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Extensions of immersions in dimension two

Vera L. Carrara Zanetic

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EXTENSIONS OF IMMERSIONS IN DIMENSION TWO

Vera Lucia Cárrara Zanetic

Introduction

Let V and N be (C^∞-)manifolds such that $\dim V \leq \dim N$ and let C be a submanifold of V. The general problem of extension of immersion is the following. Given an (C^∞-) immersion $f: C \rightarrow N$ is there an immersion $F: V \rightarrow N$ such that $F|_C = f$? We note that this problem for $\dim V = \dim N$ is quite different from the case $\dim V < \dim N$. While the second case reduces to an homotopy problem, via the Hirsh-Smale theory [8,12], the first one does not. We refer to Poenaru [11] for this discussion.

The problem we are concerned with is the above one in the case $C = \partial V$, $\dim V = \dim N = 2$, together with the classification of the extensions. Thus, we prove a theorem of classification that generalizes and simplifies previous results in this direction. Blank [1] provided in 1967, a complete solution (with classification) for the case $V = D^2$ and $N = R^2$ (*). Francis [4] extended the solution to any oriented, connected and compact V and $N = R^2$ or S^2. Marx and Ezell [2] obtained existence conditions for extensions, when N is oriented, compact and connected of any genus (**). We also mention Ezell's work [3] for

(*) An existence solution has been given by Titus [13].

(**) We should mention that Marx and Francis' works are concerned with a broader class of extensions.

the non-oriented case and Francis-Troyer's one [5] for the proper case $N = \mathbb{R}^2$, V non-compact with compact boundary and finite number of ends. In this paper we classify the proper orientation preserving extensions of $f: C \longrightarrow N$ normal and proper, with V and N oriented, connected and either compact or surfaces of finite type. A non-compact surface (=2-dimensional manifold) has finite type when it is equal to a compact one minus a finite number of points.

The combinatorial structures used for the study of extensions had their development from Blank's words and grouping of subwords up to Marx and Francis' assemblages. The assemblages, or groupings in our case of immersions [10] are based on permutations belonging to groups of permutations associated to f. Our methods are essentially the same as those used by the mentioned authors. However, we use a special kind of groupings, the (reduced) simple groupings (*), which are the ones that classify the extensions. Furthermore, using simple groupings, we eliminate the hypothesis on winding numbers that was necessary, in all previous works, to provide an (immersed) extension [1,2,4]. As a result, we obtain a completely combinatorial answer. The theorem of classification is stated and proved in section 4.

This paper forms part (**) of my Doctoral Thesis [15]. I want to thank my adviser, Prof. Mauricio Matos Peixoto, for the suggestion of the problem and for all his guidance.

(*) Among the mentioned authors, only Blank has used reduced groupings. I found it necessary to reduce the groupings in order to define the simple ones.

(**) The other part, to appear, is the application of the present results to stable maps.

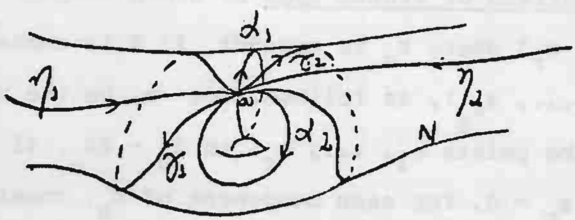
1. *Preliminaries. Normal curves.*

Most of the basic notation can be found in [9]. The category used is the C^∞-category.

A surface of finite type is a non-compact surface $X = X_o - \{e_1, \dots, e_r\}$ where X_o is compact. If X is connected, the type of X is $(a_1, \dots, a_{\delta_X})$, as follows. Let m be the number of endpoints of X (the points $e_1, \dots, e_r$) in $X_o - \partial X_o$. If $m \neq 0$, define $a_1 = \dots = a_m = 0$. For each component of X_o, consider the number of endpoints of X on it. Order the nonzero number so obtained, if any, as $a_{m-1} \leq \dots \leq a_{\delta_X}$.

Let N be a connected surface of genus g_N without boundary, which is either compact or non-compact of finite type. We are interested in some collection of curves in N. Let ∞ be an arbitrary point in N. If N is compact of positive genus then $\pi_1(N,\infty)$ has a minimal set of generators represented by a collection of simple regular closed arcs $\alpha_1, \dots, \alpha_{2g_N}$ with both origin and end at ∞, and otherwise disjoint, such that $N - \bigcup_{i=1}^{2g_N} \alpha_i$ is an open disc. Let N have finite type. Then a natural set of generators of $\pi_1(N,\infty)$ is the one represented by simple regular closed arcs $\alpha_1, \dots, \alpha_{2g_N}, \gamma_1, \dots, \gamma_{\delta_N}$ (delete the α_i-curves if $g = 0$) with both origin and end at ∞, otherwise disjoint, such that $N - ((\bigcup_{i=1}^{2g_N} \alpha_i) \cup (\bigcup_{i=1}^{\delta_N} \gamma_i))$ is a disjoint union of an open disc and δ_N homeomorphic cylinders. We orient γ_i so that the disc stands to its left-side. When $\delta_N > 0$, we also consider, for each

$i = 1, \ldots, \delta_N$, a regular arc $\eta_i = [e_i, \infty]$, that is, an arc from the endpoint e_i to the point ∞. We refer to such a collection of arcs in N as a <u>special set of arcs</u>. Note that, if we take off N a special set of arcs we get a collection of open discs.



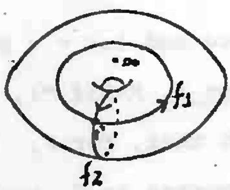
Let C be the disjoint union $C = \bigcup_{i=1}^{\rho} C_i$ of copies of S^1 and $\mathbb{R}$. A <u>normal curve</u> $f: C \to N$ is a family of proper and regular curves $f = \{f_i\}_{i=1}^{\rho}$, $f_i : C_i \to N$, with only a finite number of transverse double points as crossings, including self-intersections. Two normal curves are <u>equivalent</u>, $f \sim f'$, if there exists an orientation preserving diffeomorphism $\alpha : C \to C'$ such that $f = f' \circ \alpha$.

Let V be a bordered surface that is either compact or of finite type. Let $F: V \to N$ be a proper orientation preserving immersion. We refer to F as an <u>immersed extension of f</u> when $F|_{\partial V} = f$. Two such extensions are <u>equivalent</u>, $F \sim F'$, if there exists an orientation preserving diffeomorphism $\phi: V \to V'$ such that $F = F' \circ \phi$. We denote by $E(f)$ the set of those extensions of f with V connected and by $\mathcal{E}(f)$ the set of its equivalence classes.

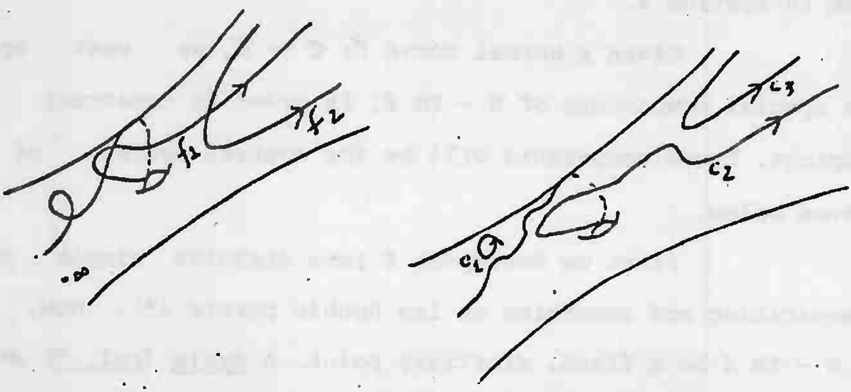
The immersed extensions of a paired normal curve will be classified by its simple transitive groupings. These will be defined in section 2. The theorem of classification will be given in section 4.

Given a normal curve $f\colon C \to N$, we must specify some special components of $N - \operatorname{Im} f$, in order to construct the groupings. Those components will be the central cycles of f, defined below.

First we decompose f into disjoint simple curves by separating and smoothing at its double points (*). Now, let $\infty \in N - \operatorname{Im} f$ be a fixed, arbitrary point. A <u>cycle</u> (<u>rel. ∞</u>) of f is any of those simple curves that borders an open, maybe unbounded, homeomorphic disc in $N - \{\infty\}$. Hence, a cycle may be an unbounded arc. Denote by <u>int c</u> that disc and let D be its closure. The cycle c is <u>positive</u> if $c = \partial D$ and <u>negative</u> if $c = -\partial D$ (the sign reverses orientation). The cycle c is a <u>central cycle</u> if $c' \not\subset \operatorname{int} c$, for any cycle c' of f. Note that, up to smoothing, a central cycle of f is either the oriented or the negatively oriented boundary of a component of $N - \operatorname{Im} f$ homeomorphic to a disc in $N - \{\infty\}$. Note also that if f has a cycle c, then f has a central cycle $c' \subset \overline{\operatorname{int} c}$. If $\operatorname{Im} f$ is contained in a disc in $N - \{\infty\}$, then all of its simple curves are cycles.



(*) When $f\colon C \to N$ is closed, that is, C is the union of copies of S^1, these simple curves are known as Gaussian circles of f[2,4].



A <u>partial cycle of f</u> is given similarly to the cycles, except that the decomposition is applied to a subset of the set of double points of f. Properties of the partial cycles as well as the properties of the cycles of f provide complete proofs of some results in this text. We mention that if f has a partial cycle c then f has a cycle $c' \subset \overline{\text{int } c}$.

2. *Groupings*

In this section we construct groupings for normal curves.

Let $f\colon C \longrightarrow N$ be a normal curve and let $\infty \in N - \operatorname{Im} f$ be a fixed, arbitrary point. A <u>set of rays for f</u>, $R=R(f,\infty)$, is a finite set of oriented simple arcs in N such that, for r, $r' \in R$, $v \in r \cap \operatorname{Im} f$, it follows that r is tranverse to f, $r \cap r' = \{\infty\}$ ir $r \neq r'$, and v is not a double point of f. Moreover, we require that $R = R_c \cup R_e$, where $r \in R_c$ is an arc with origin in a connected component of $N - \mathrm{Im}\ f$ and end at ∞. Also, $R_e = \emptyset$ if $g_N = \delta_N = \emptyset$, and otherwise R_e is a special set of arcs in N such that the bounded component of $N - \bigcup_{i=1}^{\delta_N} \gamma_i$ contains all double points of f and all rays of R_c.

A set of rays for f is <u>sufficient</u>(*) if, for each central cycle of f, R contains at least a ray with origin in int c.

A normal curve is <u>paired</u> (<u>rel.</u>R_e) if either $R_e = \emptyset$ or the number of positive and negative crossings in each closed ray $r \in R_e$ are equal. As we shall see, the normal curves which are not paired do not extend.

On the other hand, f may have extensions of non-zero degree at ∞. Thus, for each integer $\beta \geq 0$ we consider an <u>augmented curve of f</u>. This is a normal curve g^β that has for component curves those of f and also, when $\beta > 0$, β disjoint homeomorphic circles $w_1, \ldots, w_\beta$ contained in a small open disc $\mathcal{U}$ in $N - \mathrm{Im}\ f$, such that $\infty \in \mathrm{int}\ w_i \subset \mathcal{U}$ and $w_i = -\partial D_i$, where $D_i = \overline{\mathrm{int}\ w_i}$, for $i=1, \ldots, \beta$.

Let R be sufficient and let g^β be an augmented curve of f such that each $r \in R$ is a transverse to g^β. Let $Q = \mathrm{Im}\ g^\beta \cap (\bigcup_{r \in R} r)$ be the set of all crossings. Define <u>sign</u> $v = 1$, if v is a positive crossing of $r \in R$ with $\mathrm{Im}\ g^\beta$. This means that a nonzero tangent vector to r and a nonzero tangent vector to $\mathrm{Im}\ g^\beta$ form, in this order, a positive basis for the tangent space

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(*) This condition of sufficiency differs from that in the litterature [2,4].

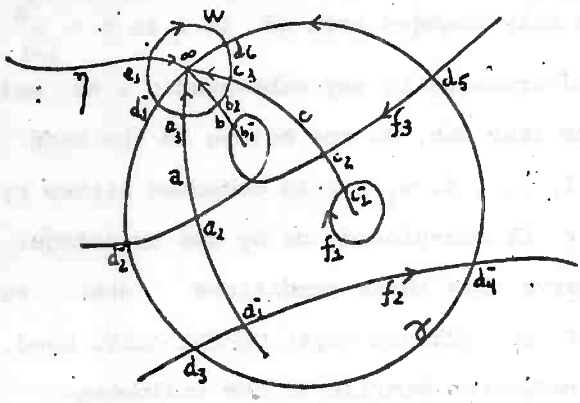
of N at v. Otherwise, define <u>sign</u> v = -1. We also define <u>index</u> v = i, if v is the $i^{\underline{th}}$ crossing in $r \in R$.

Let $\mathcal{S}_{\mathcal{C}}$ be the symmetric group on $\mathcal{C}$ endowed with the usual product of permutations [14] . Let $S \in \mathcal{S}_{\mathcal{C}}$ be the sucessor permutation: if $v \in \operatorname{Im} g_i$, $i=1, \ldots, \rho+\beta$, then $vS = v' \in \operatorname{Im} g_i$, where v' is the first crossing in Im g_i, when g_i is unbounded and v is the last crossing in it, and v' is the crossing next to v in Im g_i in all other cases. A <u>grouping permutation for g^β</u> is a permutation $P \in \mathcal{S}_{\mathcal{C}}$ given, as a product of disjoint cycles, by trivial cycles and transpositions (uv) such that $u, v \in r$, for some $r \in R$, and sign u $\neq$ sign v. We call (uv) a <u>pair of P</u>. Abstractly, we also say that P is a <u>grouping permutation on</u> (the finite set) $\underline{\mathcal{C}}$.

A <u>β-grouping for f</u>, A = A(R,f) is a pair (β,P), where β is a non-negative integer and P is a grouping permutation for g^β. We also say that A is a <u>grouping on $\mathcal{C}$</u>. Call <u>orbits of A</u> the disjoint cycles of SP.

As an example, let $f = \{f_1, f_2, f_3\} : S^1 \cup R \cup R \longrightarrow R^2$ be as in the figure. Consider the augmented curve g of f given by $g_i=f_i$, i=1,2,3 and g_4=w. Consider also the sufficient set of rays R ={γ,η,a,b,c}. Then

$$S = (c_1^-)\ (d_3 a_1^- d_4^-)\ (d_5 c_2 b_1^-\ a_2 d_2^-)\ (e_1 d_6 c_3 b_2 a_3 d_1^-).$$



Taking

$$P = (a_1^- \; a_3)\ (b_1^- \, b_3)\ (c_1^- \; c_2)\ (d_1^- \; d_3)\ (d_2^- \; d_6)\ (d_4^- \; d_5)$$

we obtain the grouping $A = (1,P)$ such that

$$SP = (c_1^- \; c_2 b_2 a_1^- \; d_5)\ (d_3 a_3)\ (b_1^- \; a_2 d_6 c_3)\ (d_4 d_1^- \; e_1^- \; d_2^-\ ).$$

Note that the orbits of A correspond to the (geometric) orbits of g (in next section). A is also effective and transitive in the following sense.

A grouping $A = (\beta,P)$ is <u>effective</u> if P moves every $u \in \mathcal{C}$ with sign $u = -1$, and all pairs of P are positive. A pair (uv) with sign $u = -1$ is <u>positive</u> if index $u <$ index v. A is <u>transitive</u> if S and P generate a transitive subgroup of $\mathcal{S}_{\mathcal{C}}$. This means that for all $v, v' \in \mathcal{C}$ there exists $Q=Q_1 \ldots Q_m \in \mathcal{S}_{\mathcal{C}}$ with $Q_j = S, S^{-1}$ or P and $v' = vQ$.

We shall consider reduced groupings instead of the defined groupings. First, we assume from now on, that the two following pre-reduction conditions are satisfied. If $\delta_N > 0$, the rays η_i, $i=1, \ldots, \delta_N$, cross only bounded arcs of Im f in $N - \bigcup_{i=1}^{\delta_N} \gamma_i$, and the first and second crossing in any unbounded g_i, as well as the last and the next to last one, do not belong to the same ray. If $\beta > 0$, then for $i = 1, \ldots, \beta$, $w_i \cap r$ is composed either by a single crossing, when r is non-closed, or by two crossings, when r is a closed ray. Observe that these conditions are easily obtained by a sliding of the involved rays. On the other hand, they might be obtained by a reduction similar to the following.

We say that S is in reduced form if, whenever the crossings u and v belong to the same ray and sign $u \neq$ sign v, then $u \neq vS$ and $v \neq uS$. If S is not in reduced form, we reduce S by sucessive elimination from of crossings u, v in the same ray, such that either $u = vS$ or $v = uS$, and sign $u \neq$ sign v. Let $\mathcal{J}^* \subset \mathcal{J}$ be the reduced set thus obtained. Let S^* be the permutation in $\mathcal{S}_{\mathcal{J}^*}$ given by $vS^* = vS^k$, where k is the least positive integer such that $vS^k \in \mathcal{J}^*$.

$A^* = (\beta, P^*)$ is a reduced grouping for f if β is a non-negative integer and P^* is a grouping permutation on the reduced set $\mathcal{J}^*$ of the set $\mathcal{J}$ of crossing of g^β and R. In what follows we delet unnecessary stars.

Two reduced groupings $A = (\beta, P)$ and $A' = (\beta', P')$ are equivalent, $A \sim A'$, if $\beta = \beta'$ and either: $\beta = 0$ and $P = P'$, or $\beta > 0$ and there exists a permutation σ on $\{1, \ldots, \beta\}$ such that $\tau_\sigma P' = P\tau_\sigma$, where $\tau_\sigma \in \mathcal{S}_{\mathcal{J}}$ is given by $v\tau_\sigma = v'$ with either $v' = v$, if $v \in$ Im f, or $v' \in w_{i\sigma} \cap r$ and sign v' = sign v, if $v \in w_i \cap r$, $r \in R$. Note that equivalence between the groupings is just a way of permuting the circles w_i, $i = 1, \ldots, \beta$. Equivalence preserves effectiveness and transitivity.

We will refer to the first and last crossings on the unbounded components of f as <u>special crossings</u> of R with f. A <u>negative orbit</u> of an effective reduced grouping A is an orbit of A that contains either some negative special crossing or some fixed (by P) crossing of a ray η_i, if $\delta_N > 0$.

An effective reduced grouping $A = (\beta, P)$ for f is <u>simple</u> if it satisfies the following properties:

(i) whenever $u, v \in r$, $r \in R$, belong to some non-negative orbit of A, then sign $u \neq$ sign v, $u \neq uP$ and $v \neq vP$;

(ii) the negative orbits of A may only contain negative crossing of rays γ_i and fixed crossings of rays η_i, if $\delta_N > 0$ and $i = 1, \ldots, \delta_N$.

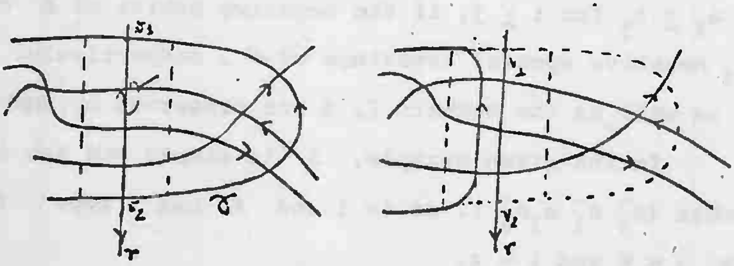
Simplicity is also preserved by equivalence.

Let $A = (\beta, P)$ be a simple grouping for f. We denote by ξ the number of orbits of A, by δ the number of its negative orbits and by ν the number of negative crossings in the reduced set $\mathcal{P}$. A simple grouping has <u>type</u> $(a_0, a_1, \ldots, a_\delta)$, with $a_0 = \delta$ and $a_i \leq a_j$ for $i \leq j$, if the negative orbits of A contain $a_1, \ldots, a_\delta$ negative special crossings of $\mathcal{P}$, respectively. The type of A, as well as the numbers ξ, δ are preserved by equivalence.

In the given example, A is simple and has a unique negative orbit $(d_4^- d_1^- e_1 d_2^-)$, so $\delta = 1$ and A has type (1, 2). Furthermore, $\nu = 6$ and $\xi = 4$.

Denote by AS (R,f) the set of transitive simple groupings and by $\mathcal{AS}(R,f)$ its equivalence classes.

Now we indicate how a reduced grouping A for f is realized as a grouping for a reduced (by R) normal curve f* of f which has S as the associated sucessor permutation. If S is in reduced form, then f* = f. Suppose not. Consider $r \in R$, $v^1, v^2 \in r$ such that index $v^1 <$ index v^2, sign $v^1 \neq$ sign v^2 and either $v^2 = v^1 S$ or $v^1 = v^2 S$. Hence, $v^1, v^2 \in \operatorname{Im} f$ and are not special crossings. Let γ be the arc of Im f from v^1 to v^2 when $v^2 = v^1 S$, and from v^2 to v^1 when $v^1 = v^2 S$. By the sufficiency of R, γ has no double point. As γ is not crossed by any ray, $\gamma \cup [v^1, v^2]$ is contained in a disc in $N - \{\infty\}$ and so borders a region Ω in that disc. Consider the continuous curve $(\operatorname{Im} f - \gamma) \cup [v^1, v^2]$ in N. We normalize it by smoothing, after pulling off r, and parallel to r, its arc over $[v^1, v^2]$ inside a closed neighborhood of $[v^1, v^2]$ like a "rectangle". We assume that $\mathcal{V} \cap r = [v^1, v^2]$ and that $\mathcal{V}$ does not contain double points of f. We construct f* inductively from f by changing each time an arc γ as above, with the following property: there are no crossings $\bar{v}^1 = v^1$, $\bar{v}^2 = v^2$ and $\bar{\gamma}$, $\bar{\Omega}$ as above, such that $\bar{\Omega} \subset \Omega$. We have only to prove that R is sufficient for f*. The following lemma provides the induction step.



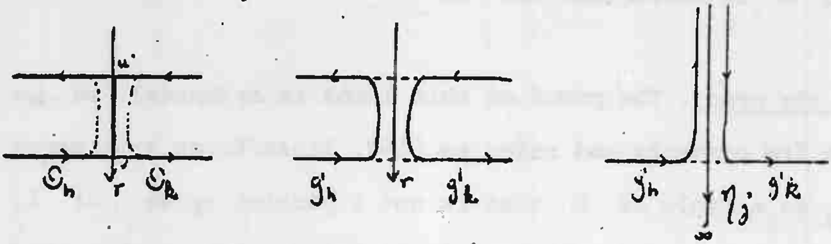
Lemma. Let R be sufficient for the normal curve f and let h be obtained from f by changing a single arc γ with the property above. Then R is sufficient for h.

Comment on the proof. The proof of this lemma is technical, so we make only a few comments and refer to [16]. It suffices to prove that, if c_1 is a cycle of h that is not a partial cycle of f, then there is a ray $r \in R_c$ with origin in int c_1. Let k be the normal curve given by h plus a loop c_2 obtained from $\gamma \cup [v^1,v^2]$ by normalizing at $[v^1,v^2]$. Using the properties of partial decompositions of h and k into simple curves, one proves that f has a cycle contained in $\overline{\text{int } c_1} \cup \mathcal{Y} \cup \overline{\text{int } c_2}$.

3. *Orbits and modified curves of a paired curve.*

Now we look at the effect of an effective grouping $A = A(R,f) = (\beta,P)$ on the paired normal curve $f: C \to N$. A decomposes the augmented curve $g = g^\beta$ of f into simpler curves but not normal ones, the orbits of g by A. In fact, consider the union of Im g with the arcs of $\bigcup_{r\in R} r$ of either the form $[\tilde{u},v]$ with $u = vP$ or $(e,v] \in n_j$, for some $j = 1, \ldots, \delta_N$, with $v = VP$. From this union, we obtain the orbits of g by reparametrizing so that an arc of Im g that either ends or starts in u or v as above is followed or preceeded, respectively, by the adequate arc of $\bigcup_{r\in R} r$. The given orientation is the one compatible with g.

Normalizing these orbits we obtain a new normal curve $g': C' \to N$, a modified curve of g by A. Precisely, we exchange each of those arcs of r by a parallel one and smooth it, as we have done to reduce a normal curve.



We denote an orbit $\mathcal{O}$ of g by $(v^o, v^1, \ldots, v^m)$ where $v^o, \ldots, v^m$ are the elements of $\mathcal{O}$ in the order in which they occur in $\mathcal{O}$, and where sign $v^o = 1$ and v^o is either special, or $v^o = v^oP \epsilon n_j$ for some $j = 1, \ldots, \delta_N$ if $\mathcal{O}$ is unbounded. Now, each orbit $\mathcal{O}$ of g is contained in the closure of an open disc in $N - \{\infty\}$, by the effectiveness of A and parity of g. Thus all of its simple curves (in the decomposition of g intc simple curves) are cycles. No ray crosses negatively a component curve of g', by the effectiveness of A. Every cycle of a component curve of g' is positive, by the effectiveness of A, by the sufficiency of R and the last remark.

We define an equivalence relation on the set of the orbits of g, in order to have a one-to-one correspondence between the resulting classes and the orbits of A. $\mathcal{O}$ will be equivalent to ' in one of the following situations: $\mathcal{O} = \mathcal{O}'$, $\mathcal{O}$ and $\mathcal{O}'$ are unbounded and there exists $v \epsilon n_j$, for some $j = 1, \ldots, \delta_N$ such that $v = vP \epsilon \mathcal{O} \cap \mathcal{O}'$, $\mathcal{O}$ and $\mathcal{O}'$ are unbounded and there are special crossings $v \epsilon \mathcal{O}$ and $v' \epsilon \mathcal{O}'$ such that either $v = v'S$ or $v' = vS$. Now, if $v \epsilon \mathcal{O}$ and sign $v = 1$, then the orbit of A associated to the class of $\mathcal{O}$ is the one in which v occurs if $v=vP$.

It is also the one in which v occurs when $[u,v] \subset \mathcal{O}$, and the one in which u occurs when $-[u,v] \subset \mathcal{O}$, if $u = vP$, $u \neq v$ ($[u,v] \subset r$).

In the example given in last section, g has the following orbits (compare with the orbits of A):

$$O_1 = (c_1^-, c_1^-, c_2, b_1^-, b_2, a_3, a_1^-, d_4^-, d_5, c_2),$$

$$O_2 = (d_3, a_1^-, a_3, d_1^-),$$

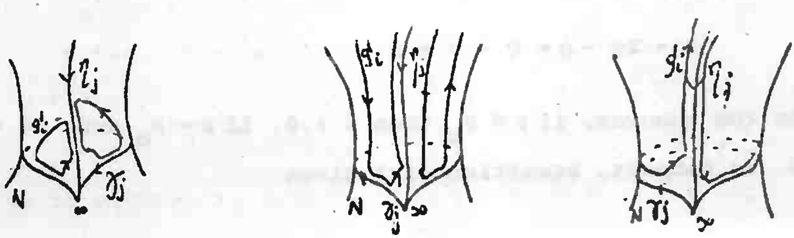
$$O_3 = (b_1^-, a_2, d_2^-, d_6, c_3, b_2),$$

$$O_4 = (d_5, d_4^-),$$

$$O_5 = (d_3, d_1^-, e_1) \quad e$$

$$O_6 = (e_1, d_6, d_6^-).$$

Suppose that S is in reduced form and that A is simple. Then, there is a non-negative orbit of A for each closed orbit of g, and vice-versa. For each negative orbit of A there is an equivalence class of unbounded orbits of g. Each component g_i', $1 \leq i \leq \xi'$, of the modified curve g' of g is positively embedded in a disc in $N - \{\infty\}$, that is, g_i' has a positive, unique cycle. This is a consequence of the properties of the decomposition of g_i' into simple curves. We illustrate the orbits of g in $N - \bigcup_{i=1}^{\delta_N} \partial_i$, in this case.



4. *The Extension Theorem*

We recall notations on previous section for the following

Theorem. Let $f\colon C \to N$ be a normal curve and let $\dot{R}$ be a sufficient set of rays for f. We have:

(i) if $E(f) \neq \emptyset$ then f is paired

(ii) if f is paired, then $E(f) \neq \emptyset$ if, and only if, $AS(R,f) \neq \emptyset$

(iii) if f is paired and $E(f) \neq \emptyset$, then there exists a natural function $\mathscr{A}\colon E(f) \to \mathscr{A}\mathscr{S}(R,f)$ and a bijection $\mathscr{E}(f) \to \mathscr{A}\mathscr{S}(R,f)$ such that diagram

$$\begin{array}{ccc} E(f) & \longrightarrow & \mathscr{A}\mathscr{S}(R,f) \\ \downarrow & & \\ \mathscr{E}(f) & & \end{array}$$

commutes,

(iv) if $F\colon V \to N$ is such that $F \in E(f)$ and if $A \in \mathscr{A}(F)$ is a β-grouping of type $(a_0, a_1, \ldots, a_\delta)$, then: V is compact if, and only if, $\delta = 0$ and V has type $(a_1, \ldots, a_\delta)$ if, and only if, $\delta > 0$. Furthermore,

$$2 - 2g - \rho = \xi - \nu + \beta.$$

Remarks. In the theorem, if $\rho \neq \rho_0$ then $\delta > 0$. If $\rho = \rho_0$ and $\delta = 0$, so that V is compact, assertion (iv) gives

$$\chi(V) = \xi - \nu + \beta,$$

where $\chi(V)$ is the Euler characteristic of V. Transitivity of $A\in\mathcal{A}(F)$, $F\in E(f)$, means that V is connected. Effectiveness of A, second condition, implies that F is orientation preserving.

Examples. In the given example, we have obtained the transitive simple grouping $A = (1,P)$ of type (1,2) and such that $\rho = 3$, $\beta = 1$, $\nu = 6$, $\xi = 4$ and $\delta = 1$. Thus, $g_V = 0$ and f has an immersed extension $F\colon V \to N$ where V has type (2) and genus 0.

Let $f\colon S^1 \to T$ (torus) be as in the 1st figure of next page. $R = \{\alpha_1, \alpha_2, r\}$ is sufficient for f and, for $\beta = 0$ we have

$$S = (c_3\ a_1^-\ b_3\ c_1^-\ b_1^-\ c_4\ a_2\ b_4\ c_2^-\ b_2^-).$$

For

$$P = (a_1^-\ a_2)\ (b_1^-\ b_3)\ (b_2^-\ b_4)\ (c_1^-\ c_3)\ (c_2^-\ c_4)$$

we get

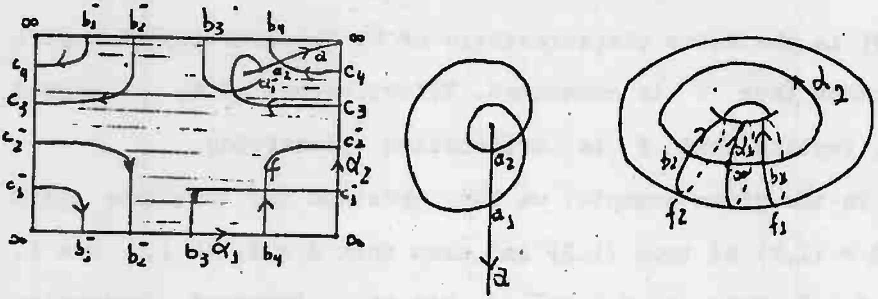
$$SP = (c_3\ a_2\ b_2^-\ c_1^-\ b_3)\ (a_1^-\ b_1^-\ c_2^-\ b_4\ c_4).$$

Hence, $A = (0,P)$ is a transitive simple grouping for f such that $\rho = 1$, $\beta = 0$, $\nu = 5$, $\xi = 2$ and $\delta = 0$. It follows that f has an immersed extension $F\colon V \to N$ where V has genus 2, that is, V is a 2-torus with one hole. Observe that f has three other non-equivalent extensions, all them to the 2-torus with one hole, as for example, the one that corresponds to $(0,P')$, where

$$P' = (a_1^-\ a_2)\ (b_1^-\ b_3)\ (b_2^-\ b_4)\ (c_1^-\ c_4)\ (c_2^-\ c_3).$$

The curves $f\colon S^1 \to R^2$ and $g\colon S^1 \to T$ with the images indicated in the figure below do not extend, although they

$P = P' = (1)$, $S = SP = (a_1a_2)$, $S' = S'P' = (b_1)\ (b_2)$. Notice that A is not simple and that g is not paired.



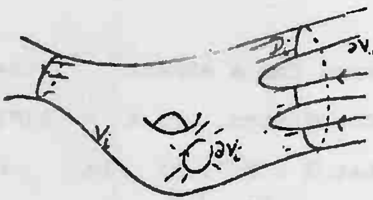
Corollary. Let $f\colon C \to N$ be proper and normal. Let R be sufficient for f. Then f has an immersed extension $F\colon V \to N$ with V non-necessarily connected if, and only f, f is paired and has a simple grouping that satisfies the following property: if $\mathcal{A}' \subset \mathcal{A}$ is an orbit of the subgroup of $\mathcal{S}_{\mathcal{A}}$ generated by S and P, then $\mathcal{A}' \not\subset \mathcal{A}^{\beta} = \mathcal{A} \cap \bigcup_{i=1}^{\beta} w_i$. Furthermore, these groupings classify the immersed extensions of f, via the given equivalence relations.

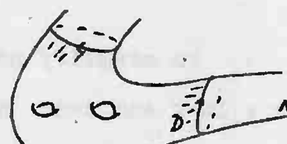
Proof of the Theorem. Suppose that $F\colon V \to N$ extend f. Let β be the cardinality of $F^{-1}(\infty)$. If $\beta > 0$, we choose neighbourhoods $\mathcal{U}' \subset \mathcal{U}$ of ∞ such that $F^{-1}(\mathcal{U})$ is the union of disjoint open sets in V with F 1-1 in each of them. We choose closed discs $D_i \subset N$, $i = 1, \ldots, \beta$, such that $\mathcal{U}' \subset D_i$ and $D_i \subset \overset{\circ}{D}_{i+1}$. We select a pre-image $\tilde{D}_i$ of each D_i to form a collection of disjoint discs in V. Set $w_i = -\partial D_i$ and $\tilde{w}_i = -\partial\tilde{D}_i$. Then $W = V - \bigcup_{i=1}^{\beta} \tilde{D}_i$ is a submanifold of V having boundary $\partial W = \partial V \cup (\bigcup_{i=1}^{\beta} \tilde{w}_i)$. Set $g = F|_{\partial W}$ and $G = F|_W$. Thus g is an augmented curve of f and $G|_{\partial W} = g$. If $\beta = 0$ we set $G = F$ and $g = f$. In any case, G is an extension of g such that $\operatorname{Im} G \neq N$ and $\infty \notin \operatorname{Im} G$. $G^{-1}(\bigcup_{r\in R} r)$ is a union of disjoint oriented simple arcs in V, of one of the three forms: (I) $[\tilde{p},\tilde{v}]$, (II) $[\tilde{u},\tilde{v}]$ or

(III) $(\tilde{e},\tilde{v}]$, where $\tilde{p} \notin \partial V$ and $G(\tilde{p}) = p$ is the origin of some ray $r \in R$, $\tilde{u} \in \partial V$ and $G(\tilde{u}) = u$ is a negative crossing on some ray r, $\tilde{e}$ is some end point of V, and where $\tilde{v} \in \partial V$ and $G(\tilde{v}) = v$ is some positive crossing on r. In case (III), $r \in R_e$ and is unbounded. If $r \in R_e$ and is closed, then we get only case (II). This proves (i). G induces, in a natural way, an effective grouping $A(G) = (O,P')$, given by $uP = V$ in case (II) and $vP = v$ otherwise. Let $g^*: C \longrightarrow N$ be a reduced curve of g, by R. One forms easily an extension G^* of g^* from G, by induction on crossings of $\operatorname{Im} g$ with R, as in the end of section 2. Setting $A(G^*) = (O,P)$ we get the effective reduced grouping (β,P) for F. We have to prove that $A(G^*)$ is transitive and simple.

To simplify notation, assume for a moment that $\operatorname{Im} F \neq N$, $\infty \in \operatorname{Im} F$ and that S is in reduced form. Let $A = A(F)$. We show that A is transitive and simple. Let $\tilde{R} \subset F^{-1}(\underset{r\in R}{\cup} r)$ be the union of the arcs of type (II) and let $f' = (f_j)_{j=1}^{\xi'}$ be a modified curve of f by $A(F)$. Each connected component V_i of $V - \tilde{R}$ contains exactly one component V_i of $V - F^{-1}(\operatorname{Im} f')$. We assert that there is exactly one V_j for each $j \in \{1, \ldots, \xi'\}$, $F|_{\partial V_j} = f'_j$ and it is an embedding, and also that each V_j is a disc in V. For, by construction, each V_i has boundary ∂V_i made up by arcs of $F^{-1}(\operatorname{Im} f')$ and arcs of ∂V, and $F|V_i: V_i \longrightarrow N$ is a normal curve $f_i'' = (f'_{ij})_{j=1}^{\xi i}$, where f'_{ij} is a component of f' and $\xi' = \Sigma\xi_i$, so that F_i given by $F|_{V_i}: V_i \longrightarrow N$ is an extension of f_i''. Now, as $A(F)$ is effective, all cycles of the components of f_i' are positive. Also, as $V_i \cap \tilde{R} = \emptyset$, then $\operatorname{Im} F_j$ is contained in a disc

$N - \{\infty\}$. Composing F_i with an orientation preserving diffeomorphism, a known result [7] shows that $\chi(V_i) = 1$, if V_i is compact. Thus $\xi_i = 1$ in this case. Now, suppose that f_i'' has some unbounded component. Then V must be non-compact. We find a compact submanifold W_o of V_i and apply the same argument to W_o, so that $\chi(W_o) = 1$, and, in particular, f_i'' has no closed component. A direct analysis will then prove that $\xi_i = 1$ in this case too. Choice of W_o: consider a maximal canonical submanifold D of N [6] that contains the image of a maximal canonical topological submanifold D_i of V_i. We illustrate these concepts. Assume that D_i contains

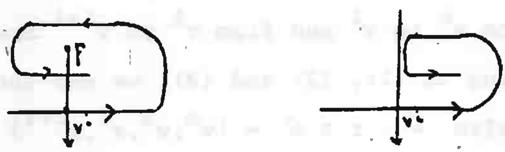




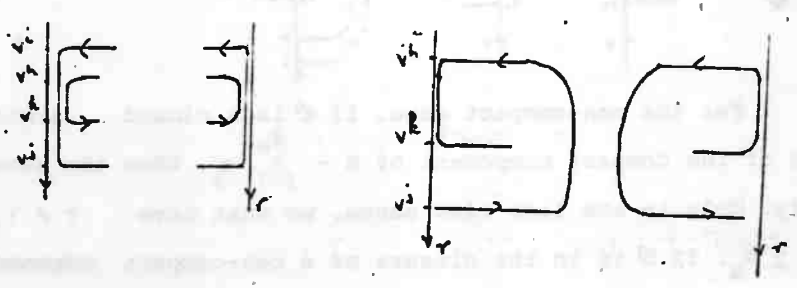
the inverse image of the closure of the neighbourhood $\mathcal{U}$ of ∞, the inverse image of the compact component of $N - \bigcup_{i=1}^{\delta_N} \gamma_i$ and all compact components of ∂V_i. Assume also that D contains all compact maximal arcs of Im f in $N - \bigcup_{i=1}^{\delta_N} \gamma_i$ and that D intersects each of the rays η_j, $j = 1, \ldots, \delta_N$ and each of the unbounded arcs of Im f in $N - \bigcup_{i=1}^{\delta_N} \gamma_i$ once and transversely. Then we get W_o by smoothing the corners of $F_i^{-1}(D)$ and normalizing $F|_{\partial W}$.

Transitivity of A follows from the connectedness of V and the celular decomposition thus obtained. For, a path from v' to $\tilde{v}' \in \partial V$ can be deformed to one that is mapped onto a path on the orbits of f. We look at the orbits of f to discover how the orbits of A are. Firstly, consider the compact case (V and N compact). Let f_j' be a component of the modified curve f' and let $\mathcal{O}$ be the corresponding orbit of f. We proved that Im f_j' is a simple curve c in a disc in $N - \{\infty\}$. Furthermore, no ray crosses Im f_j' negatively. Take $r \in R$ such that $r \cap \mathcal{O} \neq \emptyset$ and fix $v^o \in \mathcal{O} \cap r \cap \mathcal{C}$ with sign $v^o = 1$. Then $\mathcal{O} = (v^o, \ldots, v^m)$ for some $m > 0$. We make some remarks that are easy to check. We also illustrate the impossible situations.

(1) Suppose that $r \in R_c$ and let p be the origin of r. If, for some $i \in \{0,1,\ldots,m\}$, $v^i \in r$ and $v^i = v^iP$, then $p \in$ int c and the arc $[p,v^i] \subset r$ satisfies $[p,v^i] \subset \overline{\text{int } c}$.

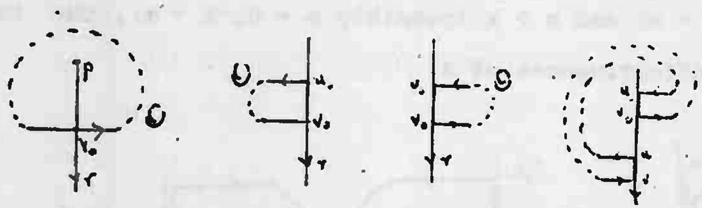


(2) Suppose that $v^i, v^j, v^h, v^k \in r$, sign v^i = sign v^h $= -1$, $v^i = v^jP$, $v^h = v^kP$ and index $v^i <$ index v^h. If either $i < j$ (possibly $i = m$, $j = 0$) and $h < k$ (possibly $h = m$, $k = 0$) or $i > j$ (possibly $i = 0$, $j = m$) and $h > k$ (possibly $h = 0$, $k = m$), then index $v^j <$ index v^h, by effectiveness of A.



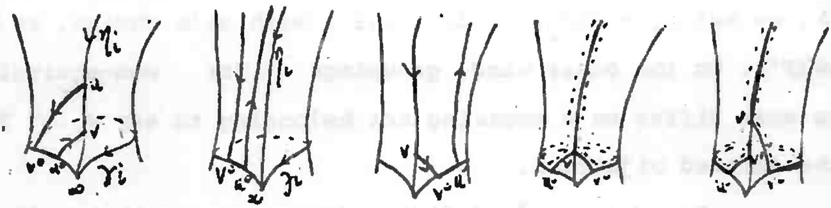
(3) Since A is reduced, if $v^j, v^h, v^k \in r$, sign v^j = sign $v^k = 1$ and $v^h = v^k P$, then $h < k$ implies $h \neq j + 1$ and $h > k$ implies $j \neq h + 1$ ($j \neq 0$ if $h = m$).

If for $v^i \in r$, sign $v^i = 1$ implies $i = 0$, then $\mathcal{O}$ has one of the following forms: (v^o), $(v^o, a^1, \ldots, a^k)$, $(v^o, u^o, a^1, \ldots, a^k)$, $(v^o, a^1, \ldots, a^k, u^o)$, where sign $u^o = -1$, $u^o = v^o P$ and $a^j \notin r$, for $1 \leq j \leq k$. Correspondingly, the orbit of A is (v^o), $(v^o a^{i_1} \ldots a^{i_h})$, $(u^o a^{i_1} \ldots a^{i_h})$, $(v^o a^{i_1} \ldots a^{i_h})$, where $\{a^{i_1}, \ldots, a^{i_h}\} \subset \{a_1, \ldots a_h\}$. Suppose that for some $i \neq 0$, $v^i \in r$ and sign $v^i = 1$. Assume that if $i > 1$, $0 < j < i$ and sign $v^j = 1$ then $v^j \notin r$. Now, P may move either both v^o and v^i, or one of them. We analyse the possibilities, according to arcs of from v^o to v^1 and from v^i to v^{i+1} are arcs of r or Im f. Now, repoorting to (1), (2) and (3), we see that both the arcs are arcs of r and also $\mathcal{O} \cap r \cap \mathcal{C} = \{v^o, v^1, v^i, v^{i+1}\}$. Setting $u^o = v^1$, $u = v^{i+1}$ and $v = v^i$, we get the following possibilities for $\mathcal{O}$: (v^o, v, u, u^o), $(v^o, v, u, a^1, \ldots, a^k, u^o)$, $(v^o, a^1, \ldots, a^k, v, u, u^o)$, $(v^o, a^1, \ldots, a^k, v, u, a^{k+1}, \ldots, a^\ell, u^o)$. The corresponding orbit of A is $(v^o u)$, $(v^o u a^{i_1} \ldots a^{i_h})$, $(v^o a^{i_1} \ldots a^{i_h} u)$, $(v^o a^{i_1} \ldots a^{i_h} u a^{j_1} \ldots a^{j_r})$.



For the non-compact case, if $\mathcal{O}$ is a closed orbit in the closure of the compact component of $N - \bigcup_{j=1}^{\delta_N} \gamma_j$, then the same arguments apply. Only in the last case above, we must have $r \neq \gamma_j$ and η_j, $1 \leq j \leq \delta_N$. If $\mathcal{O}$ is in the closure of a non-compact component of $N - \bigcup_{j=1}^{\delta_N} \gamma_j$, then a direct inspection, using section 3, shows that

$\mathcal{O}$ has one of the following forms: $(v^o u^o, v, u)$ where $v = u^o S$ and $v^o = uS$, (v^o, u^o, u, v) where $u = u^o S$ and $v^o = vS$, (v^o, u^o) where v^o and u^o are special crossings, (v^o, u^o, v) where $v = u^o S$ and v^o is a special crossing, (v, v^o, u^o) where $v^o = vS$ and u^o is a special crossing, (v, v^o, u^o, v) where $v^o = vS$ and $v = u^o S$, and (v, v^o, u^o, v') where $v^o = vS$ and $v' = u^o S$, for $u^o, v^o \in \gamma_j$ and $u, v, v' \in \eta_j$, for some $j \in \{1, \ldots, \gamma_N\}$. The corresponding orbit of A is therefore $(u^o u)$, $(u^o v)$, (u^o), $(u^o v \ldots)$, $(\ldots v u^o)$, $(v u^o)$ and $(\ldots v u^o v' \ldots)$, respectively. One sees immediatly that A is simple. This proves the necessary condition of (ii).



We now prove the sufficient condition. Let $A = A(R,f) = (\beta, P) \in AS(R,f)$. Let g^* be a reduced curve for g . Then $A' = (0,P) \in AS(R,g^*)$. We remind that the orbits of g^* by A' are simple curves, except for twice runned arcs of a ray r of either form $[e,v]$ or $[u,v]$, if any. The interiorsof each orbit is an open disc. Therefore, we define W^* and the extension $G^*: V^* \longrightarrow N$ by glueing together the closure of the interior of the orbits and the corresponding inclusion maps. The identification is made at the points of the common arcs $(e,v]$ or $[u,v]$ of adjacent orbits. We build up G from G^* inductively. To get V and F, we attach the disc bounded by w_1 to the component of ∂W that comes from W_1.

To prove (iii) we note that $E(f) = E(f^*)$, $\delta(f) = \delta(f^*)$, $AS(R,f) = AS(R,f^*)$ and $\mathcal{AS}(R,f) = \mathcal{AS}(R,f^*)$. Thus we assume that S is in reduced form. For $\beta = 0$, we have $AS(R,f) \cong \mathcal{AS}(R,f)$. Define $\mathcal{A}(F) = A(F)$, given in the proof of (ii). If $F = F' \circ \phi$, then ϕ sends arcs of either type $[\tilde{u},\tilde{v}]$ or $(\tilde{e},\tilde{v}]$ in V to arcs of the same type in V and with the same images in N. Therefore, we have a well defined function $\mathcal{E}(f) \to \mathcal{AS}(R,f)$. On the other hand, the defined extension is unique, up to equivalence of extensions. Thus we get a bijection. Suppose that $\beta > 0$. We name the choices of the discs $\tilde{D}_i$ in $F^{-1}(\mathcal{U})$ as permutations of $I_\beta = \{1,\dots,\beta\}$. Each permutation is associated to an extension $G_\beta^\sigma \in E(g^\beta)$. Let A^σ be the related grouping of f. From the definition of equivalence, we get $A^\sigma \sim A^{\sigma'}$, for all σ, σ'. If $F = F' \circ \phi$, we set $\tilde{D}'_i = \phi(\tilde{D}_i)$, $i=1, \dots, \beta$. With this choice, we have $\mathcal{A}(F) = \mathcal{A}(F')$. On the other hand, groupings for non-equivalent extensions must differ on a crossing not belonging to any w_i. Thus we have the desired bijection.

For the proof of (iv), we may suppose that S is in reduced form, $\infty \notin \operatorname{Im} f$ and that $A = A(F)$. If N is compact, we get, from the celular decompcsition of V determined by $F^{-1}(\tilde{R})$:

$$\chi(V) = 2\gamma - \Big(\sum_{i=1}^{n} [2(\nu_i+\nu'_i)+\nu'_i - \frac{1}{2}\sum_{i=1}^{n} \nu_i\Big) + \xi = \xi - \nu,$$

where ν_i is the number of edges that appear once in the face $\mathcal{F}_i$ and ν'_i is the number of edges that appear twice in $\mathcal{F}_i$.

Suppose that N is not compact. If V is compact then A has no negative orbits and so $\delta = 0$. If $\delta = 0$ then $Pv \neq v$ for $v \in \eta_i$, as A is simple. Therefore, the arcs of $F^{-1}(\eta_i)$ are bounded and thus V is compact. We set $N_o = N \cup \{e_1,\dots,e_{\delta_N}\}$ and use the compact case. For V be non-compact, let $q_i \in \eta_i$ such that the arc of η_i up to q_i do not cross Im f. Let $\beta_i \neq \emptyset$ be the cardinality of $F^{-1}(q_i)$ and $\beta' = \sum_{i=1}^{\delta_N} \beta_i$. We consider a maximal canonical topological submanifold $\tilde{D}$ of V and a maximal canonical submanifold D of N with $F(\tilde{D}) \subset D$. Assume also that ∂D is the union of arcs γ_i', $i = 1,\ldots,\delta_N$ that cross each unbounded arc of $\operatorname{Im} f$ and each arc η_i once and transversely. Let $V' = F^{-1}(D)$. Then ∂D is formed by δ closed curves $S_1,\ldots,S_\delta$, where S_i is an arc of $F^{-1}(D)$ if $a_i = 0$ and S_i has arcs of $F^{-1}(\partial D)$ and a_i arcs of unbounded components of ∂V, if $a_i > 0$. If we choose carefully a submanifold V'' of V such that V'' and $f'' = F|_{\partial V''}$ normalizes V' and $F|_{\partial V'}$, we get: for $A'' = A(F'')$,

$$\chi(V'') = \xi'' - \nu'' = [(\xi-\delta) + \rho - \rho_o + \beta'] - [\nu - \beta'].$$

Thus,

$$2 - 2g - (\rho_o + \delta) = \xi - \delta + \rho - \rho_o - \nu .$$

Comment on the proof of the corollary

For V non-connected, the groupings do not need to be transitive. But we have to remind that the curves $F^{-1}(w_i)$ are not components of ∂V. Thus, the orbits of an augmented curve g of f must connect each w_i to some component of $\operatorname{Im} f$.

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