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A WEIERSTRASS TYPE REPRESENTATION FOR SURFACES IN HYPERBOLIC SPACE WITH MEAN CURVATURE ONE

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Introduction.

A Weierstrass type formula for surfaces of prescribed mean curvature in $\mathbf{R}^3$ was given by Kenmotsu ([K]) in 1979. In 1987, R. Bryant ([B]) studied the surfaces of mean curvature one in hyperbolic space as local projections of null curves in the space of the 2×2 Hermitian symmetric matrices with its Cartan-Killing metric. Recently, Umehara and Yamada ([UY-1], [UY-2], [RUY]) produced an explicit tool to construct examples of these surfaces. They described the null curves in terms of a meromorphic function g and a holomorphic 1-form ω obtained as solutions of two ordinary differential equations.

The subject of this paper is to describe the surfaces in $\mathbf{H}^3$ with mean curvature one in a very similar manner as the minimal surfaces in $\mathbf{R}^3$. It is already well known that these surfaces have a hyperbolic holomorphic Gauss map ([B]); in our work, the function h describes the holomorphic Gauss map. Its properties will give us a Weierstrass type representation.

From the main theorem we have the immersion $X : U \subset \mathbf{C} \longrightarrow \mathbf{H}^3$ as

$$X(z) = \left(\frac{\phi_1(z) + \phi_2(z)}{2}, \Re e\, \phi_3(z), \Im m\, \phi_3(z), \frac{\phi_1(z) - \phi_2(z)}{2} \right)$$

where ϕ_j, $j = 1, 2, 3$ are solutions of the system:

$$\begin{cases} \phi_1\, \phi_2 = 1 + |\phi_3|^2 \\ \dfrac{\partial \phi_1}{\partial z} = h \dfrac{\partial \overline{\phi}_3}{\partial z} \\ \dfrac{\partial \phi_2}{\partial z} = \dfrac{1}{h} \dfrac{\partial \phi_3}{\partial z} \end{cases}$$

whose integrability condition is that of

$$\Im m\{ \overline{h} \Delta \phi_3 = 0 \}.$$

We also have a local integral representation:

$$X = \left(\Re e \int_{z_0}^{z} \left(h \frac{\partial \overline{\phi}_3}{\partial z} + \frac{1}{h} \frac{\partial \phi_3}{\partial z} \right) dz, \Re e \phi_3, \Im m \phi_3, \Re e \int_{z_0}^{z} \left(h \frac{\partial \overline{\phi}_3}{\partial z} - \frac{1}{h} \frac{\partial \phi_3}{\partial z} \right) dz \right)$$

In the last part of the paper we exhibit local solutions of this system for all functions h.

The hyperbolic Gauss map.

We consider the Lorentz space $\mathbf{L}^4 = \{x = (x_0, x_1, x_2, x_3) \in \mathbf{R}^4\}$ with the inner product

$$< x, y >= -x_0 y_0 + x_1 y_1 + x_2 y_2 + x_3 y_3.$$

The hyperbolic space is the submanifold

$$\mathbf{H}^3 = \{x \in \mathbf{L}^4 \mid < x, x >= -1, \; x_0 > 0\}.$$

In $\mathbf{H}^3$ we will consider the induced orientation from $\mathbf{L}^4$ for which the vectors v_1, v_2, v_3 in $T_p\mathbf{H}^3$ form a positive oriented basis iff $\{p, v_1, v_2, v_3\}$ forms a positive oriented basis of $\mathbf{L}^4$.

Let $X : M \longrightarrow \mathbf{H}^3$ be an isometric immersion of an orientable Riemann surface M in the hyperbolic space and $N(p)$ the oriented unitary normal vector at $p \in M$. In local isothermical coordinates $z = u + iv$ we have $||X_u|| = ||X_v|| = \lambda$, $< X_u, X_v >= 0$, and N is such that $\{X(p), \frac{1}{\lambda} X_u, \frac{1}{\lambda} X_v, N(p)\}$ is a positive basis of $T_p\mathbf{L}^4$.

We will consider the map

$$\Phi : \mathbf{H}^3 \longrightarrow D$$

$$(x_0, x_1, x_2, x_3) \longrightarrow (1, \frac{x_1}{x_0}, \frac{x_2}{x_0}, \frac{x_3}{x_0})$$

and the vector $\Phi_*(N(p))$ where

$$D = \{(x_0, x_1, x_2, x_3) \mid x_0 = 1, \; x_1^2 + x_2^2 + x_3^2 < 1\}.$$

This map is the natural isometry between $\mathbf{H}^3$ and the Klein model for the hyperbolic space given by unitary disc with the appropriated metric.

The boundary of D can be identified with the Riemann two sphere S^2.

Definition. The *hyperbolic Gauss map of an immersion* $X : M \longrightarrow \mathbf{H}^3$ is

$$n : M \longrightarrow \partial D$$

given by

$$n(p) = \Phi(X(p)) + t\Phi_*(N(p))$$

where $t > 0$ and $n(p) \in \partial D$.

It follows immediately:

Lemma 1. $n = \frac{1}{x_0 + N_0}(X + N)$.

Proof. For $X(p) = (x_0, x_1, x_2, x_3)$ and $N = (N_0, N_1, N_2, N_3)$

$$\Phi_*(N) = -\frac{N_0}{x_0^2}X + \frac{1}{x_0}N.$$

As $n(p) = \Phi(X(p)) + t\Phi_*(N(p))$ is in the cone $-x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0$ we have

$$< n, n >= -\frac{(x_0 - tN_0)^2}{X_0^4} + \frac{t^2}{X_0^2} = 0.$$

The solution t with $t > 0$ is $t = x_0/(x_0 + N_0)$. •

Remarks. 1. Since the vector $X + N$ is also in the cone there exists $\psi : U \longrightarrow \mathbf{R}$,

$$\psi(p) = -\frac{1}{< n, X >} = x_0 + N_0 = - < X + N, e_0 >, \; e_0 = (1, 0, 0, 0)$$

such that

$$\psi(p) n(p) = X(p) + N(p), \; \forall p \in U$$

and

$$N = -\frac{1}{< n, X >} n - X.$$

2. The coefficients of the second fundamental formula for the immersion X can be calculated as

$$h_{ij} = - < \nabla_{e_i} N, e_j >, \; i, j = 1, 2$$

with $e_1 = \frac{1}{\lambda}X_u$ and $e_2 = \frac{1}{\lambda}X_v$. We have

$$N_u = -h_{11}X_u - h_{12}X_v,$$

$$N_v = -h_{12}X_u - h_{22}X_v.$$

The mean curvature in the choosen normal direction and the gaussian curvature have, respectivelly, the expressions

$$H = \frac{1}{2}(h_{11} + h_{22}) \qquad \text{and} \qquad K = h_{11}h_{22} - h_{12}^2 - 1.$$

3. In isothermical parameters

$$< X_{\bar{z}z}, N >= \frac{1}{2}\lambda^2 H$$

where

$$\frac{1}{2}\lambda^2 =< X_z, X_{\bar{z}} > .$$

The mean curvature H is equal to one if and only if

$$< X_{z\bar{z}}, N >=< X_z, X_{\bar{z}} >$$

or

$$< X_{z\bar{z}}, -\frac{1}{< n, X >} n - X >=< X_z, X_{\bar{z}} > .$$

We will have $H = 1$ if and only if

$$< X_z, n_{\bar{z}} >= 0.$$

4. Taking $z = u + iv$ isothermical parameters in $U \subset \mathbf{C}$ we have the diagram:

$$\begin{array}{ccc} M & \xrightarrow{\;n\;} & \partial D \approx S^2 \\ \big\downarrow & & \big\downarrow \Pi \\ U \subset \mathbf{C} & \xrightarrow{\;h\;} & \mathbf{C} \end{array}$$

with Π the stereographic projection; then

$$n(z) = \left(1, \frac{2\Re e\, h}{|h|^2+1}, \frac{2\Im m\, h}{|h|^2+1}, \frac{|h|^2-1}{|h|^2+1}\right),$$

and n is holomorphic if and only if h is holomorphic.

This hyperbolic Gauss map behaves as the classical Gauss map for minimal surfaces in an euclidean space, that is, we have the following theorem ([B]):

Theorem 1. *Let $n : M \longrightarrow \partial D$ be the hyperbolic Gauss map of a surface $X : M \longrightarrow \mathbf{H}^3$, n non constant. The map $n : M \longrightarrow \partial D$ is conformal iff the immersion X either has mean curvature H constant and equal to one (in which case n preserves the orientation) or X is totally umbilic (in which case n reverses the orientation).*

Proof. From $n = \dfrac{1}{x_0 + N_0}(X + N)$ we have

$$n_u = \left(\frac{1}{x_0 + N_0}\right)_u (X+N) + \frac{1}{x_0+N_0}(X_u + N_u) =$$
$$= \left(\frac{1}{x_0 + N_0}\right)_u (X+N) + \frac{1}{x_0+N_0}[(1-h_{11})X_u - h_{12}X_v]$$

and

$$n_v = \left(\frac{1}{x_0 + N_0}\right)_v (X+N) + \frac{1}{x_0+N_0}(X_v + N_v) =$$
$$= \left(\frac{1}{x_0 + N_0}\right)_v (X+N) + \frac{1}{x_0+N_0}[-h_{12}X_u + (1-h_{22})X_v].$$

Consequently,

$$||n_u||^2 = \frac{\lambda^2}{(x_0+N_0)^2}[(1-h_{11})^2 + h_{12}^2]$$

$$||n_v||^2 = \frac{\lambda^2}{(x_0+N_0)^2}[(1-h_{22})^2 + h_{12}^2]$$

$$< n_u, n_v >= \frac{\lambda^2}{(x_0+N_0)^2}(-h_{12})(2-h_{11}-h_{22})$$

and for $H = 1$ or for umbilic immersions we will have $||n_u||^2 = ||n_v||^2 \geq 0$ and $< n_u, n_v >= 0$.

We also observe that $||n_u|| = ||n_v|| = 0$ if and only if the immersion is umbilical and $H = 1$; in this case we have a horosphere and the hyperbolic Gauss map n is constant.

Considering the complex differentiation

$$n_{\bar z} = \left(\frac{1}{x_0 + N_0}\right)_{\bar z} (X+N) + \frac{1}{x_0+N_0}(X_{\bar z} + N_{\bar z})$$

it follows that

(1)
$$< n_{\bar z}, n_{\bar z} >= \frac{1}{2}\frac{\lambda^2}{(x_0+N_0)^2}(H-1)[(h_{11}-h_{22}) + 2i\, h_{12}];$$

from

$$n(z) = (1, \frac{2\Re e\, h}{|h|^2+1}, \frac{2\Im m\, h}{|h|^2+1}, \frac{|h|^2-1}{|h|^2+1})$$

we have

$$n_{\bar z} = \left(\frac{1}{|h|^2+1}\right)_{\bar z} \tilde{n} + \frac{1}{|h|^2+1}\tilde{n}_{\bar z}$$

where

$$\tilde{n} = (\,|h|^2 + 1, h + \bar h, -i\,(h - \bar h\,), |h|^2 - 1)$$

and

$$\tilde{n}_{\bar z} = h_{\bar z}(\bar h, 1, -i, \bar h) + \bar h_{\bar z}(h, 1, i, h).$$

With these calculations we conclude that

(2).
$$<n_{\bar z}, n_{\bar z}> = \frac{4h_{\bar z}\ \overline{h}_{\bar z}}{(|h|^2+1)^2}$$

From (1) and (2)

$$\frac{4h_{\bar z}\ \overline{h}_{\bar z}}{(|h|^2+1)^2} = \frac{1}{2}\frac{\lambda^2}{(x_0+N_0)^2}(H-1)[(h_{11}-h_{22})+2\,i\,h_{12}]$$

and the hyperbolic Gauss map is conformal iff either $H=1$ or the immersion is umbilical. In both cases the induced metric is given by

$$<dn, dn> = <n_z\, dz + n_{\bar z}\, d\bar z, n_z\, dz + n_{\bar z}\, d\bar z> = 2 <n_z, n_{\bar z}> |dz|^2$$

or

$$<dn, dn> = \frac{\lambda^2}{(x_0+N_0)^2}[2\,H(H-1)-K]|dz|^2.$$

When $H=1$ we have

$$<dn, dn> = \frac{\lambda^2}{(x_0+N_0)^2}(-K)|dz|^2;$$

if the immersed surface is different from a horosphere $2\,H(H-1)-K>0$ and $-K>0$.

Finally we compare the orientations of $X : M \longrightarrow \mathbf{H}^3 \subset \mathbf{L}^4$ and $n : M \longrightarrow \partial D \subset \mathbf{L}^4$, when the immersion X is distinct from the horosphere.

The stereographic projection $\Pi : \partial D \approx S^2 \longrightarrow \{1\} \times \mathbf{R}^3 \subset \mathbf{L}^4$ induces a positive orientation in S^2 in which the normal vector is the internal one.

Let $\{X(p), \frac{1}{\lambda}X_u, \frac{1}{\lambda}X_v, N\}$ and $\{e_0, n_u, n_v, e_0 - n\}$, $e_0 = (1,0,0,0)$ be orthogonal frames adaptated to $X(M)$ and $n(M)$, respectivelly.

These frames are related by the matrix

$$\begin{bmatrix} x_0 & \left(\frac{1}{x_0+N_0}\right)_u & \left(\frac{1}{x_0+N_0}\right)_v & x_0 - \frac{1}{x_0+N_0} \\ \frac{1}{\lambda}<X_u, e_0> & \frac{\lambda(1-h_{11})}{x_0+N_0} & \frac{-\lambda\, h_{12}}{x_0+N_0} & \frac{1}{\lambda}<X_u, e_0> \\ \frac{1}{\lambda}<X_v, e_0> & \frac{-\lambda\, h_{12}}{x_0+N_0} & \frac{\lambda(1-h_{22})}{x_0+N_0} & \frac{1}{\lambda}<X_v, e_0> \\ -N_0 & \left(\frac{1}{x_0+N_0}\right)_u & \left(\frac{1}{x_0+N_0}\right)_v & -N_0 - \frac{1}{x_0+N_0} \end{bmatrix}$$

whose determinant is

$$\begin{vmatrix} x_0 & \left(\frac{1}{x_0+N_0}\right)_u & \left(\frac{1}{x_0+N_0}\right)_v & -\frac{1}{x_0+N_0} \\ \frac{1}{\lambda}<X_u, e_0> & \frac{\lambda(1-h_{11})}{x_0+N_0} & \frac{-\lambda\, h_{12}}{x_0+N_0} & 0 \\ \frac{1}{\lambda}<X_v, e_0> & \frac{-\lambda\, h_{12}}{x_0+N_0} & \frac{\lambda(1-h_{22})}{x_0+N_0} & 0 \\ -(x_0+N_0) & 0 & 0 & 0 \end{vmatrix} =$$

$$= -\frac{\lambda^2}{(x_0+N_0)^2}[(1-h_{11})(1-h_{22})-h_{12}^2] = \frac{\lambda^2}{(x_0+N_0)^2}[-K+2(H-1)].$$

It is easy to see that the determinant is positive if $H=1$ in which case n preserves the orientation (that is, n is holomorphic); in the umbilic case the determinant is negative, n reverses the orientation and is antiholomorphic. •

Remark. We observe that

$$< n_z, n_{\bar z} >= \frac{2}{(1+|h|^2)^2}[\,|h_z|^2 + |h_{\bar z}|^2\,].$$

When $H=1$

$$< dn, dn >= 2 < n_z, n_{\bar z} > |dz|^2 = \frac{4|h_z|^2}{(|h|^2+1)^2}|dz|^2 = -K\frac{\lambda^2}{(x_0+N_0)^2}|dz|^2,$$

and

$$-K = \frac{4|h_z|^2}{(|h|^2+1)^2}\left(\frac{\lambda^2}{(x_0+N_0)^2}\right)^{-1}. \qquad (3)$$

A Representation Theorem.

Working with a holomorphic hyperbolic Gauss map, that is, with surfaces with constant mean curvature equal to one, we have a local representation theorem similar to the Weierstrass representation for minimal surfaces in the euclidean space.

Theorem. *Let $X : M \longrightarrow \mathbf{H}^3$ be a non-umbilic immersion in $\mathbf{H}^3$ with mean curvature one and*

$$n(z) = \left(1, \frac{2\Re e h}{1+|h|^2}, \frac{2\Im m h}{1+|h|^2}, \frac{|h|^2-1}{|h|^2+1}\right)$$

its hyperbolic Gauss map. The real functions $\phi_1(z) = x_0(z) + x_3(z)$ and $\phi_2(z) = x_0(z) - x_3(z)$ and the complex function $\phi_3(z) = x_1(z) + i\,x_2(z)$ with $X(z) = (x_0(z), x_1(z), x_2(z), x_3(z))$ satisfy

$$(*) \qquad \begin{cases} \phi_1\,\phi_2 = 1 + |\phi_3|^2 \\ \dfrac{\partial \phi_1}{\partial z} = h\dfrac{\partial \bar\phi_3}{\partial z} \\ \dfrac{\partial \phi_2}{\partial z} = \dfrac{1}{h}\dfrac{\partial \phi_3}{\partial z} \end{cases}$$

Conversely, given a holomorphic non-constant function $h : U \subset \mathbf{C} \longrightarrow \mathbf{C}$, two real functions ϕ_1 and ϕ_2 ($\phi_2 > 0$) and a complex function ϕ_3 satisfying (*) in the simply connected domain U, then

$$X(z) = \left(\frac{\phi_1(z) + \phi_2(z)}{2}, \Re e\, \phi_3(z), \Im m\, \phi_3(z), \frac{\phi_1(z) - \phi_2(z)}{2} \right) .$$

defines a conformal immersion in $\mathbf{H}^3$ with constant mean curvature one and hyperbolic Gauss map n given by h as above.

Proof. First of all we observe that

$$X(z) = (x_0, x_1, x_2, x_3) \in \mathbf{H}^3 \iff -x_0^2 + x_1^2 + x_2^2 + x_3^2 = -1 \iff \phi_1\, \phi_2 = 1 + |\phi_3|^2;$$

from the first equivalence it also follows that if $\phi_2 = x_0 - x_3$ then $\phi_2 > 0$.

Given ϕ_1, ϕ_2, ϕ_3 as above we have

$$X(z) = \left(\frac{\phi_1(z) + \phi_2(z)}{2}, \Re e\, \phi_3(z), \Im m\, \phi_3(z), \frac{\phi_1(z) - \phi_2(z)}{2} \right)$$

and $< X_z, n >= 0$ if and only if

$$\frac{1}{2}\left(-1 - \frac{|h|^2 - 1}{|h|^2 + 1}\right)\frac{\partial \phi_1}{\partial z} + \frac{1}{2}\left(-1 + \frac{|h|^2 - 1}{|h|^2 + 1}\right)\frac{\partial \phi_2}{\partial z} + \frac{1}{1 + |h|^2}\left[\Re e\, h \left(\frac{\partial \phi_3}{\partial z} + \frac{\partial \overline{\phi_3}}{\partial z}\right) + i\, \Im m\, h \left(\frac{\partial \overline{\phi_3}}{\partial z} - \frac{\partial \phi_3}{\partial z}\right)\right] = 0$$

or

$$\frac{\partial \phi_1}{\partial z} + |h|^2 \frac{\partial \phi_2}{\partial z} - h\, \frac{\partial \overline{\phi_3}}{\partial z} - \overline{h}\, \frac{\partial \phi_3}{\partial z} = 0 \tag{4}$$

The assumption on the mean curvature gives us

$$H = 1 \iff < X_z, n_{\overline{z}} >=< X_z, \tilde{n}_{\overline{z}} > 0$$

where

$$n(z) = \left(1, \frac{2\Re e h}{1 + |h|^2}, \frac{2\Im m h}{1 + |h|^2}, \frac{|h|^2 - 1}{|h|^2 + 1}\right)$$

and

$$\tilde{n} = (1 + |h|^2, h + \overline{h}, -i\,(h - \overline{h}), |h|^2 - 1\,).$$

We have in this case h holomorphic and therefore

$$\tilde{n}_{\overline{z}} = (h\, \overline{h}_{\overline{z}}, \overline{h}_{\overline{z}}, i\, \overline{h}_{\overline{z}}, h\, \overline{h}_{\overline{z}}\,);$$

as h is nonconstant ($h_z \neq 0$) it follows

$$H = 1 \iff < X_z, n_{\bar{z}} >= 0 \iff$$

$$-h\left(\frac{\partial \phi_1}{\partial z} + \frac{\partial \phi_2}{\partial z}\right) + \left(\frac{\partial \phi_3}{\partial z} + \frac{\partial \bar{\phi}_3}{\partial z}\right) + h\left(\frac{\partial \bar{\phi}_1}{\partial z} - \frac{\partial \phi_2}{\partial z}\right) = 0 \iff$$

(5) $$\frac{\partial \phi_3}{\partial z} = h\,\frac{\partial \phi_2}{\partial z}.$$

Returning with this last equation in (4), finally we have

$$\frac{\partial \phi_1}{\partial z} = h\,\frac{\partial \bar{\phi}_3}{\partial z}.$$

Let $p \in M$ be a zero of h with order μ; we have from (5) that p is a zero of $\frac{\partial \phi_3}{\partial z}$ whose order is greater or equal to μ and we can write

$$\frac{\partial \phi_2}{\partial z} = \frac{1}{h}\frac{\partial \phi_3}{\partial z}.$$

Let now be

$$X(z) = \left(\frac{\phi_1(z) + \phi_2(z)}{2}, \Re e\,\phi_3(z), \Im m\,\phi_3(z), \frac{\phi_1(z) - \phi_2(z)}{2}\right)$$

with ϕ_1, ϕ_2, ϕ_3 verifying (*). It is easy to see that

(6) $$X_z = \frac{1}{2}\left[\frac{\partial \phi_3}{\partial z}\left(\frac{1}{h}, 1, -i, -\frac{1}{h}\right) + \frac{\partial \bar{\phi}_3}{\partial z}(h, 1, i, h)\right]$$

From the fact that $< X_z, X_z >= 0$ it follows that we have isothermical parameters.

Let now consider

$$\hat{n}(z) = \left(1, \frac{2\Re e h}{1 + |h|^2}, \frac{2\Im m h}{1 + |h|^2}, \frac{|h|^2 - 1}{|h|^2 + 1}\right)$$

with h the holomorphic function from (*). The vector

$$\hat{N} = -\frac{1}{< \hat{n}, X >}\hat{n} - X$$

has norm equal to one, verifies $< X_z, \hat{N} >= 0$, $< X, \hat{N} >= 0$ and

$$-\frac{1}{< \hat{n}, X >}\hat{n} = X + \hat{N}$$

therefore $\widehat{N}$ is exactly the normal vector N and $\widehat{n}$ the hyperbolic Gauss map n of the immersion X. With some calculations we obtain

$$< n_{\bar{z}}, X_z >= \frac{h_{\bar{z}}}{(1+|h|^2)} \left[\frac{\partial \overline{\phi}_3}{\partial z} - \frac{\partial \phi_3}{\partial z}\frac{\overline{h}}{h}\right].$$

From the fact that h is holomorphic it follows that $< n_{\bar{z}}, X_z >= 0$ which implies $H = 1$; h non-constant gives us a non-umbilic immersion. •

Remarks.

1. The compatibility condition for the two partial differential equations in (∗) is the same and writes

(7) $$\Im\mathrm{m}\{\overline{h}\,\Delta\phi_3\} = 0.$$

2. Choosing h and ϕ_3 such as to verify (7) we will have ϕ_1 and ϕ_2 given locally by

$$\phi_1 = 2\,\Re\mathrm{e} \int_{z_0}^{z} h\,\frac{\partial \overline{\phi}_3}{\partial z}\,dz$$

and

$$\phi_2 = 2\,\Re\mathrm{e} \int_{z_0}^{z} \frac{1}{h}\,\frac{\partial \phi_3}{\partial z}\,dz.$$

3. An integral formula can be written from (6):

(I) $$X = \left(\Re\mathrm{e} \int_{z_0}^{z} \left(h\frac{\partial \overline{\phi}_3}{\partial z} + \frac{1}{h}\frac{\partial \phi_3}{\partial z}\right) dz, \Re\mathrm{e}\,\phi_3, \Im\mathrm{m}\,\phi_3, \Re\mathrm{e} \int_{z_0}^{z} \left(h\frac{\partial \overline{\phi}_3}{\partial z} - \frac{1}{h}\frac{\partial \phi_3}{\partial z}\right) dz\right).$$

4. The metric $ds^2 = \lambda^2|dz|^2$ is such that $\lambda^2 = 2 < X_z, X_{\bar{z}} >$; from (6) we have

(8) $$\lambda^2 = \left[\left|\frac{\partial \phi_3}{\partial z}\right|^2 + \left|\frac{\partial \phi_3}{\partial \bar{z}}\right|^2 - 2\Re\mathrm{e}\left(\frac{\overline{h}}{h}\frac{\partial \phi_3}{\partial z}\frac{\partial \phi_3}{\partial \bar{z}}\right)\right]$$

and from this last expression we can conclude that p is a regular point if the derivatives $\frac{\partial \phi_3}{\partial z}$ and $\frac{\partial \phi_3}{\partial \bar{z}}$ do not vanish simultaneously at p.

We also can write:

$$\lambda^2 = \left|\frac{\partial}{\partial z}(\overline{\phi}_3 - \overline{h}\phi_2)\right|^2 = \left|\frac{\partial}{\partial \bar{z}}(\phi_3 - h\phi_2)\right|^2$$

5. From Lemma 1 we have:

$$-\frac{1}{< n, X >} = x_0 + N_0 = - < X + N, e_0 >;$$

some calculations give us:

$$1 + |\phi_3 - h\phi_2|^2 = \phi_2(\phi_1 + |h|^2\phi_2 - \bar{h}\phi_3 - h\bar{\phi}_3)$$

and

$$< n, X >= \frac{1}{|h|^2 + 1}(-\phi_1 - |h|^2\phi_2 + \bar{h}\phi_3 + h\bar{\phi}_3) = -\frac{1 + |\phi_3 - h\phi_2|^2}{\phi_2(|h|^2 + 1)}$$

The total curvature is

$$c = \int_M K\, dA$$

and from (3) it follows that

$$K = -\frac{4|h_z|^2}{(|h|^2 + 1)^2}\left(\frac{\lambda^2}{(x_0 + N_0)^2}\right)^{-1}.$$

In local coordinates

(9)
$$c = -\int \frac{4|h_z|^2}{(\phi_1 + |h|^2\phi_2 - \bar{h}\phi_3 - h\bar{\phi}_3)^2}\frac{i}{2}dz \wedge d\bar{z} =$$

$$-\int \frac{4|h_z|^2\phi_2^2}{(1 + |\phi_3 - h\phi_2|^2)^2}\frac{i}{2}dz \wedge d\bar{z} = -\int \frac{4\left|\frac{\partial}{\partial z}(\phi_3 - h\phi_2)\right|^2}{(1 + |\phi_3 - h\phi_2|^2)^2}\frac{i}{2}dz \wedge d\bar{z}$$

Examples.

To exhibit some examples we need to get two real functions ϕ_1 and ϕ_2, $\phi_2 > 0$ and a complex function ϕ_3, solutions of the the system:

(*)
$$\begin{cases} \phi_1\,\phi_2 = 1 + |\phi_3|^2 \\ \dfrac{\partial \phi_1}{\partial z} = h\dfrac{\partial \bar{\phi}_3}{\partial z} \\ \dfrac{\partial \phi_2}{\partial z} = \dfrac{1}{h}\dfrac{\partial \phi_3}{\partial z} \end{cases}$$

To find solutions, we begin with some important remarks.

1. First of all we will analyse the solutions that correspond to ϕ_3 holomorphic (or antiholomorphic); in the first case ϕ_1 (resp. ϕ_2) is constant. As $\phi_1\,\phi_2 = 1 + |\phi_3|^2$ the constant cannot be zero; it is easy to see that ϕ_1
(resp. ϕ_2) constant implies that the surface is umbilical and $x_0 + x_3$
(resp. $x_0 - x_3$) is constant; the functions x_1 and x_2 will be harmonical conjugates.

2. Given the function h we can search solutions as

$$\phi_3 = h(z)F(|z|^2),$$

with F a one real variable differentiable function.

Since

$$\overline{h}\,\Delta\phi_3 = z\,\overline{h}\,h_z\,F'(\,|z|^2) + |h|^2F'(\,|z|^2) + |z|^2\,|h|^2\,F''(\,|z|^2)$$

the compatibility condition is

$$\Im\mathrm{m}\{\,\overline{h}\,\Delta\phi_3\,\} = 0 \iff \Im\mathrm{m}\{\,z\,\overline{h}\,h'\} = 0.$$

The last condition is satisfied by all the the functions $h(z) = z^\mu$, for real μ.

3. In the case $\phi_3 = h(z)F(\,|z|^2)$, the metric (6) will be

$$\lambda^2 = |h_z|^2\;F^2(\,|z|^2). \tag{10}$$

Example 1. We have an immersion with constant mean curvature one

$$X : \mathbf{C} - \{0\} \longrightarrow \mathbf{H}^3$$

solving $(*)$ with $h(z) = z$, $F_\alpha(t) = t^\alpha$ and

$$\phi_3(z) = h(z)\,[\,A\,F_\alpha(\,|z|^2) + B\,F_\beta(\,|z|^2)\,].$$

Now, the integrability condition is satisfied (remark 2) and the solutions ϕ_1 and ϕ_2 are:

$$\phi_1(z) = \frac{\alpha}{\alpha+1}\,A\,|z|^{2\alpha+2} + \frac{\beta}{\beta+1}\,B\,|z|^{2\beta+2}$$

and

$$\phi_2(z) = \frac{\alpha+1}{\alpha}\,A\,|z|^{2\alpha} + \frac{\beta+1}{\beta}\,B\,|z|^{2\beta}$$

with α and β both distinct from zero and -1.

The condition $\phi_1 \phi_2 = 1 + |\phi_3|^2$ is verified under the restrictions:

$$\alpha + \beta = -1$$

and

(11)
$$A\,B \left(\frac{2\alpha + 1}{\alpha\,(\alpha+1)} \right)^2 = 1$$

therefore

$$2\alpha + 1 \neq 0.$$

In this case the solutions of (*) are:

(12)
$$\begin{cases} \phi_1(z) = \dfrac{\alpha}{\alpha+1}\, A\,|z|^{2\alpha+2} + \dfrac{\alpha+1}{\alpha}\, B\,|z|^{-2\alpha} \\ \phi_2(z) = \dfrac{\alpha+1}{\alpha}\, A\,|z|^{2\alpha} + \dfrac{\alpha}{\alpha+1}\, B\,|z|^{-2\alpha-2} \\ \phi_3 = z\,[\,A\,|z|^{2\alpha} + B\,|z|^{-2\alpha-2}\,] \end{cases}$$

Writing $z = r\,e^{i\theta}$:

$$\begin{cases} \phi_1(r,\theta) = \dfrac{\alpha}{\alpha+1}\, A\,r^{2\alpha+2} + \dfrac{\alpha+1}{\alpha}\, B\,r^{-2\alpha} = f_1(r) \\ \phi_2(r,\theta) = \dfrac{\alpha+1}{\alpha}\, A\,r^{2\alpha} + \dfrac{\alpha}{\alpha+1}\, B\,r^{-2\alpha-2} = f_2(r) \\ \phi_3(r,\theta) = r(\cos\theta + i\sin\theta\,)(\,A\,r^{2\alpha} + B\,r^{-2\alpha-2}) = f_3(r)\,e^{i\theta} \end{cases}$$

Now it is easy to see that all these surfaces are rotational surfaces generated by the curve

$$C(r) = (c_0(r), c_1(r), 0, c_3(r)) = \left(\frac{f_1(r) + f_2(r)}{2}, f_3(r), 0, \frac{f_1(r) - f_2(r)}{2} \right),$$

$C(r) \subset \mathbf{H}^3 \cap \mathcal{P}^3$, with $\mathcal{P}^3 = [e_0, e_1, e_3]$. We have a spherical rotation and

$$X(r,\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ 0 \\ c_3 \end{bmatrix} = \begin{bmatrix} c_0 \\ c_1\cos\theta \\ c_1\sin\theta \\ c_3 \end{bmatrix}$$

Using (9) and (11) we can have the total curvature

$$c = \int_{\mathbb{C}-\{0\}} \frac{4\alpha^2(\alpha+1)^2|z|^{4\alpha}}{(A|z|^{4\alpha+2}+B)^2} \frac{i}{2} dz d\overline{z} = -4(2\alpha+1)\pi$$

By (10) we have

$$\lambda^2 = (A|z|^{2\alpha} + B|z|^{-2\alpha-2})^2$$

and the surface is complete. These surfaces are called " catenoid cousins ".

We can find solutions of (*) for all functions h in a neighborhood of a regular point; it suffices to reparametrize the surface M by the function h.

Calling $w = h(z)$ or $z = h^{-1}(w)$, the system (*) becomes

$$(II) \qquad \begin{cases} \Phi_1 \Phi_2 = 1 + |\Phi_3|^2 \\ \dfrac{\partial \Phi_1}{\partial w} = w \dfrac{\partial \overline{\Phi}_3}{\partial w} \\ \dfrac{\partial \Phi_2}{\partial w} = \dfrac{1}{w} \dfrac{\partial \Phi_3}{\partial w} \end{cases}$$

whose solutions are

$$\Phi_j(w) = \phi_j(h^{-1}(w)),\ j = 1,2,3$$

with ϕ_j given by (12).

Example 2. The system (*) also admits solutions as

$$\phi_3(z) = F(z)G(\overline{z})$$

with F, G and h holomorphic functions satisfying $\overline{F(\overline{z})} = F(z)$, $\overline{G(\overline{z})} = G(z)$ and $\overline{h(\overline{z})} = h(z)$. In this case if

$$(13) \qquad F'(z) = h(z)G'(z)$$

the integrability condition (7) is verified.

The two last equations in (*) can be integrated and the solutions are

$$\phi_1(z) = |F(z)|^2$$

and

$$\phi_2(z) = |G(z)|^2.$$

We will modify these solutions to have the first equation satisfied; in this way, we will take F_1, G_1, F_2, G_2 as in (13), A and B real constants such that

$$\begin{cases} \phi_1 = A|F_1|^2 + B|F_2|^2 \\ \phi_2 = A|G_1|^2 + B|G_2|^2 \\ \phi_3 = A\, F_1\, \overline{G}_1 + B\, F_2\, \overline{G}_2 \end{cases}$$

with

$$AB(\overline{F}_1\overline{G}_2 - \overline{F}_2\overline{G}_1)(F_1G_2 - F_2G_1) = 1. \tag{14}$$

The surfaces called " Enneper Cousins " are corresponding to

$$h(z) = \tanh \lambda z,$$

$$G_1'(z) = \cosh \lambda z$$

$$G_2'(z) = z \cosh \lambda z;$$

consequentely, by (13) and (14),

$$F_1(z) = \frac{1}{\lambda} \cosh \lambda z$$

$$F_2(z) = \frac{1}{\lambda}\left(z\cosh \lambda z - \frac{1}{\lambda}\sinh \lambda z\right)$$

$$G_1(z) = \frac{1}{\lambda}\sinh \lambda z$$

$$G_2(z) = \frac{1}{\lambda}\left(z\sinh \lambda z - \frac{1}{\lambda}\cosh \lambda z\right)$$

and

$$AB = |\lambda|^6, \ \lambda \in \mathbf{C}.$$

The total curvature can be calculated by (9), observing that

$$\phi_1 + |h|^2\phi_2 - \overline{h}\phi_3 - h\overline{\phi}_3 = A\,|F_1 - hG_1|^2 + B\,|F_2 - hG_2|^2 = \frac{(A + B\,|z|^2)}{|\lambda|^2|\cosh z|^2}$$

and

$$K = -\int \frac{4|\lambda|^6}{A^2\left(1 + \frac{B}{A}|z|^2\right)^2}\frac{i}{2}\,dz \wedge d\overline{z} = -\int \frac{4}{(1+|w|^2)^2}\frac{i}{2}\,dw \wedge d\overline{w} = -4\pi.$$

It is also easy to see that the metric is complete.

To get new examples we have to find solutions of

$$\Im m\{\bar{h}\,\Delta\phi_3\} = 0$$

and a linear combination of this solutions in order to have

$$\phi_1\phi_2 = 1 + |\phi_3|^2$$

that is, in order to have the corresponding immersion in $\mathbf{L}^4$ contained in $\mathbf{H}^3$.

The classification of these immersions depends on the description of all the solutions of this problem.

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