of the existence and stability and/or instability for solitary waves via variational methods (the concentration–compactness principle) for the general class of equations of the type

$$u_t + u^p u_x - M u_x = 0, \tag{1.7}$$

where $u = u(x,t)$ is real valued, $x, t \in \mathbb{R}$, $p \in \mathbb{N}$, $p \geqq 1$, and M is a linear operator defined as a Fourier multiplier operator by $\widehat{Mu}(\xi) = \alpha(\xi)\widehat{u}(\xi)$. Here the symbol $\alpha(\xi)$ is a measurable, even function on $\mathbb{R}$, and satisfies

$$a_1|\xi|^{\beta_1} \leqq \alpha(\xi) \leqq a_2(1+|\xi|)^{\beta_2} \quad \text{for } \xi \in \mathbb{R} \tag{1.8}$$

where $a_1, a_2 > 0$ and $\beta_2 \geqq \beta_1 > 0$. We note that model (1.7) has the Galilean invariance of translation.

The general equation (1.7) contains several basic dispersive models of continuous study in the last decades:

(1) The generalized Korteweg–de Vries equation (GKdV henceforth): For $\alpha(\xi) = |\xi|^\beta$, $\beta > 1$, the operator M is denoted by D^β and we get the model

$$u_t + u^p u_x - D^\beta u_x = 0. \tag{1.9}$$

The case $p = 1$ and if we allow β such that $\beta \in (0, 1)$, the model in (1.9) is known as the *fractional Korteweg–de Vries equation* (fKdV henceforth). For $\beta = 2$ in (1.9), the solitary wave solutions have the following explicit profile:

$$\phi_c(\xi) = \left[\frac{(p+1)(p+2)c}{2}\right]^{1/p} \operatorname{sech}^{2/p}\left(\frac{p\sqrt{c}}{2}\xi\right). \tag{1.10}$$

For the fKdV equation explicit solutions are not known yet (see Angulo [21] and Klein and Saut [89]).

(2) The Benjamin–Ono equation (BO henceforth): For $\alpha(\xi) = |\xi|$, the operator $M = H\partial_x$ where H is the Hilbert transform, that is the convolution with the *Principal Value* of $\frac{1}{x}$: $Hf(y) = P.V.(\frac{1}{x}) * f(y)$. The model is

$$u_t + uu_x - Hu_{xx} = 0. \tag{1.11}$$

Via the Fourier transform we have that $\widehat{Hf}(\xi) = -isgn(\xi)\widehat{f}(\xi)$. The BO equation was first formally derived by Benjamin [44] (see also Ono [118]), to describe the propagation of long weakly nonlinear internal waves in a stratified fluid such that two layers of different densities are joined by a thin region where the density varies continuously (pycnocline), the lower layer being infinite.
The solitary wave solutions for the BO equation have the following algebraic profile:

$$\phi_c(\xi) = \frac{4c}{1 + c^2\xi^2}, \quad c > 0. \tag{1.12}$$

For $p \geqq 2$ and $\beta = 1$ in (1.9) (the Generalized BO equation) explicit solutions are not known.
We note that H represents a nonlocal operator, namely, the value of $Hu(x)$ depends on the whole solution u and not only on its behavior nearby the point x. For instance, for $f(x) = max\{1 - |x|, 0\}$, $x \in \mathbb{R}$, we obtain $Hf(y) = ln\left|\frac{y-1}{y+1}\right| + ln\left|\frac{y^2}{y^2-1}\right|$, $y \in \mathbb{R}$.

(3) The Benjamin equation: For $\alpha(\xi) = \xi^2 - \ell|\xi|$, where $\ell \in \mathbb{R}$, the linear differential operator M is given by $M = -\partial_x^2 - \ell\partial_x H$ and (1.7) is transformed into the pseudo-differential equation

$$u_t + uu_x + \ell Hu_{xx} + u_{xxx} = 0. \tag{1.13}$$

Benjamin [46,47], introduced this relatively new model which describes the unidirectional propagation of internal waves. This model governs approximately the evolution of waves on the interface of a two-fluid system in which *surface tension* effects can not ignored.

By considering $c > inf_{\xi\in\mathbb{R}}\alpha(\xi)$ we obtain that $M + c$ represents a positive operator. The shape of the solitary wave solution $u(x,t) = \phi_c(x - ct)$ of (1.7) must satisfy the equation

$$M\phi_c + c\phi_c - \frac{1}{p+1}\phi_c^{p+1} = 0. \tag{1.14}$$

Moreover, by the Galilean invariance of translation for (1.7) we have that for any $x_0 \in \mathbb{R}$, the profile $\phi_c(x + x_0)$ is also a solution for (1.14). The uniqueness of solutions for (1.14) (modulo this symmetry) is in general a delicate problem for dispersive models. Now, by depending on the complexity of the linear operator M we have that the existence of solutions to Eq. (1.14) can become a "major headache". Even though we can show the existence of solutions, an explicit formula for these can become a major complication. The Benjamin equation in (1.13) is a beautiful example of this situation, moreover, a result of the uniqueness of the solitary waves solutions for this model represents a very interesting open question.

The following issue what we are interested here it is related to the stability property of traveling wave profiles. For fixing ideas, we consider the model (1.7) and the existence of a sufficiently smooth curve $c \to \phi_c$ of solutions of traveling wave type $u(x,t) = \phi_c(x-ct)$, for a real profile $\phi_c : \mathbb{R} \to \mathbb{R}$ and $c \in I \subset \mathbb{R}$, with ϕ_c satisfying (1.14). Stability problems are related with one or more specific Galilean invariance associated to the model in a general form. Here we will consider the symmetry of translation, namely, if $\tau(r)$ represents the translation operator defined for ψ by $\tau(r)\psi(x) = \psi(x+r)$, we have that $\tau(r)u(x,t) = u(x+r,t)$ is a solution for (1.7) provided $u(x,t)$ is a solution. We note that if X is a functional space (by instance, Sobolev spaces) and $\phi \in X$ is a traveling wave solution of (1.7), then the map $t \to \tau_{ct}\phi$ represents a trajectory in X, called by the ϕ-*orbit generated by* ϕ, and which we will denote by Ω_ϕ, namely, $\Omega_\phi = \{\phi(\cdot + r) \ : \ r \in \mathbb{R}\}$. Define for any $\eta > 0$ the set $U_\eta \subset X$ by

$$U_\eta = \{v \in X : \inf_{y \in \mathbb{R}} \|v(\cdot) - \phi(\cdot + y)\|_X < \eta\},$$

so-called a η-neighborhood of Ω_ϕ.

Definition 1.1 Let $\phi \in X$ be a traveling wave solution to Eq. (1.7). We say that ϕ is *orbitally stable* in X by the flow of (1.7) if

(i) there is a Banach space $Y \subset X$ such that for all $u_0 \in Y$, there is a unique solution u of (1.7) in $C(\mathbb{R}; Y) \subset C(\mathbb{R}; X)$ with $u(x,0) = u_0$; and
(ii) for every $\epsilon > 0$, there exists a $\delta > 0$ such that for all $u_0 \in U_\delta \cap Y$, the solution u of (1.7) with $u(x,0) = u_0$ satisfies $u(t) \in U_\epsilon$ for all $t \in \mathbb{R}$.

In the case $u \in C((-T^*, T^*); Y) \subset C((-T^*, T^*); X)$, where T^* is the maximal time of existence of u, *the property of stability is called conditional*.

Otherwise the items (i)–(ii), we say that ϕ is X-unstable.

This notion of stability (items (i)–(ii) above) was called by Benjamin as *stability in shape* and he was the first to give a rigorous proof of stability of the solitary wave ϕ_{KdV} in (1.6) by the flow of the KdV equation (making an elaboration of Boussinesq's original ideas associated to model (1.2)).

Roughly speaking, Definition 1.1 says that the ϕ-orbit, Ω_ϕ, is stable by the flow generated by Eq. (1.7) if whenever the initial data u_0 is sufficiently near to Ω_ϕ in the X-norm, then for each instant t_0 there is a translation, $\gamma(t_0)$, such that the shape of the function $x \to u(x + \gamma(t_0), t_0)$ will begin to resemble and remain close to ϕ in the X-norm. We note that the existence of Banach space $Y \subset X$ in Definition 1.1 is required by the problem of well-posedness associated the equation in question. Sometimes a theory is not known in the space X where we want to test the stability, but it is possible to obtain information about the initial problem in some subspace Y.

With regard to the instability property of traveling wave solution for nonlinear dispersive models, we can say that there are at least three approach ways for studying this fascinant topic. The first one (and the classical ones) is that obtained via blow-up mechanics, for instance, in the case of the critical KdV equation in (1.9) ($p = 4$ and $\beta = 2$) was shown in [107–109] that the orbit generated by the solution $u(x,t) =$

$\phi_c(x-ct)$, $c>0$, is $H^1(\mathbb{R})$-unstable in the sense that there exists a sequence $\{\phi_n\}\subset H^1(\mathbb{R})$ with $\phi_n\to\phi_c$ in $H^1(\mathbb{R})$, and such that the corresponding maximal solutions u_n of the critical KdV with $u_n(0)=\phi_n$, blow-up in finite time, that means, there exists $T^*=T^*(\phi_n)>0$ such that

$$\lim_{t\uparrow T^*}\|\partial_x u_n(t)\|_1=+\infty.$$

Now, as a second instability approach, Grillakis et al. [79] set up sufficient conditions in the case of abstract Hamiltonian equations of the form

$$u_t=J\mathcal{E}'(u(t)),$$

with J *being an onto* skew-symmetric linear operator and $\mathcal{E}'$ the Frechet derivative of a functional $\mathcal{E}$ usually called of energy. So, a direct application of this theory to evolution equations of the form (1.7) (where $J=\partial_x$) is not so immediate since J is not onto. However, in the case of the following NLS equations instability approach can be applied once J is a matrix,

$$iu_t+\Delta u+|u|^{2\sigma}u=0 \tag{1.15}$$

where $u=u(x,t)\in\mathbb{C}$, $x\in\mathbb{R}^n$, $t\in\mathbb{R}$ and $\sigma>0$. An improvement of the instability theory in [79] has been made by Bona et al. [50] in the case of (1.7). As a consequence, for the gKdV in (1.9) is obtained that the solitary wave profiles for $\beta\geqq 1$ and $p>2\beta$ are $H^{\frac{\beta}{2}}(\mathbb{R})$-unstable (see End-notes subsection 2.3 below). We note that the instability of these profiles via a blow-up behavior is an open problem.

Lastly, a third instability approach is that obtained via a linear instability study. For fixing ideas with model (1.7), the strategy is the following: let ϕ_c be a solitary wave solution of Eq. (1.14) and consider the new variable based on the Galilean invariance of translation τ for (1.7)

$$w(x,t)=u(\tau(ct)x,t)-\phi_c(x).$$

Then, using Eq. (1.14) one finds that w satisfies the nonlinear equation

$$(\partial_t-c\partial_x)w+\partial_x\left(\varphi_c^p w-Mw+O\left(\|w\|^2\right)\right)=0. \tag{1.16}$$

As a leading approximation for small perturbation, we replace (1.16) by its *linearization around* ϕ_c, and hence we get the linear equation

$$(\partial_t-c\partial_x)w+\partial_x\left(\phi_c^p w-Mw\right)=0. \tag{1.17}$$

In other words, we obtain the so-called *linearized equation* around the solitary wave ϕ_c

$$\frac{dw}{dt}=J\mathcal{L}_c w, \tag{1.18}$$

$J = \partial_x$ and $\mathcal{L}_c$ is defined by

$$\mathcal{L}_c = M + c - \phi_c^p. \tag{1.19}$$

Assumptions for an instability study associated to (1.18) are not given on the generator $J\mathcal{L}_c$ but rather on the linear operator $\mathcal{L}_c$. Now, since ϕ_c depends on x but not on t, the Eq. (1.18) admits treatment by separation of variables which leads naturally to a spectral problem. Seeking particular solutions of (1.17) of the form $w(x,t) = e^{\lambda t}\psi(x)$ (so-called *growing mode solution*), where $\lambda \in \mathbb{C}$, ψ satisfies the extended eigenvalue problem

$$J\mathcal{L}_c\psi = \lambda\psi, \tag{1.20}$$

and so, we can say from (1.20) that the complex growth rate λ appears as (spectral) parameter. If Eq. (1.20) has a nonzero solution $\psi \in D(\mathcal{L}_c) \subset H^{\beta_1/2}(\mathbb{R})$ then an bootstrapping argument shows that $\psi \in H^s(\mathbb{R})$ for all $s \geqq 0$, so that (1.20) is satisfied in classical sense.

Paraphrasing classical semigroups and spectral theory nomenclature, a necessary condition for the "stability" of ϕ_c is that there are not points λ with $\mathrm{Re}(\lambda) > 0$ (which would imply the existence of a solution ψ of (1.20) that grows exponentially in time). If we write the spectrum σ of $\partial_x\mathcal{L}_c$ as the disjoint union of the essential spectrum and the discrete spectrum, $\sigma = \sigma_{esse} \cup \sigma_{disc}$, the last discussion suggests the utility of the following definition:

Definition 1.2 (*Linear stability and instability*) A solitary wave solution ϕ_c of the Eq. (1.7) is said to be linearly stable if $\sigma \subset i\mathbb{R}$. Otherwise (i.e., if σ contains point with $\mathrm{Re}(\lambda) > 0$) ϕ_c is linearly unstable.

We recall that since (1.18) is a real Hamiltonian equation, it forces certain elementary symmetries on the spectrum of σ, more exactly, since J is a skew-symmetric operator and $\mathcal{L}_c$ is a self-adjoint operator it follows σ will be symmetric with respect to reflection in the real and imaginary axes and $\sigma_{ess} \subset i\mathbb{R}$ (see [80]). Therefore, it implies that exponentially growing perturbation is always paired with exponentially decaying ones. It is the reason by which was only required in Definition 1.2 that the spectral parameter λ has to satisfy the condition $\mathrm{Re}(\lambda) > 0$.

The spectral problem in (1.20) for traveling wave solutions (by instance, solitary, standing wave or periodic waves) has been the focus of many research studies in the last two decades. In Grillakis et al. [80] were given assumptions based in that the skew-symmetric operator J is one-one and onto, and on the Morse index associated to a general self-adjoint operator $\mathcal{L}_c$. In Lopes [105] has been assumed only that J is one-to-one in order to give a convenient set of sufficient conditions for the linear (spectral) instability. Other approaches for solving (1.20) have been given in Lin [97], Kapitula and Stefanov [86], Angulo [21], Angulo and Natali [38] (the case of periodic traveling waves) and Angulo and Cavalcante [26] (stationary solutions for the Korteweg–de Vries model on metric graphs, see Sect. 6 below), among others.[1] In this

[1] An approach of nonlinear instability for solitary wave solutions, whenever these solutions are obtained via a variational approach, can be found in Gonçalves [76], Angulo [19] and recently in Corcho et al. [61]. *We note that this approach does not depend on the established ones by the theories in* [50,79,80], *like spectral conditions type.*

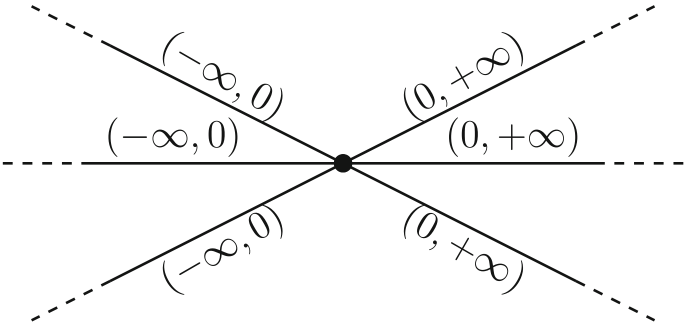

Fig. 2 A star-shaped metric graph with 6 edges

manuscript we establish a specific linear instability criterium associated to periodic traveling waves solutions for general Benjamin–Bona–Mahoney equation in (4.24).

Other focus of our study here, it is related with the dynamic of nonlinear dispersive equations on quantum star-shaped metric graphs $\mathcal{G}$. We recall that a star-shaped metric graph $\mathcal{G}$ is a structure represented by a finite or countable edges attached to a common vertex, $\nu = 0$, having each edge identified with a copy of the half-line $(-\infty, 0)$ or $(0, +\infty)$ (see Fig. 2 below). A quantum star-shaped metric graph $\mathcal{G}$ is a star-shaped metric graph with a linear Hamiltonian operator (such as Schrödinger-like operator or an Airy-like operator) suitably defined on functions which are supported on the edges.

Nonlinear dispersive models on star-shaped metric graph $\mathcal{G}$ arise as a simplification for wave propagation, for instance, in a quasi one-dimensional (e.g. meso- or nanoscale) systems that *looks like a thin neighborhood of a graph*. Quantum graph has been used to describe a variety of physical problems and applications, for instance, condensed matter, Y-josephson junction networks, polymers, optics, neuroscience, DNA, blood pressure waves in large arteries or in shallow water equation to describe a fluid network (see [49,53,55,67,94,111] and references therein). Recently, they have attracted much attention in the context of soliton transport in networks and branched structures (see [126]) since wave dynamics in networks can be modeled by nonlinear evolution equations.

In particular, the prototype of framework (graph-geometry) for the description of many phenomena has been a *star graph* $\mathcal{G}$, namely, a star-shaped metric graph with N half-lines $(0, +\infty)$ connected at a common vertex $\nu = 0$, together with a nonlinear equation suitably defined on the edges. For instance, we have the following nonlinear (vectorial) Schrödinger model ([2,3,28–30])

$$i\partial_t \mathbf{U}(x,t) - \mathcal{A}\mathbf{U}(x,t) + |\mathbf{U}(x,t)|^{p-1}\mathbf{U}(x,t) = 0, \tag{1.21}$$

where $\mathbf{U}(x,t) = (u_j(x,t))_{j=1}^N : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{C}^N$, $p > 1$, and the nonlinearity acts by components, i.e. $(|\mathbf{U}|^{p-1}\mathbf{U})_j = |u_j|^{p-1}u_j$. The function $\mathbf{U}$ can be assumed to satisfy specific boundary conditions such as either Kirchhoff, or δ, or δ' interactions at the vertex $\nu = 0$, such that the diagonal-matrix Hamiltonian operator

$$\mathcal{A} = \left(\left(-\frac{d^2}{dx^2} \right) \delta_{ij} \right)$$

remains a self-adjoint operator on $L^2(\mathcal{G})$. Here δ_{ij} represents the delta de Kronecker. In the case of a δ-interaction type at the vertex $\nu = 0$, we have $\mathcal{A}$ acting as $(\mathcal{A}\mathbf{V})(x) = (-v_j''(x))_{j=1}^N$, $x > 0$, with the domain $D_{\alpha,\delta}(\mathcal{A})$ defined by $\alpha \in \mathbb{R}$ as (see (5.30))

$$D_{\alpha,\delta}(\mathcal{A}) := \left\{ (v_j)_{j=1}^N \in H^2(\mathcal{G}) : v_1(0) = \cdots = v_N(0), \sum_{j=1}^N v_j'(0) = \alpha v_1(0) \right\}. \tag{1.22}$$

Another model of interest here, it is that of the Korteweg–de Vries equation (KdV)

$$\partial_t u_{\mathbf{e}}(x,t) = \alpha_{\mathbf{e}} \partial_x^3 u_{\mathbf{e}}(x,t) + \beta_{\mathbf{e}} \partial_x u_{\mathbf{e}}(x,t) + 2u_{\mathbf{e}}(x,t)\partial_x u_{\mathbf{e}}(x,t), \tag{1.23}$$

$x \neq 0$, $t \in \mathbb{R}$, on a star-shaped metric graph $\mathcal{G}$ with a structure represented by the set $\mathbf{E} \equiv \mathbf{E}_- \cup \mathbf{E}_+$ where $\mathbf{E}_+$ and $\mathbf{E}_-$ are finite or countable collections of semi-infinite edges $\mathbf{e}$ parametrized by $(-\infty, 0)$ or $(0, +\infty)$, respectively. The half-lines are connected at a unique vertex $\nu = 0$ (see Fig. 2). Here $(\alpha_{\mathbf{e}})$ and $(\beta_{\mathbf{e}})$ are two sequences of real numbers. As was seen in the first pages of this introduction, the KdV equation was first derived as a model for long waves propagating on a shallow water surface. Recently, the KdV model in (1.23) has been used to study blood pressure waves in large arteries. In this way, for example, in Chuiko et al. [62] was proposed a new computer model for systolic pulse waves within the cardiovascular system. Also, in Crepeau et al. [63] was showed that some particular solutions of the KdV equation, more exactly, the 2 and 3-soliton well-known solutions, seem to be good candidates to match the observed pressure pulse waves. This new applications for the KdV model suggest your study on star-shaped metric graphs (see Cavalcante [58]).

We note that branched networks provides a nice field where one can look for interesting soliton propagations and nonlinear dynamics in general, and in recent years there has been a growing interest among the scientific community in modeling and analyzing evolution problems described by partial differential equations (PDEs) on graphs. However, there are few analytic study of soliton propagation through networks and so one of the objectives of this manuscript is to provide the reader with several frontier results of this topic and new analytical tools for this study.

One of the main interest of exposition here is to establish an abstract linear instability criterium for stationary profiles on a star-shaped metric graph $\mathcal{G}$. In particular, we apply this one to the KdV model (1.23). Novel applications of our general linear instability criterium have been obtained recently in Angulo and Plaza [39,40] for the sine-Gordon equation on the framework of a Y-josephson junction

$$\begin{cases} \partial_t u_{\mathbf{e}} = v_{\mathbf{e}}, & \mathbf{e} \in \mathbf{E} = (-\infty, 0) \cup (0, +\infty) \cup (0, +\infty), \\ \partial_t v_{\mathbf{e}} = c_{\mathbf{e}}^2 \partial_x^2 u_{\mathbf{e}} - sin(u_{\mathbf{e}}), & c_{\mathbf{e}} \in \mathbb{R} - \{0\}, \end{cases} \tag{1.24}$$

which was first conceived by Nakajima et al. [115,116] as a prototype for logic circuits.

The tools used in the next sections can be classified into classical and novel for studying the dynamics of nonlinear dispersive equations. In general way, our approach will be of variational and local analysis type around the objects of our interest (solitary waves, periodic traveling waves and stationary solutions). One of the main tools in our spectral study for self-adjoint operators, will be one based on the extension theory for symmetric operators developed by Krein and von Neumann, the Sturm–Liouville Oscillation Theorems, analytic perturbation theory, mini-max principle and continuation arguments.

The paper is organized as follows. In Sect. 2 we give a geometric overview of the Lyapunov's strategy of Weinstein, Grillakis, Shatah and Strauss for orbital stability study of traveling waves. We apply this approach for studying the stability of solitary waves solutions for a Boussinesq-type system for water waves. In Sect. 3 we use the concentraction-compactness principle for showing the existence and orbital stability of solitary wave solutions for the Benjamin equation. Section 4 is dedicated to the study of the existence and orbital (in) stability of periodic traveling waves solutions. A general stability criterium is established and an application is given for the Benjamin–Ono equation. We also give a linear instability criterium for periodic traveling waves of the generalized Benjamin–Bona–Mahoney equation. Section 5 is devoted to the existence and stability of standing wave solutions for nonlinear Schrödinger models on star graphs. Section 6 is dedicated to the KdV models on star-shaped metric graphs. Here we establish a linear instability criterium for stationary waves solutions of KdV-type models. The linear instability of specific tails and bumps profiles for the KdV model on balanced star-shaped metric graphs, is also established. Moreover, each section is finished with notes which show some recent extensions of the topics covered here, together with open problems. In "Appendix" we briefly establish some tools of the extension theory of Krein and von Neumann used in our study.

Notation Let $-\infty \leq a < b \leq \infty$. We denote by $L^2(a, b)$ the Hilbert space equipped with the inner product $(u, v) = \int_a^b u(x)\overline{v(x)}dx$. By $H^n(\Omega)$ we denote the classical Sobolev spaces on $\Omega \subset \mathbb{R}$ with the usual norm. We denote by $\mathcal{G}$ the star-shaped metric graph constituted by $|\mathbf{E}_-| + |\mathbf{E}_+|$ half-lines ($|\mathbf{E}_-|$-half-lines of the form $(-\infty, 0)$ and $|\mathbf{E}_+|$-half-lines of the form $(0, +\infty)$) attached to a common vertex $\nu = 0$. On the graph we define the spaces

$$L^p(\mathcal{G}) = \bigoplus_{\mathbf{e}\in\mathbf{E}_-} L^p(-\infty, 0) \oplus \bigoplus_{\mathbf{e}\in\mathbf{E}_+} L^p(0, +\infty), \quad p > 1,$$

and

$$H^n(\mathcal{G}) = \bigoplus_{\mathbf{e}\in\mathbf{E}_-} H^n(-\infty, 0) \oplus \bigoplus_{\mathbf{e}\in\mathbf{E}_+} H^n(0, +\infty)$$

with the natural norms. For instance, for $u = (u_\mathbf{e})_{\mathbf{e}\in\mathbf{E}} \in L^p(\mathcal{G})$, $\|u\|_{L^p(\mathcal{G})} = \sum_{\mathbf{e}\in\mathbf{E}} ||u_\mathbf{e}\|_{L^p}$, with $\|\cdot\|_{L^p}$ denoting the classical $L^p(\Omega)$-norm. Moreover, for $u =$

$(u_{\mathbf{e}})_{\mathbf{e}\in\mathbf{E}}$, $v = (v_{\mathbf{e}})_{\mathbf{e}\in\mathbf{E}} \in L^2(\mathcal{G})$, we define the inner product

$$\langle u, v\rangle = \sum_{\mathbf{e}\in\mathbf{E}_-} \int_{-\infty}^{0} u_{\mathbf{e}}\overline{v_{\mathbf{e}}}dx + \sum_{\mathbf{e}\in\mathbf{E}_+} \int_{0}^{\infty} u_{\mathbf{e}}\overline{v_{\mathbf{e}}}dx.$$

Depending on the context we will use the following notations for different objects. By $||\cdot||$ we denote the norm in $L^2(\mathbb{R})$ or in $L^2(\mathcal{G})$. By $\|\cdot\|_s$ the norm in $H^s(\mathbb{R})$.

Let A be a densely defined symmetric operator on a Hilbert space H with domain $D(A)$, and let A^* be its adjoint. We denote *deficiency subspaces* of A by $\mathcal{N}_+(A) := Ker(A^*-i)$ and $\mathcal{N}_-(A) := Ker(A^*+i)$. The *deficiency indices* of A are denoted by $n_\pm(A) := \dim(\mathcal{N}_\pm(A))$. The number of negative eigenvalues counting multiplicities (Morse index) is denoted by $n(A)$. The spectrum (resp. point spectrum and essential spectrum) of A is denoted by $\sigma(\mathcal{A})$ (resp. $\sigma_p(\mathcal{A})$ and $\sigma_{ess}(\mathcal{A})$). The resolvent set of A is denoted by $\rho(A)$. By $\dim(Im(A))$ we denote the dimension of the imagem of the operator A given by $Im(A) = \{Ax : x \in D(A)\}$.

2 Orbital stability-local approach

In this section we will give the basic ideas of the theory of stability of traveling waves solutions to equations of evolution in the Hamiltonian form

$$\partial_t u(t) = JE'(u(t)), \tag{2.1}$$

developed in [79,80,129]. Here J is a skew-symmetric operator and E is a real-valued smooth functional defined on a real Hilbert space X embedding in a space of type L^2. We assume that the solutions of (2.1) are invariant under the symmetry of translations, namely, if $\tau(r)$ represents the translation operator defined to $\psi \in X$ by $\tau(r)\psi(x) = \psi(x+r)$, we have that $\tau(r)u(x,t) = u(x+r,t)$ is solution of (2.1) provided $u(x,t)$ is solution. Our interest here will be first to give a geometric overview of the Lyapunov's strategy for orbital stability and second, through of specific examples we see how this theory works for the case of solitary wave solutions (in Sect. 4, the case of periodic traveling waves will be studied). A wide variety of applications of this theory it has been obtained in the last two decades. Moreover, the theory can also be applied to systems which appear in the physical descriptions of phenomena, for instance, in the dynamic of fluid; internal waves; nonlinear interactions in shallow-water and ocean surface waves; optical; hydrodynamical systems; and plasma physics.

Our principal application here will be to establish a theory of existence and stability of solitary wave solutions for the following Boussinesq-type system class of equations (see [13]):

$$\begin{cases} u_t + v_x + uu_x = 0, \\ v_t - u_{xxx} + u_x + (uv)_x = 0, \end{cases} \tag{2.2}$$

where $u = u(x,t)$ and $v = v(x,t)$ are real valued, $x, t \in \mathbb{R}$.

In the end of this section we give some notes and very interesting open problems.

2.1 Geometric overview of the local stability theory

Suppose that (2.1) has a smooth curve $c \to \phi_c$ of solutions of type traveling wave

$$u(x,t) = \phi_c(x - ct), \tag{2.3}$$

for a real profile $\phi_c : \mathbb{R} \to \mathbb{R}$ and $c \in I \subseteq \mathbb{R}$. Moreover, we also suppose that there is another smooth functional $F : X \to \mathbb{R}$ such that E and F are invariants by translations, and are conserved quantities by the flow of (2.1). We also suppose that ϕ_c is a critical point for the functional $H = E + cF$, $H'(\phi_c) = 0$, for every c admissible. So, from the formal relation $H'(\tau(r)\phi_c) = \tau(r)H'(\phi_c)$, we obtain that

$$H'(\tau(r)\phi_c) = 0 \quad \text{for every} \quad r \in \mathbb{R}. \tag{2.4}$$

Next, since $\{\tau(r)\}_{r\in\mathbb{R}}$ represents a one-parameter group of unitary operators on X with infinitesimal generator $\tau'(0) = \frac{d}{dx}$, it follows from (2.4) that for $\mathcal{L} = H''(\phi_c)$ (the second variation of H at ϕ_c),

$$\mathcal{L}\left(\frac{d}{dx}\phi_c\right) = 0. \tag{2.5}$$

Therefore, we have that $\frac{d}{dx}\phi_c$ belongs to the kernel of the linear operator $\mathcal{L}$. That operator will be considered closed, self-adjoint, unbounded and defined on an specific dense subspace of L^2.

Next, the main point of Lyapunov's stability strategy for obtaining the property (ii) in Definition 1.1, it is to get the following property: *there is* $\eta > 0$ *and* $D > 0$ *such that*

$$E(u) - E(\phi_c) \geqq D[d(u;\Omega_{\phi_c})]^2 \tag{2.6}$$

for $d(u;\Omega_{\phi_c}) \equiv \inf_{r\in\mathbb{R}} \|u - \tau_r\phi_c\|_X < \eta$ *and* $F(u) = F(\phi_c)$. In other words, ϕ_c will be a constraint local minimum of E. So, from (2.6) and the continuity of the functionals E, F, and of the flow $t \to u(t)$ (supposing item (i) in Definition 1.1) we obtain immediately the stability of Ω_{ϕ_c} by initial perturbation in the manifold $M = \{F(u) = F(\phi_c)\}$. For general perturbations of Ω_{ϕ_c} we use the continuity of the curve $c \to \phi_c$ and the triangular inequality (see [19–79,95]).

Now, for obtaining (2.6) the analysis starts with the Taylor expansion of $E(u)$ around the profile ϕ_c together with a modulation strategy, namely, there exist a ϵ-neighborhood, U_ϵ, of the orbit Ω_ϵ and a real-valued function α such that for every $u \in U_\epsilon$ we have the orthogonality property $\tau_{\alpha(u)}u \perp \frac{d}{dx}\phi_c$ and $\alpha(\phi_c) = 0$. Thus, we obtain immediately the following equality for $L(u) = E(u) + cF(u)$ on M,

$$E(u) - E(\phi_c) = L(\tau_{\alpha(u)}u) - L(\phi_c) = \frac{1}{2}\langle \mathcal{L}\psi, \psi\rangle + o\left(\|v\|_X^2\right) \tag{2.7}$$

where $\mathcal{L} = L''(\phi_c)$, $v \equiv \tau_{\alpha(u)}u - \phi_c \equiv a\phi_c + \psi$, with $\psi \perp \phi_c$ and $a = O(\|\tau_{\alpha(u)}u - \phi_c\|_X^2)$. Moreover, we also have the main property $\psi \perp \frac{d}{dx}\phi_c$.

The next step in the analysis for obtaining (2.6) is to give conditions for the quadratic form $\langle \mathcal{L} f, f\rangle$ to be positive defined. From the calculus of variations one possible condition is that

$$\langle \mathcal{L} f, f\rangle \geqq \beta \|f\|_X^2 \quad \text{for every } f \in T_{\phi_c} M, \tag{2.8}$$

where $T_{\phi_c} M = \text{span}\{F'(\phi_c)\}$ is the tangent space to $M = \{u : F(u) = F(\phi_c)\}$ in ϕ_c. Since the curve $t \to \tau(t)\phi_c$ belongs to M and $\frac{d}{dt}\tau(t)\phi_c|_{t=0} = \frac{d}{dx}\phi_c \in T_{\phi_c} M$ then we have $Ker(\mathcal{L}) \cap T_{\phi_c} M \neq \{0\}$. Therefore condition (2.8) is not sufficient in our cases. So, the approach developed by Weinstein, Grillakis, Shatah and Strauss in [79,129] show that provided the three next conditions:[2]

- $Ker(\mathcal{L}) = \text{span}\{\frac{d}{dx}\phi_c\}$,
- the Morse index of $\mathcal{L}$ satisfies $n(\mathcal{L}) = 1$,
- for $d(c) = E(\phi_c) + cF(\phi_c)$ we have $d''(c) > 0$,

we have

$$\zeta = \inf\{\langle \mathcal{L}\psi, \psi\rangle : \psi \in [Ker(\mathcal{L})]^{\perp}, \psi \perp F'(\phi_c), \|\psi\|_X = 1\} > 0. \tag{2.9}$$

Thus, from (2.7) we get for $\|v\|_X$ small enough the property in (2.6). In the following we show that property in (2.6) implies the nonlinear stability property of the profile ϕ_c. Assume $d''(c) > 0$ and Ω_{ϕ_c} is X-unstable (see Definition 1.1). Then we can choose initial data $w_k \equiv u_k(0) \in U_{1/k} \cap Y$ and $\epsilon > 0$ such that

$$\inf_{r\in\mathbb{R}} \|w_k - \phi_c(\cdot + r)\|_X \to 0 \quad \text{but} \quad \sup_{t\in\mathbb{R}} \inf_{r\in\mathbb{R}} \|u_k(t) - \phi_c(\cdot + r)\|_X \geqq \epsilon,$$

where $u_k(t)$ is the solution of Eq. (2.1) with initial datum w_k. Now, by continuity in t, we can pick the first time t_k such that

$$\inf_{r\in\mathbb{R}} \|u_k(t_k) - \phi_c(\cdot + r)\|_X = \frac{\epsilon}{2}. \tag{2.10}$$

Now, from the continuity property of the conservation laws for (2.1), E and F, and from the translation invariance property, we get $E(u_k(t_k)) = E(w_k) \to E(\phi_c)$ and $F(u_k(t_k)) \to F(\phi_c)$ as $k \to \infty$. Next we can choose $v_k \in U_\epsilon$ so that $F(v_k) = F(\phi_c)$ and $\|v_k - u_k(t_k)\|_X \to 0$ as $k \to \infty$. By continuity $E(v_k) \to E(\phi_c)$. Choosing η sufficiently small, we may apply the analysis which gives property (2.6) to deduce that

$$0 \leftarrow E(v_k) - E(\phi_c) \geqq D\|v_k(\cdot + \alpha(v_k)) - \phi_c(\cdot)\|_X^2 = D\|v_k - \phi_c(\cdot - \alpha(v_k))\|_X^2.$$

Therefore, the inequality

$$\|u_k(t_k) - \phi_c(\cdot - \alpha(v_k))\|_X \leqq \|u_k(t_k) - v_k\|_X + \|v_k - \phi_c(\cdot - \alpha(v_k))\|_X$$

[2] For several Galilean invariance being considered in the orbital analysis, obviously the kernel of the self-adjoint operator to be studied has a bigger dimension. Moreover, it may happen that the absence of a Galilean invariance does not imply the trivial property of kernel (see [31]). The case of periodic traveling wave is a nice example for showing the situation of an one-dimensional kernel and a Morse index being bigger than one (see Theorem 4.12 below).

implies that $\|u_k(t_k) - \phi_c(\cdot - \alpha(v_k))\|_X \to 0$ as $k \to \infty$, which contradicts (2.10). So, we must have that Ω_{ϕ_c} is X-stable. This finishes the stability statement based in property (2.6).

Remark 2.1 Some comments on the function $d(c) = E(\phi_c) + cF(\phi_c)$ need to be established:

(1) since ϕ_c is a critical point for $L = E(\phi_c) + cF(\phi_c)$ we have that $d'(c) = F(\phi_c)$. So, the sufficient condition $d''(c) > 0$ in the orbital stability criterium above is reduced to

$$d''(c) = \frac{d}{dc}F(\phi_c) = \left\langle F'(\phi_c), \frac{d}{dc}\phi_c \right\rangle > 0. \tag{2.11}$$

(2) By differentiating $E'(\phi_c) + cF'(\phi_c) = 0$ with regard to c, we have $\mathcal{L}\left(\frac{d}{dc}\phi_c\right) = -F'(\phi_c)$. Therefore, from (2.11) we get $d''(c) = -\langle \mathcal{L}\left(\frac{d}{dc}\phi_c\right), \frac{d}{dc}\phi_c\rangle$. Thus, from the min-max principle $\mathcal{L}$ must have at least a Morse index bigger or equal to 1, provided $d''(c) > 0$.

(3) In general, the differential operators $\mathcal{L}$ with a dense domain on L^2-type spaces are *semi-Fredholm* (its range is closed and its kernel is finite-dimensional), thus we have $Ker(\mathcal{L})^{\perp} = R(\mathcal{L})$. Therefore, $L^2 = Ker(\mathcal{L}) \oplus R(\mathcal{L})$ and so since $F'(\phi_c) \perp \frac{d}{dx}\phi_c$ it follows the existence of $\psi \in D(\mathcal{L})$ such that $\mathcal{L}\psi = F'(\phi_c)$. Then, $\mathcal{L}\left(\frac{d}{dc}\phi_c + \psi\right) = 0$. Hence, there is $\theta \in \mathbb{R}$ such that $\frac{d}{dc}\phi_c + \psi = \theta\frac{d}{dx}\phi_c$ and therefore $\langle \psi, F'(\phi_c)\rangle = -\langle \frac{d}{dc}\phi_c, F'(\phi_c)\rangle = -d''(c)$. Then, the conclusion in (2.9) remains valid if the condition $d''(c) > 0$ is replaced by the condition

$$\text{let } \psi \in L^2, \text{ if } \mathcal{L}\psi = F'(\phi_c) \text{ then } \langle \psi, F'(\phi_c)\rangle < 0. \tag{2.12}$$

Condition (2.12) is useful in situations where the family of solitary waves ϕ_c does not depend smoothly on c (see Albert [7] and Natali et al. [117]).

2.2 Boussinesq system for water waves

In this subsection we apply the abstract orbital stability established in the last subsection to the Boussinesq system for water waves (2.2). As we will see this application needs delicate arguments to ensure that stability Definition 1.1 is fully fulfilled. We star writing this system in the following Hamiltonian form

$$\partial_t \mathbf{u} = JE'(\mathbf{u}) \tag{2.13}$$

for $\mathbf{u} = (u, v)$ (here we will identify the vector $\mathbf{u}$ with its transpose vector) and

$$J = -\begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix}, \quad E(u, v) = \frac{1}{2}\int_{\mathbb{R}} \left[(u_x)^2 + u^2 + v^2 + u^2 v\right] dx. \tag{2.14}$$

Next, for $F(u, v) = \int_{\mathbb{R}} uv dx$ we obtain that every solitary wave solution for (2.2)

$$\mathbf{u}(x, t) = (\zeta_c(x - ct), \nu_c(x - ct))$$

with the profile $\boldsymbol{\phi}_c = (\zeta_c(x), \nu_c(x))$, satisfy the relation $(E - cF)'(\boldsymbol{\phi}_c) = 0$. In other words (ζ_c, ν_c) satisfies the following system of ordinary differential equations

$$\begin{cases} -c\zeta_c = -\nu_c - \frac{1}{2}\zeta_c^2 \\ -c\nu_c = \zeta_c'' - \zeta_c - \zeta_c\nu_c. \end{cases} \tag{2.15}$$

Thus, we need to solve the nonlinear elliptic equation

$$\zeta_c'' + \frac{1}{2}\zeta_c^3 - \frac{3}{2}c\zeta_c^2 - (1 - c^2)\zeta_c = 0. \tag{2.16}$$

By using standard methods of integration and ODEs we obtain a unique solution of (2.16) (up to translation) for $|c| < 1$ in the form

$$\zeta_c(x) = \frac{2(1 - c^2)}{\cosh(\sqrt{1 - c^2}\, x) - c}. \tag{2.17}$$

Next, for $L(\mathbf{u}) = E(\mathbf{u}) - cF(\mathbf{u})$ we have the linearized operator $\mathcal{L} = L''(\boldsymbol{\phi}_c)$ defined by

$$\mathcal{L} = \begin{pmatrix} 1 - \partial_x^2 + \nu_c & \zeta_c - c \\ \zeta_c - c & 1 \end{pmatrix}. \tag{2.18}$$

Our stability theorem for solitary wave solutions of Boussinesq system (2.2) is the following.

Theorem 2.2 *The solitary wave $\boldsymbol{\phi}_c = (\zeta_c(x), \nu_c(x))$ with ζ_c defined in (2.17) and $\nu_c = c\zeta_c - \frac{1}{2}\zeta_c^2$, $|c| < 1$, are orbitally stable in $H^1(\mathbb{R}) \times L^2(\mathbb{R})$ for every admissible wave velocity c.*

We start the proof of Theorem 2.2 with a global well-posedness theory for the initial value problem associated to (2.1). This problem is actually an open question in the space $H^1(\mathbb{R}) \times L^2(\mathbb{R})$.[3] Here we will use the auxiliary space $Y = H^2(\mathbb{R}) \times H^1(\mathbb{R})$ where we can assure the global existence of solutions for (2.2). The next two theorems give the necessary information for that analysis.

Theorem 2.3 *Let $\mathbf{f} \in X^s \equiv H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$, $s > 3/2$.*

- *There exist $T = T(s, \|\mathbf{f}\|_{X^s})$ and a unique solution $\mathbf{u}$ of (1.7) such that belongs to the class $C([0, T]; X^s) \cap C^1([0, T]; X^{s-2})$ with $\mathbf{u}(0) = \mathbf{f}$.*
- *T can be chosen "independently of s".*
- *Consider T^* the greatest positive real number such that for all $T < T^*$, we have that the solution $\mathbf{u} \in C([0, T]; X^s)$. Then either, or $T^* = +\infty$ or $T^* < +\infty$ and*

$$\lim_{t \to T^*} \|\mathbf{u}(t)\|_{X^{3/2}} = +\infty.$$

Proof The proof is based on the parabolic regularization method and techniques of Bona–Smith. See [13]. □

[3] The stability statement in Theorem 2.2 is with $Y = H^2(\mathbb{R}) \times H^1(\mathbb{R})$ in Definition 1.1.

Theorem 2.4 *Let* $\mathbf{f} \in X^s \equiv H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$, $s \geqq 2$, *and* T^* *the greatest positive real number such that for all* $T < T^*$ *the solution* $\mathbf{u} = (u, v)$ *of* (2.2) *with* $\mathbf{u}(0) = \mathbf{f}$ *belongs to the class* $C([0, T]; X^s)$. *Suppose that there exists* $M > 0$ *such that* $\|u(t)\|_1 + \|v(t)\| \leqq M$ *for all* $t \in [0, T^*)$, *then* $T^* = +\infty$.

Proof The proof is based in the existence of the following conservation law G for (2.2)

$$G(u, v) = \int_{\mathbb{R}} \Big[4(u_{xx})^2 + 8(u_x)^2 + 4u^2 + 4(v_x)^2 + 4v^2 + 6u^2(u_x)^2 - 16uu_{xx}v \\ -4v(u_x)^2 + 10u^2v + 2v^3 + u^4 + 6u^2v^2 + u^4v\Big]dx, \tag{2.19}$$

and *a priori* estimative of energy. □

Proof of Theorem 2.2 We start by showing the required spectral properties for $\mathcal{L}$ with domain $D(\mathcal{L}) = H^2(\mathbb{R}) \times L^2(\mathbb{R})$: $Ker(\mathcal{L}) = \text{span}\{\frac{d}{dx}\boldsymbol{\phi}_c\}$, $n(\mathcal{L}) = 1$, and $\sigma_{ess}(\mathcal{L}) \subset (\beta, +\infty)$, for some $\beta > 0$. Indeed, by the Galilean invariance of translations for system (2.2) we obtain $\frac{d}{dx}\boldsymbol{\phi}_c \in Ker(\mathcal{L})$. Suppose $\boldsymbol{\psi} = (g, h) \in Ker(\mathcal{L})$, then for $V(x) = 1 - c^2 + 3c\zeta_c(x) - \frac{3}{2}\zeta_c^2(x)$,

$$\begin{cases} -\frac{d^2}{dx^2}g + V(x)g = 0 \\ (\zeta_c - c)g + h = 0. \end{cases} \tag{2.20}$$

From (2.16), $\frac{d}{dx}\zeta$ satisfies the first equation in (2.20). Since $g(x) \to 0$ as $|x| \to \infty$ we obtain that the Wronskian of $\frac{d}{dx}\zeta_c$ and g, is zero. Therefore, $g = \theta\frac{d}{dx}\zeta_c$ and $h = \theta\frac{d}{dx}v_c$, for $\theta \in \mathbb{R}$. Thus, zero is a simple eigenvalue of $\mathcal{L}$.

Next, we determine the Morse index of $\mathcal{L}$. Let $\mathcal{Q}$ be the quadratic form associated to $\mathcal{L}$, $\mathcal{Q}(\boldsymbol{\psi}) = \frac{1}{2}\langle\mathcal{L}\boldsymbol{\psi}, \boldsymbol{\psi}\rangle$, $\boldsymbol{\psi} = (g, h) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$. Then,

$$\mathcal{Q}(g, h) = \frac{1}{2}\int_{\mathbb{R}} (g')^2 + V(x)g^2 dx + \frac{1}{2}\int_{\mathbb{R}} [g(\zeta_c - c) + h]^2 dx \equiv \mathcal{Q}^1(g) + \frac{1}{2}\|g(\zeta_c - c) + h\|^2. \tag{2.21}$$

Now, the classical Schrödinger operator $\mathcal{H} = -\frac{d^2}{dx^2} + V(x)$ with $D(\mathcal{H}) = H^2(\mathbb{R})$ has a simple kernel with eigenfunction $\frac{d}{dx}\zeta_c$ which has exactly one node em zero, $\frac{d}{dx}\zeta_c(0) = 0$, so from the Sturm–Liouville Oscilation Theorem the Morse index of $\mathcal{H}$ is exactly one. From (2.16) and the Pohozaev identity $\frac{3}{8}\int_{\mathbb{R}} \zeta^4(x)dx - \frac{5}{4}c\int_{\mathbb{R}} \zeta^3(x)dx = (1 - c^2)\int_{\mathbb{R}} \zeta^2(x)dx$, we obtain

$$\langle\mathcal{L}\boldsymbol{\phi}_c, \boldsymbol{\phi}_c\rangle = \frac{3}{2}\int_{\mathbb{R}} \zeta_c^2(x)v_c(x)dx = -\frac{3}{10}\left[\int_{\mathbb{R}} \zeta_c^4(x) + 4(1 - c^2)\zeta_c^2(x)dx\right] < 0.$$

Hence, from mini-max principle it follows $n(\mathcal{L}) \geqq 1$. Next we show that $n(\mathcal{L}) \leqq 1$. Indeed, let $\chi \in D(\mathcal{H})$ and $\beta_0 < 0$ such that $\mathcal{H}\chi = \beta_0\chi$, then from the mini-max principle follows that for $g \perp \chi$ we have $\mathcal{Q}^1(g) \geqq 0$. Thus, using one more time the former principle, we obtain that the second eigenvalue for $\mathcal{L}$ is precisely zero since

$$\lambda_2 \geqq \inf_{\substack{(g,h)\in H^1\times L^2 \\ \langle g,\chi\rangle=0}} \frac{\mathcal{Q}^1(g)+\frac{1}{2}\|g(\zeta_c-c)+h\|^2}{\|g\|_1^2+\|h\|^2} \geqq 0.$$

Then, since zero is a simple eigenvalue follows that $\lambda_2 = 0$ and therefore $n(\mathcal{L}) \leqq 1$. Finally, to show that the rest of the spectrum is positive and bounded from below, it is sufficient to find a strictly lower bound for the third eigenvalue λ_3 for $\mathcal{L}$. Indeed, using the previous argument, in the case of the subspace generated by $\mathbf{f}_1 = (\chi, 0)$ and $\mathbf{f}_2 = (\frac{d}{dx}\zeta_c, 0)$, and the inequality $\mathcal{Q}^1(g) \geqq \alpha\|g\|^2$, for $g \perp \chi$, $g \perp \frac{d}{dx}\zeta_c$, we obtain the existence of a number $\delta > 0$ such that $\lambda_3 \geqq \delta$. Hence the required spectral properties for $\mathcal{L}$ are established.

In the following we check that for $d(c) = E(\boldsymbol{\phi}_c) - cF(\boldsymbol{\phi}_c)$ we have $d''(c) > 0$, which is equivalent to see $\frac{d}{dc}F(\boldsymbol{\phi}_c) < 0$. Using (2.17), $\nu_c = c\zeta_c - \frac{1}{2}\zeta_c^2$ and a change of variable we obtain that

$$\begin{cases} -F(\boldsymbol{\phi}_c) = 8(1-c^2)^{5/2}I_2 - 8c(1-c^2)^{3/2}I_1 \\ I_j = \int_{1-c}^{\infty} \frac{1}{x^{j+1}\sqrt{x^2+2cx+c^2-1}}dx, \quad j=1,2. \end{cases} \tag{2.22}$$

Using [1], we obtain the equality

$$-F(\boldsymbol{\phi}_c) = 4c\sqrt{1-c^2} + 4\left[sin^{-1}(c) + \frac{\pi}{2}\right],$$

and thus $-\frac{d}{dc}F(\boldsymbol{\phi}_c) = 8\sqrt{1-c^2} > 0$.

Now, let T^* be the maximal time for the solution (u, v) of (2.2) given by Theorem 2.3 for $s = 2$. From Theorem 2.4 remains to show that the solution (u, v) is uniformely bounded in the $H^1(\mathbb{R}) \times L^2(\mathbb{R})$-norm. Indeed, from the abstract stability analysis in Sect. 2.1, the local well-possednes em $H^2(\mathbb{R}) \times H^1(\mathbb{R})$ and from the former spectral analysis we have that for ϵ sufficiently small and some $r \in \mathbb{R}$, $\|\mathbf{u}(t) - \tau(r)\boldsymbol{\phi}_c\|_{H^1\times L^2} < \epsilon$, $t \in [0, T] \subset [0, T^*)$. Thus, since $\{\tau(r)\}_{r\in\mathbb{R}}$ is an unitary group in $H^1(\mathbb{R}) \times L^2(\mathbb{R})$, we obtain

$$\|\mathbf{u}(t)\|_{H^1\times L^2} < \epsilon + \|\boldsymbol{\phi}_c\|_{H^1\times L^2} \equiv M, \quad t \in [0, T] \subset [0, T^*),$$

where M *is independent of every* $T \in [0, T^*)$, and therefore $T^* = +\infty$ and the stability Definition 1.1 is completely filled. □

2.3 End-section notes

(a) For the gKdV model (1.9) with $\beta \geqq 1$, the stability condition $d''(c) > 0$ can be obtained explicitly. Indeed, by supposing a smooth curve $c \to \phi_c$ of solitary profiles satisfying (1.14), we have that $\varphi(x) = c^{-1/p}\phi_c(c^{-1/\beta}x)$ is solution of

(1.14) with $c = 1$. Thus

$$d''(c) = \frac{d}{dc}\|\phi_c\|^2 = \|\varphi\|^2 \frac{d}{dc} c^{\frac{2}{p}-\frac{1}{\beta}} = \left(\frac{2}{p} - \frac{1}{\beta}\right) c^{\frac{2}{p}-\frac{1}{\beta}-1}\|\varphi\|^2.$$

Therefore, by providing the right stability conditions established in Sect. 2.1 above, namely, the local well-posedness associated to (1.7) and the spectral conditions for $\mathcal{L}_c$ in (1.19), we have the orbital stability of the profile ϕ_c for $p < 2\beta$.

(b) For the fractional KdV model (1.9) with $\beta \in (0, 1)$ and $p = 1$, we obtain that for $c > 0$ and $\beta \leqq \frac{1}{3}$ equation in (1.14) does not admit any non-trivial solution. For $\beta \in (\frac{1}{3}, 1)$ the existence of an even and positive solution φ (ground-state) for $D^\alpha\varphi + \varphi - \varphi^2 = 0$ can be showed by solving the minimization problem (see Frank and Lenzmann [68])

$$J^\beta(\varphi) = inf\{J^\beta(v) : v \in H^{\beta/2}(\mathbb{R}) - \{0\}\} \tag{2.23}$$

where J^β is the "Weinstein" functional

$$J^\beta(v) = \frac{\left(\int_{\mathbb{R}} |D^{\frac{\beta}{2}}v|^2\right)^{\frac{1}{2\beta}} \left(\int_{\mathbb{R}} |v|^2\right)^{\frac{1}{2\beta}(\beta-1)+1}}{\int_{\mathbb{R}} |v|^3}. \tag{2.24}$$

For $\beta \in (\frac{1}{2}, 1)$, the orbital stability (conditional) of the ground-state φ has been showed in Linares et al. [99] (see also Angulo [21] and Sect. 3.2 below). For $\beta \in (\frac{1}{3}, \frac{1}{2})$, the spectral instability property of the ground-state φ has been showed by Angulo [21]. In this case, the nonlinear instability property is an open question. The case $\beta = \frac{1}{2}$ was also studied in [21] and the "stability of the blow-up" around the profile φ was obtained.
In Klein and Saut [89] were numerically constructed solitary waves for the fractional KdV equation. Moreover, they also provided a detailed numerical study of various issues associated to the dynamics of the model: blow-up in finite time versus global existence, nature of the blow-up, existence for "long" times, and the descomposition of the initial data into solitary waves plus radiation.

(c) About the condition $d''(c) < 0$, it is well known that it can imply the nonlinear (linear) instability of the orbit generated by the profile ϕ_c. From Grillakis *et al.* in [79] will be sufficient to have that J is an onto skew-symmetric operator (obviously with the right spectral conditions established in Sect. 2.1). For the case $J = \partial_x$ in (1.7) an instability approach was studied in Bona et al. [50]. In [80] a more comprenhensive nonlinear (in)stability theory was established by supposing J being one-one, onto and skew-symmetric operator.

3 Orbital stability—a global variational approach

In the latter section, we saw that the stability of solitary wave solutions to nonlinear dispersive equations has been proved by verifying an Lyapunov inequality-type of the form (2.6). This means that one must show that the solitary wave solution is a local constrained minimizer of a Hamiltonian functional associated with (2.1), and this is done basically by obtaining basic spectral information for a specific operator obtained by linearizing the solitary wave equation. In general this spectral analysis is particularly hard to perform. To avoid these difficulties an alternative method to prove the stability property of specific solitary waves solutions, which does not rely on local analysis, was developed by Cazenave and Lions [64] using Lions's method of *concentration–compactness* (Lions [100,101], Lopes [104]). In that method, instead of starting with a specific traveling wave and trying to prove that it realizes a local minimum of a constrained variational problem, one starts with the constrained variational problem and looks for global minimizers.

When the method works, it shows not only the existence of global minimizers, but also that every minimizing sequence associated with the constrained variational problem is relatively compact up to the Gallilean invariance symmetries associated. Moreover, if the functionals involved in the variational problem are conserved quantities for the evolution model (2.1), we obtain that the set of global minimizers is a stable set for the flow associated to (2.1) in the sense that a solution which starts near the set of minimizers will remain near it all the time.

According to the local methods given by Weinstein, Grillakis, Shatah and Strauss in Sect. 2, the variational approach produces (sometimes) a weaker result since it only proves the stability of a set of minimizing solutions without providing information on the structure of that set, or distinguishing among its different members called orbits. For example, it is an open problem to know if the set of solitary wave solutions obtained via the concentration–Compactness principle for the KP-I equation (see de Bouard and Saut [65])

$$(u_t + 2uu_x + u_{xxx})_x = u_{yy} \tag{3.1}$$

with $u = u(x, y, t) \in \mathbb{R}$ and $x, y, t \in \mathbb{R}$, it contains the explicit lump solitary wave profile

$$\psi_c(x, y) = 8\frac{c - \frac{x^2}{3} + \frac{y^2}{3c}}{\left(c + \frac{x^2}{3} + \frac{y^2}{3c}\right)^2}.$$

Moreover, the stability properties of the lump ψ_c is also an open problem.

We refer the reader to the work of Albert [8], where is illustrated how the concentration–compactness principle works for obtaining the stability results of solitary wave solutions of nonlinear evolution equations of the general form (1.7). We note that *this approach for proving directly stability works whenever the functionals involved in the variational analysis are conserved quantities for the evolution equation in question*. For the case of a stability-framework by considering non-conserved quantities (see Levandosky [96] and Angulo [19]). For the benefit of the reader, we establish the concentration–compactness principle which is the key tool in our analysis (see Lemma 1.1 in Lions [100]).

Lemma 3.1 (The concentration–compactness principle)
Let $\{\rho_n\}_{n\geq 1}$ be a sequence of non-negative functions in $L^1(\mathbb{R})$ satisfying $\int_{-\infty}^{\infty} \rho_n(x)dx = \lambda$ for all n and some $\lambda > 0$. Then there exists a subsequence $\{\rho_{n_k}\}_{k\geq 1}$ satisfying one of the following three conditions:

(1) *(**Compactness**) there are $y_k \in \mathbb{R}$ for $k = 1, 2, \ldots,$ such that $\rho_{n_k}(\cdot + y_k)$ is tight, i.e. for any $\epsilon > 0$, there is $R > 0$ large enough such that*

$$\int_{|x-y_k|\leq R} \rho_{n_k}(x)\,dx \geq \lambda - \epsilon;$$

(2) *(**Vanishing**) for any $R > 0$,*

$$\lim_{k\to\infty} \sup_{y\in\mathbb{R}} \int_{|x-y|\leq R} \rho_{n_k}(x)dx = 0;$$

(3) *(**Dichotomy**) there exists $\tilde{\alpha} \in (0, \lambda)$ such that for any $\epsilon > 0$, there exists $k_0 \geq 1$ and $\rho_k^1, \rho_k^2 \in L^1(\mathbb{R})$, with $\rho_k^1, \rho_k^2 \geq 0$, such that for $k \geq k_0$,*

$$\left|\rho_{n_k} - \left(\rho_k^1 + \rho_k^2\right)\right|_{L^1} \leq \epsilon,$$
$$\left|\int_{-\infty}^{\infty} \rho_k^1\,dx - \tilde{\alpha}\right| \leq \epsilon, \quad \left|\int_{-\infty}^{\infty} \rho_k^2\,dx - (\lambda - \tilde{\alpha})\right| \leq \epsilon,$$
$$supp\ \rho_k^1 \cap supp\ \rho_k^2 = \emptyset,\ dist\left(supp\ \rho_k^1, supp\ \rho_k^2\right) \to \infty \text{ as } k \to \infty.$$

Remark 3.2 In Lemma 3.1 above, the condition $\int_{-\infty}^{\infty} \rho_n(x)\,dx = \lambda$ can be replaced by $\int_{-\infty}^{\infty} \rho_n(x)\,dx = \lambda_n$ where $\lambda_n \to \lambda > 0$ as $n \to \infty$. It is enough to replace ρ_n by ρ_n/λ_n and apply the lemma.

We finish this section with some notes and very interesting open problems.

3.1 The Benjamin equation: orbital stability of solitary waves

We consider the solitary wave solutions $u(x, t) = \phi_c(x - ct)$ for the Benjamin equation (1.13). In this case the profile $\phi \equiv \phi_c$ satisfies the pseudo-differential equation

$$\phi''(\xi) + l\mathcal{H}\phi'(\xi) + \phi^2(\xi) = c\phi(\xi). \tag{3.2}$$

The existence of explicit solutions to Eq. (3.2) with $l \neq 0$ is not known yet. The problem of existence of solitary wave solutions was pioneered by Benjamin [46,47], where using a Leray–Schauder degree theory he showed the existence of even solutions for (3.2) if $\gamma = \frac{1}{2\sqrt{c}} l \in (0, 1)$, namely, if we have that $c > \frac{1}{4} l^2$. Later, Albert et al. [10], by using the Implicit Function Theorem and the existence of solutions to (3.2) when $l = 0$ (i.e. the KdV solitary wave solutions), proved the existence of a continuous curve of even solitary wave solutions to (3.2) if l is sufficiently near zero. Afterwards, Angulo [14] by using the *concentration–compactness principle* established a theory of existence of solitary wave solutions with γ close to 1.

The idea of this subsection is to give a sketch of the proof of the existence and stability of solitary waves for (3.2) given by Angulo [14]. In other words we consider the following variational problem

$$\begin{cases} \text{minimize} & V(\psi) = \frac{1}{2}\langle M\psi, \psi\rangle - \frac{1}{3}\int_{\mathbb{R}} \psi^3(x)dx \\ \text{subject to} & F(\psi) = \frac{1}{2}\int_{\mathbb{R}} \psi^2(x)dx = \lambda > 0, \end{cases} \tag{3.3}$$

where the linear operator M is defined by $M = -\partial_x^2 - l\mathcal{H}\partial_x$. The problem in (3.3) is an example out of the curve for variational problems associated to the existence of solitary waves for dispersive models. The point here is that *M has not always a nonnegative associated symbol*. This situation produces serious problems in ruling-out vanishing option in Lemma 3.1.

By starting our study, for $\lambda > 0$ we consider the following family of minimization problems

$$I_\lambda = \text{Inf}\{V(\psi) : \psi \in H^1(\mathbb{R}) \text{ and } F(\psi) = \lambda > 0\}. \tag{3.4}$$

Lemma 3.3 *For all $\lambda > 0$, we have $-\infty < I_\lambda < 0$.*

Proof Let $\psi \in H^1(\mathbb{R})$ be such that $F(\psi) = \lambda > 0$. Then $\psi_\theta(x) = \theta^{1/2}\psi_\theta(\theta x)$ for every $\theta > 0$, satisfies $F(\psi_\theta) = \lambda$ and

$$V(\psi_\theta) = \frac{\theta^2}{2}\|\psi'\|^2 - \frac{l\theta}{2}\|D^{1/2}\psi\|^2 - \frac{\theta^{1/2}}{3}|\psi|_3^3.$$

Hence, for θ small enough, we get $V(\psi_\theta) < 0$ and therefore $I_\lambda < 0$. To prove $I_\lambda > -\infty$, it is sufficient to obtain a bound of the form $\|\psi'\|^2 \leqq B(\lambda, l)$ with $F(\psi) = \lambda$ and $V(\psi) \leqq 0$. Thus, from the Gagliardo–Nirenberg-type inequality and interpolation, we obtain for $A > 0$ that $|\psi|_3^3 \leqq A|\psi'\|^{1/2}\|\psi\|^{5/2}$ and $\|D^{1/2}\psi\|^2 \leqq \|\psi'\|\|\psi\|$. Therefore,

$$\frac{1}{2}\|\psi'\|^2 \leqq \frac{l}{2}\|D^{1/2}\psi\|^2 + \frac{1}{3}|\psi|_3^3 \leqq \frac{\sqrt{2}}{2}l\lambda^{1/2}\|\psi'\| + A\lambda^{5/4}\|\psi'\|^{1/2}.$$

Since the square of $\|\psi'\|$ is bounded by smaller powers, the existence of the constant $B(\lambda, l)$ above is obtained. □

From Lemma 3.3 and its proof, it is clear that every minimizing sequence for (3.4) is bounded in $H^1(\mathbb{R})$. Next, from Plancherel's Theorem we obtain that the quadratic part for V, satisfies the following relation

$$Q(\psi) = \frac{1}{2}\int_{\mathbb{R}} (\psi')^2 - l\psi\mathcal{H}\psi' dx \geqq -\frac{l^2}{4}\lambda, \quad F(\psi) = \lambda. \tag{3.5}$$

Now, we find an upper bound for V which is the main ingredient for *ruling-out the cases of vanishing and dichotomy*.

Theorem 3.4 *There exist constants $A > 0$ and $\sigma > 0$ and an admissible function Φ such that*

$$V(\Phi) \leqq -\frac{l^2}{4}\lambda - A\lambda^{\sigma} \quad \textit{if } \lambda \textit{ is small enough.}$$

Proof (Sketch of the proof) For $\beta = \sqrt{l/2}$ we have

$$V(\Phi) + \frac{l^2}{4}\lambda \leqq \int_{\mathbb{R}} \frac{1}{2}\left[\left(\mathcal{H}\partial_x + \beta^2\right)\Phi\right]^2 - \frac{1}{3}\Phi^3 dx \equiv G(\Phi). \tag{3.6}$$

Now, the goal is to show that $G(\Phi)$ can be negative. So, we consider the following test functions $\Phi = c(\Phi_\epsilon + \delta\Psi)$, where for ϵ

$$\begin{cases} \Phi_\epsilon(x) = f(\epsilon x)cos(\beta^2 x), \quad \Psi(x) = f(\epsilon x), \quad \text{and} \\ f(y) = \frac{1}{1+y^2}. \end{cases} \tag{3.7}$$

Next, we adjust the constant c such that $F(\Psi) = \lambda$ and we also use that the parameters ϵ, δ have the form $\epsilon = \lambda^{\beta}$ and $\delta = \pm\epsilon^{\theta}$ (see [14]). So, for $\lambda \to 0$ we can obtain the estimative $G(c\Phi_\epsilon + c\delta\Psi) \leqq -A\lambda^{\sigma}$, with $A > 0$ and $\sigma = \frac{3}{2} + \beta(\theta + \frac{1}{2})$. This proves the theorem. □

As a consequence of Theorem 3.4, for λ small we may restrict our minimization problem (3.4) to the class of admissible functions ψ, such that

$$V(\psi) \leqq -\frac{l^2}{4}\lambda - A\lambda^{\sigma}. \tag{3.8}$$

In this class, we have that the nonquadratic part of V satisfies $N(\psi) \equiv V(\psi) - Q(\psi) \leqq -A\lambda^{\sigma}$.

For ruling-out dichotomy in the concentration–compactness principle we prove the sub-addivity of I_λ.

Theorem 3.5 *For all $\theta > 1$ and λ positive and small, we have*

(1) $I_{\lambda\theta} \leqq \theta I_\lambda - A\lambda^{\sigma}(\theta^{3/2} - \theta)$,
(2) $I_{\lambda\theta} < \theta I_\lambda$ *and* $I_\lambda < I_\zeta + I_{\lambda-\zeta}$ *for all* $\zeta \in (0, \lambda)$.

Proof We only prove (1) since (2) follows a standard argument. Choose $\psi \in H^1(\mathbb{R})$ with $F(\psi) = \lambda$. Then for $\psi_\theta(x) = \theta^{1/2}\psi(x)$ we have $F(\psi_\theta) = \lambda\theta$. Suppose that ψ is part of a minimizing sequence for (3.4) and $N(\psi) \leqq -A\lambda^{\sigma}$ for λ small. Then

$$V(\psi_\theta) - \theta V(\psi) \leqq (\theta^{3/2} - \theta)N(\psi) \leqq -A(\theta^{3/2} - \theta)\lambda^{\sigma}.$$

Therefore, $I_{\lambda\theta} \leqq \theta V(\psi) - A(\theta^{3/2} - \theta)\lambda^{\sigma}$ and so the proof of item (1) is completed. □

Next, from Lemma 3.3 and Theorems 3.4–3.5 comes into the game the Concentration Compactness Principle. Indeed, let $\{\psi_n\}$ be a minimizing sequence for problem (3.3). Then we have $F(\psi_n) = \lambda$ for all n, and $V(\psi_n) \to I_\lambda$ as $n \to \infty$. As we well known, because of the inclusion of $H^1(\mathbb{R})$ into $L^2(\mathbb{R})$ is not compact, one can not extract a subsequence of $\{\psi_n\}$ which converges in $L^2(\mathbb{R})$. This difficulty will be circumvented in fact by the Lions's Lemma 3.1. Overall, this principle provides that a subsequence of $\{\psi_n\}$ can be found such that, after being suitably translated (*Galilean invariance for the model in question*), each function in the subsequence is concentrated on a fixed bounded interval. Thus we can apply the compactness of the inclusion of $H^1(\Omega)$ into $L^2(\Omega)$ when Ω is a bounded set.

Now, for applying Lemma 3.1 we associate with each minimizing sequence $\{\psi_n\}$ for problem (3.4), the following sequence of nondecreasing functions: $Q_n : [0, \infty) \to [0, \lambda]$ defined by

$$Q_n(\omega) = \sup_{y\in\mathbb{R}} \int_{y-\omega}^{y+\omega} |\psi_n(x)|^2\, dx.$$

Since $\|\psi_n\|$ remains bounded then $\{Q_n\}$ comprises a uniformly bounded sequence of nondecreasing functions on $[0, \infty)$. A standard argument then implies that $\{Q_n\}$ must have a subsequence, which we denote again by $\{Q_n\}$, that converges pointwise and uniformly on compact sets to a nondecreasing limit function on $[0, \infty)$. Let Q be this limit function, and define

$$\alpha = \lim_{\omega\to\infty} Q(\omega). \tag{3.9}$$

From the assumption that $\|\psi_n\|^2 = \lambda$ it follows that $0 \leqq \alpha \leqq \lambda$. *The concentration–compactness principle distinguishes three cases:* $\alpha = \lambda$, *it called the case of compactness;* $\alpha = 0$, *it called the case of vanishing; and* $0 < \alpha < \lambda$, *it called the case of dichotomy*.

As we know, the strategy of the principle consists in ruling out vanishing and dichotomy, and so only the case of compactness can occur. It will follow, by a standard argument, that every minimizing sequence is relatively compact, after suitable translations. Thus, we start by showing that α is positive. Indeed, let λ be positive and small and suppose that $\alpha = 0$. Then $Q_n(\omega) \to 0$ for every ω. The next step is to see that $\psi_n \to 0$ in the L^p-norm, $p > 2$. In fact, the Sobolev embedding of $W^{1,1}(a, b)$ into $L^1(a, b)$, the Cauchy–Schwartz inequality, and the estimative $\|\psi_n\|_1 \leqq A_1$, assure that for fixed ω, there is a constant $A_2 = A(\omega)$ such that

$$\int_{y-\omega}^{y+\omega} |\psi_n(x)|^p\, dx \leqq A_2 \delta_n^{\frac{p}{2}-1} \int_{y-\omega}^{y+\omega} 2|\psi_n|^2 + |\psi_n'(x)|\, dx$$

with $\delta_n \to 0$ as $n \to +\infty$. Then, *covering* $\mathbb{R}$ *by intervals of radio* ω *such that every point of* $\mathbb{R}$ *is contained in at most two intervals*, we deduce

$$\int_{-\infty}^{\infty} |\psi_n(x)|^p\, dx \leqq A_3 \delta_n^{\frac{p}{2}-1} \|\psi_n'\|^2 \leqq A_4 \delta_n^{\frac{p}{2}-1},$$

and so we conclude the claim. Therefore,

$$I_\lambda = \lim_{n\to+\infty} V(\psi_n) \geqq \liminf_{n\to+\infty} Q(\psi_n) \geqq \inf Q(\psi) = -\frac{l^2}{4}\lambda.$$

On the other hand, by Theorem 3.4 we know $I_\lambda < -\frac{l^2}{4}\lambda$. This contradiction shows that vanishing cannot occur. Now from Theorem 3.5 and standard arguments we obtain that $\alpha \in (0, \lambda)$ (see Angulo [14]). So, $\alpha = \lambda$ and get the *compactness* alternative of Lions. Then the set of minimizers for I_λ

$$G_\lambda = \{\psi \in H^1(\mathbb{R}) : V(\psi) = I_\lambda \text{ and } F(\psi) = \lambda\}, \tag{3.10}$$

is nonempty for λ small. Moreover, we have the following stability theorem.

Theorem 3.6 *Let λ positive and small. Then we have the following.*

(1) *The set G_λ consists of solitary wave solutions ψ for the Benjamin equation satisfying*
$$\psi'' + l\mathcal{H}\psi' + \psi^2 = c\psi, \quad c > 0. \tag{3.11}$$
(2) *G_λ is stable in $H^1(\mathbb{R})$ by the flow of the Benjamin equation, in the following sense: For every $\epsilon > 0$ there exists $\delta > 0$ such that, if $\inf_{\psi\in G_\lambda} \|u_0 - \psi\|_1 < \delta$, then the solution $u(t)$ of (1.13) with $u(0) = u_0$ exists globally and satisfies $\inf_{\psi\in G_\lambda} \|u(t) - \psi\|_1 < \epsilon$, for all t.*
(3) *The wave speed c in (3.11) satisfies the inequality $\frac{l^2}{4} < c \leqq \frac{l^2}{4} + Al^{1/2}\lambda^{1/2}$, where A is a positive constant that does not depend on l and λ.*

Proof See Theorems 2.12 and 2.14 in [14]. □

Remark 3.7 (1) Equation (3.11) is called the Euler–Lagrange equation associated to the minimization problem (3.3) and c is the Lagrange-multiplier.
(2) The proof of item (2) in Theorem 3.6 uses strongly the invariance of the functionals V and F in (3.3) by translations and the conservation property of these functionals by the flow of the Benjamin equation.
(3) The global well-posedness theory for the Benjamin equation was established in [98].

3.2 End-section notes

(a) We note that the concentration–compactness principle can still be used to prove the stability or instability of solitary wave solutions if *the functionals involved in the variational problem are not conserved quantities*. The stability approach has been put forward by Levandosky, in [96], in which the stability of a fourth-order wave equation is studied. In Angulo's book [19] were established other applications of this stability approach, in particular, it was applied to a generalization of the BO equation in (1.11) and so it was given other demonstration of the orbital stability property of the solitary waves in (1.12). We note that Liu and Wang [102], applied this method with success to study the nonlinear stability of solitary wave solutions of a generalization of the KP-I equation (3.1). In the case of orbital instability for solitary wave solutions, a variational approach was established by Angulo [16] (see also [19]).

(b) Sometimes the property of sub-additivity can be very difficult to be showed for specific functionals, and so *ruling out the dichotomy case in the concentration–compactness principle can be a big problem*. In this case, a new approach has been developed by Lopes [103], which gives us sufficient conditions to obtain that every minimizing sequence associated with some variational problems is pre-compact (modulo translations).

(c) In the following we will use the variational characterization (2.23) of the ground-state φ for the fKdV model (1.9) (namely, φ satisfying the properties $\varphi = \varphi(|x|) > 0$ and the equation $D^\beta\varphi+\varphi-\varphi^2 = 0, \beta \in (\frac{1}{2}, 1)$), for obtaining the basic stability inequality (2.6) for ϕ_c satisfying the equation

$$D^\beta\phi_c + c\phi_c - \phi_c^2 = 0 \quad \text{for any} \quad c > 0.$$

Indeed, let φ be a minimum for the functional J^β in (2.23). Thus, the self-adjoint operator $\mathcal{L}_1 = D^\beta + 1 - 2\varphi$ satisfies the so-called non-degeneracy property, namely, $Ker(\mathcal{L}_1) = \text{span}\{\frac{d}{dx}\varphi\}$. Moreover, since $\langle \mathcal{L}_1\varphi, \varphi\rangle \leqq 0$ and $\langle \mathcal{L}_1\eta, \eta\rangle \geqq 0$ for $\eta \in C_0^\infty(\mathbb{R})$ with $\eta \perp \varphi^2$, we have $n(\mathcal{L}_1) = 1$ via the min-max principle. Now, for $R \equiv \beta\varphi + x\varphi' \in L^2(\mathbb{R})$ follows $\mathcal{L}_1 R = -\beta\varphi$ (at least in the distributional sense). Thus a bootstrapping argument shows that $R \in H^{\beta+1}(\mathbb{R})$ and so $R \in D(\mathcal{L}_1) = H^\beta(\mathbb{R})$.

Next, for any real number $\theta \neq 0$, define the dilation operator T_θ by $(T_\theta f)(x) = f(\theta x)$. Then, via the elementary scaling $\phi_c(x) = 2c\varphi(c^{1/\beta}x)$ and the relation $D^\beta(T_\theta f)(x) = \theta^\beta D^\beta f(\theta x)$, we can show that for $\theta = c^{1/\beta}$ we obtain that ϕ_c satisfies $D^\beta\phi_c + c\phi_c - \frac{1}{2}\phi_c^2 = 0$, and so, we obtain its linearized operator $\mathcal{L}_c = D^\beta + c - \phi_c$ to be studied. Now, the relation $\mathcal{L}_c = cT_\theta\mathcal{L}_1 T_\theta^{-1}$ with $\theta = c_\theta^{1/\beta}$ implies that $spec(\mathcal{L}_c) = \{cr : r \in spec(\mathcal{L}_1)\}$ and therefore $\mathcal{L}_c$ and $\mathcal{L}_1$ have the "*same structure*". Thus, ψ is an eigenfunction of $\mathcal{L}_1$ with eigenvalue λ if and only if $T_\theta\psi$ is an eigenfunction of $\mathcal{L}_c$ with eigenvalue $c\lambda$. Then, we conclude immediately that $n(\mathcal{L}_c) = 1$ and $Ker(\mathcal{L}_c) = \text{span}\{\frac{d}{dx}\phi_c\}$. Thus, since $R_c = \beta\phi_c + x\phi_c' \in D(\mathcal{L}_c)$ with $\mathcal{L}_c R_c = -\beta c\phi_c$ (where we use $D^\beta(x\varphi') = \beta D^\beta\varphi + xD^\beta\varphi'$) we obtain

$$\left\langle \mathcal{L}_c^{-1}\phi_c, \phi_c\right\rangle = -\frac{1}{\beta c}\langle R_c, \phi_c\rangle = \|\phi_c\|^2\left[\frac{1}{2\beta c} - \frac{1}{c}\right] < 0, \tag{3.12}$$

where we use integration by parts (and $x\phi_c^2(x) \to 0$ as $|x| \to +\infty$) and $\beta > \frac{1}{2}$. Hence, from regularity properties of the curve $c \to \phi_c$, condition (2.12), and from the Lyapunov property (2.6) satisfied by the energy $E(u) = \frac{1}{2}\int_{\mathbb{R}} |D^{\beta/2}u|^2 - \frac{1}{3}u^3 dx$, we obtain the orbital stability (conditional) of ϕ_c.

(d) Depending on the model under study, the concentration–compactness principle is a powerful tool to show the existence of solitary wave profiles. But this approach may be incomplete to obtain some type of information about the dynamics of the set of minimizers. One example of this situation is presented in the following Boussinesq-Full dispersion systems models for internal waves in a two-layers system

$$\begin{cases} J_b \partial_t \zeta + \mathcal{L}_{\mu_2} \partial_x v - 2\beta \partial_x(\zeta v) = 0 \\ J_d \partial_t v + (1-\gamma) J_c \partial_x \zeta - \beta \partial_x(v^2) = 0, \quad \beta > 0,\ \gamma \in (0,1), \end{cases} \tag{3.13}$$

where $\mathcal{L}_{\mu_2}$ is the self-adjoint operator defined for $\mu_2 \in (0, +\infty]$ by

$$\mathcal{L}_{\mu_2} = \frac{1}{\gamma} - \frac{\sqrt{\mu}}{\gamma^2} |D| \coth(\sqrt{\mu_2}|D|) + \frac{\mu}{\gamma}\left(a - \frac{1}{\gamma^2}\coth^2(\sqrt{\mu_2}|D|)\right)\partial_x^2,\ \mu > 0, \tag{3.14}$$

$|D| = \mathcal{H}\partial_x$, and the Bessel-type operators J_b, J_d, J_c are defined by $J_b = 1 - \mu b \partial_x^2$, $J_d = 1 - \mu d \partial_x^2$, $J_c = 1 + \mu c \partial_x^2$, with $a + b + c + d = \frac{1}{3}$, $b, d \geqq 0$, $a, c < 0$. Indeed, in Angulo and Saut [42] was showed only the existence of solitary wave profiles $\zeta(x,t) = \xi(x - \omega t)$, $v(x,t) = \nu(x - \omega t)$ for (3.13) via the concentration–compactness principle for a ω-velocity satisfying $|\omega| < (1-\gamma)\frac{|c|}{b}$. The stability of the associated set of minimizers is actually an open question.

(e) The geometry of the set of minimizers determined by a specific variational problem with constraint is not a very well understood part of the theory and more studies need to be developed. The basic problem here is that "inside the minimizers set" may exist several orbits (generated by the several Galilean invariance of the equation), each one possibly generated by different solution profiles modulo some specific invariance. In recent years the strategy to get around this problem has been to obtain results of uniqueness for the pseudo-differential Euler–Lagrange equation satisfied by the profiles, a challenging problem when dealing with pseudo-differential operators, such as those associated with the BO equation or systems like in (3.13).

4 (in)Stability of periodic traveling wave solutions

Nowadays, the study of the existence and stability of traveling waves of periodic type associated with nonlinear dispersive equations has increased significantly. A rich variety of new mathematical problems have emerged, so as well as the physical importance related to them. This subject is often studied in relation to the natural Galilean symmetries associated to the model (translation and/or rotations invariance) and to perturbations of symmetric classes, e.g., the class of periodic functions with the same minimal period as the underlying wave. However, it is possible to consider a stability study with general non-periodic perturbations, e.g., by the class of spatially localized perturbations $L^2(\mathbb{R})$ or by the class of bounded uniformly continuous perturbations $C_b(\mathbb{R})$ (see [71–73,110]).

In the case of shallow-water wave models (or long internal waves in a density-stratified ocean, ion-acoustic waves in a plasma or acoustic waves on a crystal lattice), it is well known that a formal stability theory of periodic traveling wave has started with the pioneering work of Benjamin [45] regarding to the periodic steady solutions called *cnoidal waves* found first by Korteweg and de-Vries for the KdV equation (1.4) (The Benjamin's approach was only years later completed by Angulo et al. [24] (see also [19]). The cnoidal traveling wave solution, namely, $u(x,t) = \varphi_c(x - ct)$, has a profile determined by the explicit form

$$\varphi_c(\xi) = \beta_2 + (\beta_3 - \beta_2)\mathrm{cn}^2\left(\sqrt{\frac{\beta_3 - \beta_1}{12}}\xi; k\right), \tag{4.1}$$

where $cn(\cdot; k)$ represents the Jacobi elliptic function called *cnoidal* associated with the elliptic modulus $k \in (0, 1)$, and β_i's are real constants satisfying the relations

$$\beta_1 < \beta_2 < \beta_3, \quad \beta_1 + \beta_2 + \beta_3 = 3c, \quad k^2 = \frac{\beta_3 - \beta_2}{\beta_3 - \beta_1}. \tag{4.2}$$

The cnoidal wave in (4.1) is periodic with a minimal period $T(k) = 2K(k)$, where $K(k)$ is the complete elliptic integral of the first kind (see [56]) given by

$$K(k) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}, \quad k \in (0, 1). \tag{4.3}$$

Now, according to Benjamin, Weinstein, Grillakis, Shatah, and Strauss stability framework established in Sect. 2.1 above, the existence of a non-trivial smooth curve of periodic traveling waves solutions with a same period is required. In the periodic setting this condition presents new and delicate aspects that need to be handled. For the general class of dispersive model in (1.7), by instance, the possibility of finding explicit solutions will depend naturally on the form of M: if it is a differential operator of the form $M = -\partial_x^2$, the use of the quadrature method (it means writes Eq. (1.14) in the form $[\phi_c']^2 = F(\phi_c)$), and the theory of elliptic functions has showed to be a main tool. So the solutions will depend of the Jacobian elliptic functions *snoidal, cnoidal*, and *dnoidal* type. Now, since the period of these functions depends on the complete elliptic integral $K(k)$ in (4.3), we have that the elliptic modulus k will depend on the velocity c and therefore we have that *a priori* the period of ϕ_c will also depend on c. Hence, by using the Implicit Function Theorem, it has been obtained in many cases the wanted smooth branch of periodic solutions with a fixed minimal period. We note that the procedure of quadrature method in general does not work if M is a pseudo-differential operator like the BO equation (1.11). Angulo and Natali [34] (see also [19,36]) worked on this obstacle and they used the classical *Poisson Summation Theorem* for obtaining solutions when M is a pseudo-differential operator.

With regarding to the spectral sufficient conditions for the stability study, namely, the simplicity of the zero eigenvalue and the Morse index, the problem is very delicate. The work of Angulo and Natali [34] established a new set of conditions that gives us this special spectrum structure. The analysis relies upon the theory of totally positive operators and the Fourier transform. We note that this new theory leads to a significant simplification of the proofs of stability of periodic traveling waves solutions for nonlinear dispersive equations.

In this section we establish two specific applications of the theory in [34]. The first one has relation with the stability of the periodic waves for the Benjamin–Ono equation (1.11) and the existence and (in)stability of standind waves with a periodic profile for the critical nonlinear Schrödinger equation:

$$iu_t + u_{xx} + |u|^4 u = 0. \tag{4.4}$$

We also give in this section a linear instability criterium for periodic traveling wave solutions for the general model of Benjamin–Bona–Mahoney type equation

$$u_t + u_x + (f(u))_x + (\mathcal{M}u)_t = 0,$$

with $\mathcal{M}$ being a general pseudo-differential operator (see (1.8)). In particular, we show the (non)linear instability of cnoidal waves profiles for the modified Benjamin–Bona–Mahoney equation

$$u_t + u_x + 3u^2 u_x - u_{xxt} = 0. \tag{4.5}$$

We finish this section with some notes and some very interesting open problems.

4.1 Quadrature method and Poisson Summation formula for the existence of periodic waves

In this subsection we apply the quadrature method and the Poisson Summation formula (PS formula henceforth) in the theory of the existence of periodic traveling waves solution for nonlinear dispersive model. From our point of view, the PS formula seems to be much more flexible in comparison with the quadrature method when obtaining specific formulas of periodic profiles. Moreover, PS formula approach can be used for obtaining spectral informations of linear operators in a stability study.

Here we obtain the specific profiles for the periodic traveling waves profiles associated to the BO equation (1.11) and the "critical nonlinear Schrödinger equation" (4.4).

4.1.1 Periodic traveling waves solutions to the BO equation

We will show that the BO equation (1.11) has explicit periodic traveling wave solutions of the form $u(x,t) = \varphi_c(x - ct)$ for an arbitrary fundamental period $2l$ and wave speed $c > \frac{\pi}{l}$. In fact, in [44,118] was obtained that for

$$\beta = 2c \tanh \gamma \qquad \text{and} \qquad \tanh \gamma = \frac{1}{cl}\,\pi, \tag{4.6}$$

the function

$$\varphi_c(\xi) = \frac{\beta \sinh \gamma}{\cosh(\gamma) - \cos\left(\frac{\pi\xi}{2l}\right)} \tag{4.7}$$

satisfies the pseudo-differential equation

$$\mathcal{H}\varphi_c' + c\varphi_c - \frac{1}{2}\varphi_c^2 = 0. \tag{4.8}$$

We note that the solitary wave profile (1.12) associated with the BO equation on all line may be obtained from (4.7). In fact, if we take the limit $l \to \infty$ then $\gamma \to 0$ and $\beta \to 0$ in such a way that $l\gamma \to c\pi, l\beta \to 2\pi$. Therefore, Eq. (4.7) gives us the form (1.12) at the limit.

Next, we obtain the explicit solution (4.7) by using the PS formula. The equation $\mathcal{H}\phi_\omega + \omega\phi_\omega - \frac{1}{2}\phi_\omega^2 = 0$, determines the solitary wave solutions $\phi_\omega(x) = \dfrac{4\omega}{1+\omega^2x^2}$, $\omega > 0$, and their Fourier transform is given by $\widehat{\phi_\omega}(x) = 4\pi e^{\frac{-2\pi}{\omega}|x|}$. Then, by the PS formula, we can build the following periodic profile ψ_ω with a period $2l$

$$\begin{aligned}
\psi_\omega(x) &\equiv \sum_{n=-\infty}^{+\infty} \phi_\omega(x+2ln) = \frac{2\pi}{l}\sum_{n=-\infty}^{+\infty} e^{\frac{-\pi|n|}{\omega l}} e^{\frac{\pi i n x}{l}} \\
&= \frac{2\pi}{l}\sum_{n=0}^{+\infty} \varepsilon_n e^{\frac{-\pi n}{\omega l}} \cos\left(\frac{n\pi x}{l}\right) = \frac{2\pi}{l}\mathrm{Re}\left[\coth\left(\frac{\frac{\pi}{\omega l}}{2} + \frac{i\pi x}{2l}\right)\right] \\
&= \frac{2\pi}{l}\left(\frac{\sinh\left(\frac{\pi}{\omega l}\right)}{\cosh\left(\frac{\pi}{\omega l}\right) - \cos\left(\frac{\pi x}{l}\right)}\right),
\end{aligned} \tag{4.9}$$

where $\varepsilon_n = 1$ if $n = 0$, and $\varepsilon_n = 2$ if $n = 1, 2, 3, \ldots$

Now, let Ψ_c, with $c \in \mathbb{R}$, be a smooth periodic solution of the Eq. (4.8) of period $2l$. Then Ψ_c can be expressed as a Fourier series, $\Phi_c(x) = \sum_{n=-\infty}^{+\infty} a_n e^{\frac{in\pi x}{l}}$. Substituting the expression above into the BO equation we get $\left[\frac{\pi|n|}{l} + c\right] a_n = \frac{1}{2}\sum_{m=-\infty}^{+\infty} a_{n-m}a_m$. Now, from (4.9) we consider $a_n \equiv \dfrac{2\pi}{l}e^{-\gamma|n|}$, $n \in \mathbb{Z}$, $\gamma \in \mathbb{R}$. Substituting a_n into the last identity we have

$$\sum_{m=-\infty}^{+\infty} a_{n-m}a_m = \frac{4\pi^2}{l^2}e^{-\gamma|n|}\left[|n| + 1 + 2\sum_{k=1}^{+\infty} e^{-2\gamma k}\right] = \frac{4\pi^2}{l^2}e^{-\gamma|n|}(|n| + \coth\gamma).$$

Then, we conclude that $c + \frac{\pi|n|}{l} = \frac{2\pi}{l}\cdot\frac{1}{2}(|n| + \coth\gamma)$. We denote $\gamma = \dfrac{\pi}{\omega l}$ and consider $c > \dfrac{\pi}{l}$. Then, if we choose a speed-velocity $\omega = \omega(c) > 0$ of the solitary wave ϕ_ω such that $\tanh\left(\dfrac{\pi}{\omega l}\right) = \dfrac{\pi}{cl}$, we see from (4.9) that we catch the Benjamin's periodic traveling waves profile $\Phi_c = \psi_{\omega(c)} \equiv \varphi_c$. Moreover, since we have that $\varphi_c > 0$ and $\gamma := \gamma(c) = \tanh^{-1}\left(\dfrac{\pi}{cl}\right)$ is a differentiable function for $c > \dfrac{\pi}{l}$, it follows that we obtain the following smooth curve of periodic

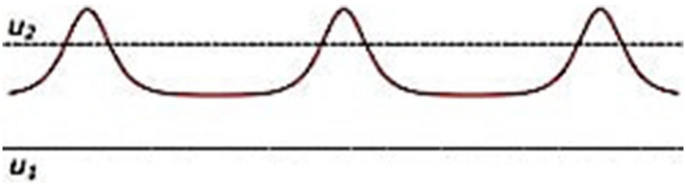

Fig. 3 The BO-periodic traveling wave solution (4.7) with $u_1 = 0$ and $u_2 = 2c$ the constant solutions when $c < \frac{\pi}{l}$

traveling waves for the BO equation, $c \in \left(\frac{\pi}{l}, +\infty\right) \mapsto \phi_c \in H^n_{per}([0, 2l])$ (see Fig. 3).

4.1.2 Critical nonlinear Schrödinger equation

In this section, we consider the periodic critical nonlinear Schrödinger (CNLS) given by equation

$$iu_t + u_{xx} + |u|^4 u = 0, \tag{4.10}$$

with $u = u(x,t) \in \mathbb{C}$, $x \in [0, L]$ and $t \in \mathbb{R}$. In what follows, we consider solutions of the form $u_\omega(x,t) = e^{i\omega t}\varphi_\omega(x)$, and we will show the existence of a smooth family of periodic waves $\varphi_\omega(x)$ with a minimal period L, $\omega \in \left(\frac{\pi^2}{L^2}, +\infty\right) \to \varphi_\omega$. If we substitute this kind of solution in (4.10) we obtain the following ODE,

$$\varphi''_\omega + \varphi^5_\omega - \omega\varphi_\omega = 0. \tag{4.11}$$

Next, we will use the quadrature method for finding a periodic profile of (4.11). Indeed, by multiplying Eq. (4.11) by φ'_ω and integrating once we obtain

$$[\varphi'_\omega]^2 = \frac{1}{3}\left(-\varphi^6_\omega + 3\omega\varphi^2_\omega + 6B_{\varphi_\omega}\right), \tag{4.12}$$

where B_{φ_ω} is a non zero constant of integration. Now, we consider the ansatz $\varphi_\omega = \psi_\omega^{\frac{1}{2}}$, this means we are considering positive solutions. Replacing φ_ω in (4.12) we obtain the following equation in quadrature form

$$[\psi'_\omega]^2 = \frac{4}{3}\left(-\psi^4_\omega + 3\omega\psi^2_\omega + 6B_{\psi_\omega}\psi_\omega\right) = \frac{4}{3}F_\psi(\psi_\omega(\xi)), \tag{4.13}$$

with $F(t) := F_\psi(t) = -t^4 + 3\omega t^2 + 6B_\psi t$. Let us consider η_1, η_2 and η_3, the non zero roots of the polynomial F_ψ such that $F(t) = t(t-\eta_1)(t-\eta_2)(\eta_3 - t)$. Because of Eq. (4.13) we must have

$$\eta_1 + \eta_2 + \eta_3 = 0, \quad \eta_1\eta_2 + \eta_2\eta_3 + \eta_1\eta_3 = -3\omega, \quad \eta_1\eta_2\eta_3 = 6B_{\psi_\omega}. \tag{4.14}$$

The first of the relations above shows us that the roots η_i, $i = 1, 2, 3$ should satisfy $\eta_1 < 0 < \eta_2 < \eta_3$. Therefore $\eta_2 \leq \psi_\omega \leq \eta_3$.

Consider a wave-speed $\omega > 0$ arbitrary but fixed. We are looking for non-constant periodic solutions ψ_ω such that its maximum value and minimum value on its period domain $[0, L]$, are given respectively, by $\psi_\omega(0) = \eta_3$ and $\psi_\omega(\sigma) = \eta_2$ for some $\sigma \in (0, L)$. The differential equation (4.13) allow us to use Leibnitz rule and conclude that

$$\int_{\psi(\xi)}^{\eta_3} \frac{dt}{\sqrt{t(\eta_3 - t)(t - \eta_2)(t - \eta_1)}} = \frac{2}{\sqrt{3}g}(\xi + M_\omega), \tag{4.15}$$

where M_ω is a constant and g is defined below. Since (4.11) is invariant under translations we can conclude from Byrd and Friedman [56] (formula 257.00) that,

$$\psi_\omega(\xi) = \frac{\eta_3(\eta_2 - \eta_1) + \eta_1(\eta_3 - \eta_2)\mathrm{sn}^2\left(\frac{2}{\sqrt{3}g}\xi; k\right)}{(\eta_2 - \eta_1) + (\eta_3 - \eta_2)\mathrm{sn}^2\left(\frac{2}{\sqrt{3}g}\xi; k\right)}, \tag{4.16}$$

where sn represent the *Jacobian elliptic function snoidal* and

$$g = \frac{2}{\sqrt{\eta_3(\eta_2 - \eta_1)}}, \quad k^2 = \frac{-\eta_1(\eta_3 - \eta_2)}{\eta_3(\eta_2 - \eta_1)}.$$

So, using the expression for k and $k^2 sn^2 + dn^2 = 1$, we arrive at the following compact form of ψ_ω:

$$\psi_\omega(\xi) = \eta_3 \left[\frac{\mathrm{dn}^2\left(\frac{2}{\sqrt{3}g}\xi; k\right)}{1 + \beta^2 \mathrm{sn}^2\left(\frac{2}{\sqrt{3}g}\xi; k\right)} \right], \tag{4.17}$$

where dn represents the *Jacobian elliptic function dnoidal*, $\beta^2 = -\eta_3 k^2/\eta_1 > 0$. Now, the periodic traveling wave solution ψ_ω has a fundamental period given by

$$T_\psi = \sqrt{3}gK(k_\psi) = \frac{2\sqrt{3}K(k_\psi)}{\sqrt{\eta_3(\eta_2 - \eta_1)}}, \quad k_\psi^2 = \frac{-\eta_1(\eta_3 - \eta_2)}{\eta_3(\eta_2 - \eta_1)}. \tag{4.18}$$

Next, by using the Implicit Function Theorem, we can obtain the following smooth family of periodic waves with a minimal period L (see Angulo and Natali [35]), $\omega \in \left(\frac{\pi^2}{L^2}, +\infty\right) \to \varphi_\omega$ with

$$\varphi_\omega(\xi) = \sqrt{\eta_3} \frac{\mathrm{dn}\left(\frac{2}{\sqrt{3}g}\xi; k\right)}{\sqrt{1 + \beta^2 \mathrm{sn}^2\left(\frac{2}{\sqrt{3}g}\xi; k\right)}}, \tag{4.19}$$

and η_3, g, k are smooth functions of ω. Lastly, we can formally see the asymptotic behavior of the solutions ψ_ω. We start with the constant solutions: indeed, it follows

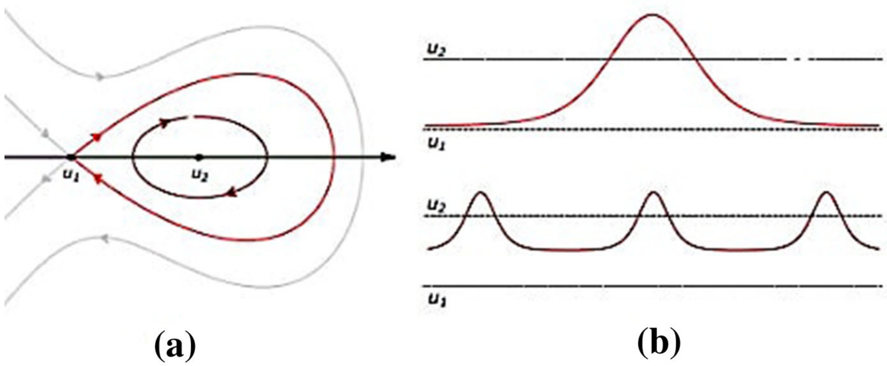

Fig. 4 In **a** we sketch a phase portrait representation $(\varphi_\omega, \varphi'_\omega)$ of (4.11) with $u_1 = 0$ and $u_2 = \omega^{1/4}$ the constant solutions. In **b** we have the smooth standing wave profile $\sqrt{f_S}$ (homoclinic orbit) and the periodic traveling waves φ_ω in (4.19) (periodic orbit)

that if $\eta_3 \to \sqrt{\omega}$ (ω fixed) then $k \to 0$ and so $\psi_\omega(z) \to \eta_3 = \sqrt{\omega}$, which gives us the non-trivial constant solution $\varphi_\omega \equiv \omega^{1/4}$ for the CNLS. On the other hand, if $\eta_3 \to \sqrt{3\omega}$ then $k \to 1$, and so $\psi_\omega(\xi) \to f_S(\xi) \equiv \sqrt{3\omega}\,\mathrm{sech}(2\sqrt{\omega}\,\xi)$. Then $\sqrt{f_S}$ is the classical standing wave profile solution, $|e^{i\omega t}\sqrt{f_S}|$, associated with the CNLS equation on whole the line.

In Fig. 4 below we show the phase portrait associated to model (4.11) (see Geyer and Quirchmayr [75]). The internal trajectories (no-changing of sign) are given by (4.19). The homoclinic trajectory is exactly the positive profile $\sqrt{f_S}$.

4.2 Angulo and Natali's stability approach

Next we establish the theory developed by Angulo and Natali [34] to study the stability of positive even periodic traveling waves solutions associated to the general dispersive equation (1.7). We start by studying the linear operator $\mathcal{L} : D(\mathcal{L}) \to L^2_{per}([-L, L])$ defined, on a dense subspace of $L^2_{per}([-L, L])$, by

$$\mathcal{L}u = (M + c)u - \phi^p u, \tag{4.20}$$

where $\phi = \phi_c$ satisfies (1.14), $p \in \mathbb{N}$, $p \geqq 1$. We will suppose that $c > -b$, where b satisfies $\alpha(k) > b$ for all $k \in \mathbb{Z}$.[4] With that condition $M + c$ represents a positive operator. Then, by using the spectral theorem for compact and self-adjoint operators we have the following characterization of the spectrum of $\mathcal{L}$ (see Theorem 11.21 in [19]).

Theorem 4.1 *The operator $\mathcal{L}$ in (4.20) is a closed, unbounded, self-adjoint operator on $L^2_{per}([0, 2L])$ whose spectrum consists of an enumerable (infinite) set of eigenvalues*

[4] Here, the pseudo-differential operator M is acting on periodic function of period $2L$. In other words, M is defined as a Fourier multiplier operator by $\widehat{Mu}(k) = \alpha(k)\widehat{u}(k)$, $k \in \mathbb{Z}$.

$\{\lambda_k\}_{k=0}^{\infty}$ *satisfying* $\lambda_0 \le \lambda_1 \le \lambda_2 \le \ldots$ *and* $\lambda_k \to \infty$ *as* $k \to \infty$*. In particular,* $\mathcal{L}$ *has zero as an eigenvalue with eigenfunction* $\frac{d}{dx}\varphi_c$*.*

Definition 4.2 We say that a sequence $\alpha = (\alpha_n)_{n\in\mathbb{Z}} \subseteq \mathbb{R}$ is in the class $PF(2)$ discrete if

$$\begin{array}{ll} \text{(i)} & \alpha_n > 0, \quad \text{for all} \quad n \in \mathbb{Z}, \\ \text{(ii)} & \alpha_{n_1-m_1}\alpha_{n_2-m_2} - \alpha_{n_1-m_2}\alpha_{n_2-m_1} > 0 \ \text{ for } \ n_1 < n_2 \text{ and } \ m_1 < m_2. \end{array} \tag{4.21}$$

The definition above is a discrete form of the continuous ones definition which appears in Karlin [87]. The main result of this section is the following,

Theorem 4.3 (Angulo and Natali) *Let* ϕ_c *be an even positive solution of* (1.14)*. Suppose that* $\widehat{\phi_c}(n) > 0$ *for every* $n \in \mathbb{Z}$*, and* $K = (\widehat{\phi_c^p}(n))_{n\in\mathbb{Z}} \in PF(2)$ *discrete. Then* $\mathcal{L}$ *in* (4.20) *has exactly one unique negative eigenvalue which is simple, and zero is a simple eigenvalue with eigenfunction* $\dfrac{d}{dx}\varphi_c$*.*

Proof The proof relies upon the theory of totally positive operators, the theory of totally positive sequences of order 2 ($PF(2)$-class) and from the spectral theory for compact self-adjoint operators (see Angulo and Natali [19,34]). □

Next we establish two applications of Theorem 4.3 for the cases of the BO and CNLS models.

4.2.1 Stability of periodic waves for the BO equation

The Poisson Summation Theorem and Theorem 4.3 will be applied with success to obtain the nonlinear stability of the periodic profiles in (4.7), with regard to the periodic flow of the BO equation. We note that in this case, Definition 4.2 needs to be changed slightly to the following one:

Definition 4.4 We say that a sequence $\alpha = (\alpha_n)_{n\in\mathbb{Z}} \subseteq \mathbb{R}$ is in the class $PF(2)$ discrete if

$$\begin{array}{ll} \text{(i)} & \alpha_n > 0, \quad \text{for all} \quad n \in \mathbb{Z}, \\ \text{(ii)} & \alpha_{n_1-m_1}\alpha_{n_2-m_2} - \alpha_{n_1-m_2}\alpha_{n_2-m_1} \geqq 0 \ \text{ for } \ n_1 \le n_2 \text{ and } m_1 < m_2, \\ \text{(iii)} & \text{strict inequality holds in (ii) whenever the intervals } (n_1, n_2) \text{ and } (m_1, m_2) \text{ intersect.} \end{array} \tag{4.22}$$

Theorem 4.5 *The periodic traveling waves* ϕ_c *in* (4.7) *with* $c > \frac{\pi}{l}$*, are orbitally stable in* $H^{\frac{1}{2}}_{per}([-l, l])$ *with regard to the periodic flow of the BO equation.*

Proof The global possednes in $H^{\frac{1}{2}}_{per}([-l, l])$ of the initial value problem for the BO equation was established in [98]. Next we have the spectral structure for the self-adjoint operator $\mathcal{L} = \mathcal{H}\partial_x + c - \phi_c$. Indeed, from (4.9) we get for γ satisfying (4.6), $\widehat{\phi_c}(n) = \frac{2\pi}{l}e^{-\gamma|n|} > 0$ for all $n \in \mathbb{Z}$. Moreover, it is easy to verify directly that the sequence $\{e^{-\gamma|n|}\}_{n\in\mathbb{Z}}$ is in $PF(2)$ discrete in the sense of Definition 4.4. Therefore, the Morse index for $\mathcal{L}$ satisfies $n(\mathcal{L})=1$ and $Ker(\mathcal{L})=\text{span}\{\frac{d}{dx}\varphi_c\}$.

For the calculus of $\frac{d}{dc}\|\phi_c\|^2$, it follows from the Parseval Theorem that

$$I=-\frac{1}{2}\frac{d}{dc}\|\phi_c\|^2=-l\frac{d}{dc}\Big(\frac{4\pi^2}{l^2}\sum_{n\in\mathbb{Z}}e^{-2\gamma|n|}\Big)=-\frac{8\pi^3}{c^2l^2-\pi^2}\sum_{n\in\mathbb{Z}}|n|e^{-2\gamma|n|}>0.$$

Hence, the orbit generated by the periodic traveling wave ϕ_c is stable in $H^{\frac{1}{2}}_{per}([-l,l])$ by the periodic flow of the BO equation provided $c>\frac{\pi}{l}$. □

Remark 4.6 The stability of the constant solutions for the BO equation, $\psi_0=2c$, in $H^{\frac{1}{2}}_{per}([-l,l])$ for any $l>0$ and $c<\frac{\pi}{l}$ was showed in Theorem 11.32 in [19].

4.2.2 (in)Stability of periodic traveling waves for the CNLS equation

Next we consider the smooth family $\omega\in\left(\frac{\pi^2}{L^2},+\infty\right)\to\varphi_\omega$ of periodic wave solutions for CNLS with φ_ω satisfying (4.11) and given in (4.19). In the study of stability properties of the solution $e^{i\omega t}\varphi_\omega$ by the flow of (4.10), the notion of stability is different from that for KdV-type equations studied in the last sections because (4.19) has at least two Galilean symmetries: translations and rotations. So, by defining the orbit generated by φ_ω via these two invariances

$$\mathcal{O}_{\varphi_\omega}=\{e^{i\theta}\varphi_\omega(\cdot+y):\ \ (y,\theta)\in\mathbb{R}\times[0,2\pi)\}, \tag{4.23}$$

we have the following definition.

Definition 4.7 We say that the orbit $\mathcal{O}_{\varphi_\omega}$ is stable by the periodic flow of the CNLS equation if for all $\varepsilon>0$, there is a $\delta(\varepsilon)>0$ such that if $\inf_{(y,\theta)\in\mathbb{R}\times[0,2\pi)}||u_0-e^{i\theta}\varphi_\omega(\cdot+y)||_{H^1_{per}}<\delta$, then the solution $u(x,t)$ associated with the CNLS equation (4.10) with initial data u_0 satisfies $\inf_{(y,\theta)\in\mathbb{R}\times[0,2\pi)}||u(\cdot,t)-e^{i\theta}\varphi_\omega(\cdot+y)||_{H^1_{per}}<\varepsilon$, for all $t\in\mathbb{R}$.

Next we establish the (in)stability results for the periodic wave solutions with profile φ_ω defined in (4.19) (see Angulo and Natali [35]).

Theorem 4.8 *The orbit $\mathcal{O}_{\varphi_\omega}$ is stable in $H^1_{per}([0,L])$ by the flow of the CNLS for $\omega\in\left(\frac{\pi^2}{L^2},\frac{r(k_0)}{L^2}\right)$ and unstable for $\omega\in\left(\frac{r(k_0)}{L^2},+\infty\right)$, where $r(k)\equiv 4K^2(k)\sqrt{k^4-k^2+1}$ and $k_0\approx 0.3823174965$. Here K represents the complete elliptic integral*

Corollary 4.9 *For $u_0\in H^1_{per}([0,L])$ sufficiently close to the orbit $\mathcal{O}_{\varphi_\omega}$ with $\omega\in\left(\frac{\pi^2}{L^2},\frac{r(k_0)}{L^2}\right)$, we see that thesolution $u=u(x,t)$ of (4.10) with $u(x,0)=u_0$ belongs to the class $C_b(\mathbb{R};H^1_{per})$.*

4.3 A linear instability criterium for periodic traveling wave solutions

In this subsection we establish a linear instability criterium of periodic traveling wave associated with some general one-dimensional dispersive models. By using analytic and asymptotic perturbation theory, we establish sufficient conditions for the existence of exponentially growing solutions to the linearized problem and so the linear instability of periodic profiles with mean zero is obtained. Application of this approach will be concerned with the linear instability of cnoidal wave solutions for the modified Benjamin–Bona–Mahony (4.5) (mBBM henceforth). Our approach can also be extended to the general model (1.7) (see [38]).

It is widely known that the spectral instability of a specific traveling wave solution of an evolution type model is a key prerequisite to show their nonlinear instability property (see [80,103,125] and references therein). Moreover, determining when the spectral instability result implies nonlinear instability is a non trivial task. In [80] was established that for getting that assertion is sufficient to show a specific inequality (see (6.2) in [80]) associated to semigroup $e^{t\mathcal{A}}$ generated by $\mathcal{A} = J\mathcal{L}_c$. In general, this inequality is a nontrivial issue to be verified because of the Spectral Mapping Theorem ($\sigma(e^{\mathcal{A}}) = e^{\sigma(\mathcal{A})}$) is present in the background (see Georgiev and Ohta [74], Cramer and Latushkin [64] and reference therein).

On the other hand, if we use Theorem 2.2, Remark 2 in Section 2 of Henry et al. [82] and the property that the mapping data-solution associated to the evolution equation in question is of class C^2 around the traveling wave solution, we can obtain that the spectral instability result will imply nonlinear instability (see Angulo and Natali [38] and Angulo et al. [33] where this kind of strategy has been used for obtaining nonlinear instability results)

4.3.1 Linear instability criterium for the generalized Benjamin–Bona–Mahony equation

The specific dispersive models, which are the focus of our study here, are the generalized Benjamin–Bona–Mahony equations (gBBM henceforth)

$$u_t + u_x + (f(u))_x + (\mathcal{M}u)_t = 0, \tag{4.24}$$

with $u = u(x,t)$, $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, is a real valued function which is periodic at the $x-$variable, $\mathcal{M}$ is a differential or pseudo-differential operator defined in the framework of periodic functions defined as a Fourier multiplier operator by $\widehat{\mathcal{M}g}(n) = \alpha(n)\widehat{g}(n)$, $n \in \mathbb{Z}$, where the symbol α satisfies relations in (1.8). For $\mathcal{M} = -\partial_x^2$ ($\alpha(n) = n^2$), and $f(u) = u^2$ and $f(u) = u^3$ in (4.24), we obtain the well-known models Benjamin–Bona–Mahoney equation and modified Benjamin–Bona–Mahoney in (4.5), respectively. In our study, we will assume $\alpha(n) \geq 0$ for $n \in \mathbb{Z}$, but our results can be modified for cases of sign changed symbols and lower bounded.

We will suppose that Eq. (4.24) admits traveling wave solutions $u(x,t) = \varphi_c(x - ct)$, with the profile $\varphi_c : \mathbb{R} \to \mathbb{R}$ being an L-periodic smooth function with mean-zero $\int_0^L \varphi_c(\xi)d\xi = 0$, and $c \in I \subset \mathbb{R}$ being called the wave-speed. Hence, from (4.24) we obtain after a first integration the pseudo-differential equation

$$(\mathcal{M}+1)\varphi_c - \frac{1}{c}(\varphi_c + f(\varphi_c)) = A_{\varphi_c}. \tag{4.25}$$

Here $c \neq 0$ and A_{φ_c} is an integration constant which will be assumed to be zero in our approach (see Alves et al. [11] and Sect. 4.4-(f) below for the case $A_{\varphi_c} \neq 0$). By considering the new variable $w(x,t) = u(x+ct,t) - \varphi_c(x)$ into the gBBM-equation and using Eq. (4.25) satisfied by φ_c, one finds that w satisfies the nonlinear equation

$$(\partial_t - c\partial_x)(w + \mathcal{M}w) + \partial_x(w + f'(\varphi_c)w + O\left(\|w\|^2\right) = 0. \tag{4.26}$$

As a leading approximation for small perturbation, we replace (4.26) by its linearization around φ_c, and hence we obtain the linear equation

$$(\partial_t - c\partial_x)(w + \mathcal{M}w) + \partial_x(w + f'(\varphi_c)w) = 0. \tag{4.27}$$

Seeking particular solutions of (4.27) of the form $w(x,t) = e^{\lambda t}\psi(x)$ (the so-called *growing mode solution*), where $\lambda \in \mathbb{C}$, ψ satisfies the linear problem

$$\partial_x \mathcal{L}_0 \psi = \lambda(\mathcal{M}+1)\psi, \tag{4.28}$$

where $\mathcal{L}_0 = c(\mathcal{M}+1) - (1+f'(\varphi_c)) : H^{\beta_2}_{per}([0,L]) \to L^2_{per}([0,L])$. Thus, we are in a similar framework to that in (1.20) and so a natural extension of Definition 1.2 can be established in the case of periodic traveling waves solutions. Since ∂_x is a skew-symmetric operator and $\mathcal{L}_0$ a self-adjoint operator we have that our eigenvalue problem (4.28) is reduced to show the existence of a spectral parameter λ com $Re(\lambda) > 0$. Moreover, we are looking for ψ in the closed subspace of mean zero, $\mathbb{V} = \left\{ f \in L^2([0,L]);\ [f] = \frac{1}{L}\int_0^L f(x)dx = 0 \right\}$.

Now we introduce our approach. If we use the expression $\frac{\partial_x}{\lambda - c\partial_x}$ as a notation for the linear operator $(\lambda - c\partial_x)^{-1}\partial_x$ with $Re(\lambda) > 0$, we obtain that the spectral problem (4.28) is equivalent to the following one

$$(\mathcal{M}+1)\psi - \frac{\partial_x}{\lambda - c\partial_x}(1+f'(\varphi_c))\psi = 0. \tag{4.29}$$

Next, we consider the orthogonal projection Q on $\mathbb{V}$ ($Q^2 = Q$ and $Q^* = Q$), namely, $Q : L^2_{per}([0,L]) \to \mathbb{V}$ defined by $Qu = u - [u]$, and the following family of closed linear operators $\mathcal{A}^\lambda : H^{\beta_2}_{per}([0,L]) \cap \mathbb{V} \longrightarrow \mathbb{V}$, $\mathrm{Re}(\lambda) > 0$, given by

$$\mathcal{A}^\lambda v \equiv (\mathcal{M}+1)v + \frac{\partial_x}{\lambda - c\partial_x} Q(v + f'(\varphi_c)v). \tag{4.30}$$

Next, in order to deduce the existence of a growing mode solution for (4.29), it is sufficient to find $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > 0$ such that the operator $\mathcal{A}^\lambda$ possesses a nontrivial kernel. Indeed, for $\psi \in H^{\beta_2}_{per}([0, L]) \cap \mathbb{V}$, $\psi \neq 0$, such that $\mathcal{A}^\lambda \psi = 0$ we obtain

$$0 = (\lambda - c\partial_x)(\mathcal{M} + 1)\psi + \partial_x \left((1 + f'(\varphi_c))\psi - \left[(1 + f'(\varphi_c))\psi\right]\right)$$
$$= (\lambda - c\partial_x)(\mathcal{M} + 1)\psi + \partial_x(u + f'(\varphi_c)\psi). \tag{4.31}$$

Our linearized instability result for the gBBM equation (4.24) is the following: For $\mathcal{X}_{\beta_2,0} \equiv H^{\beta_2}_{per}([0, L]) \cap \mathbb{V}$ we have

Theorem 4.10 (Linear instability criterium for gBBM equation) *Let $c \to \varphi_c \in \mathcal{X}_{\beta_2,0}$ be a smooth curve of periodic solution to Eq. (4.25) with $c > 1$. We assume that $Ker(Q\mathcal{L}_0) = span\left\{\frac{d}{dx}\varphi_c\right\}$. Denote by $n(Q\mathcal{L}_0)$ the number (counting multiplicity) of negative eigenvalues of the operator $Q\mathcal{L}_0$ defined on $\mathcal{X}_{\beta_2,0}$ (the Morse index). Then there is a purely growing mode $e^{\lambda t}\psi(x)$ with $\lambda > 0$, $\psi \in \mathcal{X}_{\beta_2,0} - \{0\}$, to the linearized equation (4.27) if one of the following two conditions is true:*

(i) *$n(Q\mathcal{L}_0)$ is even and $I(c) < 0$.*
(ii) *$n(Q\mathcal{L}_0)$ is odd and $I(c) > 0$.*

Here, $I(c) = -\frac{1}{||\varphi_c'||^2_{L^2_{per}}} \frac{1}{c}\frac{dF}{dc}$ with $F(c) = \frac{1}{2}\langle(\mathcal{M}+1)\varphi_c, \varphi_c\rangle$ representing the momentum evaluated in the periodic wave φ_c.

The proof of Theorem 4.10 is based in an extension of the asymptotic perturbation arguments due to Vock and Hunziker [128] and Lin [97] to the periodic case (see Angulo and Natali [38]).

4.3.2 Linear instability of cnoidal waves for the mBBM

We apply the criterium in Theorem 4.10 to obtain the linear instability of cnoidal wave profiles for the mBBM equation (4.5). Thus, for $u(x, t) = \varphi_c(x - ct)$ a periodic solution we have $-(c-1)\varphi_c + \varphi_c^3 + c\varphi_c'' = 0$. Then, we obtain the differential equation in quadrature form

$$[\varphi_c']^2 = \frac{1}{2c}\left[-\varphi_c^4 + 2(c-1)\varphi_c^2 + 4B_{\varphi_c}\right], \tag{4.32}$$

where B_{φ_c} is a nonzero integration constant. The periodic solutions related to (4.32) depend of the roots to $q(t) = -t^4 + 2(c-1)t^2 + 4B_{\varphi_c}$. Two Jacobian elliptic profiles emerge (see phase-plane (a) in Fig. 4 above), the profiles dnoidal (inner trajectory with regard to the soliton) and cnoidal (external trajectory with regard to the soliton). Indeed, if we consider the symmetric real roots $\pm\eta_1$, $\pm\eta_2$, such that $0 < \eta_2 < \eta_1$, we have positive and negative periodic solutions given explicitly by $\varphi_c(x) = \pm\eta_1 \mathrm{dn}\left(\frac{\eta_1}{\sqrt{2c}}x; k\right)$ where $dn(\cdot; k)$ represents the dnoidal Jacobi elliptic function with modulus $k \in (0, 1)$. The orbital stability of these dnoidal profiles was just established by the flow of the mBBM in [18,23] for the cubic Schrödinger equation.

Next, we suppose that q has the two symmetric real roots, $-b < 0 < b$ and two symmetric imaginary roots $\pm ia$, then we obtain for $c > 1$ a cnoidal profile solution given by (see [18,32])

$$\varphi_c(x) = b\mathrm{cn}\left(\frac{\beta}{\sqrt{c}}x; k\right). \tag{4.33}$$

Here, we have *a priori* the relations $k^2 = \dfrac{b^2}{a^2+b^2}$, $b^2 - a^2 = 2(c-1)$, $\beta = \sqrt{\frac{a^2+b^2}{2}}$. We note that for $c > 1$ we get $b^2 > 2(c-1)$ and so our cnoidal-trajectories in the phase-plane (a) in Fig. 4, lie outside the homoclinic orbit (soliton profile given by $\phi_c(x) = \sqrt{2(c-1)}sech(\sqrt{\frac{c-1}{c}}x))$. Now, since $k^2(b) = \frac{b^2}{2b^2-2(c-1)}$ we must have $k^2 \in \left(\frac{1}{2}, 1\right)$. Moreover, since the cnoidal profile has real fundamental period equal to $4K(k)$ we obtain that the fundamental period T_{φ_c} for φ_c in (4.33) can be seen as a function of b, namely,

$$T_{\varphi_c}(b) = \frac{4\sqrt{c}}{\sqrt{b^2-(c-1)}}K(k(b)). \tag{4.34}$$

Then, the Implicit Function Theorem implies the following result,

Theorem 4.11 *Let $L > 0$ be fixed and $k^2 \in \left(\frac{1}{2}, 1\right)$ satisfying $L^2 > 16K^2(k)(2k^2-1)$. Then,*

(i) *For every $c > 1$ there is a unique $b = b(c) \in (\sqrt{2(c-1)}, +\infty)$ such that the map $c \in (1, +\infty) \mapsto b(c)$ is a strictly increasing smooth function and $L = \frac{4\sqrt{c}}{\sqrt{b^2-(c-1)}}K(k)$. The modulus $k = k(c)$ is given by $k^2 = \frac{b^2}{2b^2-2(c-1)}$ and $\frac{dk}{dc} > 0$.*

(ii) *For every $c > 1$ the cnoidal wave $\varphi_c(x) = b\mathrm{cn}(\sqrt{\frac{b^2-(c-1)}{c}}x; k)$ has fundamental period L and satisfies Eq. (4.32). Moreover, the mapping $c \in (1, +\infty) \mapsto \varphi_c \in H^n_{per}([0, L])$ is a smooth function for all $n \in \mathbb{N}$.*

In Figs. 5 and 6 we show the cnoidal profiles $cn(\cdot; k)$ for $k \approx \frac{1}{\sqrt{2}}$ and $k \in (k^*, 1)$, $k^* \approx 0.909$, respectively.

Next, we consider the linearized operator $\mathcal{L}_0$ around the cnoidal wave φ_c given by $\mathcal{L}_0 = -\dfrac{d^2}{dx^2} + \left(1 - \dfrac{1}{c}\right) - \dfrac{3}{c}\varphi_c^2$. Our goal is to study initially the eigenvalue problem in $H^2_{per}([0, L])$

$$\begin{cases} \mathcal{L}_0\psi = \eta\psi, \\ \psi(0) = \psi(L), \quad \psi'(0) = \psi'(L). \end{cases} \tag{4.35}$$

In the following theorem we collect all the spectral informations needed to apply the instability criterium in Theorem 4.10.

Theorem 4.12 *It considers the eigenvalue in (4.35).*

(a) *Let $k^2 \in (\frac{1}{2}, 1)$. Then for $D(\mathcal{L}_0) = H^2_{per}([0, L])$ we have that the Morse index of $\mathcal{L}_0$ satisfies $n(\mathcal{L}_0) = 2$ and $Ker(\mathcal{L}_0) = span\{\frac{d}{dx}\varphi_c\}$.*

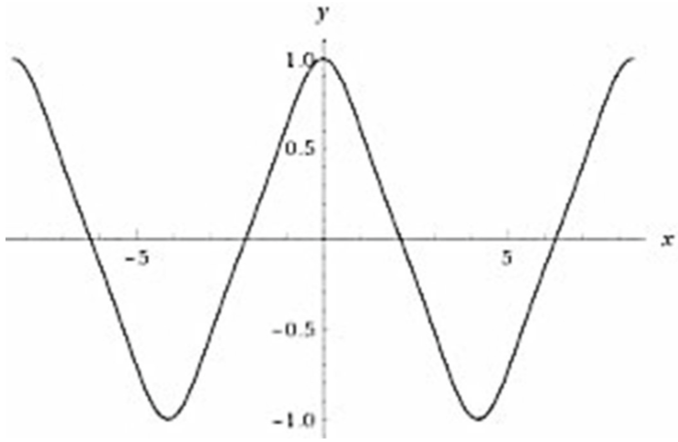

Fig. 5 Cnoidal profile $\mathrm{cn}(x; \frac{\sqrt{2}+\epsilon}{2})$

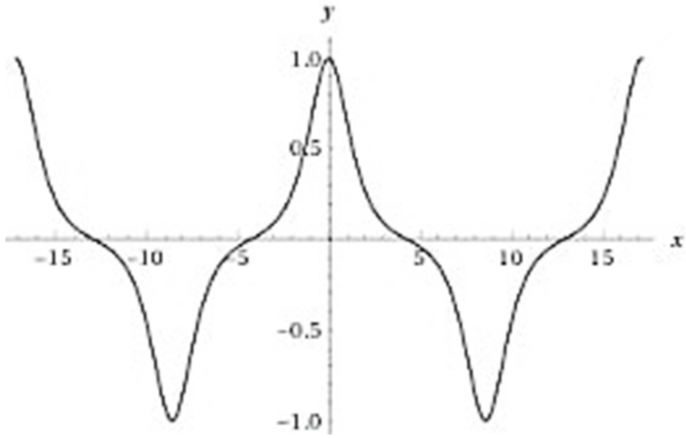

Fig. 6 Cnoidal profile $\mathrm{cn}(x; 0.909 + \epsilon)$

(b) *For* $D(\mathcal{L}_0) = H^2_{per}([0, L]) \cap \mathbb{V}$ *we have that* $Ker(Q\mathcal{L}_0) = span\{\frac{d}{dx}\varphi_c\}$ *for all* $k^2 \in (\frac{1}{2}, 1)$*, and the Morse index of* $Q\mathcal{L}_0$ *satisfies*

$$n(Q\mathcal{L}_0) = \begin{cases} 1, & for \;\; k \in \left(\frac{\sqrt{2}}{2}, k^*\right), \\ 2, & for \;\; k \in (k^*, 1) \end{cases}$$

with $k^* \approx 0.909$.

Proof Initially we have the existence of an enumerable set of eigenvalues $\{\eta_i\}_{i \geq 0}$ for (4.35). From (4.33) and the transformation $\Phi(x) = \psi(\sqrt{c}x/\beta)$ we obtain the Lame's eigenvalue problem,

$$\mathcal{P}\Phi \equiv -\Phi'' + 6k^2 sn^2(x; k)\Phi = \theta\Phi, \quad \Phi \in H^2_{per}([0, 4K(k)]). \tag{4.36}$$

Here θ will be an eigenvalue satisfying

$$\theta = \frac{1}{\beta^2}[3b^2 - (c-1) + c\eta]. \tag{4.37}$$

Now, from Floquet theory one has that the set of eigenvalues $\{\theta_i\}_{i\geq 0}$ associated to (4.36) has the distribution $\theta_0 < \theta_1 < \theta_2 < \theta_3 < \theta_4 < \theta_5 \leq \theta_6 < \cdots$. Here, inequalities mean that the first five eigenvalues are simple and all other eigenvalues have multiplicity two (see [106]). Next, the exact value of those five eigenvalues, as well as the associated eigenfunctions, will be useful for the subsequent calculations. Indeed, we have (see [18])

$$\begin{array}{ll} \theta_0 = 2[1+k^2-r(k)]; & \Phi_0(x) = k^2 sn^2(x;k) - \frac{1}{3}(1+k^2+r(k)) \\ \theta_1 = 1+k^2; & \Phi_1(x) = \partial_x sn(x;k) = cn(x;k)dn(x;k) \\ \theta_2 = 1+4k^2; & \Phi_2(x) = \partial_x cn(x;k) = -sn(x;k)dn(x;k) \\ \theta_3 = 4+k^2; & \Phi_3(x) = \partial_x dn(x;k) = -k^2 sn(x;k)cn(x;k) \\ \theta_4 = 2[1+k^2+r(k)]; & \Phi_4(x) = k^2 sn^2(x;k) - \frac{1}{3}(1+k^2-r(k)), \end{array} \tag{4.38}$$

where $r(k) = \sqrt{1-k^2+k^4}$. Moreover, for $j \neq 0$ and $j \neq 4$ we have that the eigenfunctions Φ_j has mean zero. Indeed, $\mathcal{P}\Phi_j = \theta_j \Phi_j$ implies

$$\theta_j\langle\Phi_j, 1\rangle = 6\langle k^2 sn^2, \Phi_j\rangle = 2(1+k^2)\langle\Phi_j, 1\rangle - 3\langle\Phi_j, \Phi_0+\Phi_4\rangle = 2(1+k^2)\langle\Phi_j, 1\rangle.$$

Thus, we conclude $\Phi_j \in \mathbb{V}$ for all $j \neq 0, 4$.

On the other hand, the simple eigenvalues θ_0, θ_1 and θ_2 determine the same property for η_0, η_1 and $\eta_2 = 0$ (see (4.37)), respectively, and $\eta_0 < \eta_1 < \eta_2 = 0$. Thus, on $H^2_{per}([0, L])$ we deduce that $n(\mathcal{L}_0) = 2$ and $Ker(\mathcal{L}_0) = \text{span}\{\frac{d}{dx}\varphi_c\}$.

Now, since $\frac{d}{dx}\varphi_c$ has two zeros on the interval $[0, L)$ and $Q\mathcal{L}_0(\frac{d}{dx}\varphi_c) = 0$, one has from the Floquet theory (oscillation theorem for Hill's equation, [106]) that $1 \leq n(Q\mathcal{L}_0) \leq 2$. Next, we prove that $n(Q\mathcal{L}_0) = 2$ for a specific range of the elliptic modulus k. In fact, let $\{\psi_i\}_{i\geq 0}$ be the complete orthonormal system of eigenfunctions associated with the periodic problem (4.35). We see from the analysis above that for $a_i = 1/\|\Phi_i\|_{L^2_{per}([0,4K])}$

$$\psi_i(x) = a_i\sqrt{\frac{4K}{L}}\Phi_i(\beta x/\sqrt{c}), \quad i = 0, 1, 2, \ldots, \tag{4.39}$$

where $[\psi_i] = 0$ for all $i \neq 0, 4$. Then, for $\psi \in D(Q\mathcal{L}_0) = H^2_{per}([0, L]) \cap \mathbb{V}$ such that $Q\mathcal{L}_0\psi = \lambda\psi$, we obtain the following three relations:

$$\psi = \sum_{i=0}^{\infty}\langle\psi, \psi_i\rangle\psi_i, \quad Q\mathcal{L}_0\psi = \sum_{i=0}^{\infty}\eta_i\langle\psi, \psi_i\rangle\psi_i + \frac{3}{c}\left[\varphi_c^2\psi\right] = \lambda\psi, \quad \text{and}$$

$$(\lambda - \eta_i)\langle\psi, \psi_i\rangle = \frac{3}{c}\left[\varphi_c^2\psi\right]\int_0^L \psi_i\, dx, \tag{4.40}$$

Since the sums in (4.40) are taken over the set $\langle \psi, \psi_i \rangle \neq 0$, $i \geq 0$ and $[\psi_i] = 0$ for all $i \neq 0, 4$, we obtain that $\lambda = \eta_i$, for all $i \neq 0, 4$. So, by considering $\lambda \neq \eta_i$ one has from (4.40)

$$\psi = \frac{3}{c}\left[\varphi_c^2 \psi\right] \sum_{i=0}^{\infty} \frac{[\psi_i]}{\lambda - \eta_i} \psi_i . \tag{4.41}$$

Therefore, since $\frac{3}{c}[\varphi_c^2\psi] \neq 0$ and $[\psi] = 0$, we can integrate expression (4.41) over $[0, L]$ to obtain that $\lambda \neq \eta_i$ must be a zero of the function

$$J(\lambda) = \frac{[\psi_0]^2}{\lambda - \eta_0} + \frac{[\psi_4]^2}{\lambda - \eta_4}. \tag{4.42}$$

It is important to note that J is a two variables function depending on λ and $k \in (0, 1)$, and that we are interested in the zeros of the function $J(\lambda)$. Thus, since $\lim_{\lambda \to \eta_0^+} J(\lambda) = +\infty$, $\lim_{\lambda \to \eta_4^-} J(\lambda) = -\infty$ and $J'(\lambda) < 0$, for all $\lambda \notin \{\eta_0, \eta_4\}$, we conclude that every (real) zero of J must be simple and that there is a unique zero $\lambda^* \in (\eta_0, \eta_4)$. Moreover, from [38] there is a unique k^* such that $J(0) < 0$ for $k \in (k^*, 1)$ and $J(0) \geq 0$ for $k \in (0, k^*]$ ($k^* \approx 0.909$). Next, since $\overline{\lambda} := \eta_1$ is also a negative eigenvalue related to the linear operator $Q\mathcal{L}_0$ and $\overline{\lambda} \neq \lambda^*$, we can conclude from the previous analysis that

$$n(Q\mathcal{L}_0) = \begin{cases} 2, & \text{for } \ k \in (k^*, 1), \\ 1, & \text{for } \ k \in \left(\frac{\sqrt{2}}{2}, k^*\right). \end{cases}$$

Next, we see that $Ker(Q\mathcal{L}_0) = \text{span}\{\frac{d}{dx}\varphi_c\}$. By Lemma 2.5 in [38] it is sufficient to show that if $\mathcal{L}_0 g = 1$ then $[g] \neq 0$. Suppose that $[g] = 0$. Then, from (4.40) we obtain $\frac{3}{c}[\varphi_c^2 g] = -1$ and so $\eta_i \langle g, \psi_i \rangle = -\frac{3}{c}[\varphi_c^2 g] \int_0^L \psi_i dx = L[\psi_i]$, for $i = 0, 4$. Therefore,

$$0 = \langle g, \psi_0 \rangle [\psi_0] + \langle g, \psi_4 \rangle [\psi_4] = -LJ(0). \tag{4.43}$$

This is a contradiction since $J(0) \neq 0$ for all $k \neq k^*$. This finishes the proof. □

Now, from Theorem 4.11 we have for $c > 1$ the relation $c = L^2(L^2 - 16K^2(k)(2k^2 - 1))^{-1}$ with $k^2 \in \left(\frac{1}{2}, 1\right)$. Thus we guarantee the following result.

Theorem 4.13 *The cnoidal solution φ_c obtained in Theorem 4.11 is linearly unstable for the mBBM equation (4.5), provided that the wave speed $c \in (c^*, +\infty)$, where*

$$c^* = \frac{L^2}{L^2 - 16K(k^*)^2(2k^{*2} - 1)}, \tag{4.44}$$

with $k^* \approx 0.909$. Moreover, the Ω_{φ_c}-orbit is nonlinearly unstable.

Proof From Theorems 4.10 and 4.12 we only need to see that $I(c)$ is strictly negative. Indeed, by using the explicit cnoidal profile of φ_c, Theorem 4.11 and the theory of

Jacobian elliptic functions it is possible to see that (see [38]),

$$\frac{dF(c)}{dc} = \frac{1}{2}\frac{d}{dc}||\varphi_c||^2_{H^1_{per}} > 0 \quad \text{for every} \quad c > 1 \quad (k^2 \in (1/2, 1)). \tag{4.45}$$

Thus φ_c is linearly unstable.

Now, from Theorem 3.4 in [38] we have that the initial value problem associated to (4.5) is globally well-posed in $H^1_{per}([0, L])$. Moreover, since the mapping data-solution associated to the mBBM equation is smooth we obtain from the approach of Henry *et al.* in [82] that the linear instability property of φ_c implies that the Ω_{φ_c}-orbit generated by this profile will be nonlinearly unstable (see Theorem 3.6 in [38]). This finishes the proof. □

Remark 4.14 (1) We can determine numerically that $c^* \approx \frac{L^2}{L^2-56.277}$ in Theorem 4.13 and the minimal period L in Theorem 4.11 must satisfy *a priori* the lower bounded $L^2 > 56.277$.

(2) The results due to Henry et al. in [82] and applied in Theorem 4.13, are a simpler way to get that the *spectral instability* implies *nonlinear instability*. We note that the classical instability approaches in [50,79,80] can not be applied for the generalized BBM or KdV type equations since the linear operator ∂_x in (4.28) is not invertible and the periodic framework induces serious obstacles to apply these techniques (see also Bronski and Johnson [54], Deconinck and Kapitula [66] and Haragus and Kapitula [81], and Remark 1.1 above).

Next, Theorem 4.13 shows that c^* is a threshold value for the stability problem of φ_c, namely, for $c \in (1, c^*)$ they are stable in $\mathcal{X}_{1,0}$. Indeed, from the analysis above we have $Ker(Q\mathcal{L}_0) = \text{span}\{\frac{d}{dx}\varphi_c\}$, $\frac{dF(c)}{dc} > 0$, and $n(Q\mathcal{L}_0) = 1$ for $k \in (\frac{\sqrt{2}}{2}, k^*)$, thus from the general stability framework established in Sect. 2.1 above (by changing $\mathcal{L}$ by $Q\mathcal{L}_0$ in the analysis with $D(Q\mathcal{L}_0) = H^1_{per}([0, L]) \cap \mathbb{V}$, we have the following orbital stability theorem for cnoidal profiles.

Theorem 4.15 *The cnoidal solution φ_c obtained in Theorem 4.11 is orbitally stable in $H^1_{per}([0, L]) \cap \mathbb{V}$ for $c \in (1, c^*)$ by the mBBM equation (4.5).*

Remark 4.16 (1) We note that the quantity $H(u) = \int_0^L u(x)dx$ is a conserved functional by the periodic flow of the mBBM equation. Thus, we have that the initial value problem associated to mBBM is globally well-posed in $H^1_{per}([0, L]) \cap \mathbb{V}$.

(2) From Figs. 5 and 6 we can interpret that the cnoidal waves φ_c will be stable when it represents a "cosine wave-profile" ($k \in (\frac{1}{\sqrt{2}}, k^*)$) and it will be unstable when it represents a "soliton-localized wave-profile" ($k \in (k^*, 1)$). Here, we recall that $cn(\cdot; k) \to sech(\cdot)$ as $k \to 1^+$ and sech represents the classical soliton-profile associated to the mBBM on all the line and that by the way they are orbitally stable.

4.4 End-section notes

(a) The orbital stability of cnoidal-type profile in (4.1) for the KdV model (1.5) in $H^1_{per}([0, L])$ was showed in [24] (see also [34]). The orbital stability of these cnoidal profiles in the case of perturbations that are periodic with period nL, where n is any nonzero positive integer, was proved later by Deconinck and Kapitula [66].

(b) The Poisson summation formula can also be used for obtaining the cnoidal profile in (4.1) and so to study its stability properties (see [19,24,34]). A similar approach of stability can be applied for the case of the dnoidal profiles associated to the modified Korteweg–de Vries equation $u_t + 3u^2u_x + u_{xxx} = 0$ and to the cubic Schrödinger model $iu_t + u_{xx} + |u|^2u = 0$ (see [18,19]).

(c) A study about the existence and stability of periodic traveling waves for the Korteweg–de Vries type model $u_t + 4u^3u_x + u_{xxx} = 0$ is an open question.

(d) The Intermediate Long Wave equation (Kubota et al. [93]) in a periodic framework is given by the model

$$u_t + 2uu_x + \frac{1}{\delta}u_x - \mathcal{T}_\delta(u_{xx}) = 0, \quad \delta > 0, \tag{4.46}$$

where the linear operator $\mathcal{T}_\delta$ is defined by

$$\mathcal{T}_\delta f(x) = \frac{1}{L} P.V. \int_{-L/2}^{L/2} \Gamma_{\delta,L}(x-y) f(y)dy,$$

here $P.V.$ stands for the Cauchy principal value of the integral and

$$\Gamma_{\delta,L}(\zeta) = -i \sum_{n\neq 0} coth\left(\frac{2\pi n\delta}{L}\right) e^{2in\pi\zeta/L}.$$

In Angulo et al. [25] (see also [11]) has been studied the existence and stability (linear and nonlinear) of periodic traveling waves solutions of mean zero in $H^{1/2}_{per}([0, L])$, namely, solutions for the pseudo-differential equation

$$\mathcal{T}_\delta\phi' - c\phi + \phi^2 = A_c \quad \text{with} \quad A_c = \frac{1}{L}\int_0^L \phi^2(x)dx. \tag{4.47}$$

By following Parker's arguments [120] (see also Nakamura and Matsuno [114]) it was obtained the even-periodic profile with mean zero for (4.47)

$$\phi_c(x) = \frac{2K(k)i}{L}\left[Z\left(\frac{2K(k)}{L}(x - i\delta); k\right) - Z\left(\frac{2K(k)}{L}(x + i\delta); k\right)\right], \tag{4.48}$$

where $K(k)$ denotes the complete elliptic integral of the first kind, $Z(\cdot; k)$ is the Jacobi Zeta function and $k \in (0, 1)$ (see [56]). For fixed L and δ, the wave speed c and the elliptic modulus k must satisfy specific restrictions. For the orbital stability theory (conditional), the Angulo and Natali's Theorem 4.3 and the Poisson Summation theorem were used for obtaining the spectral information of $\mathcal{L} = \mathcal{T}_\delta\partial_x - c + 2\phi$.

(e) A similar (in)stability result to that established in Theorem 4.8 for the CNLS equation in (4.10) with regard to the periodic profile (4.19), is also true to the *critical*

Korteweg–de Vries equation (CKDV) (see [34])

$$u_t + 5u^4 u_x + u_{xxx} = 0. \tag{4.49}$$

At this point in the analysis, attention should be drawn to the peculiar behavior of the positive profiles φ_ω in (4.19). Indeed, for $k \to 1^+$ we have the convergence $\varphi_\omega(\xi) \to S(\xi)$ for $S(\xi) = (3\omega)^{1/4} sech^{\frac{1}{2}}(2\sqrt{\omega}\xi)$. S represents the classical solitary wave solution for the critical Korteweg–de Vries equation on all the line. From Martel and Merle [107–109] we know that this profile is nonlinearly unstable. Therefore, this dynamic picture can tell us that the unstable behavior of the soliton S spreads in its vicinity at least to periodic traveling waves within the homoclinic orbit determined by S and vice versa for the case of the periodic profiles φ_ω with $k \approx 1$. It is an open question to know which of the two profiles of traveling waves is inducing an unstable behavior in the other one.

(f) Recently Amaral et al. [12] have determined spectral stability results of periodic waves associated with the CKDV equation (4.49). They obtained the existence of zero mean periodic waves with a cnoidal profile, $\omega \in (\frac{4\pi^2}{L^2}, +\infty) \to \phi_\omega$ (that is, periodic trajectories external to the homoclinic trajectory determined by the soliton-profile S defined in the previous item), such that there is a threshold value for the speed-velocity ω^* for which ϕ_ω is spectrally stable if $\omega \in (\frac{4\pi^2}{L^2}, \omega^*)$ and spectrally unstable if $\omega \in (\omega^*, +\infty)$.
Now, from the local well-posednes theory for (4.49) in [35] and from the proof of Theorem 4.13 we can conclude the nonlinear instability of the cnoidal profiles ϕ_ω. From item (e) above, we have the scenario that the *unstable soliton profile S is surrounded by unstable periodic profiles*.

(g) In the case of the stability of periodic traveling wave solutions for the general model (1.7) such that the profile ϕ satisfies the equation $M\phi + c\phi - \frac{1}{p+1}\phi^{p+1} + A = 0$ where A is a constant of integration not necessarily zero, has been established in Alves et al. [11]. The strategy of the method is to extend the Lyapunov approach given in Sect. 2.1 above. More exactly, for the following conserved quantities associated to (1.7)

$$E(u) = \frac{1}{2}\int_0^L uMu - \frac{1}{(p+1)(p+2)}u^{p+2}dx,$$
$$F(u) = \frac{1}{2}\int_0^L u^2 dx, \;\; M_0(u) = \int_0^L u dx,$$

we consider the new quantities $G(u) = E(u) + cF(u) + AM_0(u)$ and $Q(u) = \mu M_0(u) + \nu F(u)$ (The constants μ, ν are chosen appropriately in order to obtain the assumption (iii) below). Thus $G'(\phi) = 0$ and $G''(\phi) \equiv \mathcal{L} = M + c - \phi^p$ represents the linearization around the profile ϕ with domain $H^{\beta_2}_{per}([0, L])$. In this case, the Lyapunov function associated to the orbit Ω_ϕ is defined by $V(u) = G(u) - G(\phi) + \sigma(Q(u) - Q(\phi))^2$, for some specific $\sigma > 0$, with $V(\phi) = 0$, and so we are looking for an inequality of the form (see (2.6))

$$V(u) - V(\phi) = V(u) \geqq c[d(u;\Omega_\phi)]^2, \quad \text{for } d(u;\Omega_\phi) < \delta. \tag{4.50}$$

The sufficient conditions for obtaining the basic stability-inequality (4.50) are:

(i) The self-adjoint operator $\mathcal{L}$ has only one negative eigenvalue which is simple and zero is a simple eigenvalue whose eigenfunction is ϕ'.
(ii) There exist constants $c_1, c_2 > 0$ such that $\langle \mathcal{L}v, v\rangle \geqq c_1\|v\|^2_{\frac{\beta_2}{2}} - c_2\|v\|^2$,
(iii) There exist constants $c_3 > 0$ such that $\langle \mathcal{L}v, v\rangle \geqq c_3\|v\|^2$, for $v\perp\phi'$ and $v\perp Q'(\phi)$.

This approach has been applied for the case of the gKdV equation ($M = -\partial_x^2$) and the gBBM model (4.24) (see [11]). Moreover, the results do not depend on the parametrization of the periodic wave itself. In Stuart [127] the case of solitary and standing waves solutions have been studied.

(h) The linear instability criterium for the gBBM in Theorem 4.10 can be extended in a similar form to the general models of Korteweg–de Vries type in (1.7) (see [37,38]).

5 The nonlinear Schrödinger equation on star graphs

In this section we study the existence and orbital stability of standing wave solutions for the following vectorial nonlinear Schrödinger equation on a star graph $\mathcal{G}$,

$$i\partial_t \mathbf{U}(x,t) - \mathcal{A}\mathbf{U}(x,t) + \mathbf{F}(\mathbf{U}(x,t)) = 0 \tag{5.1}$$

where $\mathbf{U}(x,t) = (u_j(x,t))_{j=1}^N : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{C}^N$, the nonlinearity $\mathbf{F}(\mathbf{U})$ satisfies $\mathbf{F}(e^{i\theta}\mathbf{U}) = e^{i\theta}\mathbf{F}(\mathbf{U})$, $\theta \in [0, 2\pi)$. The star graph $\mathcal{G}$ will be composed by N positive half-lines attached to the common vertex $\nu = 0$, and $\mathcal{A}$ is a self-adjoint operator such that $D(\mathcal{A}) \subset L^2(\mathcal{G})$, which represents the coupling conditions in the graph-vertex.

The *standing wave solutions* for (5.1) are solutions of the form

$$\mathbf{U}(x,t) = e^{i\omega t}\mathbf{\Phi}(x), \tag{5.2}$$

with the profile $\mathbf{\Phi} \in D(\mathcal{A})$. By substituting this profile in (5.1) we arrive to the nonlinear (vectorial) system

$$\mathcal{A}\mathbf{\Phi} + \omega\mathbf{\Phi} - \mathbf{F}(\mathbf{\Phi}) = 0. \tag{5.3}$$

The equality in (5.3) should be understood in a distributional sense.

In the following subsections we study the so-called NLS-δ model in (5.1), namely, $\mathbf{F}(\mathbf{U}) = |\mathbf{U}|^{p-1}\mathbf{U}$, $p > 1$, $\mathcal{A} \equiv \mathbf{H}_\alpha^\delta$ acting for $\mathbf{V} = (v_j)_{j=1}^N$ as $(\mathbf{H}_\alpha^\delta\mathbf{V})(x) = (-v_j''(x))_{j=1}^N$, $x > 0$, and with domain $D(\mathbf{H}_\alpha^\delta) = D_{\alpha,\delta}(\mathcal{A})$ defined in (1.22) for $\alpha \in \mathbb{R}$. The nonlinearity acts componentwise, i.e. $(|\mathbf{U}|^{p-1}\mathbf{U})_j = |u_j|^{p-1}u_j$. The strategy which will be put forward can be applied to several choices of the self-adjoint operator $\mathcal{A}$ and the nonlinearity $\mathbf{F}$.

We finish this section with some notes and with some very interesting open problems.

5.1 Standing wave for NLS-δ model

We consider the NLS-δ model in (5.1). In Adami and Noja [2] was obtained the following description of all solutions to equation

$$\mathbf{H}_\alpha^\delta \mathbf{\Phi} + \omega \mathbf{\Phi} - |\mathbf{\Phi}|^{p-1}\mathbf{\Phi} = 0, \quad \mathbf{\Phi} \in D(\mathbf{H}_\alpha^\delta). \tag{5.4}$$

Theorem 5.1 *Let $[s]$ denote the integer part of $s \in \mathbb{R}$, and $\alpha \neq 0$. Then Eq. (5.4) has $\left[\frac{N-1}{2}\right] + 1$ (up to permutations of the edges of $\mathcal{G}$) vector solutions $\mathbf{\Phi}_m^\alpha = (\varphi_{m,j}^\alpha)_{j=1}^N$, $m = 0, \dots, \left[\frac{N-1}{2}\right]$, which are given by*

$$\varphi_{m,j}^\alpha(x) = \begin{cases} \left[\frac{(p+1)\omega}{2} \operatorname{sech}^2\left(\frac{(p-1)\sqrt{\omega}}{2}x - a_m\right)\right]^{\frac{1}{p-1}}, & j = 1, \dots, m; \\ \left[\frac{(p+1)\omega}{2} \operatorname{sech}^2\left(\frac{(p-1)\sqrt{\omega}}{2}x + a_m\right)\right]^{\frac{1}{p-1}}, & j = m+1, \dots, N, \end{cases} \tag{5.5}$$

where

$$a_m = \tanh^{-1}\left(\frac{\alpha}{(2m-N)\sqrt{\omega}}\right) \text{ and } \omega > \frac{\alpha^2}{(N-2m)^2}. \tag{5.6}$$

Remark 5.2 (1) Note that in the case $\alpha < 0$, the vector $\mathbf{\Phi}_m^\alpha = (\varphi_{m,j}^\alpha)_{j=1}^N$ has *m bumps* and $N - m$ *tails*. $\mathbf{\Phi}_0^\alpha$ is called the *N-tail profile*. Moreover, the N-tail profile is the only symmetric (i.e. invariant under permutations of the edges) solution of Eq. (5.4). For instance, in the case $N = 5$ we have three types of profiles: *5-tail profile* (Fig. 7), *4-tail/1-bump profile* (Fig. 8) and *3-tail/2-bump profile* (Fig. 9).

(2) In the case $\alpha > 0$, the vector $\mathbf{\Phi}_m^\alpha = (\varphi_{m,j}^\alpha)_{j=1}^N$ has *m tails* and $N - m$ *bumps* respectively. $\mathbf{\Phi}_0^\alpha$ is called the *N-bump profile*. For $N = 5$ we have: *5-bump profile* (Fig. 10), *4-bump/1-tail profile* (Fig. 11), *3-bump/ 2-tail profile* (Fig. 12).

Proof Let $\mathbf{\Phi} = (\varphi_j)_{j=1}^N \in D(\mathbf{H}_\alpha^\delta)$ satisfying the vectorial elliptic system (5.4). Thus every component of $\mathbf{\Phi}$ on every edge must seek $L^2(0, +\infty)$-solution to the equation

$$-\psi'' + \omega\psi - |\psi|^{p-1}\psi = 0, \quad \omega > 0. \tag{5.7}$$

The most general $L^2(0, +\infty)$-solution is $\psi(x) = \sigma\psi_s(x - y)$ with $\sigma \in \mathbb{C}$, $|\sigma| = 1$, $y \in \mathbb{R}$ and

$$\psi_s(x) = \left[\frac{(p+1)\omega}{2}\right]^{\frac{1}{p-1}} \operatorname{sech}^{\frac{2}{p-1}}\left(\frac{p-1}{2}\sqrt{\omega}x\right). \tag{5.8}$$

Therefore, the components φ_j are given by

$$\varphi_j(x) = \sigma_j \psi_s(x - y_j). \tag{5.9}$$

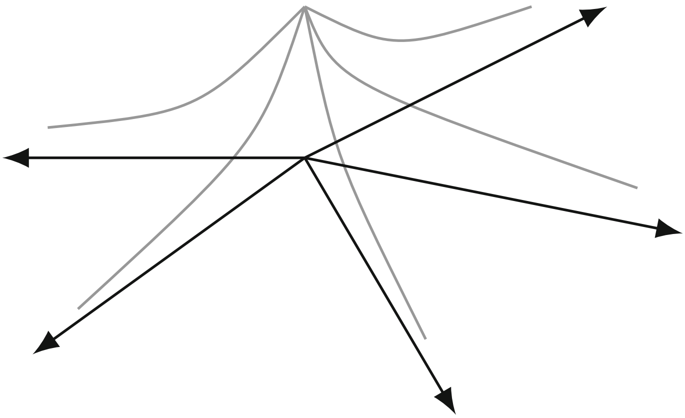

Fig. 7 5-tail profile

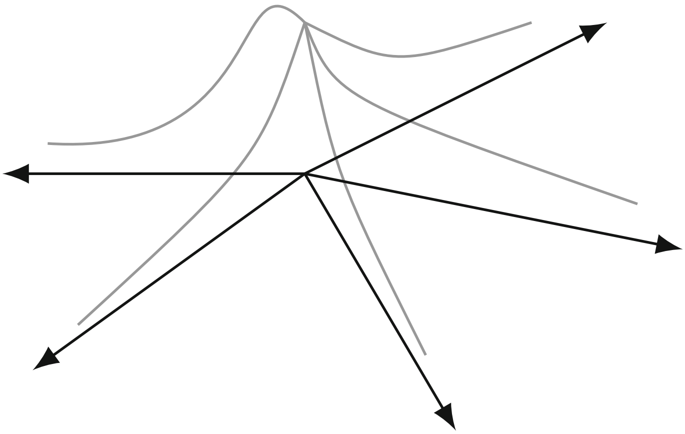

Fig. 8 4-tail/1-bump profile

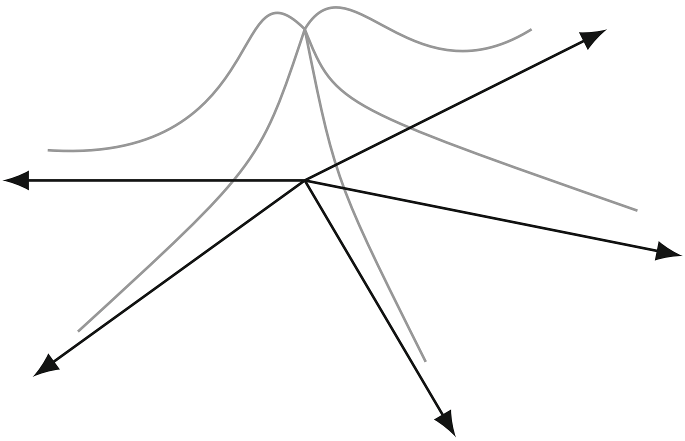

Fig. 9 3-tail/2-bump profile

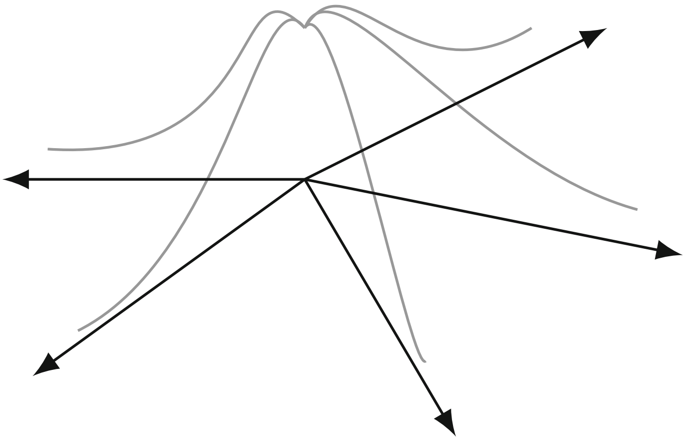

Fig. 10 5-bump profile

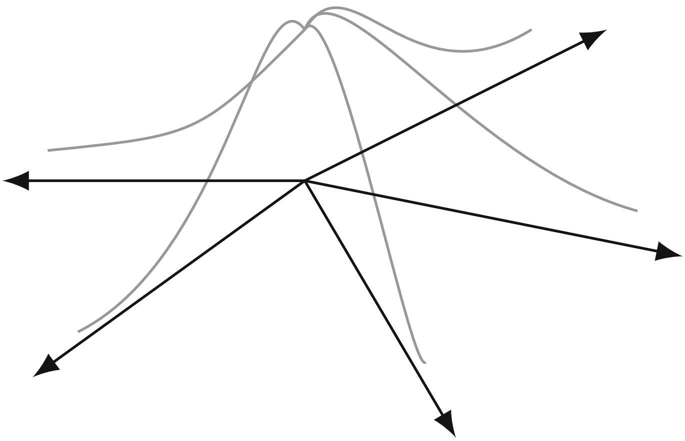

Fig. 11 4-bump/1-tail profile

In order to have a solution for (5.4) it is sufficient to impose boundary conditions in (1.22). The continuity condition in (1.22) implies $\sigma_1 = \cdots = \sigma_N$ and $y_j = \gamma_j a$ with $\gamma_j = \pm 1$ and $a > 0$. We can consider $\sigma_1 = 1$ without losing generality. Now, we determine γ_j. The second boundary condition in (1.22) (Kirchoff type-condition or $delta$-type interaction) rewrites as

$$\tanh\left(\frac{p-1}{2}\sqrt{\omega}a\right)\sum_{j=1}^{N}\gamma_j = \frac{\alpha}{\sqrt{\omega}}. \tag{5.10}$$

Equation (5.10) implies that $\sum_{j=1}^{N}\gamma_j$ must have the same sign of α. Moreover, under a choice of the set $\{\gamma_j\}_{j=1}^{N}$, condition (5.10) fixes uniquely a. Now, by referring to the bell shape of the function ψ_s, we say that in the j-th edge there is a *bump* if $y_j > 0$,

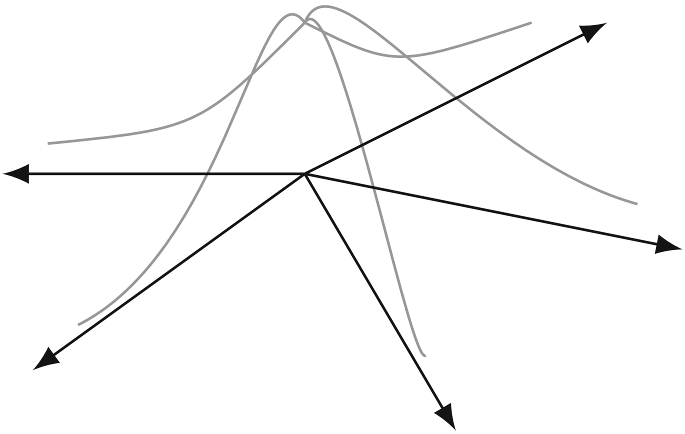

Fig. 12 3-bump/2-tail profile

that is $\gamma_j = 1$; there is a *tail* if $y_j < 0$, that is, if $\gamma_j = -1$. Then we choose to index the solutions by the number j of bumps. Thus, we obtain a unique solution to (5.10) which we call a_j. In this way we arrive at (5.5) and (5.6). This finishes the proof. $\square$

It was shown in [2] that for $-N\sqrt{\omega} < \alpha < \alpha^* < 0$, the vector tail-solution $\mathbf{\Phi}_0^\alpha = (\varphi_{0,j})_{j=1}^N$, with $\varphi_{0,j} = \varphi_{0,\alpha}$ for all j and

$$\varphi_{0,\alpha}(x) = \left[\frac{(p+1)\omega}{2}\operatorname{sech}^2\left(\frac{(p-1)\sqrt{\omega}}{2}x + \tanh^{-1}\left(\frac{-\alpha}{N\sqrt{\omega}}\right)\right)\right]^{\frac{1}{p-1}}, \quad (5.11)$$

it is the ground state associated to (5.4). The parameter α^* guarantees the minimality of the action functional

$$\mathbf{S}_\alpha(\mathbf{V}) = \tfrac{1}{2}||\mathbf{V}'||^2 + \tfrac{\omega}{2}||\mathbf{V}||^2 - \tfrac{1}{p+1}||\mathbf{V}||_{p+1}^{p+1} + \tfrac{\alpha}{2}|v_1(0)|^2, \quad (5.12)$$

for $\mathbf{V} = (v_j)_{j=1}^N \in \mathcal{E}(\mathcal{G}) = \{\mathbf{V} \in H^1(\mathcal{G}) : v_1(0) = \cdots = v_j(0)\}$ at $\mathbf{\Phi}_0^\alpha$ with the constraint given by the Nehari manifold

$$\mathcal{N} = \{\mathbf{V} \in \mathcal{E}(\mathcal{G}) \setminus \{0\} : ||\mathbf{V}'||^2 + \omega||\mathbf{V}||^2 - ||\mathbf{V}||_{p+1}^{p+1} + \alpha|v_1(0)|^2 = 0\}.$$

Note that $\mathbf{\Phi}_m^\alpha \in \mathcal{N}$ for any m. In [2] was also proved that for $m \neq 0$ and $\alpha < 0$ we have $\mathbf{S}_\alpha(\mathbf{\Phi}_0^\alpha) < \mathbf{S}_\alpha(\mathbf{\Phi}_m^\alpha) < \mathbf{S}_\alpha(\mathbf{\Phi}_{m+1}^\alpha)$. This fact justifies the name *excited states* for the stationary states $\mathbf{\Phi}_m^\alpha$, $m \neq 0$. For $\alpha > 0$ and any m nothing is known about variational properties of the profiles $\mathbf{\Phi}_m^\alpha$. In particular, one can easily verify that $\mathbf{S}(\mathbf{\Phi}_0^\alpha) > \mathbf{S}(\mathbf{\Phi}_m^\alpha) > \mathbf{S}(\mathbf{\Phi}_{m+1}^\alpha)$, $m \neq 0$.

We will see in Sect. 5.4 that when the profile $\mathbf{\Phi}_m^\alpha$ has mixed structure (i.e. has bumps and tails, excited states), they are "almost always" nonlinearly unstable.

5.2 Stability framework for the NLS on star graphs

The NLS model (5.1) is invariant under the Galilean invariance determined by the rotation-symmetry of the group $T(\theta)\mathbf{\Psi} = e^{i\theta}\mathbf{\Psi}$, for any $\theta \in [0, 2\pi)$, namely, if $\mathbf{U}$ is solution of (5.1) then $e^{i\theta}\mathbf{U}$ is also solution. Thus, the standing wave solutions in (5.2) can be written as $\mathbf{U}(x,t) = T(\omega t)\mathbf{\Phi}(x)$. We note that the classical translation-symmetry does not hold on $\mathcal{G}$. Thus, we have the following orbital stability definition.

Definition 5.3 The standing wave $\mathbf{U}(x,t) = e^{i\omega t}\mathbf{\Phi}(x)$ for model (5.1) is said to be *orbitally stable* in a Hilbert space $X \subset L^2(\mathcal{G})$ if for any $\varepsilon > 0$ there exists $\eta > 0$ with the following property: if $\mathbf{U}_0 \in X$ satisfies $||\mathbf{U}_0 - \mathbf{\Phi}||_X < \eta$, then the solution $\mathbf{U}(t)$ of (5.1) with $\mathbf{U}(0) = \mathbf{U}_0$ exists for any $t \in \mathbb{R}$ and

$$\sup_{t\in\mathbb{R}} \inf_{\theta\in\mathbb{R}} ||\mathbf{U}(t) - e^{i\theta}\mathbf{\Phi}||_X < \varepsilon.$$

Otherwise, the standing wave $\mathbf{U}(x,t) = e^{i\omega t}\mathbf{\Phi}(x)$ is said to be *orbitally unstable* in X.

In particular, for the NLS-δ model the space X coincides with the continuous energy-space $\mathcal{E}(\mathcal{G})$, where

$$\mathcal{E}(\mathcal{G}) = \left\{(v_j)_{j=1}^N \in H^1(\mathcal{G}) : v_1(0) = \cdots = v_N(0)\right\}. \tag{5.13}$$

Next, we assume the existence of a $C^2(X,\mathbb{R})$-conserved functional $E : X \to \mathbb{R}$ (interpreted as the "energy" in certain applications) i.e., for $\mathbf{V} = (v_j)_{j=1}^N \in X$

$$E(\mathbf{U}(t)) = E(\mathbf{U}_0), \quad \text{for} \;\; t \in [-T, T], \tag{5.14}$$

and $Q : L^2(\mathcal{G}) \to \mathbb{R}$ (interpreted as the "charge" in certain applications) defined by $Q(\mathbf{V}) = ||\mathbf{V}||^2$ it is also a conserved functional, i.e., $Q(\mathbf{U}(t)) = ||\mathbf{U}(t)||^2$, for $t \in [-T, T]$. Moreover, we also assume that E is invariant under the rotation-symmetry T, that is $E(T(\theta)\mathbf{V}) = E(\mathbf{V})$ for $\theta \in [0, 2\pi)$, $\mathbf{V} \in X$ (obviously Q satisfies this property).

Now, let $\Phi \in D(\mathcal{A})$ a distributional solution for (5.3) and suppose that it is a critical point of the action functional $\mathbf{S} = E + \omega Q$. For a stability study of $\mathbf{\Phi}$ a main information will be given by the second variation of $\mathbf{S}$ at $\mathbf{\Phi}$, $\mathbf{S}''(\mathbf{\Phi})$. We suppose that for $\mathbf{U} = \mathbf{U}_1 + i\mathbf{U}_2$ and $\mathbf{V} = \mathbf{V}_1 + i\mathbf{V}_2$, where the vector functions $\mathbf{U}_j, \mathbf{V}_j,\ j \in \{1, 2\}$, are assumed to be real valued, we have the following equality

$$\mathbf{S}''(\mathbf{\Phi})(\mathbf{U},\mathbf{V}) = \langle \mathbf{L}_1\mathbf{U}_1, \mathbf{V}_1\rangle + \langle \mathbf{L}_2\mathbf{U}_2, \mathbf{V}_2\rangle, \tag{5.15}$$

here $\langle\cdot,\cdot\rangle$ represents for us the inner product in $L^2(\mathcal{G})$, and $\mathbf{L}_i$ are self-adjoint operators with $D(\mathbf{L}_i) = D(\mathcal{A}) \subset L^2(\mathcal{G})$. Formally $\mathbf{S}''(\mathbf{\Phi})$ can be considered as a self-adjoint

$2N \times 2N$ matrix operator

$$\mathbf{H} = \begin{pmatrix} \mathbf{L}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{L}_2 \end{pmatrix}. \tag{5.16}$$

Next, we suppose the existence of C^1 in ω standing wave solutions for (5.3), $\omega \in J \subset \mathbb{R} \to \mathbf{\Phi}_\omega$. Define

$$p(\omega_0) = \begin{cases} 1 & \text{if } \partial_\omega ||\mathbf{\Phi}_\omega||^2 > 0 \text{ at } \omega = \omega_0, \\ 0 & \text{if } \partial_\omega ||\mathbf{\Phi}_\omega||^2 < 0 \text{ at } \omega = \omega_0. \end{cases}$$

Lastly, we suppose the well-posedness of the associated Cauchy problem for (5.1) in the energy space X. The next stability/instability result follows from Grillakis et al. [80].

Theorem 5.4 *Let $n(\mathbf{H})$ be the number of negative eigenvalues of $\mathbf{H}$ in (5.16) (the Morse index). Suppose also that*

(1) $Ker(\mathbf{L}_2) = span\{\mathbf{\Phi}_\omega\}$,
(2) $Ker(\mathbf{L}_1) = \{\mathbf{0}\}$,
(3) *the Morse index of $\mathbf{L}_1$ and $\mathbf{L}_2$ consists of a finite number of negative eigenvalues (counting multiplicities),*
(4) *the rest of the spectrum of $\mathbf{L}_1$ and $\mathbf{L}_2$ is positive and bounded away from zero. Then the following assertions hold.*
   (i) *If $n(\mathbf{H}) = p(\omega) = 1$, then the standing wave $e^{i\omega t}\mathbf{\Phi}_\omega$ is orbitally stable in the energy space X.*
   (ii) *If $n(\mathbf{H}) - p(\omega)$ is odd, then the standing wave $e^{i\omega t}\mathbf{\Phi}_\omega$ is orbitally unstable in the energy space X.*

Remark 5.5 The instability part of the above theorem needs some additional comments.

(1) It is known from [80] that when $n(\mathbf{H}) - p(\omega)$ is odd, we obtain only spectral instability of $e^{i\omega t}\mathbf{\Phi}_\omega$. Namely, that the spectrum of the linear operator

$$A \equiv \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \mathbf{H} = \begin{pmatrix} \mathbf{0} & \mathbf{L}_2 \\ -\mathbf{L}_1 & \mathbf{0} \end{pmatrix}$$

associated to the linearization of the time-dependent NLS model around $\mathbf{\Phi}_\omega$, contains an eigenvalue with positive real part. To conclude orbital instability due to [80], it is sufficient to show estimate (6.2) in [80] for the semigroup e^{tA} generated by A. In the case of Schrödinger models on star graphs it is not clear how to prove that estimate (6.2). However, we conjecture that for the operator A we have the spectral mapping theorem (that is, $\sigma(e^A) = e^{\sigma(A)}$), which would imply estimate (6.2) in [80].
(2) When $n(\mathbf{H}) = 2$ (which usually happens in many applications), we can apply the results in [119] to get the instability part of the above Theorem. We note that in this case the orbital instability follows without using spectral instability.
(3) Generally, to imply the orbital instability from the spectral one, the approach by Henry et al. [82] can be used . The key point of this method, is to use that in many

cases we have that the mapping data-solution associated to the model is of class C^2. We note in particular, for the NLS-δ model in (5.1)–(1.22), that the mapping data-solution is of class C^2 as $p > 2$ (see Theorem 5.6 below). The approach by [82] has been applied successfully in [33] for systems of KdV-type or in [38] for models of KdV-type (1.7) and of BBM-type (4.24) .

5.3 Global well-posedness for the NLS-δ on star graphs

Next we establish a local and global well-posedness theory on a star graph $\mathcal{G}$ of the initial value problem for the NLS-δ equation (5.1) in the space

$$\mathcal{E}(\mathcal{G}) = \left\{(v_j)_{j=1}^N \in H^1(\mathcal{G}) : v_1(0) = \cdots = v_N(0)\right\},$$

with $\mathbf{F}(\mathbf{U}) = |\mathbf{U}|^{p-1}\mathbf{U}$, $p > 1$, $\mathcal{A} = \mathbf{H}_\delta^\alpha$ and $D(\mathbf{H}_\delta^\alpha)$ being defined in (1.22). We note that $\mathcal{E}(\mathcal{G})$ emerges naturally as being the energy space associated to the NLS-δ equation. Thus by the Stone's Theorem we have that the linear flow,

$$\begin{cases} \frac{d}{dt}\mathbf{U}(t) = -i\mathbf{H}_\delta^\alpha \mathbf{U}(t) \\ \mathbf{U}(0) = \mathbf{U}_0 \in D(\mathbf{H}_\delta^\alpha) \end{cases} \tag{5.17}$$

is determined by the unitary group on $L^2(\mathcal{G})$, $W(t) = e^{-it\mathcal{A}}$. To determine an explicit formulation for the group $\{W(t)\}_{t\in\mathbb{R}}$ is not an easy task, because on a star graph we do not have the useful tools of Fourier analysis (Fourier transform), thus we need to use an abstract approach based on the functional calculus of operators (see [30]). Moreover, the boundary conditions on the vertex $\nu = 0$ will produce different behavior of the group in a general framework. For more general coupling conditions on a star graph such as that given by Nevanlinna pairs, the local well-posedness theories in either $L^2(\mathcal{G})$ or in the energy space can be seen in Theorem B and Theorem C of [78] (see End-section notes 5.5 below).

The following local well-possedness result in $\mathcal{E}(\mathcal{G})$ follows from standard arguments of the Banach fixed point theorem applied to non-linear Schrödinger equations (see [30,59]).

Theorem 5.6 *Let $p > 1$. Then for any $\mathbf{U}_0 \in \mathcal{E}(\mathcal{G})$ there exists $T > 0$ such that the NLS-δ model has a unique solution $\mathbf{U} \in C([-T,T], \mathcal{E}(\mathcal{G})) \cap C^1([-T,T], \mathcal{E}'(\mathcal{G}))$ satisfying $\mathbf{U}(0) = \mathbf{U}_0$. For each $T_0 \in (0,T)$ the mapping $\mathbf{U}_0 \in \mathcal{E}(\mathcal{G}) \to \mathbf{U} \in C([-T_0,T_0], \mathcal{E}(\mathcal{G}))$, is continuous. In particular, for $p > 2$ this mapping is at least of class C^2. Moreover, for $m \in \mathbb{N}$,*

$$L^2_m(\mathcal{G}) \equiv \{\mathbf{V} \in L^2(\mathcal{G}) : v_1(x) = \cdots = v_m(x), v_{m+1}(x) = \cdots = v_N(x), x > 0\}, \tag{5.18}$$

and $\mathcal{E}_m(\mathcal{G}) = \mathcal{E}(\mathcal{G}) \cap L^2_m(\mathcal{G})$, we have for $\mathbf{U}_0 \in \mathcal{E}_m(\mathcal{G})$ that $\mathbf{U}(t) \in \mathcal{E}_m(\mathcal{G})$ for all $t \in [-T,T]$.

The following global well-posedness result for the NLS-δ model is an immediate consequence of Theorem 5.6 and the existence of the conservation of charge and energy, i.e., for $\mathbf{V} = (v_j)_{j=1}^N \in \mathcal{E}(\mathcal{G})$ the quantities

$$E_\alpha(\mathbf{V}) = \tfrac{1}{2}||\mathbf{V}'||^2 - \tfrac{1}{p+1}||\mathbf{V}||_{L^{p+1}}^{p+1} + \tfrac{\alpha}{2}\,|v_1(0)|^2\,,$$

and

$$Q(\mathbf{U}(t)) = ||\mathbf{U}(t)||^2,$$

satisfy $Q(\mathbf{U}(t)) = ||\mathbf{U}_0||^2$ and $E_\alpha(\mathbf{U}(t)) = E_\alpha(\mathbf{U}_0)$, for $t \in [-T, T]$.

Theorem 5.7 *Let* $1 < p < 5$. *Then for any* $\mathbf{U}_0 \in \mathcal{E}(\mathcal{G})$, *the NLS-*$\delta$ *model has a unique global solution* $\mathbf{U} \in C(\mathbb{R}, \mathcal{E}(\mathcal{G})) \cap C^1(\mathbb{R}, \mathcal{E}'(\mathcal{G}))$ *satisfying* $\mathbf{U}(0) = \mathbf{U}_0$. *Similarly for* $\mathbf{U}_0 \in \mathcal{E}_k(\mathcal{G})$.

Remark 5.8 (1) Using the Sobolev embedding theorem and Gagliardo–Nirenberg inequality on star graphs, one can see that $E_\alpha : \mathcal{E}(\mathcal{G}) \to \mathbb{R}$ is well defined and $E_\alpha \in C^2(\mathcal{E}(\mathcal{G}), \mathbb{R})$ since $p > 1$.

(2) The property of the data-solution mapping to be of class C^2 for $p > 2$, it will deep-tool for showing that the linear instability property of standing wave solutions for the NLS-δ equation (and other models) will be indeed nonlinear instability (see Theorems 5.9 and 5.10 and Remarks 5.5 and 5.21).

5.4 Stability theory for the NLS-δ on star graphs

In this subsection we study the orbital stability of the standing wave $\mathbf{U}(x, t) = e^{i\omega t}\mathbf{\Phi}(x)$ of the Schrödinger model (5.1) for the case of the NLS-δ model (1.22), with the profile $\mathbf{\Phi}_m^\alpha$, $m = 0, \dots, \left[\frac{N-1}{2}\right]$ determined in Theorem 5.1. We will investigate orbital stability in the energy space $X = \mathcal{E}(\mathcal{G})$ defined in (5.13). Thus the functional $E_\alpha : \mathcal{E}(\mathcal{G}) \to \mathbb{R}$ defined for $\mathbf{V} = (v_j)_{j=1}^N \in \mathcal{E}(\mathcal{G})$ by

$$E_\alpha(\mathbf{V}) = \tfrac{1}{2}||\mathbf{V}'||^2 - \tfrac{1}{p+1}||\mathbf{V}||_{L^{p+1}}^{p+1} + \tfrac{\alpha}{2}\,|v_1(0)|^2\,, \tag{5.19}$$

is well defined by the Sobolev embedding theorem and Gagliardo–Nirenberg inequality. Moreover, by using Theorem 5.6 (continuous dependence property) it follows that E_α and $Q(\mathbf{V}) = \|\mathbf{V}\|^2$ are conservation laws for the NLS-δ. Moreover, for the action $\mathbf{S}_\alpha = E_\alpha + \omega Q$ follows from (5.4) the critical point property of $\mathbf{\Phi}_m^\alpha$, $\mathbf{S}'_\alpha(\mathbf{\Phi}_m^\alpha) = 0$, for any $\alpha \neq 0$ and $m = 0, \dots, \left[\frac{N-1}{2}\right]$. Also, for $\mathbf{\Phi}_m^\alpha = (\varphi_{m,j})_{j=1}^N$, we consider the following two self-adjoint diagonal matrix operators

$$\mathbf{L}_{1,m,\alpha} = \left(\left(-\frac{d^2}{dx^2} + \omega - p(\varphi_{m,j})^{p-1}\right)\delta_{i,j}\right),$$
$$\mathbf{L}_{2,m,\alpha} = \left(\left(-\frac{d^2}{dx^2} + \omega - (\varphi_{m,j})^{p-1}\right)\delta_{i,j}\right),$$

$$D(\mathbf{L}_{1,m,\alpha}) = D(\mathbf{L}_{2,m,\alpha}) = D_{\alpha,\delta}, \tag{5.20}$$

where $D_{\alpha,\delta} = D(\mathbf{H}^{\alpha}_{\delta})$ is defined in (1.22) and $\delta_{i,j}$ is the Kronecker symbol. The operators $\mathbf{L}_{i,m,\alpha}$ are associated with the second variation $\mathbf{S}''_{\alpha}(\mathbf{\Phi}^{\alpha}_{m})$ and satisfy the relation in (5.15).

We will see below that when the profile $\mathbf{\Phi}^{\alpha}_{m}$ has mixed structure (i.e. excited states), they are "almost always" unstable. More exactly, for the space $\mathcal{E}_m(\mathcal{G}) = \mathcal{E}(\mathcal{G}) \cap L^2_m(\mathcal{G})$, with $L^2_m(\mathcal{G})$ defined in Theorem 5.6, we obtain the following orbital stability/instability result of the excited states. The case of tail and bump profiles ($m = 0$) will be established separately.

Theorem 5.9 *Let $\alpha \neq 0$, $m \in \left\{1, \dots, \left[\frac{N-1}{2}\right]\right\}$ and $\omega > \frac{\alpha^2}{(N-2m)^2}$. Let also the profile $\mathbf{\Phi}^{\alpha}_{m}$ be defined by (5.5), we consider the spaces $\mathcal{E} = \mathcal{E}(\mathcal{G})$ and $\mathcal{E}_m = \mathcal{E}_m(\mathcal{G})$. Then the following assertions hold.*

(i) *Let $\alpha < 0$, then*

  (1) *for $1 < p \leq 5$ the standing wave $e^{i\omega t}\mathbf{\Phi}^{\alpha}_{m}$ is orbitally unstable in $\mathcal{E}$;*
  (2) *for $p > 5$ there exists $\omega^*_m > \frac{\alpha^2}{(N-2m)^2}$ such that the standing wave $e^{i\omega t}\mathbf{\Phi}^{\alpha}_{m}$ is orbitally unstable in $\mathcal{E}$ with $\omega \in (\frac{\alpha^2}{(N-2m)^2}, \omega^*_m)$.*

(ii) *Let $\alpha > 0$, then*

  (1) *for $1 < p \leq 3$ the standing wave $e^{i\omega t}\mathbf{\Phi}^{\alpha}_{m}$ is orbitally stable in $\mathcal{E}_m$;*
  (2) *for $3 < p < 5$ there exists $\hat{\omega}_m > \frac{\alpha^2}{(N-2m)^2}$ such that the standing wave $e^{i\omega t}\mathbf{\Phi}^{\alpha}_{m}$ is orbitally unstable in $\mathcal{E}$ with $\omega \in (\frac{\alpha^2}{(N-2m)^2}, \hat{\omega}_m)$, and $e^{i\omega t}\mathbf{\Phi}^{\alpha}_{m}$ is orbitally stable in $\mathcal{E}_m$ with $\omega \in (\hat{\omega}_m, \infty)$;*
  (3) *for $p \geq 5$ the standing wave $e^{i\omega t}\mathbf{\Phi}^{\alpha}_{m}$ is orbitally unstable in $\mathcal{E}$.*

In the case of $p > 5$, $\alpha < 0$ and $\omega > \omega^*_m$ our approach does not provide any information about the stability of the excited states $\mathbf{\Phi}^{\alpha}_{m}$. The proof of Theorem 5.9 is based on the extension theory of symmetric operators, the analytic perturbations theory and on Weinstein–Grillakis–Shatah–Strauss approach established in Theorem 5.4.

Next we establish the results of stability for the cases of tail and bump profiles.

Theorem 5.10 *Let $\alpha \neq 0$ and $\mathbf{\Phi}^{\alpha}_{0}$ be defined by (5.11).*

(1) ***Bump case**: Let $\alpha > 0$, $1 < p < 5$ and $\omega > \frac{\alpha^2}{N^2}$. Then the following assertions hold.*

  (i) *If $1 < p \leq 3$, then $e^{i\omega t}\mathbf{\Phi}^{\alpha}_{0}$ is orbitally unstable in $\mathcal{E}(\mathcal{G})$.*
  (ii) *If $3 < p < 5$, then there exists $\omega_2 > \frac{\alpha^2}{N^2}$ such that $e^{i\omega t}\mathbf{\Phi}^{\alpha}_{0}$ is orbitally unstable in $\mathcal{E}(\mathcal{G})$ for $\omega > \omega_2$.*

(2) ***Tail case**: Let $\alpha < 0$ and $\omega > \frac{\alpha^2}{N^2}$. Then the following assertions hold.*

  (i) *If $1 < p \leq 5$, then $e^{i\omega t}\mathbf{\Phi}^{\alpha}_{0}$ is orbitally stable in $\mathcal{E}(\mathcal{G})$.*

(ii) *If* $p > 5$*, then there exists* $\omega_1 > \frac{\alpha^2}{N^2}$ *such that* $e^{i\omega t}\boldsymbol{\Phi}_0^\alpha$ *is orbitally stable in* $\mathcal{E}(\mathcal{G})$ *for* $\omega < \omega_1$*, and* $e^{i\omega t}\boldsymbol{\Phi}_0^\alpha$ *is orbitally unstable in* $\mathcal{E}(\mathcal{G})$ *for* $\omega > \omega_1$*.*

In the bump case, we have that for the conditions $p \geq 5$ and $3 < p < 5$ (with $\omega \in (\frac{\alpha^2}{N^2}, \omega_2)$), our method does not provide any information about orbital stability of $e^{i\omega t}\boldsymbol{\Phi}_0^\alpha$.

The proof of Theorems 5.9 and 5.10 will be developed in the following subsections. We start with the conditions in Theorem 5.4.

5.4.1 Kernel of operators $\mathbf{L}_{i,m,\alpha}$, $i = 1, 2$, in (5.20)

Let the profile $\boldsymbol{\Phi}_m \equiv \boldsymbol{\Phi}_m^\alpha$ be defined by (5.5), including the case $m = 0$ (tail and bump profiles), and we consider the domain $D_{\alpha,\delta}$ defined in (1.22).

Proposition 5.11 *Let* $\alpha \neq 0$*,* $m \in \left\{0, 1, \ldots, \left[\frac{N-1}{2}\right]\right\}$ *and* $\omega > \frac{\alpha^2}{(N-2m)^2}$*. Then the following assertions hold for* $\mathbf{L}_{i,\alpha} = \mathbf{L}_{i,m,\alpha}$*.*

(i) $Ker(\mathbf{L}_{2,\alpha}) = span\{\boldsymbol{\Phi}_m\}$ *and* $\mathbf{L}_{2,\alpha} \geq 0$*.*
(ii) $Ker(\mathbf{L}_{1,\alpha}) = \{\mathbf{0}\}$*.*
(iii) *The positive part of the spectrum of the operators* $\mathbf{L}_{i,\alpha}$*,* $i = 1, 2$*, is bounded away from zero.*

Proof (i) It is clear that $\boldsymbol{\Phi}_m = (\varphi_{m,j})_{j=1}^N \in Ker(\mathbf{L}_{2,\alpha})$. To show the equality $Ker(\mathbf{L}_{2,\alpha}) = \text{span}\{\boldsymbol{\Phi}_m\}$ let us note that any $\mathbf{V} = (v_j)_{j=1}^N \in H^2(\mathcal{G})$ satisfies the following identity

$$-v_j'' + \omega v_j - \varphi_{m,j}^{p-1} v_j = \frac{-1}{\varphi_{m,j}} \frac{d}{dx}\left[\varphi_{m,j}^2 \frac{d}{dx}\left(\frac{v_j}{\varphi_{m,j}}\right)\right], \quad x > 0.$$

Thus, for $\mathbf{V} \in D_{\alpha,\delta}$ we obtain

$$\langle \mathbf{L}_{2,\alpha}\mathbf{V}, \mathbf{V}\rangle = \sum_{j=1}^N \int_0^\infty (\varphi_{m,j})^2 \left|\frac{d}{dx}\left(\frac{v_j}{\varphi_{m,j}}\right)\right|^2 dx + \sum_{j=1}^N \left[v_j'(0)v_j(0) - |v_j(0)|^2 \frac{\varphi_{m,j}'(0)}{\varphi_{m,j}(0)}\right].$$

Using boundary conditions (1.22), we get

$$\sum_{j=1}^N \left[v_j'(0)v_j(0) - |v_j(0)|^2 \frac{\varphi_{m,j}'(0)}{\varphi_{m,j}(0)}\right] = \alpha|v_1(0)|^2 + \sqrt{\omega}|v_1(0)|^2(N-2m)\frac{\alpha}{(2m-N)\sqrt{\omega}} = 0,$$

which induces $\langle \mathbf{L}_{2,\alpha}\mathbf{V}, \mathbf{V}\rangle \geqq 0$. Moreover, since $\langle \mathbf{L}_{2,\alpha}\mathbf{V}, \mathbf{V}\rangle = 0$ if and only if $\mathbf{V} = c\boldsymbol{\Phi}_m$ we obtain immediate that $Ker(\mathbf{L}_{2,\alpha}) = \text{span}\{\boldsymbol{\Phi}_m\}$ and $\mathbf{L}_{2,\alpha} \geqq 0$.

(ii) Concerning the kernel of $\mathbf{L}_{1,\alpha}$, the only $L^2(\mathbb{R}_+)$-solution of the equation $-v_j'' + \omega v_j - p\varphi_{m,j}^{p-1} v_j = 0$ is $v_j = \varphi_{m,j}'$ up to a factor (see [48]). Thus, any element of $Ker(\mathbf{L}_{1,\alpha})$ has the form $\mathbf{V} = (v_j)_{j=1}^N = (c_j\varphi_{m,j}')_{j=1}^N$, $c_j \in \mathbb{R}$. Continuity condition $v_1(0) = \cdots = v_N(0)$ induces that $c_1 = \cdots = c_N$, i.e.

$$v_j(x) = c \begin{cases} -\varphi'_{m,j}, & j = 1, \ldots, m; \\ \varphi'_{m,j}, & j = m+1, \ldots, N \end{cases}, \quad c \in \mathbb{R}.$$

Condition $\sum_{j=1}^N v'_j(0) = \alpha v_j(0)$ is equivalent to the equality

$$c\left(\frac{\omega(1-p)}{2} + \frac{p-1}{2}\frac{\alpha^2}{(N-2m)^2}\right) = 0.$$

The last equality induces that either $\omega = \frac{\alpha^2}{(N-2m)^2}$ (which is impossible) or $c = 0$, and therefore $\mathbf{V} \equiv \mathbf{0}$.

(iii) By Weyl's theorem (see [122]) the essential spectrum of $\mathbf{L}_{i,\alpha}$ coincides with $[\omega, \infty)$. Thus, there can be only finitely many isolated eigenvalues in $(-\infty, \omega')$ for any $\omega' < \omega$. Then (iii) follows easily.

□

5.4.2 Morse index for $\mathbf{L}_{1,m,\alpha}$ in (5.20) with $m \neq 0$

Let $L^2_m(\mathcal{G})$ be defined in (5.18) and consider the matrix operator $\mathbf{H}$ defined in (5.16) associated with operators $\mathbf{L}_{i,m,\alpha} \equiv \mathbf{L}_{i,\alpha}$ in (5.20). The main theorem of this subsection is the following.

Theorem 5.12 *Let $\alpha \neq 0$, $m \in \left\{1, \ldots, \left[\frac{N-1}{2}\right]\right\}$ and $\omega > \frac{\alpha^2}{(N-2m)^2}$. Then the following assertions hold.*

(i) *If $\alpha < 0$, then $n(\mathbf{H}) = 2$ in $L^2_m(\mathcal{G})$, i.e. $n(\mathbf{H}|_{L^2_m(\mathcal{G})}) = 2$.*
(ii) *If $\alpha > 0$, then $n(\mathbf{H}) = 1$ in $L^2_m(\mathcal{G})$,i.e. $n(\mathbf{H}|_{L^2_m(\mathcal{G})}) = 1$.*

The proof of Theorem 5.12 will be based on the perturbation analytic theory and the extension theory of symmetric operators. For this purpose let us define the following self-adjoint matrix Schrödinger operator on $L^2(\mathcal{G})$ with Kirchhoff condition at $\nu = 0$

$$\mathbf{L}^0_1 = \left(\left(-\frac{d^2}{dx^2} + \omega - p\varphi_0^{p-1}\right)\delta_{i,j}\right),$$
$$D(\mathbf{L}^0_1) = \left\{\mathbf{V} \in H^2(\mathcal{G}) : v_1(0) = \cdots = v_N(0),\ \sum_{j=1}^N v'_j(0) = 0\right\}, \tag{5.21}$$

where φ_0 represents the half-soliton solution for the classical NLS model,

$$\varphi_0(x) = \left[\frac{(p+1)\omega}{2}\,\text{sech}^2\left(\frac{(p-1)\sqrt{\omega}}{2}x\right)\right]^{\frac{1}{p-1}}. \tag{5.22}$$

Next, from definition of the profiles $\mathbf{\Phi}_m$ in (5.5) it follows $\mathbf{\Phi}_m = \mathbf{\Phi}^\alpha_m \to \mathbf{\Phi}_0$, as $\alpha \to 0$, on $H^1(\mathcal{G})$, where $\mathbf{\Phi}_0 = (\varphi_0, \ldots, \varphi_0)$. As we intend to study negative spectrum of $\mathbf{L}_{1,\alpha}$, we first need to describe spectral properties of $\mathbf{L}_1^0$ (which is "limit value" of $\mathbf{L}_{1,\alpha}$ as $\alpha \to 0$).

Theorem 5.13 *Let $\mathbf{L}_1^0$ be defined by (5.21) and $m \in \left\{1, \dots, \left[\frac{N-1}{2}\right]\right\}$. Then*

(i) *$Ker(\mathbf{L}_1^0) = span\{\hat{\mathbf{\Phi}}_{0,1}, \dots, \hat{\mathbf{\Phi}}_{0,N-1}\}$, where $\hat{\mathbf{\Phi}}_{0,j} = (0, \dots, 0, \underset{\mathbf{j}}{\varphi_0'}, \underset{\mathbf{j+1}}{-\varphi_0'}, 0, \dots, 0)$.*

(ii) *In the space $L_m^2(\mathcal{G})$ we have $Ker(\mathbf{L}_1^0) = span\{\widetilde{\mathbf{\Phi}}_{0,m}\}$, i.e. for any m $Ker(\mathbf{L}_1^0|_{L_m^2(\mathcal{G})}) = span\{\widetilde{\mathbf{\Phi}}_{0,m}\}$, where*

$$\widetilde{\mathbf{\Phi}}_{0,m} = \left(\underset{\mathbf{1}}{\tfrac{N-m}{m}\varphi_0'}, \dots, \underset{\mathbf{m}}{\tfrac{N-m}{m}\varphi_0'}, \underset{\mathbf{m+1}}{-\varphi_0'}, \dots, \underset{\mathbf{N}}{-\varphi_0'}\right). \tag{5.23}$$

(iii) *The operator $\mathbf{L}_1^0$ has one simple negative eigenvalue in $L^2(\mathcal{G})$, i.e. $n(\mathbf{L}_1^0) = 1$. Moreover, $\mathbf{L}_1^0$ has one simple negative eigenvalue in $L_m^2(\mathcal{G})$ for any m, i.e. $n(\mathbf{L}_1^0|_{L_m^2(\mathcal{G})}) = 1$.*

(iv) *The positive part of the spectrum of $\mathbf{L}_1^0$ is bounded away from zero.*

Proof The proof repeats the one of Theorem 3.6 in [30]. We give here the highlight points of the analysis.

(i) Any element of $Ker(\mathbf{L}_1^0)$ has the form $\mathbf{V} = (v_j)_{j=1}^N = (c_j\varphi_0')_{j=1}^N$, $c_j \in \mathbb{R}$. Condition $\sum_{j=1}^N v_j'(0) = 0$ gives rise to $(N-1)$-dimensional kernel of $\mathbf{L}_1^0$. It is obvious that functions $\hat{\mathbf{\Phi}}_{0,j}$, $j = 1, \dots, N-1$ form basis there.

(ii) Arguing as in the previous item, we can see that $Ker(\mathbf{L}_1^0)$ is one-dimensional in $L_m^2(\mathcal{G})$, and it is spanned on $\widetilde{\mathbf{\Phi}}_{0,m}$.

(iii) Consider the symmetric operator $(\mathbf{L}_0^0, D(\mathbf{L}_0^0))$ with $\mathbf{L}_0^0 = \left(\left(-\frac{d^2}{dx^2} + \omega - p\varphi_0^{p-1}\right)\delta_{i,j}\right)$, and

$$D(\mathbf{L}_0^0) = \left\{\mathbf{V} \in H^2(\mathcal{G}) : v_1(0) = \dots = v_N(0) = 0, \ \sum_{j=1}^N v_j'(0) = 0\right\},$$

where deficiency indices satisfy $n_\pm(\mathbf{L}_0^0) = 1$. Moreover, $(\mathbf{L}_1^0, D(\mathbf{L}_1^0))$ in (5.21) belongs to the one-parameter family of self-adjoint extension of the symmetric operator $(\mathbf{L}_0^0, D(\mathbf{L}_0^0))$. Next, we show that operator $\mathbf{L}_0^0$ is non-negative on $D(\mathbf{L}_0^0)$. First, note that every component of the vector $\mathbf{V} = (v_j)_{j=1}^N \in H^2(\mathcal{G})$ satisfies the following identity

$$-v_j'' + \omega v_j - p\varphi_0^{p-1}v_j = \frac{-1}{\varphi_0'}\frac{d}{dx}\left[(\varphi_0')^2\frac{d}{dx}\left(\frac{v_j}{\varphi_0'}\right)\right], \quad x > 0.$$

Using the above equality and integrating by parts, we get for $\mathbf{V} \in D(\mathbf{L}_0^0)$

$$\langle \mathbf{L}_0^0\mathbf{V}, \mathbf{V}\rangle = \sum_{j=1}^N \int_0^\infty (\varphi_0')^2 \left|\frac{d}{dx}\left(\frac{v_j}{\varphi_0'}\right)\right|^2 dx + \sum_{j=1}^N \left[-v_j'v_j + v_j^2\frac{\varphi_0''}{\varphi_0'}\right]_0^\infty \geq 0,$$

where the non-integral term becomes zero by the boundary conditions for $\mathbf{V}$ and from the fact that $x = 0$ is the first-order zero for φ_0' (i.e. $\varphi_0''(0) \neq 0$). Indeed,

$$\sum_{j=1}^{N}\left[-v_j'v_j + v_j^2\frac{\varphi_0''}{\varphi_0'}\right]_0^\infty = -\sum_{j=1}^{N}\lim_{x\to 0+}\frac{2v_j(x)v_j'(x)\varphi_0''(x)+v_j^2(x)\varphi_0'''(x)}{\varphi_0''(x)} = 0.$$

Since $n_\pm(\mathbf{L}_0^0) = 1$, by Proposition 7.3 in "Appendix", it follows that $n(\mathbf{L}_1^0) \leq 1$. Taking into account that $\langle \mathbf{L}_1^0\mathbf{\Phi}_0, \mathbf{\Phi}_0\rangle = -(p-1)||\mathbf{\Phi}_0||_{L^{p+1}}^{p+1} < 0$, we arrive via the mini–max principle at $n(\mathbf{L}_1^0) = 1$. Finally, since $\mathbf{\Phi}_0 \in L_m^2(\mathcal{G})$ for any m, we have $n(\mathbf{L}_1^0|_{L_m^2(\mathcal{G})}) = 1$. Item (iv) follows from Weyl's theorem.

□

Remark 5.14 Observe that, when we deal with deficiency indices, the operator $\mathbf{L}_0^0$ is assumed to act on complex-valued functions which however does not affect the analysis of negative spectrum of $\mathbf{L}_1^0$ acting on real-valued functions.

Theorem 5.13 gives us a good framework for applying tools from analytic perturbation theory on space $L_m^2(\mathcal{G})$ for operator $\mathbf{L}_1^0$, and so the main point will be to determine which is the direction that the simple eigenvalue zero for $\mathbf{L}_1^0$ will jump, to the right or to the left (we recall from Proposition 5.11 that $Ker(\mathbf{L}_{1,\alpha})$ is trivial for any $\alpha \neq 0$).

We start our analytic perturbation theory framework with another characterization to the self-adjoint operators in (5.20). Indeed, for $\mathbf{U}, \mathbf{V} \in \mathcal{E}$ written like $\mathbf{U} = \mathbf{U}_1 + i\mathbf{U}_2$ and $\mathbf{V} = \mathbf{V}_1 + i\mathbf{V}_2$, it is easy seen that $S''(\mathbf{\Phi}_m)(\mathbf{U}, \mathbf{V})$ can be formally rewritten as

$$S''(\mathbf{\Phi}_m)(\mathbf{U}, \mathbf{V}) = B_{1,m}^\alpha(\mathbf{U}_1, \mathbf{V}_1) + B_{2,m}^\alpha(\mathbf{U}_2, \mathbf{V}_2). \tag{5.24}$$

Here the bilinear forms $B_{1,m}^\alpha$ and $B_{2,m}^\alpha$ are defined for $\mathbf{F} = (f_j)_{j=1}^N$, $\mathbf{G} = (g_j)_{j=1}^N \in \mathcal{E}$ by

$$\begin{aligned} B_{1,m}^\alpha(\mathbf{F}, \mathbf{G}) &= \sum_{j=1}^{N}\int_0^\infty \left(f_j'g_j' + \omega f_j g_j - p(\varphi_{m,j})^{p-1}f_j g_j\right)dx + \alpha f_1(0)g_1(0),\\ B_{2,m}^\alpha(\mathbf{F}, \mathbf{G}) &= \sum_{j=1}^{N}\int_0^\infty \left(f_j'g_j' + \omega f_j g_j - (\varphi_{m,j})^{p-1}f_j g_j\right)dx + \alpha f_1(0)g_1(0). \end{aligned} \tag{5.25}$$

Next, we determine the self-adjoint operators associated with the forms $B_{j,m}^\alpha$ in order to establish a self-contained analysis. First note that the forms $B_{j,m}^\alpha$, $j \in \{1, 2\}$, are bilinear bounded from below and closed. Thus, it appears self-adjoint operators $\mathbb{L}_{1,m,\alpha}$ and $\mathbb{L}_{2,m,\alpha}$ associated (uniquely) with $B_{1,m}^\alpha$ and $B_{2,m}^\alpha$ by the First Representation Theorem (see Kato [88]), namely,

$$\mathbb{L}_{j,m,\alpha}\mathbf{V} = \mathbf{W}, \qquad j \in \{1, 2\},$$

$$D(\mathbb{L}_{j,m,\alpha}) = \{\mathbf{V} \in \mathcal{E} : \exists \mathbf{W} \in L^2(\mathcal{G}) \;\; s.t. \;\; \forall \mathbf{Z} \in \mathcal{E}, \quad B^{\alpha}_{j,m}(\mathbf{V}, \mathbf{Z}) = \langle \mathbf{W}, \mathbf{Z} \rangle\}. \tag{5.26}$$

In the following theorem we describe the operators $\mathbb{L}_{1,m,\alpha}$ and $\mathbb{L}_{2,m,\alpha}$, in more explicit form, and its relation with the operators $\mathbf{L}_{i,m,\alpha}$, $j = 1, 2$, in (5.20), which emerge from the extension theory (see [30]).

Theorem 5.15 *The operators $\mathbb{L}_{1,m,\alpha}$ and $\mathbb{L}_{2,m,\alpha}$ defined by (5.26) are given on the domain $D(\mathbb{L}_{j,m,\alpha}) = D_{\alpha,\delta}$ and by $\mathbb{L}_{1,m,\alpha} = \mathbf{L}_{1,m,\alpha}$, $\mathbb{L}_{2,m,\alpha} = \mathbf{L}_{2,m,\alpha}$.*

The following lemma states the analyticity of the family of operators $\mathbf{L}_{1,m,\alpha}$.

Lemma 5.16 *As function of α, $(\mathbf{L}_{1,m,\alpha})$ is real-analytic family of self-adjoint operators of type (B) in the sense of Kato.*

Proof By Theorem 5.15 and [88, Theorem VII-4.2], it suffices to prove that the family of bilinear forms $(B^{\alpha}_{1,m})$ defined in (5.25) is real-analytic of type (B). Indeed, it is immediate that it is bounded from below and closed. Moreover, the decomposition of $B^{\alpha}_{1,m} = B^{\alpha} + B_{1,m}$,

$$B^{\alpha}(\mathbf{U}, \mathbf{V}) = \sum_{j=1}^{N} \int_0^{\infty} u'_j v'_j dx + \alpha u_1(0) v_1(0), \;\; B_{1,m}(\mathbf{U}, \mathbf{V}) = \sum_{j=1}^{N} \int_0^{\infty} (\omega - p(\varphi_{m,j})^{p-1}) u_j v_j dx.$$

implies that $\alpha \to B^{\alpha}_{1,m}(\mathbf{V}, \mathbf{V})$ is analytic. □

Combining Lemma 5.16 and Theorem 5.13 in the framework of the perturbation theory, we obtain the following proposition.

Proposition 5.17 *Let $m \in \left\{1, \ldots, \left[\frac{N-1}{2}\right]\right\}$. Then there exist $\alpha_0 > 0$ and two analytic functions $\lambda_m : (-\alpha_0, \alpha_0) \to \mathbb{R}$ and $\mathbf{F}_m : (-\alpha_0, \alpha_0) \to L^2_m(\mathcal{G})$ such that*

(i) *$\lambda_m(0) = 0$ and $\mathbf{F}_m(0) = \widetilde{\mathbf{\Phi}}_{0,m}$, where $\widetilde{\mathbf{\Phi}}_{0,m}$ is defined by (5.23).*
(ii) *For all $\alpha \in (-\alpha_0, \alpha_0)$, $\lambda_m(\alpha)$ is the simple isolated second eigenvalue of $\mathbf{L}_{1,m,\alpha}$ in $L^2_m(\mathcal{G})$, and $\mathbf{F}_m(\alpha)$ is the associated eigenvector for $\lambda_m(\alpha)$.*
(iii) *α_0 can be chosen small enough to ensure that for $\alpha \in (-\alpha_0, \alpha_0)$ the spectrum of $\mathbf{L}_{1,m,\alpha}$ in $L^2_m(\mathcal{G})$ is positive, except at most the first two eigenvalues.*

Proof See [30]. □

Now we investigate how the perturbed second eigenvalue moves depending on the sign of α. Indeed, by using Taylor's theorem for the two analytic functions λ_m and $\mathbf{F}$ we obtain the following theorem (see [30] for the proof).

Theorem 5.18 *There exists $0 < \alpha_1 < \alpha_0$ such that $\lambda_m(\alpha) < 0$ for any $\alpha \in (-\alpha_1, 0)$, and $\lambda_m(\alpha) > 0$ for any $\alpha \in (0, \alpha_1)$. Thus, in $L^2_m(\mathcal{G})$ for α small, we have $n(\mathbf{L}_{1,m,\alpha}) = 2$ as $\alpha < 0$, and $n(\mathbf{L}_{1,m,\alpha}) = 1$ as $\alpha > 0$.*

Now we can count the number of negative eigenvalues of $\mathbf{L}_{1,m,\alpha}$ in $L^2_m(\mathcal{G})$ for any α, using a classical continuation argument based on the Riesz projection.

Theorem 5.19 *Let $m \in \left\{1, \ldots, \left[\frac{N-1}{2}\right]\right\}$ and $\omega > \frac{\alpha^2}{(N-2m)^2}$. Then the following assertions hold.*

(i) *If $\alpha > 0$, then $n(\mathbf{L}_{1,m,\alpha}|_{L^2_m(\mathcal{G})}) = 1$.*
(ii) *If $\alpha < 0$, then $n(\mathbf{L}_{1,m,\alpha}|_{L^2_m(\mathcal{G})}) = 2$.*

Proof We consider the case $\alpha < 0$. Recall that $Ker(\mathbf{L}_{1,m,\alpha}) = \{\mathbf{0}\}$ by Proposition 5.11. Define α_∞ by

$$\alpha_\infty = \inf\{\tilde{\alpha} < 0 : \mathbf{L}_{1,m,\alpha} \text{ has exactly two negative eigenvalues for all } \alpha \in (\tilde{\alpha}, 0)\}.$$

Theorem 5.18 implies that α_∞ is well defined and $\alpha_\infty \in [-\infty, 0)$. We claim that $\alpha_\infty = -\infty$. Suppose that $\alpha_\infty > -\infty$. Let $M = n(\mathbf{L}_{1,m,\alpha_\infty})$ and Γ be a closed curve (for example, a circle or a rectangle) such that $0 \in \Gamma \subset \rho(\mathbf{L}_{1,m,\alpha_\infty})$, and all the negative eigenvalues of $\mathbf{L}_{1,m,\alpha_\infty}$ belong to the inner domain of Γ. The existence of such Γ can be deduced from the lower semi-boundedness of the quadratic form $B^\alpha_{1,m}$ in (5.25) associated to $\mathbf{L}_{1,m,\alpha_\infty}$. Next, from Lemma 5.16 it follows that there is $\epsilon > 0$ such that for $\alpha \in [\alpha_\infty - \epsilon, \alpha_\infty + \epsilon]$ we have $\Gamma \subset \rho(\mathbf{L}_{1,m,\alpha})$ and for $\xi \in \Gamma$, $\alpha \to (\mathbf{L}_{1,m,\alpha} - \xi)^{-1}$ is analytic. Therefore, the existence of an analytic family of Riesz-projections $\alpha \to P(\alpha)$ given by

$$P(\alpha) = -\frac{1}{2\pi i} \oint_\Gamma (\mathbf{L}_{1,m,\alpha} - \xi)^{-1} d\xi$$

implies that $\dim(Im(P(\alpha))) = \dim(Im(P(\alpha_\infty))) = n(\mathbf{L}_{1,m,\alpha_\infty}) = M$ for all $\alpha \in [\alpha_\infty - \epsilon, \alpha_\infty + \epsilon]$ (recall $\mathbf{L}_{1,m,\alpha_\infty}$ is self-adjoint). Next, by definition of α_∞, $\mathbf{L}_{1,m,\alpha_\infty+\epsilon}$ has two negative eigenvalues and $M = 2$, hence $\mathbf{L}_{1,m,\alpha}$ has two negative eigenvalues for $\alpha \in (\alpha_\infty - \epsilon, 0)$, which contradicts the definition of α_∞. Therefore, $\alpha_\infty = -\infty$. □

Lastly, we evaluate $p(\omega)$ defined in Sect. 5.2 (see [30]).

Proposition 5.20 *Let $\alpha \neq 0$, $m \in \left\{1, \ldots, \left[\frac{N-1}{2}\right]\right\}$ and $\omega > \frac{\alpha^2}{(N-2m)^2}$. Let also $J_m(\omega) = \partial_\omega ||\mathbf{\Phi}^\alpha_m||^2_2$. Then the following assertions hold*

(i) *Let $\alpha < 0$, then*
   (1) *for $1 < p \leq 5$, we have $J_m(\omega) > 0$;*
   (2) *for $p > 5$, there exists ω^*_m such that $J_m(\omega^*_m) = 0$, and $J_m(\omega) > 0$ for $\omega \in \left(\frac{\alpha^2}{(N-2m)^2}, \omega^*_m\right)$, while $J_m(\omega) < 0$ for $\omega \in (\omega^*_m, \infty)$.*

(ii) *Let $\alpha > 0$, then*
   (1) *for $1 < p \leq 3$, we have $J_m(\omega) > 0$;*
   (2) *for $3 < p < 5$, there exists $\hat{\omega}_m$ such that $J_m(\hat{\omega}_m) = 0$, and $J_m(\omega) < 0$ for $\omega \in \left(\frac{\alpha^2}{(N-2m)^2}, \hat{\omega}_m\right)$, while $J_m(\omega) > 0$ for $\omega \in (\hat{\omega}_m, \infty)$;*
   (3) *for $p \geq 5$, we have $J_m(\omega) < 0$.*

5.4.3 Proof of Theorems 5.9 and 5.10

In this subsection we proof the Theorems 5.9 and 5.10 via the framework established in Sect. 5.2.

Proof of Theorem 5.9 From Theorem 5.6 we obtain the local well-posed in $\mathcal{E}$ and $\mathcal{E}_m$ of the Cauchy problem for the NLS-δ for any $p > 1$.

(i) Let $\alpha < 0$. From Theorem 5.12, we have $n(\mathbf{H}) = 2$ in $L^2_m(\mathcal{G})$. Therefore, by Proposition 5.20-item (i) we obtain $n(\mathbf{H}^\alpha_m) - p(\omega) = 1$ for $1 < p \leq 5$, $\omega > \frac{\alpha^2}{(N-2m)^2}$, and for $p > 5$, $\omega \in (\frac{\alpha^2}{(N-2m)^2}, \omega^*_m)$. Thus, from Theorem 5.4 we get the assertions $(i) - 1)$ and $(i) - 2)$ in $\mathcal{E}_m$. Since $\mathcal{E}_m \subset \mathcal{E}$, we get the results in $\mathcal{E}$.

(ii) Let $\alpha > 0$. Due to Theorem 5.12, we have $n(\mathbf{H}) = 1$ in $L^2_m(\mathcal{G})$. Therefore, by Proposition 5.20-item (ii) we obtain $n(\mathbf{H}) - p(\omega) = 1$ for $p \geq 5$, $\omega > \frac{\alpha^2}{(N-2m)^2}$ and $3 < p < 5$, $\omega \in (\frac{\alpha^2}{(N-2m)^2}, \hat{\omega}_m)$. Therefore, we obtain instability of $e^{i\omega t}\mathbf{\Phi}^\alpha_m$ in $\mathcal{E}_m$ and consequently in $\mathcal{E}$. From the other hand, for $1 < p \leq 3$, $\omega > \frac{\alpha^2}{(N-2m)^2}$ and $3 < p < 5$, $\omega \in (\hat{\omega}_m, \infty)$, we have $n(\mathbf{H}) = p(\omega)$, which yields together with Theorem 5.7, the stability of $e^{i\omega t}\mathbf{\Phi}^\alpha_m$ in $\mathcal{E}_m$. Thus, (ii) is proved. This finishes the proof. □

Proof of Theorem 5.10 By following the same strategy as in Theorems 5.18 and 5.19 we have for $\mathbf{L}_{1,0,\alpha} = \mathbf{L}_{1,\alpha}$ associated to the tail and bump profiles $\mathbf{\Phi}^\alpha_0 = (\varphi_{0,\alpha})^N_{j=1}$ defined in (5.11) and $m \in \left\{1, \dots, [\frac{N-1}{2}]\right\}$, $\alpha \neq 0$ with $\omega > \frac{\alpha^2}{N^2}$, that for $\alpha > 0$ we have $n(\mathbf{L}_{1,\alpha}) = 2$ in $L^2_m(\mathcal{G})$, and that for $\alpha < 0$, we have $n(\mathbf{L}_{1,\alpha}) = 1$ in $L^2_m(\mathcal{G})$ and in $L^2(\mathcal{G})$ (in this last case, we use the extension theory for symmetric operator as in the proof of Theorem 5.13). Next, for $m = 0$, $\omega > \frac{\alpha^2}{N^2}$ and $J(\omega) = \partial_\omega ||\mathbf{\Phi}^\alpha_0||^2$, we have from [30] that the following assertions hold,

(i) Let $\alpha < 0$, then

  (1) for $1 < p \leq 5$, we have $J(\omega) > 0$;
  (2) for $p > 5$, there exists ω_1 such that $J(\omega_1) = 0$, and $J(\omega) > 0$ for $\omega \in \left(\frac{\alpha^2}{N^2}, \omega_1\right)$, while $J(\omega) < 0$ for $\omega \in (\omega_1, \infty)$.

(ii) Let $\alpha > 0$, then

  (1) for $1 < p \leq 3$, we have $J(\omega) > 0$;
  (2) for $3 < p < 5$, there exists ω_2 such that $J(\omega_2) = 0$, and $J(\omega) < 0$ for $\omega \in \left(\frac{\alpha^2}{N^2}, \omega_2\right)$, while $J(\omega) > 0$ for $\omega \in (\omega_2, \infty)$;
  (3) for $p \geq 5$, we have $J(\omega) < 0$.

Thus, for $\alpha > 0$ we obtain $n(\mathbf{H}|_{L^2_m(\mathcal{G})}) - p(\omega) = 1$ as $p \in (1, 3]$, $\omega > \frac{\alpha^2}{N^2}$, and $p \in (3, 5)$, $\omega > \omega_2$. Then, from Theorem 5.4 we get orbital instability of $e^{i\omega t}\mathbf{\Phi}^\alpha_0$ in $\mathcal{E}_m(\mathcal{G})$ and consequently in $\mathcal{E}(\mathcal{G})$. For $\alpha < 0$, we obtain the orbital stability $e^{i\omega t}\mathbf{\Phi}^\alpha_0$ for $1 < p \leqq 5$ and any $\omega > \frac{\alpha^2}{N^2}$, and for $p > 5$ and $\omega \in (\frac{\alpha^2}{N^2}, \omega_1)$ (in this case the global existence of solutions $\mathbf{U}(t)$ for $\mathbf{U}(0) = \mathbf{U}_0 \approx \Omega_{\mathbf{\Phi}^\alpha_0}$ is deduced from Theorem 5.6, and from the uniform estimative $\|\mathbf{U}(t)\|_{H^1} < \epsilon + \|\mathbf{\Phi}_0^{\alpha}\|_{H^1}$ for $t \in [0, T_{max})$). Moreover, we have the orbital instability of $e^{i\omega t}\mathbf{\Phi}_0^{\alpha}$ for $p > 5$ and $\omega > \omega_1$. This finishes the proof. □

Remark 5.21 From Remark 5.5 we recall that when $n(\mathbf{H}) - p(\omega)$ is odd, we obtain initially from [80] only spectral instability of $e^{i\omega t}\mathbf{\Phi}_m^{\alpha}$. To conclude orbital instability from spectral instability we use from Theorem 5.6 that the mapping data-solution is of class C^2 for $p > 2$.

5.5 End-section notes

(a) The study of the NLS model (5.1) for other choices of $\mathcal{A}$ and $\mathbf{F}$ have been made in [28–30]. By instance, for the NLS-log-δ where $\mathcal{A}$ is the δ-interaction defined by (1.22) and $\mathbf{F} = \mathbf{U}Log|\mathbf{U}|^2$ or for $\mathcal{A}$ being a δ'-interaction, namely, $(\mathcal{A}\mathbf{V})(x) = (-v_j''(x))_{j=1}^N$, $x > 0$,

$$\mathbf{D}_{\lambda,\delta'}(\mathcal{A}) = \left\{ \mathbf{V} \in H^2(\mathcal{G}) : v_1'(0) = \cdots = v_N'(0),\ \sum_{j=1}^{N} v_j(0) = \lambda v_1'(0) \right\}, \tag{5.27}$$

and $\mathbf{F} = |\mathbf{U}|^{p-1}\mathbf{U}$, $p > 1$.

(b) In this section we have seen some of the classical results of the extension theory for symmetric operators developed by von Neumann and Krein, and several applications have been given for the Laplacian operator on metric star graphs, where the matching (boundary) conditions at the vertex $\nu = 0$, were of δ-interaction type in (1.22). Next, we will see other way to parametrize all self-adjoint realizations L of $-\Delta$ in $L^2(\mathcal{G})$-space on a metric star graph $\mathcal{G}$ with N half-lines of the form $(0, +\infty)$ attached to the common vertex $\nu = 0$. We will make use of the notion of Nevanlinna pairs given in the following definition [91],

Definition 5.22 A pair $\{A, B\}$ of $N \times N$ matrices is said to be a Nevanlinna pair if;

(a) $AB^* = BA^*$,
(b) The horizontally concatenated $N \times 2N$ matrix $[A, B]$ has maximal rank N.

Next, let $u : \mathcal{G} \to \mathbb{C}$ and it writes u as a column-vector $u = (u_1, \ldots, u_N)^t$ where each u_j is defined on the interval $(0, +\infty)$. We express the conditions at the vertex $\nu = 0$ as $u(0) = (u_1(0+), \ldots, u_N(0+))^t$ and $u'(0) = (u_1'(0+), \ldots, u_N'(0+))^t$. In the following we introduce the Laplacian $-\Delta(A, B)$ as

$$\begin{aligned} -\Delta(A, B)u &= \left(-u_1'', \ldots, -u_N''\right)^t, \\ D(-\Delta(A, B)) &= \{u \in H^2(\mathcal{G}) : Au(0) + Bu'(0) = 0\}. \end{aligned} \tag{5.28}$$

A crucial result concerning the parametrization of all self-adjoint extensions of the Laplace operator in $L^2(\mathcal{G})$ in terms of the boundary conditions, it was obtained in [91]. Indeed, we have the following proposition.

Proposition 5.23 *Let A, B be $N \times N$ matrices. The next two assertions are equivalent:*

(1) *The operator $-\Delta(A, B)$ defined in (5.28) is self-adjoint;*
(2) *$\{A, B\}$ is a Nevanlinna pair.*

For instance, the δ-coupling in (1.22) is obtained for the Nevanlinna pair $\{A, B\}$ defined by $A = A_\delta$, $B = B_\delta$ as

$$A = \begin{pmatrix} 1 & -1 & 0 & \dots & 0 \\ 0 & 1 & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & -1 \\ \frac{\alpha}{N} & \frac{\alpha}{N} & \frac{\alpha}{N} & \dots & \frac{\alpha}{N} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & \dots & 0 \\ 0 & & 0 \\ \vdots & & \vdots \\ & & \\ -1 & \dots & -1 \end{pmatrix}. \tag{5.29}$$

(c) The existence and stability of standing waves profiles for (5.1) with $\mathcal{A} = -\Delta(A, B)$ in (5.28) and arbitrary Nevanlinna pair $\{A, B\}$, is an open question.
(d) $N = 2$ in (5.1) and $\mathcal{G} = (-\infty, 0) \cup (0, +\infty)$ represent the simplest case of the nonlinear Schrödinger equation posed on a metric graph. The case of $F(u) = |u|^p u$ and a δ-condition at the vertice, $\mathcal{A} = \mathcal{A}_{\delta,\alpha}$, is defined by

$$\begin{cases} \mathcal{A}_{\delta,\alpha} v(x) = -v''(x), \quad x \neq 0, \\ D(\mathcal{A}_{\delta,\alpha}) = \left\{v \in H^1(\mathbb{R}) \cap H^2(\mathbb{R} - \{0\}) : v'(0+) - v'(0-) = \alpha v(0)\right\}. \end{cases} \tag{5.30}$$

The operator $\mathcal{A}_{\delta,\alpha}$ is formally defined by the expression $\mathcal{A}_{\delta,\alpha} = -\frac{d^2}{dx^2} + \alpha\delta(x)$, where $\delta(x)$ is the Dirac delta distribution centered at $x = 0$. This case has been studied in a series de papers (see [5,6,20,22,41,57,69,70,77,83–85,95]). Moreover, we have only two standing wave profiles; for $\alpha < 0$ the tail-profile and for $\alpha > 0$ the bump-profile. In this case, Theorem 5.10 with $\mathcal{E} = H^1(\mathbb{R})$ recovers the conditions-parameters of (in)stability obtained in [95] for these profiles.
(e) For $N = 2$ in (5.1) and $\mathcal{G} = (-\infty, 0) \cup (0, +\infty)$, in Adami et al.[4] and Angulo et al. [31], was studied the NLS with $F(u) = |u|^p u$ and a δ'-interaction [43], namely, the case of $\mathcal{A} = \mathcal{A}_{\delta',-\beta}$ with

$$\begin{cases} \mathcal{A}_{\delta',-\beta} v(x) = -v''(x), \quad x \neq 0, \\ D(\mathcal{A}_{\delta',-\beta}) = \left\{H^2(\mathbb{R} - \{0\}) : v(0+) - v(0-) = -\beta v'(0), v'(0+) = v'(0-)\right\}. \end{cases}$$

The study of the (in)stability of the odd discontinuous tail-like profile with $\beta > 0$ (Fig. 13) was established in Adami et al. [4]. The case of the odd discontinuous bump-like profile with $\beta < 0$ (see Fig. 14) was recently studied in Angulo et.al [31].

6 The KdV equation on metric graphs

In this section we continue our study of nonlinear dispersive equations on metric graphs. Our focus will be the model Korteweg-de Vries in (1.23) defined on a star-shaped metric graph $\mathcal{G}$ with the notation for the edge's set $\mathbf{E}$ as $\mathbf{E} = \mathbf{E}_- \cup \mathbf{E}_+$, where

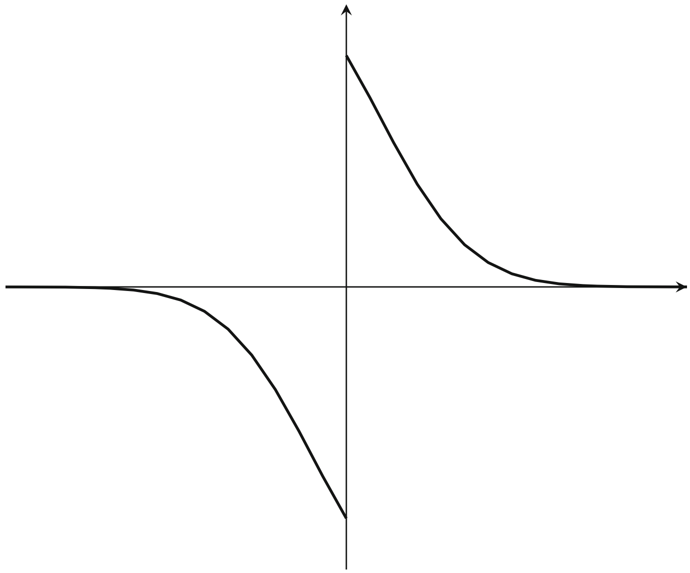

Fig. 13 NLS-δ' tail-profile, $\beta > 0$

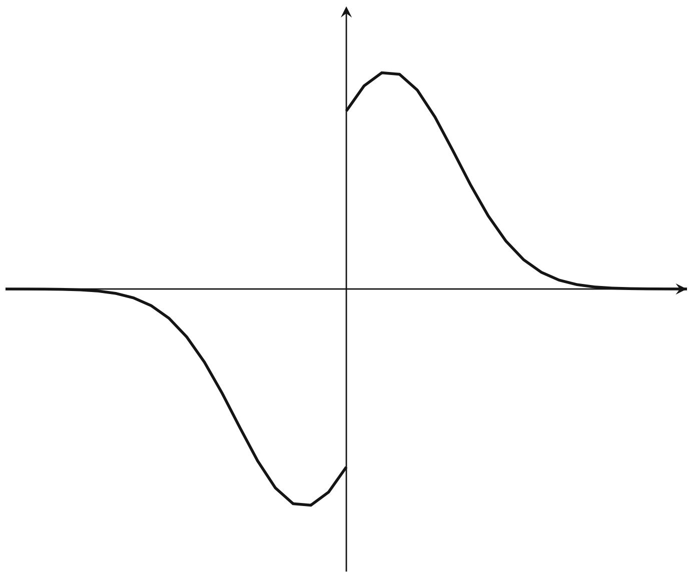

Fig. 14 NLS-δ' bump-profile, $\beta < 0$

$\mathbf{E}_-$ represents the collection of negative semi-infinite edges $(-\infty, 0)$, and $\mathbf{E}_+$ the positive semi-infinite edges $(0, +\infty)$. We use the notation $|\mathbf{E}_\pm|$ for the number of edges. For the case $|\mathbf{E}_-| = |\mathbf{E}_+|$, the metric graph is called a *balanced star-shaped metric graph*.

The main interest of exposition here is to establish a linear instability criterium for stationary profiles of the KdV model on general graph $\mathcal{G}$. A starting point is to determine when the Airy type operator

$$A_0 : (u_\mathbf{e})_{\mathbf{e}\in\mathbf{E}} \to \left(\alpha_\mathbf{e} \frac{d^3}{dx^3} u_\mathbf{e} + \beta_\mathbf{e} \frac{d}{dx} u_\mathbf{e}\right)_{\mathbf{e}\in\mathbf{E}} \tag{6.1}$$

will have extensions A_{ext} on $L^2(\mathcal{G})$ such that the dynamics induced by the linear evolution problem

$$\begin{cases} z_t = A_{ext} z, \\ z(0) = u_0 \in D(A_{ext}), \end{cases} \tag{6.2}$$

is given by a C_0-group. In this point the theory in Mugnolo et al. [112] and Schubert et al. [124] give that properties of the induced dynamics can be obtained by studying boundary operators in the corresponding boundary space induced by the vertice of the graph. In Sect. 6.1 below we give a brief description of this theory. In particular, we construct a family of skew-self-adjoint extensions $(H_Z, D(H_Z))_{Z\in\mathbb{R}}$ of δ-type interaction for A_0 in the case of a general balanced star-shaped graph, such that in the case of two half-lines is defined by:

$$\begin{cases} H_Z u = \left(\alpha_\mathbf{e} \frac{d^3}{dx^3} u_\mathbf{e} + \beta_\mathbf{e} \frac{d}{dx} u_\mathbf{e}\right)_{\mathbf{e}\in\mathbf{E}}, \quad u = (u_\mathbf{e})_{\mathbf{e}\in\mathbf{E}} \\ D(H_Z) = \Big\{u = (u_-, u_+) \in H^3(-\infty, 0) \oplus H^3(0, +\infty) : u_-(0-) = u_+(0+), \\ \qquad u'_+(0+) - u'_-(0-) = Z u_-(0-),\ \frac{Z^2}{2} u_-(0-) + Z u'_-(0-) = u''_+(0+) - u''_-(0-)\Big\}, \end{cases} \tag{6.3}$$

where for $\alpha_\mathbf{e} = (\alpha_-, \alpha_+) \in \mathbb{R}^+ \times \mathbb{R}^+$ and $\beta_\mathbf{e} = (\beta_-, \beta_+) \in \mathbb{R} \times \mathbb{R}$ we need to have $\alpha_- = \alpha_+$ and $\beta_- = \beta_+$.

The first formal study of stationary solutions for the KdV model (1.23) and its dynamic, was made in Angulo and Cavalcanti [26]. A stationary solution for (1.23) is a solution of the form

$$(u_\mathbf{e}(x, t))_{\mathbf{e}\in\mathbf{E}} = (\phi_\mathbf{e}(x))_{\mathbf{e}\in\mathbf{E}},$$

where for $\mathbf{e} \in \mathbf{E}_-$ the profile $\phi_\mathbf{e} : (-\infty, 0) \to \mathbb{R}$ satisfies $\phi_\mathbf{e}(-\infty) = 0$, and for $\mathbf{e} \in \mathbf{E}_+$, $\phi_\mathbf{e} : (0, \infty) \to \mathbb{R}$ satisfies $\phi_\mathbf{e}(+\infty) = 0$. The existence of profiles of stationary type for the KdV, namely, solutions of the following nonlinear elliptic equation

$$\alpha_\mathbf{e} \frac{d^2}{dx^2} \phi_\mathbf{e}(x) + \beta_\mathbf{e} \phi_\mathbf{e}(x) + \phi_\mathbf{e}^2(x) = 0, \quad \mathbf{e} \in \mathbf{E}, \tag{6.4}$$

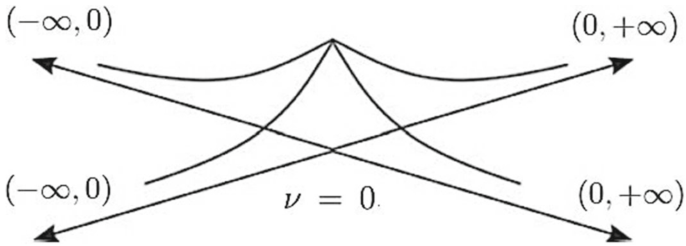

Fig. 15 Tail profiles for the KdV

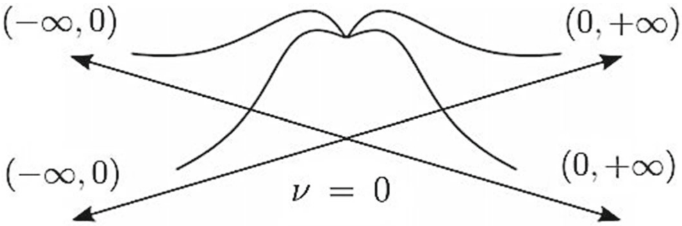

Fig. 16 Bumps profiles for the KdV

are well know and will depend of the profile of the classical soliton associated to the KdV on the full line (see (1.6)), namely, for specific conditions on $\alpha_{\mathbf{e}}$ and $\beta_{\mathbf{e}}$, we have

$$\phi_{\mathbf{e}}(x) = c(\alpha_{\mathbf{e}}, \beta_{\mathbf{e}})sech^2(d(\alpha_{\mathbf{e}}, \beta_{\mathbf{e}})x + p_{\mathbf{e}}), \quad \mathbf{e} \in \mathbf{E}. \tag{6.5}$$

The specific value of the shift $p_{\mathbf{e}}$ will depend on other (or others) condition(s) imposed on the profile $\phi_{\mathbf{e}}$ in the vertex of the graph $\nu = 0$. In [26] was considered the case for $(\phi_{\mathbf{e}})_{\mathbf{e}\in\mathbf{E}} \in D(H_Z)$ on balanced star-shaped metric graph and it was showed that the tails and bumps stationary profiles in Figs. 15 and 16, respectively, are linearly unstable.

Novel applications of our general linear instability criterium in Sect. 6.3 have been obtained recently in the case of kink-profile for the sine-Gordon equation (1.24) on Y-josephson junction in Angulo and Plaza [39,40] (see End-notes of this section)

6.1 Extension theory and the KdV model on metric graphs

We consider the Airy operator A_0 in (6.1) as an unbounded operator on a certain Hilbert space belonging to $L^2(\mathcal{G})$. Here we want to obtain skew-self-adjoint extensions $(A_{ext}, D(A_{ext}))$ of A_0 and for then the generated dynamics induced by (6.2) is given by a C_0-unitary group. Since the Airy operator A_0 has odd order, changing the sign of each constant $\alpha_{\mathbf{e}}$ it is equivalent to exchange the positive and negative half line and so without loss of generality we can choose $\alpha_{\mathbf{e}} > 0$ for every $\mathbf{e} \in \mathbf{E} = \mathbf{E}_- \cup \mathbf{E}_+$.

The following proposition from [112] gives us an answer about the problem associated to (6.2).

Proposition 6.1 *Let $\mathcal{G}$ be a star-shaped metric graph consisting of finitely many half-lines $\mathbf{E} \equiv \mathbf{E}_- \cup \mathbf{E}_+$ and let $(\alpha_{\mathbf{e}})_{\mathbf{e}\in\mathbf{E}}$, $(\beta_{\mathbf{e}})_{\mathbf{e}\in\mathbf{E}}$ be two sequences of real numbers with $\alpha_{\mathbf{e}} > 0$ for all $\mathbf{e} \in \mathbf{E}$. Consider the operator A_0 defined in (6.1) with*

$$D(A_0) \equiv \bigoplus_{\mathbf{e}\in\mathbf{E}_-} C_c^\infty(-\infty, 0) \oplus \bigoplus_{\mathbf{e}\in\mathbf{E}_+} C_c^\infty(0, +\infty).$$

Then, iA_0 is a densely defined symmetric operator on the Hilbert space

$$L^2(\mathcal{G}) = \bigoplus_{\mathbf{e}\in\mathbf{E}_-} L^2(-\infty, 0) \oplus \bigoplus_{\mathbf{e}\in\mathbf{E}_+} L^2(0, +\infty),$$

with deficiency indices $(n_+(A_0), n_-(A_0)) = (2|\mathbf{E}_-|+|\mathbf{E}_+|, |\mathbf{E}_-|+2|\mathbf{E}_+|)$. Therefore, A_0 has skew-self-adjoint extension on $L^2(\mathcal{G})$ if and only if $|\mathbf{E}_-| = |\mathbf{E}_+|$ ($n_+(A_0) = n_-(A_0)$).

We recall from the classical Krein–von Neumann extension theory for symmetric operators (see Chapter 4 in Naimark [113] and Theorem X.2 in Reed and Simon [122]) and Proposition 6.1, that the operator $(A_0, D(A_0))$, on the case of balanced star graphs, admits a $9|\mathbf{E}_+|^2$-parameter family of skew-self-adjoint extension generating each one a unitary dynamics on $L^2(\mathcal{G})$ associated to the linear evolution equation (6.2). We note that every skew-self-adjoint extension, $(A, D(A))$, is obtained as a restriction of $(-A_0^*, D(A_0^*))$ with $-A_0^* = A_0$ and

$$D(A_0^*) \equiv \bigoplus_{\mathbf{e}\in\mathbf{E}_-} H^3(-\infty, 0) \oplus \bigoplus_{\mathbf{e}\in\mathbf{E}_+} H^3(0, +\infty). \tag{6.6}$$

The main idea in [112] (see also [124]) to parametrize the skew-self-adjoint extensions of $(A_0, D(A_0))$ is through relations between boundary values, a strategy very similar to that established in the case of the Laplacian operator via Nevanlinna pairs (see End-notes section 5). Indeed, for $u = (u_{\mathbf{e}})_{\mathbf{e}\in\mathbf{E}} \in D(A_0^*)$ we denote $u(0-) \equiv (u_{\mathbf{e}}(0-))_{\mathbf{e}\in\mathbf{E}_-}$ and $u(0+) \equiv (u_{\mathbf{e}}(0+))_{\mathbf{e}\in\mathbf{E}_+}$, and so we consider the following space of vectors boundary values in $\mathbb{C}^{3n}$, $(u(0-), u'(0-), u''(0-))$ and $(u(0+), u'(0+), u''(0+))$, spanning respectively subspaces $\mathbb{G}_-$ and $\mathbb{G}_+$, with $n = |\mathbf{E}_\pm|$. The boundary form of the operator A_0 is easily seen for $u, v \in D(A_0^*)$ to be (where we are identifying a vector with its transpose)

$$\langle A_0^* u, v\rangle + \langle u, A_0^* v\rangle = \left(B_- \begin{pmatrix} u(0-) \\ u'(0-) \\ u''(0-) \end{pmatrix}, \begin{pmatrix} u(0-) \\ u'(0-) \\ u'(0-) \end{pmatrix}\right)_{\mathbb{G}_-} - \left(B_+ \begin{pmatrix} u(0+) \\ u'(0+) \\ u''(0+) \end{pmatrix}, \begin{pmatrix} u(0+) \\ u'(0+) \\ u'(0+) \end{pmatrix}\right)_{\mathbb{G}_+} \tag{6.7}$$

where for $I = I_{n\times n}$ representing the identity matrix of order $n \times n$, we have

$$B_- = \begin{pmatrix} -I\beta_- & 0 & -I\alpha_- \\ 0 & I\alpha_- & 0 \\ -I\alpha_- & 0 & 0 \end{pmatrix}, \quad B_+ = \begin{pmatrix} -I\beta_+ & 0 & -I\alpha_+ \\ 0 & I\alpha_+ & 0 \\ -I\alpha_+ & 0 & 0 \end{pmatrix} \tag{6.8}$$

and $\alpha_\pm = (\alpha_{\mathbf{e}})_{\mathbf{e}\in\mathbf{E}_\pm}$, $\beta_\pm = (\beta_{\mathbf{e}})_{\mathbf{e}\in\mathbf{E}_\pm}$. Thus by considering the (indefinite) inner product $\langle\cdot|\cdot\rangle_\pm : \mathbb{G}_\pm \times \mathbb{G}_\pm \to \mathbb{C}$ by $\langle x|y\rangle_\pm \equiv (B_\pm x, y)_{\mathbb{G}_\pm}$, $x, y \in \mathbb{G}_\pm$, we obtain that $(\mathbb{G}_\pm, \langle\cdot|\cdot\rangle_\pm)$ are Krein spaces and $\langle\cdot|\cdot\rangle_\pm$ is non-degenerate (for $x \in \mathbb{G}_\pm$ with $\langle x|x\rangle_\pm = 0$ follows $x = 0$). Then from Theorem 3.8 of [112] we have that for a linear operator $L : \mathbb{G}_- \to \mathbb{G}_+$, the operator $(A_L, D(A_L))$ defined by

$$\begin{cases} A_L u = -A_0^* u = A_0 u \\ D(A_L) = \{u \in D(A_0^*) : L(u(0-), u'(0-), u''(0-)) = (u(0+), u'(0+), u''(0+))\}, \end{cases} \tag{6.9}$$

is a skew-self-adjoint extension of $(A_0, D(A_0))$ if and only if L is $(\mathbb{G}_-, \mathbb{G}_+)$-unitary, that means,

$$\langle Lx|Ly\rangle_+ = (B_+ Lx, Ly)_{\mathbb{G}_+} = \langle x|y\rangle_- = (B_- x, y)_{\mathbb{G}_-}, \tag{6.10}$$

or equivalently, $\boxed{L^* B_+ L = B_-}$. Indeed, For $u, v \in D(A_L)$ it follows from (6.7)

$$\begin{aligned} \langle -A_L u, v\rangle + \langle u, -A_L v\rangle = \langle A_0^* u, v\rangle + \langle u, A_0^* v\rangle &= \langle u(0-)|v(0-)\rangle_- - \langle u(0+)|v(+)\rangle_+ \\ &= \langle u(0-)|v(0-)\rangle_- - \langle Lu(0-)|Lv(-)\rangle_+. \end{aligned}$$

Then, $(A_L)^* = -A_L$ if and only L is $(\mathbb{G}_-, \mathbb{G}_+)$-unitary.

Next, we establish our family of interest of skew-self-adjoint extension of $(A_0, D(A_0))$ on a balanced metric star graph $\mathcal{G}$ with $|\mathbf{E}_\pm| = n$, $n \geqq 1$, and with a δ-interaction type at the vertex. It will be used in the existence and stability of specific stationary solutions for the KdV model. We consider $B_\pm$ in (6.8) and $L \equiv L_{3n\times 3n} : \mathbb{G}_- \to \mathbb{G}_+$ of order $3n \times 3n$, $Z \in \mathbb{R}$, defined by

$$L \equiv \begin{pmatrix} I & 0 & 0 \\ ZI & I & 0 \\ \frac{Z^2}{2} I & ZI & I \end{pmatrix}. \tag{6.11}$$

Thus, for $\alpha_\pm = (\alpha_{\mathbf{e}})_{\mathbf{e}\in\mathbf{E}_\pm}$, $\beta_\pm = (\beta_{\mathbf{e}})_{\mathbf{e}\in\mathbf{E}_\pm}$, we obtain $L^* B_+ L = B_-$ if and only if $\alpha_+ = \alpha_-$ and $\beta_+ = \beta_-$. Then, in this case (and only in this one) we obtain that L is $(\mathbb{G}_-, \mathbb{G}_+)$-unitary. Therefore, the operators $(H_Z, D(H_Z))$, $Z \in \mathbb{R}$, defined by

$$\begin{cases} H_Z u = -A_0^* u = A_0 u \\ D(H_Z) = \left\{u \in D(A_0^*) : L(u(0-), u'(0-), u''(0-)) = (u(0+), u'(0+), u''(0+))\right\} \end{cases} \tag{6.12}$$

will be skew-self-adjoint extensions for $(A_0, D(A_0))$, where for $u = (u_{\mathbf{e}})_{\mathbf{e}\in\mathbf{E}} \in D(H_Z)$ we use the abbreviations

$$u(0-) = (u_{\mathbf{e}}(0-))_{\mathbf{e}\in\mathbf{E}_-}, \quad u'(0-) = (u'_{\mathbf{e}}(0-))_{\mathbf{e}\in\mathbf{E}_-}, \quad u''(0-) = (u''_{\mathbf{e}}(0-))_{\mathbf{e}\in\mathbf{E}_-},$$

(similarly for the terms $u(0+)$, $u'(0+)$ and $u''(0\pm)$). Thus, we obtain the following system of conditions for $u = (u_{\mathbf{e}})_{\mathbf{e}\in\mathbf{E}} \in D(H_Z)$

$$u(0-) = u(0+), \quad u'(0+) - u'(0-) = Zu(0-), \quad (\delta\text{-interaction type for every two half-lines})$$
$$\frac{Z^2}{2}u(0-) + Zu'(0-) = u''(0+) - u''(0-). \tag{6.13}$$

It will be of our interest to consider stationary-profile $\phi_Z \in D(H_Z) \cap \mathcal{C}$ with

$$\mathcal{C} = \left\{(u_{\mathbf{e}})_{\mathbf{e}\in\mathbf{E}} \in L^2(\mathcal{G}) : u_{1,-}(0-) = \cdots = u_{n,-}(0-) = u_{1,+}(0+) = \cdots = u_{n,+}(0+)\right\}, \tag{6.14}$$

that is, the continuity of ϕ_Z at the vertex $\nu = 0$. Therefore, from (6.13) follows that for $u = (u_{\mathbf{e}})_{\mathbf{e}\in\mathbf{E}} \in D(H_Z) \cap \mathcal{C}$ we have the δ-type condition

$$u(0-) = u(0+) \ \text{ and } \quad \sum_{\mathbf{e}\in\mathbf{E}_+} u'_{\mathbf{e}}(0+) - \sum_{\mathbf{e}\in\mathbf{E}_-} u'_{\mathbf{e}}(0-) = Znu_{1,+}(0+). \tag{6.15}$$

Lastly, specific formulas for unitary groups associated to the possible skew-self-adjoint extensions of the Airy operator A_0 is an open problem in general. In Lemmas 7.5–7.6 and Propositions 7.8–7.10 in Angulo and Cavalcante [26], the Green functions and specific formulas for the unitary groups associated to the one-parameter family of skew-self-adjoint extensions of δ-type $(H_Z, D(H_Z))$, were established.

6.1.1 Stationary solutions for the KdV model with a δ-type interaction

Next, we build a family of continuous (at zero) stationary profiles for the KdV model on balanced graph $\mathcal{G}$ such that $\phi = (\phi_{\mathbf{e}})_{\mathbf{e}\in\mathbf{E}} \in D(H_Z)$. We consider initially the case of constants sequences $(\alpha_{\mathbf{e}})_{\mathbf{e}\in\mathbf{E}} = (\alpha_+)$, $(\beta_{\mathbf{e}})_{\mathbf{e}\in\mathbf{E}} = (\beta_+)$, with $\alpha_+ > 0$ and $\beta_+ < 0$. Thus we obtain from (6.5) for $Z \neq 0$ and $-\frac{\beta_+}{\alpha_+} > \frac{Z^2}{4}$, the half-soliton profile $\phi_{+,Z}$

$$\phi_{+,Z}(x) = -\frac{3\beta_+}{2} sech^2\left[\frac{\sqrt{-\beta_+}}{2\sqrt{\alpha_+}}x - tanh^{-1}\left(\frac{Z\sqrt{\alpha_+}}{2\sqrt{-\beta_+}}\right)\right], \quad x > 0 \tag{6.16}$$

and $\phi_{-,Z}(x) \equiv \phi_{+,Z}(-x)$ for $x < 0$, such that satisfy (6.4) for $x \neq 0$.

Then, the following constants sequences of functions $(\phi_{\mathbf{e}}) \equiv (\phi_{-,Z})$ for $\mathbf{e} \in \mathbf{E}_-$ and $(\phi_{\mathbf{e}}) \equiv (\phi_{+,Z})$ for $\mathbf{e} \in \mathbf{E}_+$ satisfy the continuity in zero and $\phi_Z = (\phi_{-,Z}, \phi_{+,Z})_{\mathbf{e}\in\mathbf{E}} \in D(H_Z)$. ϕ_Z represents a family of bumps profiles for $Z > 0$ (see Fig. 16) and for $Z < 0$ the family of tails profiles (see Fig. 15).

For the general case of sequences $(\alpha_{\mathbf{e}})_{\mathbf{e}\in\mathbf{E}} = (\alpha_1, \alpha_2)$ and $(\beta_{\mathbf{e}})_{\mathbf{e}\in\mathbf{E}} = (\beta_1, \beta_2)$, with $\alpha_2 = (\alpha_{1,+}, \ldots, \alpha_{n,+}) = \alpha_1 = (\alpha_{1,-}, \ldots, \alpha_{n,-})$ and $\beta_2 = (\beta_{1,+}, \ldots \beta_{n,+}) = \beta_1 = (\beta_{1,-}, \ldots \beta_{n,-})$ see [26].

6.2 Angulo and Cavalcante's linear instability for KdV on a start graph

In this section we establish a linear instability criterium of stationary solutions for the KdV model (1.23) on a start-shaped metric graph $\mathcal{G}$ with $|\mathbf{E}_+| = n$ and $|\mathbf{E}_-| = m$. Let us consider an extension $(A_{ext}, D(A_{ext}))$ of the Airy operator A_0 in (6.1) on $L^2(\mathcal{G})$, such that the dynamic induced by the linear evolution problem (6.2) is given by a C_0-group (see [112]).

Suppose for $(\phi_{\mathbf{e}})_{\mathbf{e}\in\mathbf{E}} \in D(A_{ext})$ we have that each component of $(\tilde{u}_{\mathbf{e}}(x,t))_{\mathbf{e}\in\mathbf{E}} = (\phi_{\mathbf{e}}(x))_{\mathbf{e}\in\mathbf{E}}$ is a nontrivial solution of (6.4). Next, we suppose for $\mathbf{e} \in \mathbf{E}$, that $u_{\mathbf{e}}$ satisfies formally equality in (1.23) and it defines

$$v_{\mathbf{e}}(x,t) \equiv u_{\mathbf{e}}(x,t) - \phi_{\mathbf{e}}(x). \tag{6.17}$$

Then, for $(v_{\mathbf{e}})_{\mathbf{e}\in\mathbf{E}} \in D(A_{ext})$ we have for each $\mathbf{e} \in \mathbf{E}$ the system

$$\partial_t v_{\mathbf{e}}(x,t) = \alpha_{\mathbf{e}}\partial_x^3 v_{\mathbf{e}}(x,t) + \beta_{\mathbf{e}}\partial_x v_{\mathbf{e}}(x,t) + 2\partial_x(v_{\mathbf{e}}(x,t)\phi_{\mathbf{e}}(x)), \tag{6.18}$$

which represents the linearized equation for (1.23) around $\phi_{\mathbf{e}}$. Our objective in the following will be to give sufficient conditions for obtaining that the trivial solution $v_{\mathbf{e}} \equiv 0$, $\mathbf{e} \in \mathbf{E}$, is unstable by the linear flow of (6.18). More exactly, we are interested in finding a *growing mode solution* of (6.18) with the form $v_{\mathbf{e}}(x,t) = e^{\lambda t}\psi_{\mathbf{e}}$ and $\mathrm{Re}(\lambda) > 0$. In other words, we need to solve the formal system for $\mathbf{e} \in \mathbf{E}$,

$$\lambda\psi_{\mathbf{e}} = -\partial_x \mathcal{L}_{\mathbf{e}}\psi_{\mathbf{e}}, \qquad \mathcal{L}_{\mathbf{e}} = -\alpha_{\mathbf{e}}\frac{d^2}{dx^2} - \beta_{\mathbf{e}} - 2\phi_{\mathbf{e}} \tag{6.19}$$

with $\psi_{\mathbf{e}} \in D(\partial_x \mathcal{L}_{\mathbf{e}})$.

Now, we write our eigenvalue problem in (6.19) in a matrix form. Indeed, let $\psi = (\psi_{\mathbf{e}})_{\mathbf{e}\in\mathbf{E}}$ and write $\psi = (\psi_-, \psi_+)^t$ with $\psi_- = (\psi_{\mathbf{e}})_{\mathbf{e}\in\mathbf{E}_-}$ and $\psi_+ = (\psi_{\mathbf{e}})_{\mathbf{e}\in\mathbf{E}_+}$, then we can write (6.19) as

$$NE\psi = \lambda\psi, \quad Re(\lambda) > 0, \quad \psi \in D(E), \tag{6.20}$$

with N and E being $(m+n)\times(m+n)$-diagonal matrix defined by

$$N = \begin{pmatrix} -\partial_x I_m & 0 \\ 0 & -\partial_x I_n \end{pmatrix}, \quad E = \begin{pmatrix} \mathcal{L}_- & 0 \\ 0 & \mathcal{L}_+ \end{pmatrix} \tag{6.21}$$

where I_k denotes the identity matrix of order k, and

$$\mathcal{L}_- = \mathrm{diag}\left(-\alpha_{1,-}\frac{d^2}{dx^2} - \beta_{1-} - 2\phi_{1,-}, \ldots, -\alpha_{m,-}\frac{d^2}{dx^2} - \beta_{m,-} - 2\phi_{m,-}\right) \tag{6.22}$$

where $(\alpha_{\mathbf{e}})_{\mathbf{e}\in\mathbf{E}_-} \equiv (\alpha_{1,-}, \ldots, \alpha_{m,-})$, $(\beta_{\mathbf{e}})_{\mathbf{e}\in\mathbf{E}_-} \equiv (\beta_{1,-}, \ldots, \beta_{m,-})$, and $(\phi_{\mathbf{e}})_{\mathbf{e}\in\mathbf{E}_-} \equiv (\phi_{1,-}, \ldots, \phi_{m,-})$. $\mathcal{L}_+$ is defined similarly for $(\alpha_{\mathbf{e}})_{\mathbf{e}\in\mathbf{E}_+}$, $(\beta_{\mathbf{e}})_{\mathbf{e}\in\mathbf{E}_+}$ and $(\phi_{\mathbf{e}})_{\mathbf{e}\in\mathbf{E}_+}$.

Thus, we are in a similar framework to that in (1.20) and so a natural extension of Definition 1.2 can be established in the case of stationary vector solutions for the KdV model (1.23) on metric graphs (Fig. 17).

Next, we establish our theoretical framework and assumptions for obtaining a non-trivial solution to problem in (6.20):

(S_1) Let $(A_{ext}, D(A_{ext}))$ be an extension of $(A_0, D(A_0))$ such that the solution of the linearized KdV model (6.2) is given by a C_0-group.

(S_2) Suppose $0 \neq \phi = (\phi_{\mathbf{e}})_{\mathbf{e}\in\mathbf{E}} \in D(A_{ext})$ such that $(\tilde{u}_{\mathbf{e}}(x,t))_{\mathbf{e}\in\mathbf{E}} = (\phi_{\mathbf{e}}(x))_{\mathbf{e}\in\mathbf{E}}$ is a stationary solution for the KdV model (1.23).

(S_3) Let E be defined on a domain $D(E) \subset L^2(\mathcal{G})$ on which E is self-adjoint and such that $D(A_{ext}) \subset D(E)$.

(S_4) Since for every $u \in D(A_{ext})$ we have $Eu \in D(N)$, we suppose $\langle NEu, \phi\rangle = 0$ for every $u \in D(A_{ext})$.

(S_5) Suppose $E : D(E) \to L^2(\mathcal{G})$ is invertible with Morse index $n(E)$ such that:

(a) for $n(E) = 1$, $\sigma(E) = \{\lambda_0\} \cup J_0$ with $J_0 \subset [r_0, +\infty)$ for $r_0 > 0$ and $\lambda_0 < 0$,

(b) for $n(E) = 2$, $\sigma(E) = \{\lambda_1, \lambda_2\} \cup J$ with $J \subset [r, +\infty)$ for $r > 0$ and $\lambda_1, \lambda_2 < 0$. Moreover, for $\Phi_1, \Phi_2 \in D(E) - \{0\}$ with $E\Phi_i = \lambda_i \Phi_i$ $(i = 1, 2)$ we have $\langle N\phi, \Phi_1\rangle \neq 0$ or $\langle N\phi, \Phi_2\rangle \neq 0$.

(S_6) For $\psi \in D(E)$ with $E\psi = \phi$, we have $\langle \psi, \phi\rangle \neq 0$.

(S_7) Suppose the operator $N : D(N) \cap D(E) \to L^2(\mathcal{G})$ is a skew-symmetric operator and also that N on $D(N)$ is one-to-one.

Next, we give the preliminaries for establishing our instability criterium in Theorem 6.2 below. The main idea in the following is to reduce our eigenvalue problem (6.20) to the orthogonal subspace $[\phi]^\perp$ by Assumption S_4). Thus we consider the orthogonal projection $Q : L^2(\mathcal{G}) \to L^2(\mathcal{G})$,

$$Q(u) = u - \langle u, \phi\rangle \frac{\phi}{\|\phi\|^2} \tag{6.23}$$

associated to the nontrivial stationary solution ϕ, and we also consider $X_2 = Q(L^2(\mathcal{G})) = \{f \in L^2(\mathcal{G}) : f \perp \phi\} = [\phi]^\perp$. We also define the closed skew-adjoint operator $N_0 : D(N_0) \subset X_2 \to X_2$, $D(N_0) \equiv D(N) \cap X_2$, for $f \in D(N_0)$ by

$$N_0 f \equiv QNf = Nf - \langle Nf, \phi\rangle \frac{\phi}{\|\phi\|^2}, \tag{6.24}$$

and the reduced self-adjoint operator for E, $F : D(F) \to X_2$, $D(F) = D(E) \cap X_2$ by

$$Ff \equiv QEf = Ef - \langle Ef, \phi\rangle \frac{\phi}{\|\phi\|^2}. \tag{6.25}$$

Now, we have the following basic assumption in the case $n(E) = 2$ in Assumption (S_5).

(H) There is a real number η, satisfying $\eta > 0$, such that $F : D(F) \to X_2$, $D(F) = D(E) \cap X_2$, it is invertible and with Morse index equal to one. Moreover, all the remainder of the spectrum is contained in $[\eta, +\infty)$.

Theorem 6.2 (1) *Suppose the Assumptions* (S_1)–(S_7) *hold with* $n(E) = 2$ *in Assumption* (S_5)*, and the basic Assumption* (H)*. Then the operator* NE *has a real positive and a real negative eigenvalue.*

(2) *Suppose* (S_1), (S_2), (S_3), (S_5), (S_7) *hold with* $n(E) = 1$. *Then the operator* NE *has a real positive and a real negative eigenvalue.*

The proof of Theorem 6.2 is based in ideas from Lopes [105] and from Krasnoelskii's result on closed convex cone in [92] (see Angulo and Cavalcante [26,27]).

6.3 Linear instability of tail and bump solutions for a balanced general star graph $\mathcal{G}$

The main result of this subsection is the following.

Theorem 6.3 *It considers the profiles* $\phi_\pm$ *in* (6.16) *with* $\alpha_+ = 1$, $\beta_+ = -1$ *and* $1 > \frac{Z^2}{4}$. *It defines* $\phi_Z = (\phi_{\mathbf{e}})_{\mathbf{e}\in\mathbf{E}} \in D(H_Z)$ *with* $\phi_{\mathbf{e}} = \phi_-$ *for* $\mathbf{e} \in \mathbf{E}_-$ *and* $\phi_{\mathbf{e}} = \phi_+$ *for* $\mathbf{e} \in \mathbf{E}_+$. *Then,*

$$\Phi_Z(x,t) = \phi_Z(x)$$

defines a family of linearly unstable stationary solutions for the Korteweg–de Vries model (1.23).

The linear instability of the *continuous (at zero) tail and bump profiles* ϕ_Z, $Z \neq 0$, in Theorem 6.3, it will be a consequence of Theorems 6.2 applied with a framework determined by the space $D(H_Z) \cap \mathcal{C}$.

For the general case of the sequences $(\alpha_{\mathbf{e}})_{\mathbf{e}\in\mathbf{E}} = (\alpha_1, \alpha_2)$ and $(\beta_{\mathbf{e}})_{\mathbf{e}\in\mathbf{E}} = (\beta_1, \beta_2)$, with $\alpha_2 = (\alpha_{1,+}, \dots, \alpha_{n,+}) = \alpha_1 = (\alpha_{1,-}, \dots, \alpha_{n,-})$ and $\beta_2 = (\beta_{1,+}, \dots \beta_{n,+}) = \beta_1 = (\beta_{1,-}, \dots \beta_{n,-})$, see [26].

We start our analysis by considering the $2n \times 2n$-matrix skew-symmetric operator N in (6.21) and the $2n \times 2n$-matrix Schrödinger operator

$$\mathcal{E}_Z = \begin{pmatrix} \mathcal{L}_{Z,-} & 0 \\ 0 & \mathcal{L}_{Z,+} \end{pmatrix} \tag{6.26}$$

with

$$\mathcal{L}_{Z,\pm} = \operatorname{diag}\left(-\frac{d^2}{dx^2} + 1 - 2\phi_\pm, \dots, -\frac{d^2}{dx^2} + 1 - 2\phi_\pm\right) \tag{6.27}$$

being $n \times n$-diagonal matrices. Via the extension theory for symmetric operators $(\mathcal{E}_Z, D(\mathcal{E}_Z))$ is a family of self-adjoint operators whose domain is $D(\mathcal{E}_Z) = D_{Z,\delta} \cap \mathcal{C} \subset H^2(\mathcal{G})$, with (see (6.15))

$$u \in D_{Z,\delta} \Leftrightarrow u(0-) = u(0+), \quad \sum_{\mathbf{e}\in\mathbf{E}_+} u'_{\mathbf{e}}(0+) - \sum_{\mathbf{e}\in\mathbf{E}_-} u'_{\mathbf{e}}(0-) = Znu_{1,+}(0+).$$

It is immediate from (6.13) that $D(H_Z) \cap \mathcal{C} \subset D(\mathcal{E}_Z)$ and so Assumption (S_3) holds. We also we obtain Assumption (S_4). Assumption (S_7) is immediate by the continuity property at zero of each element in $D(\mathcal{E}_Z)$. Moreover, from Proposition 7.10 in [26] we have that subspace $D(H_Z) \cap \mathcal{C}$ is invariant by the unitary group $\{W(t)\}_{t\in\mathbb{R}}$ generated by H_Z.

The following lemma shows part of Assumption (S_5).

Lemma 6.4 *Let $Z \neq 0$ and the operator $\mathcal{E}_Z : D(\mathcal{E}_Z) \to L^2(\mathcal{G})$ defined in (6.26) with $D(\mathcal{E}_Z) = D_{Z,\delta} \cap \mathcal{C}$. Then, $\mathcal{E}_Z$ is invertible with $\sigma_{ess}(\mathcal{E}_Z) = [1, +\infty)$.*

Proof It is sufficient to consider the case of two half-lines, $\mathbf{E} = (-\infty, 0) \cup (0, \infty)$. Thus, let $u = (u_-, u_+) \in D(\mathcal{E}_Z)$, $\mathcal{E}_Z u = 0$. Since $\mathcal{L}_\pm \phi'_\pm = 0$, we need to have $u_-(x) = a\phi'_-(x)$, $x < 0$, and $u_+(x) = b\phi'_+(x)$, $x > 0$ (see [48]). Next, from the continuity property at zero for u, $\phi'_+(0+) = -\phi'_-(0-)$, and $\phi''_+(0+) = \phi''_-(0-)$ it follows

$$a = -b \quad \text{and} \quad -2a\phi''_+(0+) = Zu_+(0+) = Zu_-(0-) = Za\phi'_-(0-) = -\frac{Z}{2}a\phi_+(0+). \tag{6.28}$$

Suppose $a \neq 0$. Then, from (6.28) we have $\phi''_+(0+) = \frac{Z^2}{4}\phi_+(0+)$ and so from (6.4) and (6.16) we arrive to

$$1 - \phi_+(0+) = \frac{Z^2}{4} \Longrightarrow Z^2 = 4,$$

which does not happen ($1 > \frac{Z^2}{4}$). Then, $a = b = 0$ and $u \equiv 0$.

Next, by Weyl's theorem (see Theorem XIII.14 of [122]), the essential spectrum of $\mathcal{E}_Z$ coincides with $[1, +\infty)$. Then $\mathcal{E}_Z$ is an invertible operator. This finishes the proof. □

Proposition 6.5 *Let $\mathcal{E}_Z : D(\mathcal{E}_Z) \to L^2(\mathcal{G})$ defined in (6.26) with $D(\mathcal{E}_Z) = D_{Z,\delta} \cap \mathcal{C}$. Define the following closed subspace on $L^2(\mathcal{G})$,*

$$L^2_n(\mathcal{G}) = \{u = (u_\mathbf{e})_{\mathbf{e}\in\mathbf{E}} : u_\mathbf{e} = f \ \textit{ for all } \mathbf{e} \in \mathbf{E}_-, \ \ \textit{and} \ \ u_\mathbf{e} = g \ \textit{ for all } \mathbf{e} \in \mathbf{E}_+\}$$

Then, $n(\mathcal{E}_Z|_{L^2_n(\mathcal{G})}) = 2$ for $Z > 0$, and $n(\mathcal{E}_Z|_{L^2_n(\mathcal{G})}) = 1$ for $Z < 0$.

The proof of Proposition 6.5 will based in the analytic perturbation theory and the extension theory of symmetric operators. We note that in the case $Z < 0$ (tail case) can be given an argument based exclusively in the extension theory of symmetric operators to be obtained that $n(\mathcal{E}_Z) = 1$ on $L^2(\mathcal{G})$ (see Angulo and Cavalcante [26]).

In the case $n = 1$ in Proposition 6.5 (two half-lines), we have $L^2_1(\mathcal{G}) = L^2(\mathcal{G})$. The proof of Proposition 6.5 will be divided in several lemmas.

Lemma 6.6 *Define the following self-adjoint matrix Schrödinger operator in $L^2(\mathcal{G})$ with Kirchhoff's type condition at $\nu = 0$*

$$\mathcal{E}_0 = \begin{pmatrix} \mathcal{L}_{0,-} & 0 \\ 0 & \mathcal{L}_{0,+} \end{pmatrix} \tag{6.29}$$

where

$$\mathcal{L}_{0,\pm} = diag\left(-\frac{d^2}{dx^2} + 1 - 2\phi_0, \ldots, -\frac{d^2}{dx^2} + 1 - 2\phi_0\right), \tag{6.30}$$

being $n \times n$-diagonal matrices, ϕ_0 is the classical one soliton solution for the KdV equation on the full line,

$$\phi_0(x) = \frac{3}{2}sech^2\left(\frac{1}{2}x\right) \quad x \in \mathbb{R}, \tag{6.31}$$

and

$$D(\mathcal{E}_0) = \left\{ u \in H^2(\mathcal{G}) \cap \mathcal{C} : u(0-) = u(0+), \sum_{\mathbf{e}\in\mathbf{E}_+} u'_{\mathbf{e}}(0+) - \sum_{\mathbf{e}\in\mathbf{E}_-} u'_{\mathbf{e}}(0-) = 0 \right\}. \tag{6.32}$$

(1) *In the space $L^2_n(\mathcal{G})$ we have $Ker(\mathcal{E}_0) = span\{\Phi'_0\}$, where $\Phi'_0 = (\phi'_0)_{\mathbf{e}\in\mathbf{E}}$.*
(2) *The operator $(\mathcal{E}_0, D(\mathcal{E}_0))$ has one simple negative eigenvalue in $L^2(\mathcal{G})$. Moreover, we also have $n(\mathcal{E}_0|_{L^2_n(\mathcal{G})}) = 1$.*
(3) *The rest of the spectrum of $\mathcal{E}_0$ is positive and bounded away from zero.*

Proof The proof of item (1) follows from a similar analysis as in Lemma 6.4.

For item (2), we use extension theory for symmetric operators. Indeed, we consider the $2n \times 2n$-diagonal matrix operator

$$\mathcal{F}_0 = diag\left(-\frac{d^2}{dx^2}, \dots, -\frac{d^2}{dx^2}\right), \tag{6.33}$$

with domain

$$D(\mathcal{F}_0) = \left\{ u \in H^2(\mathcal{G}) : u(0-) = u(0+) = 0, \sum_{\mathbf{e}\in\mathbf{E}_+} u'_{\mathbf{e}}(0+) - \sum_{\mathbf{e}\in\mathbf{E}_-} u'_{\mathbf{e}}(0-) = 0 \right\}. \tag{6.34}$$

Then $(\mathcal{F}_0, D(\mathcal{F}_0))$ represents a closed symmetric operator densely defined on $L^2(\mathcal{G})$ (we note that $\bigoplus_{\mathbf{e}\in\mathbf{E}_-} C_c^\infty(-\infty, 0) \oplus \bigoplus_{\mathbf{e}\in\mathbf{E}_+} C_c^\infty(0, +\infty) \subset D(\mathcal{F}_0)$). Moreover, the adjoint operator $(\mathcal{F}_0^*, D(\mathcal{F}_0^*))$ is given by (see Proposition 7.4 in "Appendix")

$$\mathcal{F}_0^* = \mathcal{F}_0, \quad D(\mathcal{F}_0^*) = \{u \in H^2(\mathcal{G}) : u \in \mathcal{C}\}. \tag{6.35}$$

Next, from (6.35), the deficiency indices for $(\mathcal{F}_0, D(\mathcal{F}_0))$ are $n_\pm(\mathcal{F}_0) = 1$. Then, from the Krein–von Neumann extension theory for symmetric operators we obtain that all self-adjoint extension of $(\mathcal{F}_0, D(\mathcal{F}_0))$, denoted by $(\mathcal{L}_Z, D(\mathcal{L}_Z))$, can be parametrized by $Z \in \mathbb{R}$ as $\mathcal{L}_Z = \mathcal{F}_0$, and $u \in D(\mathcal{L}_Z)$ if and only if $u \in \mathcal{C}$ and satisfies (6.15). Next, we define the following bounded operator on $L^2(\mathcal{G})$

$$\mathcal{B}_0 = \begin{pmatrix} M_{0,+} & 0 \\ 0 & M_{0,-} \end{pmatrix}, \quad M_{0,\pm} = \text{diag}\,(1 - 2\phi_0, \dots, 1 - 2\phi_0)$$

with $M_{0,\pm}$ being $n \times n$-diagonal matrices. Then, from [113]-Chapter IV-Theorem 6, it follows that the symmetric operators $\mathcal{F}_0$ and $\tilde{\mathcal{F}}_0 = \mathcal{F}_0 + \mathcal{B}_0$ with $D(\tilde{\mathcal{F}}_0) = D(\mathcal{F}_0)$, have the same deficiency indices, $n_\pm(\tilde{\mathcal{F}}_0) = n_\pm(\mathcal{F}_0) = 1$. Thus $(\mathcal{E}_0, D(\mathcal{E}_0))$ belongs to the family of the self-adjoint extensions of $\tilde{\mathcal{F}}_0$.

Now we see that the symmetric operator $\tilde{\mathcal{F}}_0$ with domain $D(\tilde{\mathcal{F}}_0) = D(\mathcal{F}_0)$ in (7.3) is non-negative. Indeed, it is easy to verify that for $u = (u_{\mathbf{e}})_{\mathbf{e}\in\mathbf{E}} \in H^2(\mathcal{G})$ the following identity holds

$$- u''_{\mathbf{e}} + u_{\mathbf{e}} - 2\phi_0 u_{\mathbf{e}} = -\frac{1}{\phi'_0}\frac{d}{dx}\left[(\phi'_0)^2 \frac{d}{dx}\left(\frac{u_{\mathbf{e}}}{\phi'_0}\right)\right], \tag{6.36}$$

for $x < 0$ if $\mathbf{e} \in \mathbf{E}_-$, $x > 0$ if $\mathbf{e} \in \mathbf{E}_+$. Using the above equality and integrating by parts, we get for $u = (u_{\mathbf{e}})_{\mathbf{e}\in\mathbf{E}} \in D(\tilde{\mathcal{F}}_0)$ that

$$\begin{aligned}\langle \tilde{\mathcal{F}}_0 u, u\rangle &= \sum_{\mathbf{e}\in\mathbf{E}_-}\int_{-\infty}^{0} (\phi'_0)^2 \left|\frac{d}{dx}\left(\frac{u_{\mathbf{e}}}{\phi'_0}\right)\right|^2 dx + \sum_{\mathbf{e}\in\mathbf{E}_+}\int_{0}^{+\infty} (\phi'_0)^2 \left|\frac{d}{dx}\left(\frac{u_{\mathbf{e}}}{\phi'_0}\right)\right|^2 dx \\ &\quad - \sum_{\mathbf{e}\in\mathbf{E}_-}\left[\frac{u_{\mathbf{e}}}{\phi'_0}\left[(\phi'_0)^2\frac{d}{dx}\left(\frac{u_{\mathbf{e}}}{\phi'_0}\right)\right]\right]_{-\infty}^{0-} - \sum_{\mathbf{e}\in\mathbf{E}_+}\left[\frac{u_{\mathbf{e}}}{\phi'_0}\left[(\phi'_0)^2\frac{d}{dx}\left(\frac{u_{\mathbf{e}}}{\phi'_0}\right)\right]\right]_{0+}^{+\infty}.\end{aligned} \tag{6.37}$$

The integral terms in (6.37) are non-negative and equal zero if and only if $u \equiv 0$. Since $u(0-) = u(0+) = 0$ and $\phi''_0(0\pm) \neq 0$, the non-integral term vanishes and we obtain $\tilde{\mathcal{F}}_0 \geqq 0$.

Due to Proposition 7.3 in “Appendix”, we have that the self-adjoint extension $\mathcal{E}_0$ of $\tilde{\mathcal{F}}_0$ satisfies $n(\mathcal{E}_0) \leqq 1$. Taking into account the notation $\Phi_0 = (\phi_0)_{\mathbf{e}\in\mathbf{E}}$ for the “solitary wave profile” we have $\mathcal{E}_0\Phi_0 = \Psi$, $\Psi = (-\phi_0^2)_{\mathbf{e}\in\mathbf{E}}$, and so

$$\langle \mathcal{E}_0\Phi_0, \Phi_0\rangle = -n\int_{-\infty}^{0} \phi^3_{0,-}(x)dx - n\int_{0}^{+\infty} \phi^3_{0,+}(x)dx < 0,$$

then from minimax principle one gets $n(\mathcal{E}_0) = 1$. Moreover, since $\Phi_0 = (\phi_0)_{\mathbf{e}\in\mathbf{E}} \in L^2_n(\mathcal{G})$ we get $n(\mathcal{E}_0|_{L^2_n(\mathcal{G})}) = 1$.

Item 3) is an immediate consequence of Weyl’s theorem (see [122]). This finishes the proof. □

Combining Lemma 6.6 and the framework of the perturbation theory as in Sect. 5.4.2 above (see [26]) we obtain the next result. We note initially that is not difficult to see the convergence $\phi_Z = (\phi_-, \phi_+)_{\mathbf{e}\in\mathbf{E}} \to \Phi_0 = (\phi_0)_{\mathbf{e}\in\mathbf{E}}$ as $Z \to 0$, in $H^1(\mathcal{G}) \cap L^2_n(\mathcal{G})$.

Lemma 6.7 *There exist $Z_0 > 0$ and two analytic functions $\Theta : (-Z_0, Z_0) \to \mathbb{R}$ and $\Upsilon : (-Z_0, Z_0) \to L^2_n(\mathcal{G})$ such that*

(i) $\Theta(0) = 0$ *and* $\Upsilon(0) = \Phi'_0$, *where* $\Phi'_0 = (\phi'_0)_{\mathbf{e}\in\mathbf{E}}$.
(ii) *For all $Z \in (-Z_0, Z_0)$, $\Theta(Z)$ is the simple isolated second eigenvalue of $\mathcal{E}_Z$ in $L^2_n(\mathcal{G})$, and $\Upsilon(Z)$ is the associated eigenvector for $\Theta(Z)$.*

(iii) *Z_0 can be chosen small enough to ensure that for $Z \in (-Z_0, Z_0)$ the spectrum of $\mathcal{E}_Z$ in $L^2_n(\mathcal{G})$ is positive, except at most the first two eigenvalues.*
(iv) *Since* $\lim_{Z\to 0}\langle N\phi_Z, \Upsilon(Z)\rangle = \|\Phi_0\|^2 > 0$ *we obtain that*

$$\langle N\phi_Z, \Upsilon'(Z)\rangle \neq 0 \tag{6.38}$$

at least for Z small. Thus, a continuation argument shows (6.38) for all Z.

We note that relation (6.38) shows part (b) in Assumption (S_5). Now, by using the Taylor's theorem and by following a similar argument as in Lemma 5.3 in [26] we establish how the perturbed second eigenvalue moves depending on the sign of Z.

Lemma 6.8 *There exists $0 < Z_1 < Z_0$ such that $\Theta(Z) > 0$ for any $Z \in (-Z_1, 0)$ and $\Theta(Z) < 0$ for any $Z \in (0, Z_1)$. Thus, in the space $L^2_n(\mathcal{G})$ for Z small, we have $n(\mathcal{E}_Z) = 1$ as $Z < 0$, and $n(\mathcal{E}_Z) = 2$ as $Z > 0$.*

Proof of Proposition 6.5 From Lemma 6.8 we have for Z small that $n(\mathcal{E}_Z) = 1$ as $Z < 0$, and $n(\mathcal{E}_Z) = 2$ as $Z > 0$. Thus for counting the same Morse index of $\mathcal{E}_Z$ for any Z we use a classical continuation argument based on the Riesz-projection as in Theorem 5.19 (see Lemma 5.3 in [26]). This finishes the proof. □

The following lemma shows Assumption (S_6). Initially, we consider $\omega = -\beta_+ \approx 1$ in (6.16). Then, we have the differentiable family of stationary solutions a one-parameter $\phi_{Z,\omega} = (u_{-,Z,\omega}, u_{+,Z,\omega})_{\mathbf{e}\in\mathbf{E}}$ with $u_{-,Z,\omega} = (\phi_{-,Z,\omega})_{\mathbf{e}\in\mathbf{E}_-}$ and $u_{+,Z,\omega} = (\phi_{+,Z,\omega})_{\mathbf{e}\in\mathbf{E}_+}$, with $\phi_{+,Z,\omega}$ defined in (6.16), $\phi_{-,Z,\omega}(x) \equiv \phi_{+,Z,\omega}(-x)$, $x < 0$ and $\omega > \frac{Z^2}{4}$. Then, for $\varphi_\omega = (-\frac{d}{d\omega}u_{-,Z,\omega}, -\frac{d}{d\omega}u_{+,Z,\omega})_{\mathbf{e}\in\mathbf{E}}$ we have $\varphi \equiv \varphi_\omega|_{\omega=1} \in D(\mathcal{E}_Z)$ and $\mathcal{E}_Z\varphi = \phi_Z$. Thus with the former notation, we obtain immediately the following result.

Lemma 6.9 *Let $Z \neq 0$. The smooth curve of profiles $\omega \in (\frac{Z^2}{4}, +\infty) \to \phi_{Z,\omega} = (\phi_{-,Z,\omega}, \phi_{+,Z,\omega})_{\mathbf{e}\in\mathbf{E}}$ satisfies for $\varphi \equiv -\frac{d}{d\omega}\phi_{Z,\omega}|_{\omega=1}$ the relations*

$$\mathcal{E}_Z\varphi = \phi_Z \quad \textit{and} \quad \langle \varphi, \phi_Z\rangle < 0. \tag{6.39}$$

Proof of Theorem 6.3 Let $Z > 0$. From Lemmas 6.4–6.9, Proposition 6.5, relation (6.38) and Theorem 6.2 we obtain the linear instability property of the bump's profiles ϕ_Z for the KdV model (1.23). Let $Z < 0$, then from Lemmas 6.4–6.9 and Proposition 6.5 we obtain via Theorem 6.3-(2) the linear instability of the tail's profiles ϕ_Z. This finishes the proof. □

6.4 End-section notes

(a) The case $n = 1$ in Theorem 6.3 can be showed via the analytic perturbations theory of operators, while the case of $n \geqq 2$, analytic perturbation and the extension theory of symmetric operators of Krein and von Neumann are required.
(b) The orbital instability implication from the spectral one established by Theorem 6.3, is a open problem.

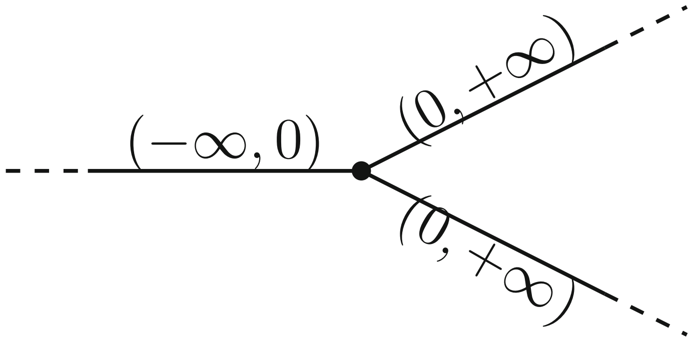

Fig. 17 Y-Josephson junction

(c) The existence and stability of other families of stationary profiles for the KdV model (1.23) defined on a different graph-geometry (balance or non-balanced graphs) are the objective of several works in progress. Moreover, the existence and stability of stationary profiles for the generalized KdV model $\partial_t u_{\mathbf{e}} = \alpha_{\mathbf{e}}\partial_x^3 u_{\mathbf{e}} + \beta_{\mathbf{e}}\partial_x u_{\mathbf{e}} + p u_{\mathbf{e}}^{p-1}\partial_x u_{\mathbf{e}}$, $\mathbf{e} \in \mathbf{E}$, $p \in \mathbb{N}$, $p \geqq 2$, are also very interesting problems.

(d) In Angulo and Plaza [39,40] was studied recently the sine-Gordon model (1.24) in the case of Y-Josephson junction geometry (see Fig. 16). We recall that model (1.24) was first conceived by Nakajima et al. [115,116] as a prototype for logic circuits. In [39,40] was obtained the linear instability (also nonlinear instability) of kink-profile for (1.24) of the form $(u_{\mathbf{e}}(x,t))_{\mathbf{e}\in\mathbf{E}} = (\phi_{\mathbf{e}}(x))_{\mathbf{e}\in\mathbf{E}}$ and $(v_{\mathbf{e}}(x,t))_{\mathbf{e}\in\mathbf{E}} = (0)_{\mathbf{e}\in\mathbf{E}}$, with

$$\phi_{\mathbf{e}}(x) = 4arctan\left(e^{\frac{1}{|c_{\mathbf{e}}|}(x+a_{\mathbf{e}})}\right) \quad \mathbf{e} \in \mathbf{E}, \tag{6.40}$$

satisfying $-c_{\mathbf{e}}^2\phi_{\mathbf{e}}'' + sin(\phi_{\mathbf{e}}) = 0$, $\mathbf{e} \in \mathbf{E} = (-\infty, 0) \cup (0, +\infty) \cup (0, +\infty)$. Here, the shift $a_{\mathbf{e}}$ will depend of the conditions determined on the vertex $\nu = 0$ of the Y-junction. Between the several domains worked in [39,40], we have the δ-type interaction in the vertex $\nu = 0$,

$$\left\{(v_j)_{j=1}^3 \in H^2(\mathcal{G}) : v_1(0-) = v_2(0+) = v_3(0+), \sum_{j=2}^3 c_j^2 v_j'(0+) - c_1^2 v_1'(0-) = Z v_1(0-)\right\} \tag{6.41}$$

with $Z \in \mathbb{R}$.

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7 Appendix

Next, for convenience of the reader and because of non-standard results used in the body of this manuscript we formulate the following results of the extension theory (see [113]). The first one reads as follows.

Theorem 7.1 (von-Neumann decomposition) *Let A be a closed, symmetric operator, then*

$$D(A^*) = D(A) \oplus \mathcal{N}_{-i} \oplus \mathcal{N}_{+i}, \tag{7.1}$$

with $\mathcal{N}_{\pm i} = Ker(A^* \mp iI)$. Therefore, for $u \in D(A^*)$ and $u = x + y + z \in D(A) \oplus \mathcal{N}_{-i} \oplus \mathcal{N}_{+i}$,

$$A^*u = Ax + (-i)y + iz. \tag{7.2}$$

Remark 7.2 The direct sum in (7.1) is not necessarily orthogonal.

Our second result of the extension theory of symmetric operators, it gives us a strategy for estimating the Morse-index of the self-adjoint extensions.

Proposition 7.3 *Let A be a densely defined lower semi-bounded symmetric operator (that is, $A \geq mI$) with finite deficiency indices $n_\pm(A) = k < \infty$ in the Hilbert space $\mathcal{H}$, and let $\widetilde{A}$ be a self-adjoint extension of A. Then the spectrum of $\widetilde{A}$ in $(-\infty, m)$ is discrete and consists of at most k eigenvalues counting multiplicities.*

The following result was used in the proof of Lemma 6.6 (see [26]).

Proposition 7.4 *Let $\mathcal{G}$ be a balanced star-shaped metric graph with a structure represented by the set $\mathbf{E} \equiv \mathbf{E}_- \cup \mathbf{E}_+$ and $|\mathbf{E}_-| = |\mathbf{E}_+| = n$. The $2n \times 2n$-diagonal-matrix Schrödinger operator on $L^2(\mathcal{G})$*

$$\mathcal{F}_0 = \left(\left(-\frac{d^2}{dx^2}\right)\delta_{i,j}\right)$$

with domain

$$D(\mathcal{F}_0) = \left\{u \in H^2(\mathcal{G}) : u(0-) = u(0+) = 0, \sum_{\mathbf{e}\in\mathbf{E}_+} u'_{\mathbf{e}}(0) - \sum_{\mathbf{e}\in\mathbf{E}_-} u'_{\mathbf{e}}(0) = 0\right\}, \tag{7.3}$$

is a densely defined symmetric operator with deficiency indices $n_\pm(\mathcal{F}_0) = 1$. Therefore, all the self-adjoint extensions of $(\mathcal{F}_0, D(\mathcal{F}_0))$ can be parametrized by $Z \in \mathbb{R}$, namely, $(\mathcal{L}_Z, D(\mathcal{L}_Z))$, with the action $\mathcal{L}_Z \equiv \mathcal{F}_0$ and $u \in D(\mathcal{L}_Z)$ if and only if $u \in \mathcal{C} \cap D_{Z,\delta}$,

$$D_{Z,\delta} = \left\{u \in H^2(\mathcal{G}) : u(0-) = u(0+), \sum_{\mathbf{e}\in\mathbf{E}_+} u'_{\mathbf{e}}(0+) - \sum_{\mathbf{e}\in\mathbf{E}_-} u'_{\mathbf{e}}(0-) = Znu_{1,+}(0+)\right\}. \tag{7.4}$$

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