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Beatification: Flattening the Poisson bracket for two-dimensional fluid and plasma theories

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A perturbative method called *beatification* is presented for a class of two-dimensional fluid and plasma theories. The Hamiltonian systems considered, namely, the Euler, Vlasov-Poisson, Hasegawa-Mima, and modified Hasegawa-Mima equations, are naturally described in terms of noncanonical variables. The beatification procedure amounts to finding the correct transformation that removes the explicit variable dependence from a noncanonical Poisson bracket and replaces it with a fixed dependence on a chosen state in the phase space. As such, beatification is a major step toward casting the Hamiltonian system in its canonical form, thus enabling or facilitating the use of analytical and numerical techniques that require or favor a representation in terms of canonical, or beatified, Hamiltonian variables. *Published by AIP Publishing.*
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I. INTRODUCTION

The Hamiltonian formalism is a fundamental pillar of theoretical physics, as the time evolution of every isolated system is expected to possess Hamiltonian structure. Although most literature on Hamiltonian systems uses canonical variables, numerous physical theories are Hamiltonian yet naturally described in terms of noncanonical variables,[1] such as many prominent fluid and plasma models,[2–13] the generalized coherent-state approach to semiclassical dynamics,[14–17] and even the time-dependent Schrödinger equation itself.[18]

For such noncanonical representations, the question then arises: How does one obtain a global transformation from noncanonical to canonical variables in infinite-dimensional Hamiltonian systems, which are generally described by sets of partial differential equations? In the present paper, this problem is partially addressed by using an analytical method known as *beatification*.[19,20]

Beatification is a perturbative procedure through which the explicit variable dependence of a noncanonical Poisson bracket is replaced by a fixed dependence on a chosen state in phase space, designated as the *reference state*. As a result of the beatifying transformation, the Hamiltonian functional undergoes an increase in its degree of nonlinearity, so that the removal of the dynamical variable in the Poisson bracket is compensated.[19,20] Another important consequence of beatification is that it greatly facilitates the search for canonical variables, as can be readily verified in the case of a finite-dimensional Hamiltonian system.[19,20] Therefore, beatification can be seen as a preparatory step toward canonization, as implied by its name.

Considering both finite- and infinite-dimensional Hamiltonian systems, the beatification method was first presented in Ref. 19, where the beatifying transformation was derived, to its lowest perturbative order, by using an equilibrium as reference state. In a subsequent work,[20] considering specifically the Poisson bracket for vorticity-like variables in two-dimensional fluid and plasma theories, beatification was extended to second order and generalized to take into account completely arbitrary reference states. In the present paper, considering again the case of two-dimensional fluid and plasma models, the beatifying transformation is further extended to infinite perturbative order, also termed *complete beatification*. In this way, an appropriate setting is established for an in-depth investigation of some important Hamiltonian models, namely, the two-dimensional Euler equation,[9] the standard[10,11,21,22] and modified[12,23] versions of the Hasegawa-Mima equation, and the one-degree-of-freedom Vlasov-Poisson equation.[24,25] We mention that, in essence, the equivalent of complete beatification was obtained for the special case of expansion about shear flow equilibria in Ref. 26 and similar ideas were explored for the Vlasov-Poisson system by introduction of a generating function in Ref. 25.

A beatified Hamiltonian system exhibits some significant analytical advantages over its original noncanonical form. First, due to the removal of the explicit variable dependence from the Poisson bracket, the beatified system can have its degrees of freedom truncated without the loss of Hamiltonian structure.[20] Such a reduction process is very useful for obtaining a finite set of dynamical equations from an infinite-dimensional system, while retaining the Hamiltonian properties of the latter. Potential uses for this Hamiltonian truncation procedure include constructing low-dimensional models for describing specific physical mechanisms[27–44] and obtaining semi-discrete schemes for numerical integration of partial differential equations, as an alternative to techniques used or derived, for example, in Refs. 45–52. Second, as a collateral effect of beatification, all Casimir invariants[53] of a system become linear in the dynamical variables. As an evident consequence, analytical manipulation of these constants of motion is greatly simplified. Third, as a perturbative approach, beatification can be

used to simplify or emphasize the local dynamics about a particular phase-space point, chosen as reference state.

The remainder of this paper is organized as follows. Section II briefly introduces the Hamiltonian formalism for two-dimensional fluid and plasma theories. In this context, the pertinent Poisson brackets, Hamiltonian functionals, and equations of motion are presented. In Section III, the Casimir invariants of the relevant theories are derived. Section IV presents the beatification procedure itself, which is composed of two key steps. First, the field variable is rewritten as the sum of a reference state and a perturbative field. Second, an additional transformation is imposed on the perturbative field so that the explicit variable dependence is eliminated in the Poisson bracket. At the end of Section IV, the effect of the beatifying transformation on the Casimir invariants is examined in detail. Section V analyzes the particular case of a finite-order beatification procedure and presents a recurrence formula for the inverse of the beatifying transformation. In Section VI, the main findings of the paper are summarized and potential applications are discussed. Finally, in the Appendix, the suppression of the variable dependence in the Poisson bracket as a result of the beatifying transformation is demonstrated.

II. HAMILTONIAN FORMULATION

In this section, we present the main elements of the Hamiltonian formalism for four important fluid and plasma models, namely, the two-dimensional Euler equation, the standard and modified versions of the Hasegawa-Mima equation, and the one-degree-of-freedom Vlasov-Poisson equation. As a first step, we introduce the fundamental Poisson bracket that connects all the Hamiltonian systems considered[54]

$$\{F,G\} = \int_{\mathcal{D}} \mathrm{d}^2 r \, \frac{\delta F}{\delta \omega} \mathcal{J}(\omega) \frac{\delta G}{\delta \omega}, \tag{1}$$

where $\omega = \omega(x, y; t)$ is a vorticity-like scalar field on the two-dimensional domain $\mathcal{D}$, $\mathrm{d}^2 r = \mathrm{d}x\mathrm{d}y$, F and G are two arbitrary functionals of ω, and $\mathcal{J}$ is the *Poisson operator*, which is defined as

$$\mathcal{J}(f)g = -[f,g], \tag{2}$$

for two arbitrary functions f and g on the domain $\mathcal{D}$ and $[f,g] = (\partial_x f)(\partial_y g) - (\partial_y f)(\partial_x g)$. Notice that, except for a minus sign, the Poisson operator is just the Jacobian determinant $\partial(f,g)/\partial(x,y)$ or, equivalently, the z component of the cross product between gradients, that is, $(\nabla f \times \nabla g)_z$. Also presented in Equation (1), the functional derivatives are defined as usual

$$\delta F[\omega;\delta\omega] = \frac{\mathrm{d}}{\mathrm{d}\varepsilon} F[\omega + \varepsilon\delta\omega]\big|_{\varepsilon=0} = \int_{\mathcal{D}} \mathrm{d}^2 r \, \frac{\delta F}{\delta \omega} \delta\omega. \tag{3}$$

For simplicity, from now on, we choose the domain $\mathcal{D}$ as a normalized 2-torus, so that $x, y \in [0, 1)$ and periodic boundary conditions are implied. This choice allows us to promptly get rid of boundary terms whenever an integration by parts over $\mathcal{D}$ is performed.

Once the Poisson brackets of the relevant systems are known, in order to conclude the presentation of their Hamiltonian formalism, we also have to establish their Hamiltonian functionals. We shall see shortly that the required functionals are contained in the following general formula:[55]

$$H[\omega] = \int_{\mathcal{D}} \mathrm{d}^2 r \, h_1(r)\omega(r;t) + \frac{1}{2}\int_{\mathcal{D}} \mathrm{d}^2 r \int_{\mathcal{D}} \mathrm{d}^2 r' \, \omega(r;t) h_2(r,r')\omega(r';t), \tag{4}$$

for $r = (x, y)$. The quantities $h_1(r)$ and $h_2(r, r')$ of Equation (4) describe, respectively, the free-motion and two-point-interaction energies of the system. For the Vlasov-Poisson equation, considering particles with normalized mass and charge, the Hamiltonian functional is given with $h_1(r) = y^2/2$ and $h_2(r, r') = V(|x - x'|)$, where $V(|x - x'|)$ is the Green's function for the Poisson equation. In the cases of the Euler, Hasegawa-Mima, and modified Hasegawa-Mima equations, the Hamiltonian functionals are jointly specified by the expressions $h_1(r) = \mathcal{L}^{-1}\lambda$ and $h_2(r, r') = -\delta^{(2)}(r - r')\mathcal{L}^{-1}$, where $\lambda = \lambda(x, y)$ is a specified function on the domain $\mathcal{D}$ and $\mathcal{L}$ is a linear operator, which is also required to be self-adjoint with respect to the following scalar product between functions on the domain $\mathcal{D}$:

$$\langle f, g\rangle = \int_{\mathcal{D}} fg \, \mathrm{d}^2 r. \tag{5}$$

In each of the Hamiltonian models considered in this paper, the quantities ω, λ, and $\mathcal{L}$ perform different roles. In the case of the two-dimensional Euler equation, ω stands for the usual scalar vorticity, $\lambda = 0$, and $\mathcal{L}$ is the two-dimensional Laplacian operator, that is, $\mathcal{L} = \Delta = \partial_x^2 + \partial_y^2$. For the standard form of the Hasegawa-Mima equation, ω is a vorticity-like field related to the electrostatic potential ϕ by the transformation $\omega = \mathcal{L}\phi + \lambda$, λ is a function depending on the electron density at equilibrium,[11] and $\mathcal{L} = \Delta - 1$. For the modified Hasegawa-Mima equation, ω and λ have the same meanings as for the standard version, while $\mathcal{L} = \Delta - 1 + \mathcal{P}$, where the operator $\mathcal{P}$ denotes integration over the y-axis, that is, $\mathcal{P}f = \int_0^1 f \mathrm{d}y$ for any function f on the domain $\mathcal{D}$.[56] For the one-degree-of-freedom Vlasov-Poisson equation, ω is the phase-space probability distribution of a one-species plasma and the quantities λ and $\mathcal{L}$ are not defined.

A distinguishing feature in the description of the one-degree-of-freedom Vlasov-Poisson equation is that the two-dimensional domain $\mathcal{D}$ stands for the phase space of a charged particle restricted to a one-dimensional configuration space, unlike the three other systems considered, in which $\mathcal{D}$ is the actual space occupied by the fluid or plasma. In other words, for the Euler, Hasegawa-Mima, and modified Hasegawa-Mima equations, the coordinates x and y denote the spatial position of a fluid or plasma infinitesimal element, whereas, in the case of the Vlasov-Poisson equation, x and y represent, respectively, the position and linear momentum variables of a phase-space probability density function.

For completeness, we now work out the equations of motion for the four models considered here. To this end, we write down the usual Hamiltonian relation between the time variation of the field ω and the Poisson bracket (1)

$$\frac{\partial \omega}{\partial t} = \{\omega, H\} = \mathcal{J}(\omega)\frac{\delta H}{\delta \omega}. \tag{6}$$

From Equation (4), we can readily calculate the derivative of the Hamiltonian functional, which is found to be $\delta H/\delta\omega = h_1(r) + \int_{\mathcal{D}} \mathrm{d}^2 r'\, h_2(r, r')\omega(r'; t)$. By substituting this result into identity (6) with the appropriate values for h_1 and h_2, we first obtain a general expression for the Euler, Hasegawa-Mima, and modified Hasegawa-Mima equations

$$\frac{\partial \omega}{\partial t} = \left[\omega, \mathcal{L}^{-1}(\omega - \lambda)\right]. \tag{7}$$

Then, by considering the suitable choices for h_1 and h_2, we derive the one-degree-of-freedom Vlasov-Poisson equation

$$\frac{\partial \omega}{\partial t} + y\frac{\partial \omega}{\partial x} - \frac{\partial \phi}{\partial x}\frac{\partial \omega}{\partial y} = 0 \tag{8}$$

in which $\phi(x) = \int_{\mathcal{D}} V(|x - x'|)\omega(r')\mathrm{d}^2 r'$ is the electrostatic potential.

III. CASIMIR INVARIANTS

The Casimir invariants associated with a certain Poisson bracket are defined as the quantities whose functional derivatives belong to the null space of the corresponding Poisson operator. Therefore, in the case of the bracket (1), the Casimir invariants are determined by the following identity:

$$\mathcal{J}(\omega)\frac{\delta \mathrm{C}}{\delta \omega} = 0. \tag{9}$$

According to Equations (1) and (9), note that the Poisson bracket between a Casimir invariant $C[\omega]$ and an arbitrary functional $F[\omega]$ is identically zero, that is, $\{F, \mathrm{C}\} = 0$ for any functional F of the field ω. As a direct consequence, the Casimir invariants are constants of motion for any choice of Hamiltonian functional:

$$\frac{\mathrm{dC}}{\mathrm{d}t} = \{\mathrm{C}, H\} = 0. \tag{10}$$

By employing definition (2), we can readily demonstrate that the Poisson operator $\mathcal{J}$ satisfies the following equation:

$$\mathcal{J}(\omega)g(\omega) = 0, \tag{11}$$

where $g(\omega)$ denotes an arbitrary function of the vorticity-like field ω. By comparing identities (9) and (11) for $g(\omega) = \mathrm{d}f(\omega)/\mathrm{d}\omega$, we conclude that the Casimir invariants for the Poisson bracket (1) must take the form

$$\mathrm{C}[\omega] = \int_{\mathcal{D}} f(\omega)\, \mathrm{d}^2 r \tag{12}$$

as the functional derivative of the above expression is simply given by $\delta C/\delta\omega = \mathrm{d}f(\omega)/\mathrm{d}\omega$.

As a concluding remark to this section, we observe that, since the null space of a canonical Poisson operator is trivial, the existence of Casimir invariants is an exclusive property of noncanonical Hamiltonian systems.

IV. BEATIFICATION

Note that, excluding the possible dependence on the field ω arising from the derivatives of the functionals F and G, the Poisson bracket (1) has its own dependence on the field variable, which is contained in the Poisson operator $\mathcal{J}(\omega)$. The removal of this explicit variable dependence constitutes the primary purpose of the perturbative method known as *beatification*. In this section, we present the main result of the paper, namely, the infinite-order beatifying transformation for the Poisson bracket (1).

The beatification procedure is composed of two stages. First, the field ω is rewritten as the sum of a reference state ω_0 and a perturbative variable μ. Second, a nonlinear change of variables is performed on μ, which is recast in terms of a new perturbative field η. As a consequence of this second transformation, the Poisson operator becomes independent of the field variable.

The first step in the beatification procedure amounts to the following shift of the field ω:

$$\omega(x, y; t) = \omega_0(x, y) + \varepsilon\mu(x, y; t). \tag{13}$$

Here, the quantity ω_0, designated as the *reference state*, is an arbitrary time-independent function on the domain $\mathcal{D}$. In general, a specific choice for the state ω_0 is determined by physical or mathematical features we want to introduce or emphasize in the Hamiltonian system under study. In Equation (13), we have also presented the new dynamical variable μ and the perturbative parameter ε, which is used to keep track of terms from different perturbative orders during the beatification process.

As a preparation for our future developments, it is necessary to recast the Poisson bracket (1) in terms of the variable μ. For this purpose, we shall make use of the *chain rule for functional derivatives*[9]

$$\frac{\delta F}{\delta \chi} = \left(\frac{\delta \xi}{\delta \chi}\right)^{\dagger}\frac{\delta F}{\delta \xi}, \tag{14}$$

where χ and ξ are two field variables related by the transformation $\xi = \xi\{\chi\}$.[57] The quantity $\delta\xi/\delta\chi$ denotes the linear operator that, when applied on the variation $\delta\chi$, results in the corresponding variation $\delta\xi$. The operator $(\delta\xi/\delta\chi)^{\dagger}$ symbolizes the adjoint of $\delta\xi/\delta\chi$ with respect to the scalar product (5).

By applying the chain rule (14), we obtain that the functional derivatives with respect to the fields ω and μ are related by $\delta F/\delta\omega = \varepsilon^{-1}(\delta F/\delta\mu)$. Upon substitution of this result into Equation (1), a new form for the Poisson bracket is achieved

$$\varepsilon^2\{F, G\} = \int_{\mathcal{D}} \mathrm{d}^2 r \frac{\delta F}{\delta \mu}\left[\mathcal{J}(\omega_0) + \varepsilon\mathcal{J}(\mu)\right]\frac{\delta G}{\delta \mu}. \tag{15}$$

As we can see in the above equation, transformation (13) divided the original Poisson operator $\mathcal{J}(\omega)$ into two terms. The first one is simply the operator $\mathcal{J}(\omega)$ calculated at the reference state, that is, $\mathcal{J}(\omega_0)$. The second term, given by $\mathcal{K}(\mu) = \varepsilon\mathcal{J}(\mu)$, constitutes the perturbative part of the Poisson operator and is now responsible for all explicit variable dependence of the Poisson bracket.

The second step of the beatification procedure corresponds to finding a transformation $\eta = \eta\{\mu\}$ which eliminates the term $\mathcal{K}(\mu)$ in Equation (15) while preserving the operator $\mathcal{J}(\omega_0)$. In this way, the ultimate goal of beatification is attained, namely, removing the dependence on the dynamical field from the Poisson operator $\mathcal{J}(\omega)$ and replacing it with the reference state ω_0.

According to the functional chain rule, presented in Equation (14), the derivatives with respect the perturbative variable μ and the beatified field η are related by $\delta F/\delta\mu = (\delta\eta/\delta\mu)^\dagger \delta F/\delta\eta$, in which $\delta\eta/\delta\mu$ is the linear operator that transforms an infinitesimal variation $\delta\mu$ into a corresponding variation $\delta\eta$. By applying the chain rule to both functional derivatives in Equation (15), we rewrite the Poisson bracket in terms of the field η

$$\varepsilon^2\{F, G\} = \int_{\mathcal{D}} \mathrm{d}^2 r \frac{\delta F}{\delta\eta}\, \tilde{\mathcal{J}}\, \frac{\delta G}{\delta\eta}, \tag{16}$$

where the transformed Poisson operator is given by

$$\tilde{\mathcal{J}} = \frac{\delta\eta}{\delta\mu}\left[\mathcal{J}(\omega_0) + \varepsilon\mathcal{J}(\mu)\right]\left(\frac{\delta\eta}{\delta\mu}\right)^\dagger, \tag{17}$$

for $\mu = \mu\{\eta\}$. As previously mentioned, the beatifying transformation $\eta = \eta\{\mu\}$ is found by demanding that the right-hand side of Equation (17) be reduced to the value of the Poisson operator $\mathcal{J}(\omega)$ at the reference state. In short, the beatifying transformation is defined by the following identity:

$$\tilde{\mathcal{J}} = \mathcal{J}(\omega_0). \tag{18}$$

By employing the above equation, the beatifying transformation can be derived through an order-by-order perturbative process, whose zeroth-order term is taken as the identity transformation.[19] In the present paper, we do not follow this approach. Instead, we simply propose an expression for the beatifying transformation and then prove that it indeed satisfies Equation (18). In accordance with this plan of action, we now present the infinite-order beatifying transformation for the Poisson bracket (1)

$$\eta = \sum_{j=0}^{\infty} \frac{\varepsilon^j}{(j+1)!}\mathcal{B}^j\mu^{j+1}, \tag{19}$$

where, for notational simplicity, we have also defined an auxiliary operator

$$\mathcal{B}f = -\frac{1}{2}\left(\partial_x \frac{f}{\partial_x\omega_0} + \partial_y \frac{f}{\partial_y\omega_0}\right), \tag{20}$$

for any function f on the domain $\mathcal{D}$.[58] As expected from such perturbative expansion, expression (19) constitutes a near-identity transformation, that is, the perturbative series approaches the identity transformation $\eta = \mu$ as the parameter ε goes to zero.

Due to its relatively high complexity and length, the proof that transformation (19) satisfies identity (18) is left to the Appendix. In the remainder of this section, we shall discuss the Casimir invariants of the beatified Poisson operator and their relation to the original Casimir functionals, presented in Equation (12).

Analogous to Equation (9), the Casimir invariants associated with the Poisson operator (18) are defined by the following identity:

$$\mathcal{J}(\omega_0)\frac{\delta\tilde{C}[\eta]}{\delta\eta} = 0, \tag{21}$$

where we have introduced the tilde notation to specifically denote the Casimir functionals of the beatified field, that is, $\tilde{C} = \tilde{C}[\eta]$. By employing definition (2) in a completely similar way to Equation (11), we can readily show that

$$\mathcal{J}(\omega_0)g(\omega_0) = 0, \tag{22}$$

for any function g of the state ω_0. By comparing Equations (21) and (22), we conclude that the Casimir invariants $\tilde{C}[\eta]$ must be linear in the field η. For this reason, we define the following general expression for the beatified Casimir functionals:

$$\tilde{C}[\eta] = \int_{\mathcal{D}} \mathrm{d}^2 r\left[f(\omega_0) + \varepsilon\frac{\mathrm{d}f(\omega_0)}{\mathrm{d}\omega_0}\eta\right] \tag{23}$$

in which f corresponds to a second arbitrary function of ω_0. For the purpose of demonstrating the validity of the above definition, we present the functional derivative of this expression

$$\frac{\delta\tilde{C}[\eta]}{\delta\eta} = \varepsilon\frac{\mathrm{d}f(\omega_0)}{\mathrm{d}\omega_0}. \tag{24}$$

As a direct consequence of Equations (22) and (24) for $g(\omega_0) = \varepsilon[\mathrm{d}f(\omega_0)/\mathrm{d}\omega_0]$, we observe that definition (23) indeed satisfies identity (21).

Equations (12) and (23) provide general expressions for the Casimir invariants associated, respectively, with the Poisson operators $\mathcal{J}(\omega)$ and $\mathcal{J}(\omega_0)$. However, despite knowing the transformations connecting the fields ω and η, the direct relation between the functionals $C[\omega]$ and $\tilde{C}[\eta]$ has not yet been identified. In order to precisely relate these two equivalent sets of dynamical invariants, we now substitute transformation (19) into definition (23)

$$\tilde{C}[\eta] = \int_{\mathcal{D}} \mathrm{d}^2 r\left\{f(\omega_0) + \sum_{j=0}^{\infty}\frac{\varepsilon^{j+1}}{(j+1)!} \times \left[(\mathcal{B}^\dagger)^j\frac{\mathrm{d}f(\omega_0)}{\mathrm{d}\omega_0}\right]\mu^{j+1}\right\}. \tag{25}$$

In the above equation, we have introduced the adjoint of operator (20), which is explicitly given by

$$\mathcal{B}^{\dagger} f = \frac{1}{2}\left(\frac{\partial_x f}{\partial_x \omega_0} + \frac{\partial_y f}{\partial_y \omega_0}\right), \tag{26}$$

for any function f on the domain $\mathcal{D}$. Equation (25) can be greatly simplified with the aid of an important property of the operator $\mathcal{B}^{\dagger}$

$$\mathcal{B}^{\dagger} g(\omega_0) = \frac{\mathrm{d}g(\omega_0)}{\mathrm{d}\omega_0}, \tag{27}$$

which follows directly from identity (26) for any function g of the state ω_0. By making successive uses of property (27), Equation (25) reduces to

$$\begin{aligned}\tilde{C}[\eta] &= \int_{\mathcal{D}} \mathrm{d}^2 r \left[f(\omega_0) + \sum_{j=1}^{\infty} \frac{\varepsilon^j}{j!} \frac{\mathrm{d}^j f(\omega_0)}{\mathrm{d}\omega_0^j} \mu^j \right], \\ &= \int_{\mathcal{D}} f(\omega_0 + \varepsilon\mu)\, \mathrm{d}^2 r = \int_{\mathcal{D}} f(\omega)\, \mathrm{d}^2 r, \\ &= C[\omega]. \end{aligned} \tag{28}$$

The transition from the first to the second line of the above equation has been accomplished by identifying the Taylor series of the function $f(\omega_0 + \varepsilon\mu)$. As evidenced by identity (28), the function f found in definition (23) is exactly the same as that of Equation (12). In this way, the effect of the infinite-order beatifying transformation on the Casimir invariants becomes completely known.

V. FINITE-ORDER BEATIFICATION AND INVERSE TRANSFORMATION

In many situations of practical interest, such as the derivation of low-dimensional Hamiltonian models,[19,20] the beatification procedure is more appropriately used as a finite-order perturbative method. In this case, the beatifying transformation (19) is truncated at a predetermined power of the parameter ε

$$\eta^{(n)}\{\mu\} = \sum_{j=0}^{n} \frac{\varepsilon^j}{(j+1)!} \mathcal{B}^j \mu^{j+1}, \tag{29}$$

where $\eta^{(n)}$ is the n-th order beatified field. By using the above equation instead of the complete transformation (19), the Poisson operator (17) takes the following form:

$$\tilde{\mathcal{J}} = \mathcal{J}(\omega_0) + O(\varepsilon^{n+1}), \tag{30}$$

that is, the beatified Poisson operator of Equation (18) is again obtained, but this time the result holds only up to a certain perturbative order. The proof that the finite-order beatifying transformation provides the Poisson operator (30) is obtained by retracing the steps of the Appendix, where the proof for the complete beatification is given.

As a consequence of identity (30), the Casimir functionals $\tilde{C}[\eta^{(n)}]$ must have leading order terms, excluding constants, which are linear in the field $\eta^{(n)}$, in complete analogy with Equation (23). Therefore, by repeating the reasoning of Section IV with the finite-order transformation (29), we can show that the original Casimir invariants $C[\omega]$, presented in Equation (12), are related to the following beatified functionals:

$$\tilde{C}[\eta^{(n)}] = \int_{\mathcal{D}} \mathrm{d}^2 r \left[f(\omega_0) + \varepsilon \frac{\mathrm{d}f(\omega_0)}{\mathrm{d}\omega_0} \eta^{(n)} \right] + O(\varepsilon^{n+2}). \tag{31}$$

According to the above expression, the Casimir invariants $C[\omega]$ are also linearized by the finite-order beatifying transformation, but only if the terms of order ε^{n+2} and higher are discarded.

An actual application of the finite-order beatification procedure to the investigation of a specific dynamical system often relies on the knowledge of the inverse of transformation (29). Among other possible purposes, the inverse transformation is particularly necessary in calculating the beatified form of Hamiltonian functionals.

The inverse of the finite-order beatifying transformation can be obtained through a recursive order-by-order process. That is, given that the j-th order inverse transformation $\mu^{(j)} = \mu^{(j)}\{\eta\}$ is known for $j = 0, 1, \ldots, (n-1)$, the n-th order inverse transformation is determined by the following recurrence relation:

$$\mu^{(n)}\{\eta\} = \eta - \sum_{j=1}^{n} \frac{\varepsilon^j}{(j+1)!} \mathcal{B}^j \left[\mu^{(n-j)}\{\eta\}\right]^{j+1}, \tag{32}$$

where it is implied that $\mu^{(0)}\{\eta\} = \eta$. Note that the exponentiation of $\mu^{(n-j)}\{\eta\}$ on the right-hand side of Equation (32) can give rise to terms of order greater than ε^n. These spurious terms must be discarded so that the recurrence formula produces a consistent result for $\mu^{(n)}\{\eta\}$.

Notice that, upon substitution of expression (16) for $n > 0$, a Hamiltonian functional $H[\omega] = H[\omega_0 + \varepsilon\mu]$ experiences an increase in its degree of nonlinearity. This effect constitutes a compensation for the suppression of the variable dependence in the Poisson bracket so that the corresponding equations of motion are not subjected to a decrease in their degree of nonlinearity.

VI. CONCLUSION

Beatification is a perturbative method with the primary purpose of eliminating the variable dependence of a noncanonical Poisson operator by replacing it with a chosen reference state. As the main result of this paper, we present the infinite-order beatifying transformation for the fundamental Poisson bracket of four important fluid and plasma Hamiltonian models, namely, the two-dimensional Euler equation, the standard and modified versions of the Hasegawa-Mima equation, and the one-degree-of-freedom Vlasov-Poisson equation. This work builds on previous studies[19,20] by extending the beatification procedure to infinite perturbative order.

The noncanonical Hamiltonian formalism for two-dimensional fluid and plasma theories was briefly outlined in Section II. Although this discussion was focused on four particular models, we would like to point out that the

applicability of our central results is not restricted to the dynamical systems explicitly considered, as the Poisson bracket (1) takes part in the Hamiltonian description of a fairly broad class of continuous media theories. As clearly stated by the defining relation (18), the beatification procedure is independent of the specific choice for the Hamiltonian functional, since it depends only on the form of the Poisson operator. Consequently, the main results of Section IV, such as the beatifying transformation (19) and the beatified Casimir invariants (23), have geometrical nature and are not affected by purely dynamical aspects of a particular Hamiltonian system.

An interesting secondary effect of beatification is the linearization of the Casimir functionals, which is a direct consequence of definition (18). At the end of Section IV, by employing the infinite-order beatifying transformation, the precise relation between the original Casimir invariants, presented in Section III, and the corresponding beatified functionals was derived. The Casimir invariants have many potential applications in the practical study of noncanonical Hamiltonian systems. For example, the time-independent value of a Casimir functional can be used to validate a numerical solution.

A very common procedure in theoretical physics is the dimensional reduction of large dynamical systems with the purpose of enabling the application of numerical methods or generating low-dimensional models for the description of specific physical mechanisms. As already discussed in a previous work,[20] a direct reduction in the degrees of freedom of a noncanonical Hamiltonian system can be quite problematic, as the truncated version of a noncanonical Poisson bracket generally does not satisfy the Jacobi identity. That is, excluding special and accidental cases, the Hamiltonian structure of a noncanonical system is eliminated by dimensional reduction. As a consequence, many fundamental properties of the dynamical system are possibly lost, such as the incompressibility of phase-space volumes, which prevents the occurrence of attractors.[59] Beatification is a useful tool in preparing a noncanonical Hamiltonian system for proper dimensional reduction, since the Jacobi identity is preserved when a variable-independent Poisson operator undergoes a truncation process.

Another relevant application of the beatification method is as an intermediate step toward the canonization of Hamiltonian systems. More than this, the preliminary use of the beatification procedure can be seen as a significant part of a systematic approach for obtaining canonical variables in complex Hamiltonian systems. As such, beatification provides access to a wide array of analytical and numerical methods requiring a canonical Hamiltonian representation.

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APPENDIX: BEATIFICATION PROOF

Here, we present the proof that transformation (19) reduces expression (17) to the beatified Poisson operator $\mathcal{J}(\omega_0)$, as anticipated by Equation (18). As a first step toward this goal, we perform the first variation of identity (19)

$$\delta\eta = \sum_{j=0}^{\infty} \frac{\varepsilon^j}{j!} \mathcal{B}^j \mu^j \delta\mu. \tag{A1}$$

According to the above equation, variations in the fields μ and η are related by the following linear operator:

$$\frac{\delta\eta}{\delta\mu} f = \sum_{j=0}^{\infty} \frac{\varepsilon^j}{j!} \mathcal{B}^j \mu^j f, \tag{A2}$$

where f is an arbitrary function on the domain $\mathcal{D}$. In preparation for manipulating identity (17), we now present some special properties of the operators $\mathcal{J}$, $\mathcal{B}$, and $\mathcal{B}^\dagger$. First, as a direct consequence of Equation (26), note that the adjoint operator $\mathcal{B}^\dagger$ satisfies the Leibniz's rule, that is, $\mathcal{B}^\dagger fg = f\mathcal{B}^\dagger g + g\mathcal{B}^\dagger f$ for any two functions f and g on the domain $\mathcal{D}$. By performing successive applications of this rule, an important identity for the powers of the operator $\mathcal{B}^\dagger$ is obtained

$$(\mathcal{B}^\dagger)^m fg = f(\mathcal{B}^\dagger)^m g + \sum_{n=1}^{m} (\mathcal{B}^\dagger)^{m-n} (\mathcal{B}^\dagger f)(\mathcal{B}^\dagger)^{n-1} g. \tag{A3}$$

On account of its high relevance to our subsequent developments, we also present an interesting relation connecting the operators $\mathcal{J}$, $\mathcal{B}$, and $\mathcal{B}^\dagger$

$$\mathcal{B} f \mathcal{J}(\omega_0) g = -\mathcal{J}(f) g - \mathcal{J}(\omega_0) f \mathcal{B}^\dagger g, \tag{A4}$$

which is valid for any functions f, g, and ω_0 on the domain $\mathcal{D}$, under the condition that the operators $\mathcal{B}$ and $\mathcal{B}^\dagger$ are defined in terms of the state ω_0, in accordance with Equations (20) and (26). Notice that, in the particular case of a function f with constant value, identity (A4) reduces to $\mathcal{B}\mathcal{J}(\omega_0) g = -\mathcal{J}(\omega_0)\mathcal{B}^\dagger g$.

By employing Equations (A2) and (A4), we manipulate the following product of operators:

$$\begin{aligned}
\frac{\delta\eta}{\delta\mu}\mathcal{J}(\omega_0) &= \mathcal{J}(\omega_0) + \sum_{j=1}^{\infty} \frac{\varepsilon^j}{j!} \mathcal{B}^j \mu^j \mathcal{J}(\omega_0), \\
&= \mathcal{J}(\omega_0) - \sum_{j=1}^{\infty} \frac{\varepsilon^j}{j!} \mathcal{B}^{j-1} \mathcal{J}\left(\mu^j\right) \\
&\quad - \sum_{j=1}^{\infty} \frac{\varepsilon^j}{j!} \mathcal{B}^{j-1} \mathcal{J}(\omega_0) \mu^j \mathcal{B}^\dagger, \\
&= \mathcal{J}(\omega_0) - \sum_{j=1}^{\infty} \frac{\varepsilon^j}{(j-1)!} \mathcal{B}^{j-1} \mu^{j-1} \mathcal{J}(\mu) \\
&\quad + \sum_{j=1}^{\infty} (-1)^j \frac{\varepsilon^j}{j!} \mathcal{J}(\omega_0) (\mathcal{B}^\dagger)^{j-1} \mu^j \mathcal{B}^\dagger,
\end{aligned} \tag{A5}$$

which is found on the right-hand side of Equation (17). In elaborating the above result, we have also made use of the identity $\mathcal{J}(\mu^j) = j\mu^{j-1}\mathcal{J}(\mu)$, which directly follows from definition (2). Upon substitution of expression (A5), the Poisson operator (17) is readily reformulated as follows:

$$\tilde{\mathcal{J}} = \mathcal{J}(\omega_0)\left[1+\sum_{j=1}^{\infty}(-1)^j\frac{\varepsilon^j}{j!}(\mathcal{B}^\dagger)^{j-1}\mu^j\mathcal{B}^\dagger\right]\left(\frac{\delta\eta}{\delta\mu}\right)^\dagger. \tag{A6}$$

By inserting the adjoint of Equation (A2) into the above identity, further manipulations are performed on the transformed Poisson operator

$$\begin{aligned}\tilde{\mathcal{J}} &= \mathcal{J}(\omega_0)\left[1+\sum_{j=1}^{\infty}(-1)^j\frac{\varepsilon^j}{j!}(\mathcal{B}^\dagger)^{j-1}\mu^j\mathcal{B}^\dagger\right]\left[1+\sum_{j=1}^{\infty}\frac{\varepsilon^j}{j!}\mu^j(\mathcal{B}^\dagger)^j\right]\\ &= \mathcal{J}(\omega_0)\left[1+\sum_{j=1}^{\infty}\frac{\varepsilon^j}{j!}\mu^j(\mathcal{B}^\dagger)^j+\sum_{j=1}^{\infty}(-1)^j\frac{\varepsilon^j}{j!}(\mathcal{B}^\dagger)^{j-1}\mu^j\mathcal{B}^\dagger+\sum_{j,k=1}^{\infty}(-1)^j\frac{\varepsilon^{j+k}}{j!k!}(\mathcal{B}^\dagger)^{j-1}\mu^j\mathcal{B}^\dagger\mu^k(\mathcal{B}^\dagger)^k\right]\\ &= \mathcal{J}(\omega_0)+\mathcal{J}(\omega_0)\left\{\sum_{j=2}^{\infty}\frac{\varepsilon^j}{j!}\left[(-1)^j(\mathcal{B}^\dagger)^{j-1}\mu^j+\mu^j(\mathcal{B}^\dagger)^{j-1}\right]+\sum_{j=2}^{\infty}\sum_{k=1}^{j-1}(-1)^{j-k}\frac{\varepsilon^j}{(j-k)!k!}(\mathcal{B}^\dagger)^{j-k-1}\mu^{j-k}\mathcal{B}^\dagger\mu^k(\mathcal{B}^\dagger)^{k-1}\right\}\mathcal{B}^\dagger\\ &= \mathcal{J}(\omega_0)+\mathcal{J}(\omega_0)\mathcal{O}\mathcal{B}^\dagger,\end{aligned} \tag{A7}$$

where, for notational convenience, we have introduced the operator $\mathcal{O}$, whose explicit expression is given within the curly brackets on the right-hand side of the third equality. For the purpose of simplifying Equation (A7), we now make use of identity (A3) in obtaining two new auxiliary relations

$$(\mathcal{B}^\dagger)^{j-1}\mu^j f = \mu^j(\mathcal{B}^\dagger)^{j-1}f + j\sum_{k=1}^{j-1}(\mathcal{B}^\dagger)^{j-k-1}\mu^{j-1}(\mathcal{B}^\dagger\mu)(\mathcal{B}^\dagger)^{k-1}f, \tag{A8a}$$

$$\begin{aligned}&(\mathcal{B}^\dagger)^{j-k-1}\mu^{j-k}\mathcal{B}^\dagger\mu^k(\mathcal{B}^\dagger)^{k-1}f\\ &\quad= (\mathcal{B}^\dagger)^{j-k-1}\mu^j(\mathcal{B}^\dagger)^k f + k(\mathcal{B}^\dagger)^{j-k-1}\mu^{j-1}(\mathcal{B}^\dagger\mu)(\mathcal{B}^\dagger)^{k-1}f\\ &\quad= \mu^j(\mathcal{B}^\dagger)^{j-1}f + k(\mathcal{B}^\dagger)^{j-k-1}\mu^{j-1}(\mathcal{B}^\dagger\mu)(\mathcal{B}^\dagger)^{k-1}f\\ &\qquad+j\sum_{m=k+1}^{j-1}(\mathcal{B}^\dagger)^{j-m-1}\mu^{j-1}(\mathcal{B}^\dagger\mu)(\mathcal{B}^\dagger)^{m-1}f,\end{aligned} \tag{A8b}$$

in which f is again an arbitrary function on the domain $\mathcal{D}$. In deriving the above equations, we have also used the identity $\mathcal{B}^\dagger\mu^j = j\mu^{j-1}\mathcal{B}^\dagger\mu$, which follows from the Leibniz's rule for the operator $\mathcal{B}^\dagger$. By employing Equations (A8), we recast the operator $\mathcal{O}$ in a more convenient form

$$\begin{aligned}\mathcal{O} = &\sum_{j=2}^{\infty}\varepsilon^j\left[\frac{(-1)^j+1}{j!}+(-1)^j\sum_{k=1}^{j-1}\frac{(-1)^k}{(j-k)!k!}\right]\mu^j(\mathcal{B}^\dagger)^{j-1}\\ &+\sum_{j=2}^{\infty}(-\varepsilon)^j\sum_{k=1}^{j-1}\left[\frac{1}{(j-1)!}+\frac{(-1)^k}{(j-k)!(k-1)!}\right]\\ &\times(\mathcal{B}^\dagger)^{j-k-1}\mu^{j-1}(\mathcal{B}^\dagger\mu)(\mathcal{B}^\dagger)^{k-1}\\ &+\sum_{j=2}^{\infty}(-\varepsilon)^j\sum_{k=1}^{j-1}\sum_{m=k+1}^{j-1}\frac{(-1)^k j}{(j-k)!k!}\\ &\times(\mathcal{B}^\dagger)^{j-m-1}\mu^{j-1}(\mathcal{B}^\dagger\mu)(\mathcal{B}^\dagger)^{m-1}.\end{aligned} \tag{A9}$$

With the aid of the binomial theorem, the terms within square brackets in the first line of the above equation are shown to cancel one another. After making use of this fact, the operator $\mathcal{O}$ can be further simplified by switching the labels and the order of the summations over the indices k and m in the last two lines of Equation (A9). In this way, we obtain

$$\begin{aligned}\mathcal{O} = &\sum_{j=2}^{\infty}\sum_{k=1}^{j-1}\left[\frac{(-1)^k}{(j-k)!(k-1)!}+\sum_{m=0}^{k-1}\frac{(-1)^m j}{(j-m)!m!}\right]\\ &\times(-\varepsilon)^j(\mathcal{B}^\dagger)^{j-k-1}\mu^{j-1}(\mathcal{B}^\dagger\mu)(\mathcal{B}^\dagger)^{k-1}.\end{aligned} \tag{A10}$$

The terms within square brackets in the above expression also cancel one another, as can be straightforwardly demonstrated by induction on the index k. Therefore, identity (A10) simply reduces to $\mathcal{O}=0$. By substituting this result into Equation (A7), we finally arrive at the defining relation (18), thus concluding the proof that the beatification of the Poisson bracket (1) is provided by transformation (19).

[1]The term *noncanonical* is used to indicate a complete or overcomplete set of dynamical variables in terms of which a Hamiltonian system is not described by the canonical form of Hamilton's equations, but by a Poisson-bracket form with preserved algebraic properties. For further information on the fundamentals of noncanonical Hamiltonian systems, we refer the reader to Refs. 5 and 9.

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[57]The curly-bracket notation indicates that ξ depends on χ in an arbitrary manner, which may include, for example, derivatives of the latter.
[58]In the particular case where the state ω_0 depends on only one coordinate, the definition of the operator $\mathcal{B}$ needs to be slightly modified. For $\omega_0 = \omega_0(x)$, the correct expression is given by $\mathcal{B}f = -\partial_x(f/\partial_x\omega_0)$. The corresponding expression for $\omega_0 = \omega_0(y)$ is evident.
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