Extensions of immersions in dimension two

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Introduction

Let $V$ and $N$ be $(C^\infty)$-manifolds such that $\dim V < \dim N$ and let $C$ be a submanifold of $V$. The general problem of extension of immersion is the following. Given an $(C^\infty)$ immersion $f: C \to N$ is there an immersion $F: V \to N$ such that $F|_C = f$? We note that this problem for $\dim V = \dim N$ is quite different from the case $\dim V < \dim N$. While the second case reduces to an homotopy problem, via the Hirsh-Smale theory [8,12], the first one does not. We refer to Poenaru [11] for this discussion.

The problem we are concerned with is the above one in the case $C = \partial V$, $\dim V = \dim N = 2$, together with the classification of the extensions. Thus, we prove a theorem of classification that generalizes and simplifies previous results in this direction. Blank [1] provided in 1967, a complete solution (with classification) for the case $V = D^2$ and $N = R^2$ (*). Francis [4] extended the solution to any oriented, connected and compact $V$ and $N = R^2$ or $S^2$. Marx and Ezell [2] obtained existence conditions for extensions, when $N$ is oriented, compact and connected of any genus (**). We also mention Ezell's work [3] for

(*) An existence solution has been given by Titus [13].

(**) We should mention that Marx and Francis' works are concerned with a broader class of extensions.
the non-oriented case and Francis-Troyer's one [5] for the proper case \( N = \mathbb{R}^2 \), \( V \) non-compact with compact boundary and finite number of ends. In this paper we classify the proper orientation preserving extensions of \( f: C \to N \) normal and proper, with \( V \) and \( N \) oriented, connected and either compact or surfaces of finite type. A non-compact surface (=2-dimensional manifold) has finite type when it is equal to a compact one minus a finite number of points.

The combinatorial structures used for the study of extensions had their development from Blank's words and grouping of subwords up to Marx and Francis' assemblages. The assemblages, or groupings in our case of immersions [10] are based on permutations belonging to groups of permutations associated to \( f \). Our methods are essentially the same as those used by the mentioned authors. However, we use a special kind of groupings, the (reduced) simple groupings (*), which are the ones that classify the extensions. Furthermore, using simple groupings, we eliminate the hypothesis on winding numbers that was necessary, in all previous works, to provide an (immersed) extension [1,2,4]. As a result, we obtain a completely combinatorial answer. The theorem of classification is stated and proved in section 4.

This paper forms part (**) of my Doctoral Thesis [15]. I want to thank my adviser, Prof. Mauricio Matos Peixoto, for the suggestion of the problem and for all his guidance.

(*) Among the mentioned authors, only Blank has used reduced groupings. I found it necessary to reduce the groupings in order to define the simple ones.

(**) The other part, to appear, is the application of the present results to stable maps.

Most of the basic notation can be found in [9]. The category used is the $C^\infty$-category.

A surface of finite type is a non-compact surface $X = X_0 \setminus \{e_1, \ldots, e_r\}$ where $X_0$ is compact. If $X$ is connected, the type of $X$ is $(a_1, \ldots, a_6)_X$, as follows. Let $m$ be the number of endpoints of $X$ (the points $e_1, \ldots, e_r$) in $X_0 \setminus 3X$. If $m \neq 0$, define $a_1 = \cdots = a_m = 0$. For each component of $X_0$, consider the number of endpoints of $X$ on it. Order the nonzero number so obtained, if any, as $a_{m-1} \leq \cdots \leq a_6$.

Let $N$ be a connected surface of genus $g_N$ without boundary, which is either compact or non-compact of finite type. We are interested in some collection of curves in $N$. Let $\infty$ be an arbitrary point in $N$. If $N$ is compact of positive genus then $\pi_1(N, \infty)$ has a minimal set of generators represented by a collection of simple regular closed arcs $a_1, \ldots, a_{2g_N}$ with both origin and end at $\infty$, and otherwise disjoint, such that $N - \bigcup_{i=1}^{2g_N} a_i$ is an open disc. Let $N$ have finite type. Then a natural set of generators of $\pi_1(N, \infty)$ is the one represented by simple regular closed arcs $a_1, \ldots, a_{2g_N}$, $\gamma_1, \ldots, \gamma_{6g_N}$ (delete the $a_1$-curves if $g = 0$) with both origin and end at $\infty$, otherwise disjoint, such that $N - \left( \bigcup_{i=1}^{2g_N} a_i \right) \cup \left( \bigcup_{i=1}^{\delta_N} \gamma_i \right)$ is a disjoint union of an open disc and $\delta_N$ homeomorphic cylinders. We orient $\gamma_1$ so that the disc stands to its left-side. When $\delta_N > 0$, we also consider, for each
i = 1, \ldots, \delta_N$, a regular arc $\eta_i = (e_i, m_i]$ that is, an arc from the endpoint $e_i$ to the point $m_i$. We refer to such a collection of arcs in $N$ as a special set of arcs. Note that, if we take off $N$, a special set of arcs we get a collection of open discs.

Let $C$ be the disjoint union $C = \bigcup_{i=1}^{\rho} C_i$ of copies of $S^1$ and $\mathbb{R}$. A normal curve $f: C \to N$ is a family of proper and regular curves $f = \{f_i\}_{i=1}^{\rho}, f_i: C_i \to N$, with only a finite number of transverse double points as crossings, including self-intersections. Two normal curves are equivalent, $f \sim f'$, if there exists an orientation preserving diffeomorphism $\alpha: C \to C'$ such that $f = f' \circ \alpha$.

Let $V$ be a bordered surface that is either compact or of finite type. Let $F: V \to N$ be a proper orientation preserving immersion. We refer to $F$ as an immersed extension of $f$ when $F|_{\partial V} = f$. Two such extensions are equivalent, $F \sim F'$, if there exists an orientation preserving diffeomorphism $\phi: V \to V'$ such that $F = F' \circ \phi$. We denote by $\mathcal{E}(f)$ the set of those extensions of $f$ with $V$ connected and by $\mathcal{G}(f)$ the set of its equivalence classes.

The immersed extensions of a paired normal curve will be classified by its simple transitive groupings. These will
be defined in section 2. The theorem of classification will be given in section 4.

Given a normal curve \( f: C \rightarrow N \), we must specify some special components of \( N - \text{Im} f \), in order to construct the groupings. Those components will be the central cycles of \( f \), defined below.

First we decompose \( f \) into disjoint simple curves by separating and smoothing at its double points \( (*) \). Now, let \( \omega \in N - \text{Im} f \) be a fixed, arbitrary point: A cycle (rel. \( \omega \)) of \( f \) is any of those simple curves that borders an open, maybe unbounded, homeomorphic disc in \( N - \{\omega\} \). Hence, a cycle may be an unbounded arc. Denote by \( \text{int} c \) that disc and let \( D \) be its closure. The cycle \( c \) is positive if \( c \subseteq D \) and negative if \( c \subseteq -D \) (the sign reverses orientation). The cycle \( c \) is a central cycle if \( c' \not\subseteq \text{int} c \), for any cycle \( c' \) of \( f \). Note that, up to smoothing, a central cycle of \( f \) is either the oriented or the negatively oriented boundary of a component of \( N - \text{Im} f \) homeomorphic to a disc in \( N - \{\omega\} \). Note also that if \( f \) has a cycle \( c \), then \( f \) has a central cycle \( c' \subseteq \text{int} c \). If \( \text{Im} f \) is contained in a disc in \( N - \{\omega\} \), then all of its simple curves are cycles.

\( (*) \) When \( f: C \rightarrow N \) is closed, that is, \( C \) is the union of copies of \( S^1 \), these simple curves are known as Gaussian circles of \( f[2,4] \).
A partial cycle of $f$ is given similarly to the cycles, except that the decomposition is applied to a subset of the set of double points of $f$. Properties of the partial cycles as well as the properties of the cycles of $f$ provide complete proofs of some results in this text. We mention that if $f$ has a partial cycle $c$ then $f$ has a cycle $c' = \text{Int } c$.

2. Groupings

In this section we construct groupings for normal curves.

Let $f: C \to N$ be a normal curve and let $\omega \in N - \text{Im } f$ be a fixed, arbitrary point. A set of rays for $f$, $R = R(f, \omega)$, is a finite set of oriented simple arcs in $N$ such that, for $r, r' \in R$, $v \in r \cap \text{Im } f$, it follows that $r$ is transverse to $f$, $rr' = \{\omega\}$ if $r \neq r'$, and $v$ is not a double point of $f$. Moreover, we require that $R = R_c \cup R_e$, where $r \in R_c$ is an arc with origin in a connected
component of $N - \text{Im } f$ and end at $\infty$. Also, $R_e = \emptyset$ if $g_N = \delta_N = \emptyset$, and otherwise $R_e$ is a special set of arcs in $N$ such that the bounded component of $N - \bigcup_{i=1}^{\delta_N} Y_i$ contains all double points of $f$ and all rays of $R_e$.

A set of rays for $f$ is **sufficient**(*) if, for each central cycle of $f$, $R$ contains at least a ray with origin in $\text{int } c$.

A normal curve is **paired** (rel. $R_e$) if either $R_e = \emptyset$ or the number of positive and negative crossings in each closed ray $r \in R_e$ are equal. As we shall see, the normal curves which are not paired do not extend.

On the other hand, $f$ may have extensions of non-zero degree at $\infty$. Thus, for each integer $\beta > 0$ we consider an **augmented curve** of $f$. This is a normal curve $g^\beta$ that has for component curves those of $f$ and also, when $\beta > 0$, $\beta$ disjoint homeomorphic circles $w_1, \ldots, w_\beta$ contained in a small open disc $\mathcal{U}$ in $N - \text{Im } f$, such that $\infty \notin \text{int } w_i \subset \mathcal{U}$ and $w_i = -\partial D_i$, where $D_i = \text{int } w_i$, for $i=1, \ldots, \beta$.

Let $R$ be sufficient and let $g^\beta$ be an augmented curve of $f$ such that each $r \in R$ is a transverse to $g^\beta$. Let $Q = \text{Im } g^\beta \cap (\bigcup_{r \in R} r)$ be the set of all crossings. Define $\text{sign } v = 1$, $r \in R$ if $v$ is a positive crossing of $r \in R$ with $\text{Im } g^\beta$. This means that a nonzero tangent vector to $r$ and a nonzero tangent vector to $\text{Im } g^\beta$ form, in this order, a positive basis for the tangent space

(*) This condition of sufficiency differs from that in the literature [2,4].
of \( N \) at \( v \). Otherwise, define \( \text{sign} \, v = -1 \). We also define index \( v = \text{index of the } i\text{th crossing in } r \in R \).

Let \( \mathcal{G} \) be the symmetric group on \( G \) endowed with the usual product of permutations [14]. Let \( S \in \mathcal{G} \) be the successor permutation: if \( v \in \text{Im} \, g_i, \, i=1, \ldots, \rho + \beta \), then \( vS = v' \in \text{Im} \, g_i \), where \( v' \) is the first crossing in \( \text{Im} \, g_i \), when \( g_i \) is unbounded and \( v \) is the last crossing in it, and \( v' \) is the crossing next to \( v \) in \( \text{Im} \, g_i \) in all other cases. A grouping permutation for \( g^S \) is a permutation \( P \in \mathcal{G} \) given, as a product of disjoint cycles, by trivial cycles and transpositions \((uv)\) such that \( u, \, v \in r \), for some \( r \in R \), and \( \text{sign} \, u \neq \text{sign} \, v \). We call \((uv)\) a pair of \( P \). Abstractly, we also say that \( P \) is a grouping permutation on \((G)\).

A \( \beta \)-grouping for \( f \), \( A = A(R, f) \) is a pair \((\beta, P)\), where \( \beta \) is a non-negative integer and \( P \) is a grouping permutation for \( g^S \). We also say that \( A \) is a grouping on \( G \). Call orbits of \( A \) the disjoint cycles of \( SP \).

As an example, let \( f = \{f_1, f_2, f_3\} : S^1 \cup R \cup R \rightarrow R^2 \) be as in the figure. Consider the augmented curve \( g \) of \( f \) given by \( g_i = f_i \), \( i=1,2,3 \) and \( g_4 = w \). Consider also the sufficient set of rays \( R = \{\gamma, \eta, a, b, c\} \). Then

\[
S = (c_1^-) \, (d_3^{-} a_4^{-}) \, (d_5 c_2 b_1^{-} a_2^{-}) \, (e_1 d_6 c_3 b_2 a_3 d_1^-).
\]
Taking
\[ P = (a_1^{-1} a_3) (b_1^{-1} b_3) (c_1^{-1} c_2) (d_1^{-1} d_3) (d_2^{-1} d_6) (d_4^{-1} d_5) \]
we obtain the grouping \( A = (1, P) \) such that

\[ SP = (c_1^{-1} c_2 b_2 a_1^{-1} d_5) (d_3 a_3) (b_1^{-1} a_2 d_6 c_3) (d_4 d_1 e_1^{-1} d_2^{-1}) \]

Note that the orbits of \( A \) correspond to the (geometric) orbits of \( g \) (in next section). \( A \) is also effective and transitive in the following sense.

A grouping \( A = (\beta, P) \) is effective if \( P \) moves every \( u \in \mathcal{G} \) with sign \( u = -1 \), and all pairs of \( P \), are positive. A pair \((uv)\) with sign \( u = -1 \) is positive if index \( u < \) index \( v \). A is transitive if \( S \) and \( P \) generate a transitive subgroup of \( \mathcal{G} \). This means that for all \( v, v' \in \mathcal{G} \) there exists \( Q = Q_1 \cdots Q_m \in \mathcal{G} \) with \( Q_j = S, S^{-1} \) or \( P \) and \( v' = vQ \).

We shall consider reduced groupings instead of the defined groupings. First, we assume from now on, that the two fol-
lowing pre-reduction conditions are satisfied. If $\delta_N > 0$, the rays $\eta_i, i=1, \ldots, \delta_N$, cross only bounded arcs of $\text{Im} f$ in $N - \frac{\delta_N}{\delta_i} \gamma_i$, and the first and second crossing in any unbounded $g_1$, as well as the last and the next to last one, do not belong to the same ray. If $S > 0$, then for $i = 1, \ldots, \beta$, $w_i \cap r$ is composed either by a single crossing, when $r$ is non-closed, or by two crossings, when $r$ is a closed ray. Observe that these conditions are easily obtained by a sliding of the involved rays. On the other hand, they might be obtained by a reduction similar to the following.

We say that $S$ is in reduced form if, whenever the crossings $u$ and $v$ belong to the same ray and $\text{sign} u \neq \text{sign} v$, then $u \neq vs$ and $v \neq us$. If $S$ is not in reduced form, we reduce $S$ by successive elimination from of crossings $u, v$ in the same ray, such that either $u = vs$ or $v = us$, and $\text{sign} u \neq \text{sign} v$. Let $\mathcal{J}^* \subset \mathcal{J}$ be the reduced set thus obtained. Let $S^*$ be the permutation in $\mathcal{J}^*$ given by $vs^* = vs^k$, where $k$ is the least positive integer such that $vs^k \in \mathcal{J}^*$.

$A^* = (\beta, P^*)$ is a reduced grouping for $f$ if $\beta$ is a non-negative integer and $P^*$ is a grouping permutation on the reduced set $\mathcal{J}^*$ of the set $\mathcal{J}$ of crossing of $g^\beta$ and $R$. In what follows we delete unnecessary stars.

Two reduced groupings $A = (\beta, P)$ and $A' = (\beta', P')$ are equivalent, $A \sim A'$, if $\beta = \beta'$ and either $\beta = 0$ and $P = P'$, or $\beta > 0$ and there exists a permutation $\sigma$ on $\{1, \ldots, \beta\}$ such that $\tau^\sigma_{P'} = P\tau^\sigma$, where $\tau^\sigma \in \mathcal{S}$ is given by $v\tau^\sigma = v'$ with either $v' = v$, if $v \in \text{Im} f$, or $v' \in w_1 \cap r$ and $\text{sign} v' = \text{sign} v$, if $v \in w_1 \cap r$, $r \in R$. Note that equivalence between the groupings is just a way of
permuting the circles \( w_i, \ i = 1, \ldots, \delta \). Equivalence preserves effectiveness and transitivity.

We will refer to the first and last crossings on the unbounded components of \( f \) as \textit{special crossings} of \( R \) with \( f \). A \textbf{negative orbit} of an effective reduced grouping \( A \) is an orbit of \( A \) that contains either some negative special crossing or some fixed (by \( P \)) crossing of a ray \( \eta_i \), if \( \delta_N > 0 \).

An effective reduced grouping \( A = (\beta, P) \) for \( f \) is \textbf{simple} if it satisfies the following properties:

(i) whenever \( u, v \in r, r \in R \), belong to some non-negative orbit of \( A \), then \( \text{sign } u \neq \text{sign } v, \ u \neq uP \) and \( v \neq vP \);

(ii) the negative orbits of \( A \) may only contain negative crossing of rays \( \gamma_i \) and fixed crossings of rays \( \eta_i \), if \( \delta_N > 0 \) and \( i = 1, \ldots, \delta_N \).

Simplicity is also preserved by equivalence.

Let \( A = (\beta, P) \) be a simple grouping for \( f \). We denote by \( \xi \) the number of orbits of \( A \), by \( \delta \) the number of its negative orbits and by \( \nu \) the number of negative crossings in the reduced set \( \phi \). A simple grouping has type \( (a_0, a_1, \ldots, a_\delta) \), with \( a_0 = \delta \) and \( a_i \leq a_j \) for \( i \leq j \), if the negative orbits of \( A \) contain \( a_1, \ldots, a_\delta \) negative special crossings of \( \phi \), respectively. The type of \( A \), as well as the numbers \( \xi, \delta \) are preserved by equivalence.

In the given example, \( A \) is simple and has a unique negative orbit \( (d_4^-, d_1^- e_1 d_2^-) \), so \( \delta = 1 \) and \( A \) has type \( (1, 2) \). Furthermore, \( \nu = 6 \) and \( \xi = 4 \).

Denote by \( \text{AS} (R, f) \) the set of transitive simple
groupings and by $\mathcal{A}(R, f)$ its equivalence classes.

Now we indicate how a reduced grouping $A$ for $f$ is realized as a grouping for a reduced (by $R$) normal curve $f^*$ of $f$ which has $S$ as the associated successor permutation. If $S$ is in reduced form, then $f^* = f$. Suppose not. Consider $r \in R$, $v^1, v^2 \in r$ such that index $v^1 < \text{index } v^2$, sign $v^1 \neq \text{sign } v^2$ and either $v^2 = v^1 S$ or $v^1 = v^2 S$. Hence, $v^1, v^2 \in \text{Im } f$ and are not special crossings. Let $\gamma$ be the arc of $\text{Im } f$ from $v^1$ to $v^2$ when $v^2 = v^1 S$, and from $v^2$ to $v^1$ when $v^1 = v^2 S$. By the sufficiency of $R$, $\gamma$ has no double point. As $\gamma$ is not crossed by any ray, $\gamma \cup [v^1, v^2]$ is contained in a disc in $N - \{\infty\}$ and so borders a region $\Omega$ in that disc. Consider the continuous curve $(\text{Im } f - \gamma) \cup [v^1, v^2]$ in $N$. We normalize it by smoothing, after pulling off $r$, and parallel to $r$, its arc over $[v^1, v^2]$ inside a closed neighborhood of $[v^1, v^2]$ like a "rectangle". We assume that $\gamma \cap r = [v^1, v^2]$ and that $\gamma$ does not contain double points of $f$. We construct $f^*$ inductively from $f$ by changing each time an arc $\gamma$ as above, with the following property: there are no crossings $\overline{v}^1 = v^1, \overline{v}^2 = v^2$ and $\overline{\gamma}, \overline{\Omega}$ as above, such that $\overline{\Omega} \subset \Omega$. We have only to prove that $R$ is sufficient for $f^*$. The following lemma provides the induction step.
Lemma. Let $R$ be sufficient for the normal curve $f$ and let $h$ be obtained from $f$ by changing a single arc $\gamma$ with the property above. Then $R$ is sufficient for $h$.

Comment on the proof. The proof of this lemma is technical, so we make only a few comments and refer to [16]. It suffices to prove that, if $c_1$ is a cycle of $h$ that is not a partial cycle of $f$, then there is a ray $r \in R_c$ with origin in $\text{int} \ c_1$. Let $k$ be the normal curve given by $h$ plus a loop $c_2$ obtained from $\gamma [v^1, v^2]$ by normalizing at $[v^1, v^2]$. Using the properties of partial decompositions of $h$ and $k$ into simple curves, one proves that $f$ has a cycle contained in $\text{int} \ c_1 \cup \gamma \cup \text{int} \ c_2$.

3. Orbits and modified curves of a paired curve.

Now we look at the effect of an effective grouping $A = A(R, f) = (\beta, P)$ on the paired normal curve $f: C \to N$. $A$ decomposes the augmented curve $g = g^\beta$ of $f$ into simpler curves but not normal ones, the orbits of $g$ by $A$. In fact, consider the union of $\text{Im} \ g$ with the arcs of $u \in R$ of either the form $[u, v]$ with $u = VP$ or $[v, v] \in \eta_j$, for some $j = 1, \ldots, 6N$, with $v = VP$. From this union, we obtain the orbits of $g$ by reparametrizing so that an arc of $\text{Im} \ g$ that either ends or starts in $u$ or $v$ as above is followed or preceded, respectively, by the adequate arc of $u \in R$. The given orientation is the one compatible with $g$.

Normalizing these orbits we obtain a new normal curve $g': C' \to N$, a modified curve of $g$ by $A$. Precisely, we exchange each of those arcs of $r$ by a parallel one and smooth it,
as we have done to reduce a normal curve.

We denote an orbit $\mathcal{O}$ of $g$ by $(v^0, v^1, \ldots, v^m)$ where $v^0, \ldots, v^m$ are the elements of $\mathcal{G}$ in the order in which they occur in $\mathcal{O}$, and where sign $v^0 = 1$ and $v^0$ is either special, or $v^0 = v^0_j \in \eta_j$ for some $j = 1, \ldots, \delta_N$ if $\mathcal{O}$ is unbounded. Now, each orbit $\mathcal{O}$ of $g$ is contained in the closure of an open disc in $N - \{\infty\}$, by the effectiveness of $A$ and parity of $g$. Thus all of its simple curves (in the decomposition of $g$ into simple curves) are cycles. No ray crosses negatively a component curve of $g'$, by the effectiveness of $A$. Every cycle of a component curve of $g'$ is positive, by the effectiveness of $A$, by the sufficiency of $R$ and the last remark.

We define an equivalence relation on the set of the orbits of $g$, in order to have a one-to-one correspondence between the resulting classes and the orbits of $A$. $\mathcal{O}$ will be equivalent to $\mathcal{O}'$ in one of the following situations: $\mathcal{O} = \mathcal{O}'$, $\mathcal{O}$ and $\mathcal{O}'$ are unbounded and there exists $v \in \eta_j$, for some $j = 1, \ldots, \delta_N$ such that $v = v^p \in \mathcal{O} \cap \mathcal{O}'$, $\mathcal{O}$ and $\mathcal{O}'$ are unbounded and there are special crossings $v \in \mathcal{O}$ and $v' \in \mathcal{O}'$ such that either $v = v'S$ or $v' = vS$. Now, if $v \in \mathcal{O}$ and sign $v = 1$, then the orbit of $A$ associated to the class of $\mathcal{O}$ is the one in which $v$ occurs if $v = v^p$. 

\[ \begin{array}{c}
\mathcal{O}_h \quad \mathcal{O}_k \\
\mathcal{O}_h' \quad \mathcal{O}_k'
\end{array} \]
It is also the one in which $v$ occurs when $[u,v] \subset \emptyset$, and the one in which $u$ occurs when $-[u,v] \subset \emptyset$, if $u = vP$, $u \neq v$ ($[u,v] \subset r$).

In the example given in last section, $g$ has the following orbits (compare with the orbits of $A$):

\begin{align*}
O_1 &= (c_1^-, c_1^+, c_2, b_1^-, b_2, a_3, a_1^-, d_4^-, d_5, c_2), \\
O_2 &= (d_3, a_1^-, a_3, d_1^-), \\
O_3 &= (b_1^-, a_2, d_2^-, d_6, c_3, b_2), \\
O_4 &= (d_5, d_4^-), \\
O_5 &= (d_3, d_1^-, e_1) e \\
O_6 &= (e_1, d_6, d_6^-).
\end{align*}

Suppose that $S$ is in reduced form and that $A$ is simple. Then, there is a non-negative orbit of $A$ for each closed orbit of $g$, and vice-versa. For each negative orbit of $A$ there is an equivalence class of unbounded orbits of $g$. Each component $g_i'$, $1 \leq i \leq \xi'$, of the modified curve $g'$ of $g$ is positively embedded in a disc in $N - \{\infty\}$, that is, $g_i'$ has a positive, unique cycle. This is a consequence of the properties of the decomposition of $g_i'$ into simple curves. We illustrate the orbits of $g$ in $N - \delta_i$, in this case.
4. The Extension Theorem

We recall notations on previous section for the following.

Theorem. Let \( f: C \to N \) be a normal curve and let \( R \) be a sufficient set of rays for \( f \). We have:

(i) if \( E(f) \neq \emptyset \) then \( f \) is paired.

(ii) if \( f \) is paired, then \( E(f) \neq \emptyset \) if, and only if,

\[
\text{AS}(R,f) \neq \emptyset
\]

(iii) if \( f \) is paired and \( E(f) \neq \emptyset \), then there exists

a natural function \( \mathcal{A}: E(f) \to \mathcal{S}(R,f) \) and a

bijection \( \mathcal{G}(f) \to \mathcal{S}(R,f) \) such that diagram

\[
\begin{array}{ccc}
E(f) & \xrightarrow{\mathcal{A}} & \mathcal{S}(R,f) \\
\downarrow & & \\
\mathcal{G}(f) & \end{array}
\]

commutes,

(iv) if \( F: V \to N \) is such that \( F \in E(f) \) and if

\( A \in \mathcal{S}(F) \) is a \( \beta \)-grouping of type \( (a_0, a_1, \ldots, a_6) \),

then: \( V \) is compact if, and only if, \( \delta = 0 \) and \( V \)

has type \( (a_1, \ldots, a_6) \) if, and only if, \( \delta > 0 \).

Furthermore,

\[
2 - 2g - \rho = \xi - \nu + \beta.
\]

Remarks. In the theorem, if \( \rho \neq \rho_0 \) then \( \delta > 0 \). If \( \rho = \rho_0 \) and \( \delta = 0 \),

so that \( V \) is compact, assertion (iv) gives...
\[ \chi(V) = \xi - \nu + \beta, \]

where \( \chi(V) \) is the Euler characteristic of \( V \). Transitivity of \( G(F) \), \( F \epsilon E(f) \), means that \( V \) is connected. Effectiveness of \( A \), second condition, implies that \( F \) is orientation preserving.

**Examples.** In the given example, we have obtained the transitive simple grouping \( A = (1,P) \) of type \((1,2)\) and such that \( \rho = 3, \beta = 1, \nu = 6, \xi = 4 \) and \( \delta = 1 \). Thus, \( q_V = 0 \) and \( f \) has an immersed extension \( F: V \to N \) where \( V \) has type \((2)\) and genus \( 0 \).

Let \( f: S^1 \to T \) (torus) be as in the 1st figure of next page. \( R = \{a_1, a_2, r\} \) is sufficient for \( f \) and, for \( \beta = 0 \) we have

\[ S = (c_3 \ a_1 b_3 \ c_1 b_1 c_4 a_2 b_4 c_2 b_2). \]

For

\[ P = (a_1^- a_2) \ (b_1^- b_3) \ (b_2^- b_4) \ (c_1^- c_3) \ (c_2^- c_4), \]

we get

\[ SP = (c_3 \ a_2 b_2 c_1 b_3) \ (a_1^- b_1^- c_2^- b_4 c_4). \]

Hence, \( A = (0,P) \) is a transitive simple grouping for \( f \) such that \( \rho = 1, \beta = 0, \nu = 5, \xi = 2 \) and \( \delta = 0 \). It follows that \( f \) has an immersed extension \( F: V \to N \) where \( V \) has genus \( 2 \), that is, \( V \) is a 2-torus with one hole. Observe that \( f \) has three other non-equivalent extensions, all them to the 2-torus with one hole, as for example, the one that corresponds to \((0,P')\), where

\[ P' = (a_1^- a_2) \ (b_1^- b_3) \ (b_2^- b_4) \ (c_1^- c_4) \ (c_2^- c_3). \]

The curves \( f: S^1 \to \mathbb{R}^2 \) and \( g: S^1 \to T \) with the images indicated in the figure below do not extend, although they
$P = P' = (l), S = SP = (a_1a_2), S' = S'P' = (b_1)(b_2)$. Notice that
A is not simple and that $g$ is not paired.

Corollary. Let $f: C \to N$ be proper and normal. Let $R$ be sufficient
for $f$. Then $f$ has an immersed extension $F: V \to N$ with $V$ non-
necessarily connected if, and only $f$, $f$ is paired and has a simple
grouping that satisfies the following property: if $\mathcal{C} \subseteq \mathcal{C}$ is
an orbit of the subgroup of $\mathcal{G}$ generated by $S$ and $P$, then
$\mathcal{C} \not\subseteq \mathcal{G}^\beta = \bigcap_{i=1}^\infty \mathcal{V}$. Furthermore, these groupings classify the
immersed extensions of $f$, via the given equivalence relations.

Proof of the Theorem. Suppose that $F: V \to N$ extend $f$. Let $\beta$ be the
cardinality of $F^{-1}(\infty)$. If $\beta > 0$, we choose neighbourhoods $\mathcal{W}' \subseteq \mathcal{W}$
of $\infty$ such that $F^{-1}(\mathcal{W})$ is the union of disjoint open sets in $V$ with
$F$ in each of them. We choose closed discs $D_i \subseteq N, i = 1, \ldots, \beta$
such that $\mathcal{W}' \subseteq D_i$ and $D_i \subseteq D_{i+1}$. We select a pre-image $\tilde{D}_i$
of each $D_i$ to form a collection of disjoint discs in $V$. Set
$w_i = -\partial D_i$
and $\tilde{w}_i = -\partial \tilde{D}_i$. Then $W = V - \bigcup_{i=1}^\beta D_i$ is a submanifold of $V$
having boundary $\partial W = \partial V \cup (\cup_{i=1}^\beta \tilde{w}_i)$. Set $g = F|_{\partial W}$ and $G = F|_W$. Thus $g$ is an
augmented curve of $f$ and $G|_{\partial W} = g$. If $\beta = 0$ we set $G = F$
and $g = f$. In any case, $G$ is an extension of $g$ such that $\text{Im } G \neq N$
and $\not\subseteq \text{Im } G$. $G^{-1}(u,v)$ is a union of disjoint oriented simple
arcs in $V$, of one of the three forms: (I) $[\tilde{p},\tilde{v}]$, (II) $[\tilde{u},\tilde{v}]$ or
(III) \((\bar{e}, \bar{v})\), where \(\bar{p} \notin \partial V\) and \(G(\bar{p}) = p\) is the origin of some ray \(r \in R\), \(\bar{u} \in \partial V\) and \(G(\bar{u}) = u\) is a negative crossing on some ray \(r\), \(\bar{e}\) is some end point of \(V\), and where \(\bar{v} \in \partial V\) and \(G(\bar{v}) = v\) is some positive crossing on \(r\). In case (III), \(r \in R_e\) and is unbounded. If \(r \in R_e\) and is closed, then we get only case (II). This proves (i).

\(G\) induces, in a natural way, an effective grouping \(A(G) = (O, P')\), given by \(uP = V\) in case (II) and \(vP = v\) otherwise. Let \(g^* : C \to N\) be a reduced curve of \(g\), by \(R\). One forms easily an extension \(G^*\) of \(g^*\) from \(G\), by induction on crossings of \(\text{Im } g\) with \(R\), as in the end of section 2. Setting \(A(G^*) = (O, P)\) we get the effective reduced grouping \((\beta, P)\) for \(F\). We have to prove that \(A(G^*)\) is transitive and simple.

To simplify notation, assume for a moment that \(\text{Im } F \neq N\), \(\epsilon \in \text{Im } F\) and that \(S\) is in reduced form. Let \(A = A(F)\). We show that \(A\) is transitive and simple. Let \(R \subset F^{-1}(\epsilon)\) be the union of the arcs of type (II) and let \(f' = (f_j)_{j=1}^{\xi'}\) be a modified curve of \(f\) by \(A(F)\). Each connected component \(V_i\) of \(V - R\) contains exactly one component \(V_i'\) of \(V - F^{-1}(\text{Im } f')\). We assert that there is exactly one \(V_j\) for each \(j \in \{1, \ldots, \xi'\}\), \(F|_{\partial V_j} = f_j\) and it is an embedding, and also that each \(V_j\) is a disc in \(V\). For, by construction, each \(V_i\) has boundary \(\partial V_i\) made up by arcs of \(F^{-1}(\text{Im } f')\) and arcs of \(\partial V\), and \(F|_{V_i} : V_i \to N\) is a normal curve \(f_i' = (f_{ij}')_{j=1}^{\xi_i}\), where \(f_{ij}'\) is a component of \(f'\) and \(\xi_i = \sum_{j=1}^{\xi_i}\), so that \(F_i\) given by \(F|_{V_i} : V_i \to N\) is an extension of \(f_i'\). Now, as \(A(F)\) is effective, all cycles of the components of \(f_i'\) are positive. Also, as \(V_i \cap R = \emptyset\), then \(\text{Im } F_j\) is contained in a disc.
N = {∞}. Composing $F_1$ with an orientation preserving diffeomorphism, a known result [7] shows that $\chi(V_1) = 1$, if $V_1$ is compact. Thus $\xi_1 = 1$ in this case. Now, suppose that $f_1'$ has some unbounded component. Then $V$ must be non-compact. We find a compact submanifold $W_0$ of $V_1$ and apply the same argument to $W_0$, so that $\chi(W_0) = 1$, and, in particular, $f_1'$ has no closed component. A direct analysis will then prove that $\xi_1 = 1$ in this case too. Choice of $W_0$: consider a maximal canonical submanifold $D$ of $N$ [6] that contains the image of a maximal canonical topological submanifold $D_1$ of $V_1$. We illustrate these concepts. Assume that $D_1$ contains

the inverse image of the closure of the neighbourhood $\mathcal{V}$ of $\infty$, the inverse image of the compact component of $N - \bigcup_{i=1}^{N-\gamma_1}$ and all compact components of $\partial V_1$. Assume also that $D$ contains all compact maximal arcs of $\text{Im} \ f$ in $N - \bigcup_{i=1}^{N-\gamma_1}$ and that $D$ intersects each of the rays $\eta_j$, $j = 1, \ldots, N$ and each of the unbounded arcs of $\text{Im} \ f$ in $N - \bigcup_{i=1}^{N-\gamma_1}$ once and transversely. Then we get $W_0$ by smoothing the corners of $F_1^{-1}(D)$ and normalizing $F|_{\partial W}$.

Transitivity of $A$ follows from the connectedness of $V$ and the cellular decomposition thus obtained. For, a path from $v'$ to $\tilde{v}' \in \partial V$ can be deformed to one that is mapped onto a path on the orbits of $f$. We look at the orbits of $f$ to discover how the
orbits of $A$ are. Firstly, consider the compact case ($V$ and $N$ compact).
Let $f_j^i$ be a component of the modified curve $f'$ and let $\mathcal{O}$ be the corresponding orbit of $f$. We proved that $\text{Im } f_j^i$ is a simple curve $c$ in a disc in $N - \{m\}$. Furthermore, no ray crosses $\text{Im } f_j^i$ negatively. Take $r \in \mathbb{R}$ such that $r \cap \partial \mathcal{O} \neq \emptyset$ and fix $v^0 \in \partial \mathcal{O}$ and $\mathcal{C}$ with sign $v^0 = -1$. Then $\mathcal{O} = (v^0, \ldots, v^m)$ for some $m > 0$. We make some remarks that are easy to check. We also illustrate the impossible situations.

1. Suppose that $r \not\subset r_c$ and let $p$ be the origin of $r$. If, for some $i \in \{0, 1, \ldots, m\}$, $v^i \subset r$ and $v^i = v^i_p$, then $p \in \text{int } c$ and the arc $[p, v^i] \subset r$ satisfies $[p, v^i] \subset \text{int } c$.

2. Suppose that $v^i, v^j, v^h, v^k \subset r$, sign $v^i = \text{sign } v^h = -1$, $v^i = v^j_p$, $v^h = v^k_p$ and index $v^i < \text{index } v^h$. If either $i < j$ (possibly $i = m$, $j = 0$) and $h < k$ (possibly $h = m$, $k = 0$) or $i > j$ (possibly $i = 0$, $j = m$) and $h > k$ (possibly $h = 0$, $k = m$), then index $v^j < \text{index } v^h$, by effectiveness of $A$. 

![Diagram](image-url)
(3) Since $A$ is reduced, if $v_j^j, v_h^i, v_k^i \in r$, sign $v_j^j =$ sign $v_k^i = 1$ and $v_h^i = v_k^i \in \rho$, then $h < k$ implies $h \neq j + 1$ and $h > k$ implies $j \neq h + 1$ ($j \neq 0$ if $h = m$).

If for $v_i^i \in r$, sign $v_i^i = 1$ implies $i = 0$, then $\mathcal{O}$ has one of the following forms: $(v_0^0), (v_0^0, a_1^1, \ldots, a_k^k), (v_0^0, a_1^1, \ldots, a_k^k, u_0^0)$, where sign $u_0^0 = -1$, $u_0^0 = v_0^0 \rho$ and $a_1^1 \in r$, for $1 \leq j \leq k$. Correspondingly, the orbit of $A$ is $(v_0^0), (v_0^0, a_1^1, \ldots, a_k^k), (v_0^0, a_1^1, \ldots, a_k^k, u_0^0)$, where $\{a_1^1, \ldots, a_k^k\} \subseteq \{a_1, \ldots, a_n\}$. Suppose that for some $i \neq 0$, $v_i^i \in r$ and sign $v_i^i = 1$. Assume that if $i > 1$, $0 < j < i$ and sign $v_j^j = 1$ then $v_j^j \in r$. Now, $P$ may move either both $v_0^0$ and $v_i^i$, or one of them. We analyse the possibilities, according to arcs of from $v_0^0$ to $v_1^1$ and from $v_i^i$ to $v_{i+1}^i$ are arcs of $r$ or $Im \, \mathfrak{f}$. Now, reporting to (1), (2) and (3), we see that both the arcs are arcs of $r$ and also $\mathcal{O} \cap r \cap \mathcal{O} = \{v_0^0, \, v_1^1, \, v_i^i, \, v_{i+1}^i\}$. Setting $u_0^0 = v_1^1$, $u = v_{i+1}^i$ and $v = v_i^i$, we get the following possibilities for $\mathcal{O}$: $(v_0^0, v_1^1, u_0^0), (v_0^0, v_i^i, a_1^1, \ldots, a_k^k, u_0^0), (v_0^0, a_1^1, \ldots, a_k^k, v_i^i, u_0^0), (v_0^0, a_1^1, \ldots, a_k^k, v_i^i, a_{k+1}^1, \ldots, a_{i+1}^i, u_0^0)$. The corresponding orbit of $A$ is $(v_0^0, v_1^1, a_1^1, \ldots, a_k^k, u_0^0), (v_0^0, a_1^1, \ldots, a_k^k, a_{k+1}^1, \ldots, a_{i+1}^i, u_0^0)$.

For the non-compact case, if $\mathcal{O}$ is a closed orbit in the closure of the compact component of $N - \bigcup_{j=1}^{\delta_N} \gamma_j$, then the same arguments apply. Only in the last case above, we must have $r \neq \gamma_j$ and $\eta_j^j, 1 \leq j \leq \delta_N$. If $\mathcal{O}$ is in the closure of a non-compact component of $N - \bigcup_{j=1}^{\delta_N} \gamma_j$, then a direct inspection, using section 3, shows that
\( O \) has one of the following forms: \((v^0u^0,v,u)\) where \( v = u^0s \) and \( v^0 = u^0s \), \((v^0,u^0,u,v)\) where \( u = u^0s \) and \( v^0 = vs \), \((v^0,u^0)\) where \( v^0 \) and \( u^0 \) are special crossings, \((v^0,u^0,v)\) where \( v = u^0s \) and \( v^0 \) is a special crossing, \((v,v^0,u^0)\) where \( v^0 = vs \) and \( u^0 \) is a special crossing, \((v,v^0,u^0,v)\) where \( v^0 = vs \) and \( v = u^0s \), and \((v,v^0,u^0,v')\) where \( v^0 = vs \) and \( v' = u^0s \), for \( u^0,v^0 \in \gamma_j \) and \( u,v,v' \in \eta_j \), for some \( j \in \{1, \ldots, \gamma_N\} \). The corresponding orbit of \( A \) is therefore \((u^0u)\), \((u^0v)\), \((u^0)\), \((u^0v)\) \((u^0v)\) \((u^0\nu)\), \((u^0\nu)\), \((u^0\nu)\), \((u^0\nu)\) \((u^0\nu)\) \((u^0\nu)\) \((u^0\nu)\), respectively.

One sees immediately that \( A \) is simple. This proves the necessary condition of (ii).

We now prove the sufficient condition. Let \( A = A(R,f) = (\beta,P) \in AS(R,f) \). Let \( g^* \) be a reduced curve for \( g \). Then \( A' = (O,P) \in AS(R,g^*) \). We remind that the orbits of \( g^* \) by \( A' \) are simple curves, except for twice runned arcs of a ray \( r \) of either form \([e,v]\) or \([u,v]\), if any. The interior of each orbit is an open disc. Therefore, we define \( W^* \) and the extension \( G^* : V^* \to N \) by glueing together the closure of the interior of the orbits and the corresponding inclusion maps. The identification is made at the points of the common arcs \([e,v]\) or \([u,v]\) of adjacent orbits. We build up \( G \) from \( G^* \) inductively. To get \( V \) and \( F \), we attach the disc bounded by \( w_1 \) to the component of \( \partial W \) that comes from \( W_1 \).

To prove (iii) we note that \( E(f) = E(f^*), \psi(f) = \psi(f^*) \), \( AS(R,f) = AS(R,f^*) \) and \( AS^*(R,f) = AS^*(R,f^*) \). Thus we assume that \( S \) is
in reduced form. For \( \beta = 0 \), we have \( AS(R, f) = \mathcal{N}(R, f) \).

Define \( \mathcal{A}(F) = A(F) \), given in the proof of (ii). If \( F = F' \circ \phi \), then \( \phi \) sends arcs of either type \([u, v]\) or \([e, v]\) in \( V \) to arcs of the same type in \( V \) and with the same images in \( N \). Therefore, we have a well defined function \( \mathcal{A}(f) \rightarrow \mathcal{A}(R, f) \). On the other hand, the defined extension is unique, up to equivalence of extensions. Thus we get a bijection.

Suppose that \( \beta > 0 \). We name the choices of the discs \( \mathcal{B}_i \) in \( F^{-1}(\mathcal{B}) \) as permutations of \( I_\beta = \{1, \ldots, \beta\} \). Each permutation is associated to an extension \( G_\beta \in E(g^\beta) \). Let \( A^\sigma \) be the related grouping of \( f \). From the definition of equivalence, we get \( A^\sigma = A^{\sigma'} \) for all \( \sigma, \sigma' \). If \( F = F' \circ \phi \), we set \( \mathcal{B}_i = \phi(\mathcal{B}_i), i = 1, \ldots, \beta \). With this choice, we have \( \mathcal{A}(F) = \mathcal{A}(F') \). On the other hand, groupings for non-equivalent extensions must differ on a crossing not belonging to any \( \mathcal{W}_i \). Thus we have the desired bijection.

For the proof of (iv), we may suppose that \( S \) is in reduced form, \( \neq \) \( \text{Im} f \) and that \( A = A(F) \). If \( N \) is compact, we get, from the cellular decomposition of \( V \) determined by \( F^{-1}(\mathcal{B}) \):

\[
\chi(V) = 2\gamma - \left( \sum_{1=1}^{n} (2v_1 + v'_1) + v_1 - \frac{1}{2}\sum_{1=1}^{n} v_1 \right) + \xi = \xi - \nu,
\]

where \( v_1 \) is the number of edges that appear once in the face \( \mathcal{F}_i \) and \( v'_1 \) is the number of edges that appear twice in \( \mathcal{F}_i \).

Suppose that \( N \) is not compact. If \( V \) is compact then \( A \) has no negative orbits and so \( \delta = 0 \). If \( \delta = 0 \) then \( PV \neq V \) for \( v \in \eta_1 \), as \( A \) is simple. Therefore, the arcs of \( F^{-1}(\eta_1) \) are bounded and thus \( V \) is compact. We set \( N_0 = N \cup \{e_1, \ldots, e_{\delta_0}\} \) and use the compact case. For \( V \) be non-compact, let \( q_1 \in \eta_1 \) such that the arc of \( \eta_1 \) up to \( q_1 \) do not cross \( \text{Im} f \). Let \( \beta_1 \neq \emptyset \) be the
cardinality of $F^{-1}(q_1)$ and $\beta' = \sum_{i=1}^{\delta} \beta_i$. We consider a maximal canonical topological submanifold $D$ of $V$ and a maximal canonical submanifold $D$ of $N$ with $F(D) \subset D$. Assume also that $2D$ is the union of arcs $\gamma_i$, $i = 1, \ldots, \delta_N$ that cross each unbounded arc of $\text{Im } f$ and each arc $\eta_i$ once and transversely. Let $V' = F^{-1}(D)$. Then $2D$ is formed by $\delta$ closed curves $S_1, \ldots, S_\delta$, where $S_1$ is an arc of $F^{-1}(D)$ if $a_1 = 0$ and $S_1$ has arcs of $F^{-1}(2D)$ and $a_1$ arcs of unbounded components of $\partial V$, if $a_1 > 0$. If we choose carefully a submanifold $V''$ of $V$ such that $V''$ and $F'' = F|_{\partial V''}$ normalizes $V'$ and $F|_{\partial V''}$, we get: for $A'' = A(F'')$,

$$\chi(V''') = \xi'' - v'' = [(\xi - \delta) + \rho - \rho_0 + \beta'] - [v - \beta'].$$

Thus,

$$2 - 2g - (\rho_0 + \delta) = \xi - \delta + \rho - \rho_0 - v.$$

Comment on the proof of the corollary

For $V$ non-connected, the groupings do not need to be transitive. But we have to remind that the curves $F^{-1}(w_1)$ are not components of $\partial V$. Thus, the orbits of an augmented curve $g$ of $f$ must connect each $w_1$ to some component of $\text{Im } f$.

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