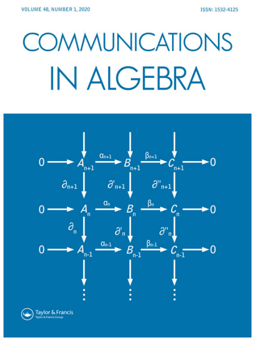

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Multiplicative Lie-type derivations on alternative rings

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ABSTRACT

Let $\mathfrak{R}$ be an alternative ring containing a nontrivial idempotent and $\mathfrak{D}$ be a multiplicative Lie-type derivation from $\mathfrak{R}$ into itself. Under certain assumptions on $\mathfrak{R}$, we prove that $\mathfrak{D}$ is almost additive. Let $p_n(x_1, x_2, \ldots, x_n)$ be the $(n-1)$-th commutator defined by n indeterminates $x_1, \ldots, x_n$. If $\mathfrak{R}$ is a unital alternative ring with a nontrivial idempotent and is $\{2, 3, n-1, n-3\}$-torsion free, it is shown under certain condition of $\mathfrak{R}$ and $\mathfrak{D}$ that $\mathfrak{D} = \delta + \tau$, where δ is a derivation and $\tau : \mathfrak{R} \to \mathcal{Z}(\mathfrak{R})$ such that $\tau(p_n(a_1, \ldots, a_n)) = 0$ for all $a_1, \ldots, a_n \in \mathfrak{R}$.

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1. Introduction and preliminaries

Let $\mathfrak{A}$ be an associative ring. We define the *Lie product* $[x, y] := xy - yx$ and *Jordan product* $x \circ y := xy + yx$ for all $x, y \in \mathfrak{A}$. Then $(\mathfrak{A}, [,])$ becomes a Lie algebra and $(\mathfrak{A}, \circ)$ is a Jordan algebra. It is a fascinating topic to study the connection between the associative, Lie, and Jordan structures on $\mathfrak{A}$. In this field, two classes of mappings are of crucial importance. One of them consists of mappings, preserving a type of product, for example, Jordan homomorphisms and Lie homomorphisms. The other one is formed by differential operators, satisfying a type of Leibniz formulas, such as Jordan derivations and Lie derivations. In the AMS Hour Talk of 1961, Herstein proposed many problems concerning the structure of Jordan and Lie mappings in associative simple and prime rings [17]. Roughly speaking, he conjectured that these mappings are all of the proper or standard forms. The renowned Herstein's Lie-type mapping research program was formulated since then. Martindale [25] gave a major force in this program under the assumption that the rings contain some nontrivial idempotents. The first idempotent-free result on Lie-type mappings was obtained by Brešar [4]. Recently, several new articles have also studied the additivity of maps that maintain new products and derivable maps about new products among them we can mention references [22–24, 29, 30]. Also the structures of derivations, Jordan derivations, and Lie derivations on (non-)associative rings were studied systematically by many people (cf. [1–4, 5–7, 9–15, 17–21, 25, 28, 31]). It is obvious that every derivation is a Lie derivation. But the converse is in general not true. A basic question toward Lie derivations of the associative algebras is that whether they can be decomposed into the sum of a derivation and a central-valued mapping, see [1–4, 15, 20, 21, 25] and references therein. In this article, we will address the structure of Lie derivations without additivity on alternative rings.

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Let $\mathfrak{R}$ and $\mathfrak{R}'$ be two rings (not necessarily associative) and $\varphi : \mathfrak{R} \to \mathfrak{R}'$ be a mapping, we call φ is *additive* if $\varphi(a+b) = \varphi(a) + \varphi(b)$, *almost additive* if $\varphi(a+b) - \varphi(a) - \varphi(b) \in \mathcal{Z}(\mathfrak{R})$, *multiplicative* if $\varphi(ab) = \varphi(a)\varphi(b)$, for all $a, b \in \mathfrak{R}$. Let $\mathfrak{R}$ be a ring with commutative center $\mathcal{Z}(\mathfrak{R})$ and $[x_1, x_2] = x_1x_2 - x_2x_1$ denote the usual Lie product of x_1 and x_2. Let us define the following sequence of polynomials:

$$p_1(x) = x \quad \text{and} \quad p_n(x_1, x_2, ..., x_n) = [p_{n-1}(x_1, x_2, ..., x_{n-1}), x_n]$$

for all integers $n \geq 2$. Thus, $p_2(x_1, x_2) = [x_1, x_2], p_3(x_1, x_2, x_3) = [[x_1, x_2], x_3]$, etc. Let $n \geq 2$ be an integer. A mapping (not necessarily additive) $\mathfrak{D} : \mathfrak{R} \to \mathfrak{R}$ is called a *multiplicative Lie n-derivation* if

$$\mathfrak{D}(p_n(x_1, x_2, ..., x_n)) = \sum_{i=1}^{n} p_n(x_1, x_2, ..., x_{i-1}, \mathfrak{D}(x_i), x_{i+1}, ..., x_n). \tag{1}$$

Lie n-derivations were introduced by Abdullaev [1], where the form of Lie n-derivations of a certain von Neumann algebra was described. According to the definition, each multiplicative Lie derivation is a multiplicative Lie 2-derivation and each multiplicative Lie triple derivation is a multiplicative Lie 3-derivation. Fošner et al. [15] showed that every multiplicative Lie n-derivation from an associative algebra $\mathcal{A}$ into itself is a multiplicative Lie $(n + k(n-1))$-derivation for each $k \in \mathbb{N}_0$. Multiplicative Lie 2-derivations, Lie 3-derivations, and Lie n-derivations are collectively referred to as *multiplicative Lie-type derivations*.

A ring $\mathfrak{R}$ is said to be *alternative* if $(x, x, y) = 0 = (y, x, x)$ for all $x, y \in \mathfrak{R}$, and *flexible* if $(x, y, x) = 0$ for all $x, y \in \mathfrak{R}$, where $(x, y, z) = (xy)z - x(yz)$ is the associator of $x, y, z \in \mathfrak{R}$. It is known that alternative rings are flexible. An alternative ring $\mathfrak{R}$ is called *k-torsion free* if $k\ x = 0$ implies $x = 0$, for any $x \in \mathfrak{R}$, where $k \in \mathbb{Z}$, $k > 0$, and *prime* if $\mathfrak{A}\mathfrak{B} \neq 0$ for any two nonzero ideals $\mathfrak{A}, \mathfrak{B} \subseteq \mathfrak{R}$. The *nucleus* $\mathcal{N}(\mathfrak{R})$ and the *commutative center* $\mathcal{Z}(\mathfrak{R})$ are defined by:

$$\mathcal{N}(\mathfrak{R}) = \{r \in \mathfrak{R} \mid (x, y, r) = 0 = (x, r, y) = (r, x, y) \text{ for all } x, y \in \mathfrak{R}\} \text{ and}$$
$$\mathcal{Z}(\mathfrak{R}) = \{r \in \mathfrak{R} \mid [r, x] = 0 \text{ for all } x \in \mathfrak{R}\}.$$

By [8, Theorem 1.1] we have the following.

Theorem 1.1. *Let $\mathfrak{R}$ be a 3-torsion free alternative ring. So $\mathfrak{R}$ is a prime ring if and only if $a\mathfrak{R} \cdot b = 0$ (or $a \cdot \mathfrak{R}b = 0$) implies that $a = 0$ or $b = 0$ for $a, b \in \mathfrak{R}$.*

A nonzero element $e_1 \in \mathfrak{R}$ is called an *idempotent* if $e_1^2 = e_1$ and the idempotent e_1 is a *nontrivial idempotent* if e_1 is not the multiplicative identity element of $\mathfrak{R}$. Let us consider $\mathfrak{R}$ an alternative ring and fix a nontrivial idempotent $e_1 \in \mathfrak{R}$. Let $e_2 : \mathfrak{R} \to \mathfrak{R}$ and $e_2' : \mathfrak{R} \to \mathfrak{R}$ be linear operators given by $e_2(a) = a - e_1a$ and $e_2'(a) = a - ae_1$. Clearly, $e_2^2 = e_2 \circ e_2 = e_2$, $(e_2')^2 = e_2'$. Note that if $\mathfrak{R}$ has a unity, then $e_2 = 1 - e_1 \in \mathfrak{R}$. Let us denote $e_2(a)$ by e_2a and $e_2'(a)$ by ae_2. It is easy to see that $e_ia \cdot e_j = e_i \cdot ae_j$ $(i, j = 1, 2)$ for all $a \in \mathfrak{R}$. By [16] we know that $\mathfrak{R}$ has a Peirce decomposition

$$\mathfrak{R} = \mathfrak{R}_{11} \oplus \mathfrak{R}_{12} \oplus \mathfrak{R}_{21} \oplus \mathfrak{R}_{22},$$

where $\mathfrak{R}_{ij} = e_i\mathfrak{R}e_j$ $(i, j = 1, 2)$, satisfying the following multiplicative relations:

(i) $\mathfrak{R}_{ij}\mathfrak{R}_{jl} \subseteq \mathfrak{R}_{il}(i, j, l = 1, 2)$;
(ii) $\mathfrak{R}_{ij}\mathfrak{R}_{ij} \subseteq \mathfrak{R}_{ji}(i, j = 1, 2)$;
(iii) $\mathfrak{R}_{ij}\mathfrak{R}_{kl} = 0$, if $j \neq k$ and $(i, j) \neq (k, l), (i, j, k, l = 1, 2)$;
(iv) $x_{ij}^2 = 0$, for all $x_{ij} \in \mathfrak{R}_{ij}(i, j = 1, 2;\ i \neq j)$.

The first result about the additivity of mappings on rings was given by Martindale [26], he established a condition on a ring $\mathfrak{R}$ such that every multiplicative isomorphism on $\mathfrak{R}$ is additive.

In [20, 21], Li and his coauthors also considered the almost additivity of maps for the case of Lie multiplicative mappings and Lie 3-derivation on associative rings. They proved the following.

Theorem 1.2. *Let $\mathfrak{R}$ be an associative ring containing a nontrivial idempotent e_1 and satisfying the following condition: ($\mathbb{Q}$) If $A_{11}B_{12} = B_{12}A_{22}$ for all $B_{12} \in \mathfrak{R}_{12}$, then $A_{11} + A_{22} \in \mathcal{Z}(\mathfrak{R})$. Let $\mathfrak{R}'$ be another ring. Suppose that a bijection map $\Phi : \mathfrak{R} \to \mathfrak{R}'$ satisfies*

$$\Phi([A, B]) = [\Phi(A), \Phi(B)]$$

for all $A, B \in \mathfrak{R}$. Then $\Phi(A + B) = \Phi(A) + \Phi(B) + Z'_{A,B}$ for all $A, B \in \mathfrak{R}$, where $Z'_{A,B}$ is an element in the commutative center $\mathcal{Z}(\mathfrak{R}')$ of $\mathfrak{R}'$ depending on A and B.

Theorem 1.3. *Let $\mathfrak{R}$ be an associative ring containing a nontrivial idempotent e_1 and satisfying the following condition: ($\mathbb{Q}$) If $A_{11}B_{12} = B_{12}A_{22}$ for all $B_{12} \in \mathfrak{R}_{12}$, then $A_{11} + A_{22} \in \mathcal{Z}(\mathfrak{R})$. Suppose that a mapping $\delta : \mathfrak{R} \to \mathfrak{R}$ satisfies*

$$\delta([[A, B], C]) = [[\delta(A), B], C] + [[A, \delta(B)], C] + \big[[A, B], \delta(C)\big]$$

for all $A, B, C \in \mathfrak{R}$. Then there exists a $Z_{A,B}$ (depending on A and B) in $\mathcal{Z}(\mathfrak{R})$ such that $\delta(A + B) = \delta(A) + \delta(B) + Z_{A,B}$.

In [13], Ferreira and Guzzo investigated the additivity of Lie triple derivations. They obtained the following result.

Theorem 1.4. *Let $\mathfrak{R}$ be an alternative ring. Suppose that $\mathfrak{R}$ is a ring containing a nontrivial idempotent e_1 which satisfies*

(i) *If $[a_{11} + a_{22}, \mathfrak{R}_{12}] = 0$, then $a_{11} + a_{22} \in \mathcal{Z}(\mathfrak{R})$,*
(ii) *If $[a_{11} + a_{22}, \mathfrak{R}_{21}] = 0$, then $a_{11} + a_{22} \in \mathcal{Z}(\mathfrak{R})$.*

Then each multiplicative Lie triple derivation $\mathfrak{D}$ of $\mathfrak{R}$ into itself is almost additive.

In a recent article, Ferreira and Guzzo [9] studied the characterization of Lie 2-derivation on alternative rings. They showed that

Theorem 1.5. *Let $\mathfrak{R}$ be a unital 2,3-torsion free alternative ring with nontrivial idempotents e_1, e_2 and with associated Peirce decomposition $\mathfrak{R} = \mathfrak{R}_{11} \oplus \mathfrak{R}_{12} \oplus \mathfrak{R}_{21} \oplus \mathfrak{R}_{22}$. Suppose that $\mathfrak{R}$ satisfies the following conditions:*

(1) *If $x_{ij}\mathfrak{R}_{ji} = 0$, then $x_{ij} = 0$ ($i \neq j$);*
(2) *If $x_{11}\mathfrak{R}_{12} = 0$ or $\mathfrak{R}_{21}x_{11} = 0$, then $x_{11} = 0$;*
(3) *If $\mathfrak{R}_{12}x_{22} = 0$ or $x_{22}\mathfrak{R}_{21} = 0$, then $x_{22} = 0$;*
(4) *If $z \in \mathcal{Z}(\mathfrak{R})$ with $z \neq 0$, then $z\mathfrak{R} = \mathfrak{R}$.*

Let $\mathfrak{D} : \mathfrak{R} \to \mathfrak{R}$ be a multiplicative Lie derivation of $\mathfrak{R}$. Then $\mathfrak{D}$ is the form $\delta + \tau$, where δ is an additive derivation of $\mathfrak{R}$ and τ is a mapping from $\mathfrak{R}$ into the commutative center $\mathcal{Z}(\mathfrak{R})$, which maps commutators into the zero if and only if

(a) $e_2\mathfrak{D}(\mathfrak{R}_{11})e_2 \subseteq \mathcal{Z}(\mathfrak{R})e_2$,
(b) $e_1\mathfrak{D}(\mathfrak{R}_{22})e_1 \subseteq \mathcal{Z}(\mathfrak{R})e_1$.

Inspired by the aforementioned results, we are planning to extend Theorem 1.4 to an arbitrary mulitplicative Lie-type derivations in Section 2. In Section 3, we give the characterization of multiplicative Lie-type derivations on alternative rings and study the structure of multiplicative Lie-type derivations on alternative rings, which can be considered as a natural generalization of Theorem 1.5.

2. Almost additivity of multiplicative Lie-type derivations

We shall prove as follows the first main result of this article.

Theorem 2.1. *Let $\mathfrak{R}$ be an alternative ring with nontrivial idempotent e_1, $\mathcal{Z}(\mathfrak{R})$ be the commutative center of $\mathfrak{R}$ and $\mathfrak{D}$ be a multiplicative Lie-type derivation of $\mathfrak{R}$. Suppose that $\mathfrak{R}$ satisfies the following conditions:*

(i) *If $[a_{11}+a_{22}, \mathfrak{R}_{12}] = 0$, then $a_{11}+a_{22} \in \mathcal{Z}(\mathfrak{R})$,*
(ii) *If $[a_{11}+a_{22}, \mathfrak{R}_{21}] = 0$, then $a_{11}+a_{22} \in \mathcal{Z}(\mathfrak{R})$.*

Then $\mathfrak{D}$ is almost additive.

As our goal is to generalize the result obtained in [13], the following Lemmas are generalizations of Lemmas that appear in [13]. The hypotheses of the following lemmas are the same as Theorem 2.1.

It is easy to see that $\mathfrak{D}(0) = 0$.

Lemma 2.2. *For any $a_{11} \in \mathfrak{R}_{11}, b_{ij} \in \mathfrak{R}_{ij}$, with $i \neq j$ there exist $z_{a_{11}, b_{ij}} \in \mathcal{Z}(\mathfrak{R})$ such that, $\mathfrak{D}(a_{11} + b_{ij}) = \mathfrak{D}(a_{11}) + \mathfrak{D}(b_{ij}) + z_{a_{11}, b_{ij}}$.*

Proof. We only prove the case of $i = 1$, $j = 2$ because the demonstration of the other cases is rather similar by using the condition (i) of the Theorem 2.1. Let us set $t = \mathfrak{D}(a_{11}+b_{12}) - \mathfrak{D}(a_{11}) - \mathfrak{D}(b_{12})$. Then we get $p_n(t, e_1, \ldots, e_1) = 0$, which is due to the fact

$$\begin{aligned}\mathfrak{D}(p_n(a_{11}+b_{12}, e_1, \ldots, e_1)) &= \mathfrak{D}((-1)^{n+1} b_{12}) \\ &= \mathfrak{D}(p_n(a_{11}, e_1, \ldots, e_1)) \\ &\quad + \mathfrak{D}(p_n(b_{12}, e_1, \ldots, e_1)).\end{aligned}$$

In view of the definition of $\mathfrak{D}$, we have $(-1)^{n+1} t_{12} + t_{21} = 0$. Now we will use the condition (ii) of the Theorem 2.1. For any $c_{21} \in \mathfrak{R}_{21}$, we know that

$$\begin{aligned}\mathfrak{D}(p_n(a_{11}+b_{12}, c_{21}, e_1, \ldots, e_1)) &= \mathfrak{D}(-c_{21} a_{11}) \\ &= \mathfrak{D}(p_n(a_{11}, c_{21}, e_1, \ldots, e_1)) \\ &\quad + \mathfrak{D}(p_n(b_{12}, c_{21}, e_1, \ldots, e_1)).\end{aligned}$$

Now using the definition of $\mathfrak{D}$ and $D(0) = 0$, we obtain $[t_{11} + t_{22}, c_{21}] = p_n(t, c_{21}, e_1, \ldots, e_1) = 0$. Therefore by condition (ii) of the Theorem 2.1 we have $t_{11} + t_{22} \in \mathcal{Z}(\mathfrak{R})$. And hence $\mathfrak{D}(a_{11} + b_{12}) = \mathfrak{D}(a_{11}) + \mathfrak{D}(b_{12}) + z_{a_{11}, b_{12}}$. □

Lemma 2.3. *For any $a_{12} \in \mathfrak{R}_{12}$ and $b_{21} \in \mathfrak{R}_{21}$, we have $\mathfrak{D}(a_{12} + b_{21}) = \mathfrak{D}(a_{12}) + \mathfrak{D}(b_{21})$.*

Proof. First, observe that $(-1)^{n+1} a_{12} + b_{21} = p_n(e_1 + a_{12}, e_1 - b_{21}, e_1, \ldots, e_1)$ for all $a_{12} \in \mathfrak{R}_{12}$ and $b_{21} \in \mathfrak{R}_{21}$. By invoking Lemma 2.2, we arrive at

$$\begin{aligned}\mathfrak{D}((-1)^{n+1}a_{12}+b_{21}) &= \mathfrak{D}(p_n(e_1+a_{12},e_1-b_{21},e_1,\dots,e_1))\\ &= p_n(\mathfrak{D}(e_1+a_{12}),e_1-b_{21},e_1,\dots,e_1)\\ &+ p_n(e_1+a_{12},\mathfrak{D}(e_1-b_{21}),e_1,\dots,e_1)\\ &\quad+\sum_{i=3}^{n}p_n(e_1+a_{12},e_1-b_{21},e_1,\dots,\mathfrak{D}(e_1),\dots,e_1)\\ &= \mathfrak{D}(p_n(e_1,e_1,e_1,\dots,e_1))\\ &+ \mathfrak{D}(p_n(e_1,-b_{21},e_1,\dots,e_1))\\ &+ \mathfrak{D}(p_n(a_{12},e_1,e_1,\dots,e_1))\\ &+ \mathfrak{D}(p_n(a_{12},-b_{21},e_1,\dots,e_1))\\ &= \mathfrak{D}((-1)^{n+1}a_{12})+\mathfrak{D}(b_{21}).\end{aligned}$$

In the case of n is odd, then $\mathfrak{D}(-a_{12}+b_{21})=\mathfrak{D}(-a_{12})+\mathfrak{D}(b_{21})$. However, this clearly implies that $\mathfrak{D}(a_{12}+b_{21})=\mathfrak{D}(a_{12})+\mathfrak{D}(b_{21})$. □

Lemma 2.4. *For any $a_{ij},b_{ij}\in\mathfrak{R}_{ij}$ with $i\neq j$, we have $\mathfrak{D}(a_{ij}+b_{ij})=\mathfrak{D}(a_{ij})+\mathfrak{D}(b_{ij})$.*

Proof. Here we shall only prove the case $i=2,\ j=1$ because the proofs of the other cases are similar. Note that $x_{ij}^2=0,$ for all $x_{ij}\in\mathfrak{R}_{ij}(i,j=1,2;\ \ i\neq j)$. Thus we have

$$a_{21}+b_{21}+2(-1)^{n+1}a_{21}b_{21}=p_n(e_1+a_{21},e_1-b_{21},e_1,\dots,e_1).$$

Now making use of Lemmas 2.2 and 2.3 we get

$$\begin{aligned}\mathfrak{D}(a_{21}+b_{21})+\mathfrak{D}(2(-1)^{n+1}a_{21}b_{21}) &= \mathfrak{D}(a_{21}+b_{21}+2(-1)^{n+1}a_{21}b_{21})\\ &= \mathfrak{D}(p_n(e_1+a_{21},e_1-b_{21},e_1,\dots,e_1))\\ &= p_n(\mathfrak{D}(e_1+a_{21}),e_1-b_{21},e_1,\dots,e_1)\\ &+ p_n(e_1+a_{21},\mathfrak{D}(e_1-b_{21}),e_1,\dots,e_1)\\ &+\sum_{i=3}^{n}p_n(e_1+a_{21},e_1-b_{21},e_1,\dots,\mathfrak{D}(e_1),\dots,e_1)\\ &= p_n(\mathfrak{D}(e_1)+\mathfrak{D}(a_{21}),e_1-b_{21},e_1,\dots,e_1)\\ &+ p_n(e_1+a_{21},\mathfrak{D}(e_1)+\mathfrak{D}(-b_{21}),e_1\dots,e_1)\\ &+\sum_{i=3}^{n}p_n(e_1+a_{21},e_1-b_{21},e_1,\dots,\mathfrak{D}(e_1),\dots,e_1)\\ &= \mathfrak{D}(p_n(e_1,e_1,\dots,e_1))+\mathfrak{D}(p_n(e_1,-b_{21},e_1,\dots e_1))\\ &+ \mathfrak{D}(p_n(a_{21},e_1,e_1,\dots,e_1))\\ &+ \mathfrak{D}(p_n(a_{21},-b_{21},e_1,\dots,e_1))\\ &= \mathfrak{D}(a_{21})+\mathfrak{D}(b_{21})+\mathfrak{D}((-1)^{n+1}2a_{21}b_{21}).\end{aligned}$$

For the case $i=1,\ j=2$, we only need to use

$$(-1)^{n+1}(a_{12}+b_{12})+2a_{12}b_{12}=p_n(e_1+a_{12},e_1-b_{12},e_1,\dots,e_1)$$

together with Lemmas 2.2 and 2.3. □

Lemma 2.5. *For any* $a_{ii}, b_{ii} \in \mathfrak{R}_{ii}$, $i = 1, 2$, *there exists a* $z_{a_{ii}, b_{ii}} \in \mathcal{Z}(\mathfrak{R})$ *such that*

$$\mathfrak{D}(a_{ii} + b_{ii}) = \mathfrak{D}(a_{ii}) + \mathfrak{D}(b_{ii}) + z_{a_{ii}, b_{ii}}.$$

Proof. Let us set $t = \mathfrak{D}(a_{ii} + b_{ii}) - \mathfrak{D}(a_{ii}) - \mathfrak{D}(b_{ii})$. On the one hand,

$$\begin{aligned} 0 &= \mathfrak{D}(0) \\ &= \mathfrak{D}(p_n(a_{ii} + b_{ii}, e_1, ..., e_1)) \\ &= p_n(\mathfrak{D}(a_{ii} + b_{ii}), e_1, ..., e_1) + \sum_{i=2}^{n} p_n(a_{ii} + b_{ii}, e_1, ..., \mathfrak{D}(e_1), ..., e_1). \end{aligned}$$

On the other hand,

$$\begin{aligned} 0 &= \mathfrak{D}(0) + \mathfrak{D}(0) \\ &= \mathfrak{D}(p_n(a_{ii}, e_1, ..., e_1)) + \mathfrak{D}(p_n(b_{ii}, e_1, ..., e_1)) \\ &= p_n(\mathfrak{D}(a_{ii}) + \mathfrak{D}(b_{ii}), e_1, ..., e_1) + \sum_{i=2}^{n} p_n(a_{ii} + b_{ii}, e_1, ..., \mathfrak{D}(e_1), ..., e_1). \end{aligned}$$

This implies that $p_n(t, e_1, ..., e_1) = 0$. That is $t_{12} = t_{21} = 0$. For any $c_{ij} \in \mathfrak{R}_{ij}$, with $i \neq j$, by Lemma 2.4, we obtain

$$\begin{aligned} \mathfrak{D}((-1)^{n+1}(a_{ii} + b_{ii})c_{ij}) &= \mathfrak{D}((-1)^{n+1}a_{ii}c_{ij}) + \mathfrak{D}((-1)^{n+1}b_{ii}c_{ij}) \\ &= \mathfrak{D}(p_n(c_{ij}, a_{ii}, e_1, ..., e_1)) \\ &+ \mathfrak{D}(p_n(c_{ij}, b_{ii}, e_1, ..., e_1)) \\ &= p_n(\mathfrak{D}(c_{ij}), a_{ii} + b_{ii}, e_1, ..., e_1) \\ &+ p_n(c_{ij}, \mathfrak{D}(a_{ii}) + \mathfrak{D}(b_{ii}), e_1, ..., e_1) \\ &+ \sum_{i=3}^{n} p_n(c_{ij}, a_{ii} + b_{ii}, e_1, ..., \mathfrak{D}(e_1), ..., e_1). \end{aligned}$$

Now we also have,

$$\begin{aligned} \mathfrak{D}((-1)^{n+1}(a_{ii} + b_{ii})c_{ij}) &= \mathfrak{D}(p_n(c_{ij}, a_{ii} + b_{ii}, e_1, ..., e_1)) \\ &= p_n(\mathfrak{D}(c_{ij}), a_{ii} + b_{ii}, e_1, ..., e_1) \\ &+ p_n(c_{ij}, \mathfrak{D}(a_{ii} + b_{ii}), e_1, ..., e_1) \\ &+ \sum_{i=3}^{n} p_n(c_{ij}, a_{ii} + b_{ii}, e_1, ..., \mathfrak{D}(e_1), ..., e_1). \end{aligned}$$

Hence $p_n(c_{ij}, t, e_1, ..., e_1) = 0$. This give $[t_{11} + t_{22}, c_{ij}] = 0$ for all $c_{ij} \in \mathfrak{R}_{ij}$ with $i \neq j$. By the conditions of Theorem 2.1, we get $t_{11} + t_{22} \in \mathcal{Z}(\mathfrak{R})$. Therefore, $\mathfrak{D}(a_{ii} + b_{ii}) = \mathfrak{D}(a_{ii}) + \mathfrak{D}(b_{ii}) + z_{a_{ii}, b_{ii}}$. □

Lemma 2.6. *For any* $a_{11} \in \mathfrak{R}_{11}, b_{12} \in \mathfrak{R}_{12}, c_{21} \in \mathfrak{R}_{21}, d_{22} \in \mathfrak{R}_{22}$, *there exists a* $z_{a_{11}, b_{12}, c_{21}, d_{22}} \in \mathcal{Z}(\mathfrak{R})$ *such that*

$$\mathfrak{D}(a_{11} + b_{12} + c_{21} + d_{22}) = \mathfrak{D}(a_{11}) + \mathfrak{D}(b_{12}) + \mathfrak{D}(c_{21}) + \mathfrak{D}(d_{22}) + z_{a_{11}, b_{12}, c_{21}, d_{22}}.$$

Proof. Let us write $t = \mathfrak{D}(a_{11} + b_{12} + c_{21} + d_{22}) - \mathfrak{D}(a_{11}) - \mathfrak{D}(b_{12}) - \mathfrak{D}(c_{21}) - \mathfrak{D}(d_{22})$. By the definition of $\mathfrak{D}$ and Lemma 2.3 we know that $p_n(t, e_1, ..., e_1) = 0$. Indeed,

$$\begin{aligned}
p_n(t, e_1, ..., e_1) &= p_n(\mathfrak{D}(a_{11} + b_{12} + c_{21} + d_{22}) - \mathfrak{D}(a_{11}) \\
&\quad - \mathfrak{D}(b_{12}) - \mathfrak{D}(c_{21}) - \mathfrak{D}(d_{22}), e_1, ..., e_1) \\
&= p_n(\mathfrak{D}(a_{11} + b_{12} + c_{21} + d_{22}), e_1, ..., e_1) - p_n(\mathfrak{D}(a_{11}), e_1, ..., e_1) \\
&\quad - p_n(\mathfrak{D}(b_{12}), e_1, ..., e_1) - p_n(\mathfrak{D}(c_{21}), e_1, ..., e_1) - p_n(\mathfrak{D}(d_{22}), e_1, ..., e_1) \\
&= \mathfrak{D}(p_n(a_{11} + b_{12} + c_{21} + d_{22}, e_1, ..., e_1)) \\
&\quad - \sum_{i=2}^{n} p_n(a_{11} + b_{12} + c_{21} + d_{22}, e_1, ..., \mathfrak{D}(e_1), ..., e_1) \\
&\quad - \left\{ \mathfrak{D}(p_n(a_{11}, e_1, ..., e_1)) - \sum_{i=2}^{n} p_n(a_{11}, e_1, ..., \mathfrak{D}(e_1), ..., e_1) \right\} \\
&\quad - \left\{ \mathfrak{D}(p_n(b_{12}, e_1, ..., e_1)) - \sum_{i=2}^{n} p_n(b_{12}, e_1, ..., \mathfrak{D}(e_1), ..., e_1) \right\} \\
&\quad - \left\{ \mathfrak{D}(p_n(c_{21}, e_1, ..., e_1)) - \sum_{i=2}^{n} p_n(c_{21}, e_1, ..., \mathfrak{D}(e_1), ..., e_1) \right\} \\
&\quad - \left\{ \mathfrak{D}(p_n(d_{22}, e_1, ..., e_1)) - \sum_{i=2}^{n} p_n(d_{22}, e_1, ..., \mathfrak{D}(e_1), ..., e_1) \right\} \\
&= \mathfrak{D}((-1)^{n+1} b_{12} + c_{21}) - \mathfrak{D}((-1)^{n+1} b_{12}) - \mathfrak{D}(c_{21}) \\
&= 0.
\end{aligned}$$

As $p_n(t, e_1, ..., e_1) = 0$, we conclude that $(-1)^{n+1} t_{12} + t_{21} = 0$. Now for all $x_{12} \in \mathfrak{R}_{12}$, by Lemmas 2.3 and 2.4 we get

$$\begin{aligned}
&p_n(\mathfrak{D}(a_{11} + b_{12} + c_{21} + d_{22}), x_{12}, e_1, ..., e_1) \\
&+ p_n(a_{11} + b_{12} + c_{21} + d_{22}, \mathfrak{D}(x_{12}), e_1, ..., e_1) \\
&+ \sum_{i=3}^{n} p_n(a_{11} + b_{12} + c_{21} + d_{22}, x_{12}, e_1, ..., \mathfrak{D}(e_1), ..., e_1) \\
&= \mathfrak{D}(p_n(a_{11} + b_{12} + c_{21} + d_{22}, x_{12}, e_1, ..., e_1)) \\
&= \mathfrak{D}((-1)^{n+1} x_{12} d_{22} + (-1)^n a_{11} x_{12} + (-1)^n b_{12} x_{12}) \\
&= \mathfrak{D}((-1)^{n+1} x_{12} d_{22} + (-1)^n a_{11} x_{12}) + \mathfrak{D}((-1)^n b_{12} x_{12}) \\
&= \mathfrak{D}((-1)^{n+1} x_{12} d_{22}) + \mathfrak{D}((-1)^n a_{11} x_{12}) + \mathfrak{D}((-1)^n b_{12} x_{12}) \\
&= \mathfrak{D}(p_n(a_{11}, x_{12}, e_1, ..., e_1)) + \mathfrak{D}(p_n(b_{12}, x_{12}, e_1, ..., e_1)) \\
&+ \mathfrak{D}(p_n(c_{21}, x_{12}, e_1, ..., e_1)) + \mathfrak{D}(p_n(d_{22}, x_{12}, e_1, ..., e_1)) \\
&= p_n(\mathfrak{D}(a_{11}) + \mathfrak{D}(b_{12}) + \mathfrak{D}(c_{21}) + \mathfrak{D}(d_{22}), x_{12}, e_1, ..., e_1) \\
&+ p_n(a_{11} + b_{12} + c_{21} + d_{22}, \mathfrak{D}(x_{12}), e_1, ..., e_1) \\
&+ \sum_{i=3}^{n} p_n(a_{11} + b_{12} + c_{21} + d_{22}, x_{12}, e_1, ..., \mathfrak{D}(e_1), ..., e_1).
\end{aligned}$$

We therefore have $p_n(\mathfrak{D}(a_{11} + b_{12} + c_{21} + d_{22}), x_{12}, e_1, ..., e_1) = p_n(\mathfrak{D}(a_{11}) + \mathfrak{D}(b_{12}) + \mathfrak{D}(c_{21}) + \mathfrak{D}(d_{22}), x_{12}, e_1, ..., e_1)$. That is, $[t_{11} + t_{22}, x_{12}] = p_n(t, x_{12}, e_1, ..., e_1) = 0$. Applying the condition (i) of Theorem 2.1 yields $t = t_{11} + t_{22} \in \mathcal{Z}(\mathfrak{R})$. Thus, $\mathfrak{D}(a_{11} + b_{12} + c_{21} + d_{22}) = \mathfrak{D}(a_{11}) + \mathfrak{D}(b_{12}) + \mathfrak{D}(c_{21}) + \mathfrak{D}(d_{22}) + z_{a_{11}, b_{12}, c_{21}, d_{22}}$, where $z_{a_{11}, b_{12}, c_{21}, d_{22}} \in \mathcal{Z}(\mathfrak{R})$.

We are ready to prove our Theorem 2.1. □

Proof of Theorem 2.1. Let $a, b \in \mathfrak{R}$ with $a = a_{11} + a_{12} + a_{21} + a_{22}$ and $b = b_{11} + b_{12} + b_{21} + b_{22}$. By previous Lemmas we obtain

$$\begin{aligned}
\mathfrak{D}(a+b) &= \mathfrak{D}(a_{11} + a_{12} + a_{21} + a_{22} + b_{11} + b_{12} + b_{21} + b_{22}) \\
&= \mathfrak{D}((a_{11} + b_{11}) + (a_{12} + b_{12}) + (a_{21} + b_{21}) + (a_{22} + b_{22})) \\
&= \mathfrak{D}(a_{11} + b_{11}) + \mathfrak{D}(a_{12} + b_{12}) + \mathfrak{D}(a_{21} + b_{21}) + \mathfrak{D}(a_{22} + b_{22}) + z_1 \\
&= \mathfrak{D}(a_{11}) + \mathfrak{D}(b_{11}) + z_2 + \mathfrak{D}(a_{12}) \\
&+ \mathfrak{D}(b_{12}) + \mathfrak{D}(a_{21}) + \mathfrak{D}(b_{21}) + \mathfrak{D}(a_{22}) + \mathfrak{D}(b_{22}) \\
&\quad + z_3 + z_1 \\
&= (\mathfrak{D}(a_{11}) + \mathfrak{D}(a_{12}) + \mathfrak{D}(a_{21}) + \mathfrak{D}(a_{22})) + (\mathfrak{D}(b_{11}) \\
&+ \mathfrak{D}(b_{12}) + \mathfrak{D}(b_{21}) + \mathfrak{D}(b_{22})) \\
&\quad + (z_1 + z_2 + z_3) \\
&= \mathfrak{D}(a_{11} + a_{12} + a_{21} + a_{22}) - z_4 + \mathfrak{D}(b_{11} \\
&+ b_{12} + b_{21} + b_{22}) - z_5 + (z_1 + z_2 + z_3) \\
&= \mathfrak{D}(a) + \mathfrak{D}(b) + (z_1 + z_2 + z_3 - z_4 - z_5) \\
&= \mathfrak{D}(a) + \mathfrak{D}(b) + z_{a,b}.
\end{aligned}$$

This finishes the proof of Theorem 2.1.

Corollary 2.7. *Let $\mathfrak{R}$ be an alternative ring. Suppose that $\mathfrak{R}$ is a ring containing a nontrivial idempotent e_1 which satisfies:*

(i) *If $[a_{11} + a_{22}, \mathfrak{R}_{12}] = 0$, then $a_{11} + a_{22} \in \mathcal{Z}(\mathfrak{R})$,*
(ii) *If $[a_{11} + a_{22}, \mathfrak{R}_{21}] = 0$, then $a_{11} + a_{22} \in \mathcal{Z}(\mathfrak{R})$.*

Then every Lie 3-derivation $\mathfrak{D}$ of $\mathfrak{R}$ into itself is almost additive.

Corollary 2.8. *Let $\mathfrak{R}$ be a 3-torsion free prime alternative ring. Suppose that $\mathfrak{R}$ is an alternative ring containing a nontrivial idempotent e_1. Then every Lie 3-derivation $\mathfrak{D}$ of $\mathfrak{R}$ into itself is almost additive.*

Proof. In [13] the authors showed that any prime alternative ring satisfies the conditions of the Theorem 2.1. Hence the result holds true for $n = 3$. □

3. Characterization of Lie-type derivations on alternative rings

In this section, we will characterize multiplicative Lie-type derivations on alternative rings and provide an essential structure theorem for multiplicative Lie-type derivations. Henceforth, let $\mathfrak{R}$ be a $\{2, 3, (n-1), (n-3)\}$-torsion free alternative ring satisfying the following conditions:

(1) If $x_{ij}\mathfrak{R}_{ji} = 0$, then $x_{ij} = 0$ $(i \neq j)$;
(2) If $x_{11}\mathfrak{R}_{12} = 0$ or $\mathfrak{R}_{21}x_{11} = 0$, then $x_{11} = 0$;
(3) If $\mathfrak{R}_{12}x_{22} = 0$ or $x_{22}\mathfrak{R}_{21} = 0$, then $x_{22} = 0$;
(4) If $z \in \mathcal{Z}$ with $z \neq 0$, then $z\mathfrak{R} = \mathfrak{R}$.

We refer the reader to [9] about the proofs of the following propositions.

Proposition 3.1. *Any prime alternative ring satisfies conditions* (1), (2), (3).

Proposition 3.2. *Let $\mathfrak{R}$ be a 2, 3-torsion free alternative ring satisfying the conditions* (1), (2), *and* (3).

(♠) *If* $[a_{11}+a_{22}, \mathfrak{R}_{12}]=0$, *then* $a_{11}+a_{22}\in \mathcal{Z}(\mathfrak{R})$,
(♣) *If* $[a_{11}+a_{22}, \mathfrak{R}_{21}]=0$, *then* $a_{11}+a_{22}\in \mathcal{Z}(\mathfrak{R})$.

Proposition 3.3. If $\mathcal{Z}(\mathfrak{R}_{ij})=\{a\in\mathfrak{R}_{ij} \mid [a,\mathfrak{R}_{ij}]=0\}$, then $\mathcal{Z}(\mathfrak{R}_{ij})\subseteq\mathfrak{R}_{ij}+\mathcal{Z}(\mathfrak{R})$ with $i\neq j$.

The main result in this section reads as follows.

Theorem 3.4. *Let $\mathfrak{R}$ be a unital $\{2,3,n-1,n-3\}$-torsion free alternative ring with nontrivial idempotents e_1, e_2 and with associated Peirce decomposition $\mathfrak{R}=\mathfrak{R}_{11}\oplus\mathfrak{R}_{12}\oplus\mathfrak{R}_{21}\oplus\mathfrak{R}_{22}$. Suppose that $\mathfrak{R}$ satisfies the following conditions:*

(1) *If $x_{ij}\mathfrak{R}_{ji}=0$, then $x_{ij}=0$ $(i\neq j)$;*
(2) *If $x_{11}\mathfrak{R}_{12}=0$ or $\mathfrak{R}_{21}x_{11}=0$, then $x_{11}=0$;*
(3) *If $\mathfrak{R}_{12}x_{22}=0$ or $x_{22}\mathfrak{R}_{21}=0$, then $x_{22}=0$;*
(4) *If $z\in\mathcal{Z}(\mathfrak{R})$ with $z\neq 0$, then $z\mathfrak{R}=\mathfrak{R}$.*

Let $\mathfrak{D}:\mathfrak{R}\to\mathfrak{R}$ be a multiplicative Lie-type derivation of $\mathfrak{R}$. Then $\mathfrak{D}$ is the form $\delta+\tau$, where δ is an additive derivation of $\mathfrak{R}$ and τ is a mapping from $\mathfrak{R}$ into the commutative center $\mathcal{Z}(\mathfrak{R})$, such that $\tau(p_n(a_1,a_2,\ldots,a_n))=0$ for all $a_1,a_2,\ldots,a_n\in\mathfrak{R}$ if and only if

(a) $e_2\mathfrak{D}(\mathfrak{R}_{11})e_2\subseteq\mathcal{Z}(\mathfrak{R})e_2$,
(b) $e_1\mathfrak{D}(\mathfrak{R}_{22})e_1\subseteq\mathcal{Z}(\mathfrak{R})e_1$,
(c) $\mathfrak{D}(\mathfrak{R}_{ij})\subseteq\mathfrak{R}_{ij}, 1\leq i\neq j\leq 2$.

The following Lemmas have the same hypotheses of Theorem 3.4 and we need these Lemmas for the proof of the first part this Theorem.

First, assume that the multiplicative Lie-type derivation $\mathfrak{D}:\mathfrak{R}\to\mathfrak{R}$ satisfies the conditions (a), (b), and (c). Let e_1 be a nontrivial idempotent of $\mathfrak{R}$. We start with the following lemma.

Lemma 3.5. $\mathfrak{D}(e_1)-f_{y,z}(e_1)\in\mathcal{Z}(\mathfrak{R})$, *with* $y=\mathfrak{D}(e_1)_{12}+\mathfrak{D}(e_1)_{21}$, $z=e_1$ *where* $f_{y,z}:=[L_y,L_z]+[L_y,R_z]+[R_y,R_z]$ *and L, R are left and right multiplication operators, respectively.*

Proof. In the case of n is even, we have

$$\begin{aligned}\mathfrak{D}(a_{12})&=\mathfrak{D}(p_n(e_1,a_{12},e_1,\ldots,e_1))\\&=p_n(e_1,\mathfrak{D}(a_{12}),e_1,\ldots,e_1)+\sum_{i=2}^{n}p_n(e_1,a_{12},e_1,\ldots,\mathfrak{D}(e_1),\ldots,e_1)\\&=-a_{12}\mathfrak{D}(e_1)e_1+e_1\mathfrak{D}(e_1)a_{12}-a_{12}\mathfrak{D}(e_1)+e_1\mathfrak{D}(a_{12})-\mathfrak{D}(a_{12})e_1\\&\quad+\sum_{i=3}^{n}p_n(e_1,a_{12},e_1,\ldots,\mathfrak{D}(e_1),\ldots,e_1).\end{aligned}$$

Multiplying the left and right sides in the above equation by e_1 and e_2, respectively, we obtain

$$\begin{aligned}e_1\mathfrak{D}(a_{12})e_2&=e_1\mathfrak{D}(e_1)a_{12}-a_{12}\mathfrak{D}(e_1)e_2+e_1\mathfrak{D}(a_{12})e_2\\&\quad+\sum_{i=3}^{n}(-1)^{n-1}\left[\mathfrak{D}(e_1)_{11}+\mathfrak{D}(e_1)_{22},a_{12}\right].\end{aligned}$$

This implies

$$-(n-3)\left[\mathfrak{D}(e_1)_{11}+\mathfrak{D}(e_1)_{22},a_{12}\right]+2\mathfrak{D}(e_1)_{12}a_{12}=0$$

for all $a_{12}\in\mathfrak{R}_{12}$. In light of condition (♠) of Proposition 3.2, we assert that $\mathfrak{D}(e_1)_{11}+\mathfrak{D}(e_1)_{22}\in\mathcal{Z}(\mathfrak{R})$. Taking $y=\mathfrak{D}(e_1)_{12}+\mathfrak{D}(e_1)_{21}$ and $z=e_1$ we see that $\mathfrak{D}(e_1)-f_{y,z}(e_1)=\mathfrak{D}(e_1)_{11}+\mathfrak{D}(e_1)_{22}\in\mathcal{Z}(\mathfrak{R})$.

In the case of n is odd, we get

$$\begin{aligned}\mathfrak{D}(a_{12})&=\mathfrak{D}(p_n(a_{12},e_1,\ldots,e_1))\\&=p_n(\mathfrak{D}(a_{12}),e_1,\ldots,e_1)+\sum_{i=2}^{n}p_n(a_{12},e_1,\ldots,\mathfrak{D}(e_1),\ldots,e_1)\\&=\mathfrak{D}(a_{12})e_1-2e_1\mathfrak{D}(a_{12})e_1+e_1\mathfrak{D}(a_{12})\\&+\sum_{i=2}^{n}p_n(a_{12},e_1,\ldots,\mathfrak{D}(e_1),\ldots,e_1).\end{aligned}$$

Multiplying the left and right sides in the above equation by e_1 and e_2, respectively, we arrive at

$$\begin{aligned}e_1\mathfrak{D}(a_{12})e_2&=e_1\mathfrak{D}(a_{12})e_2+\sum_{i=2}^{n}e_1p_n(a_{12},e_1,\ldots,\mathfrak{D}(e_1),\ldots,e_1)e_2\\&=e_1\mathfrak{D}(a_{12})e_2-(n-1)\left[\mathfrak{D}(e_1)_{11}+\mathfrak{D}(e_1)_{22},a_{12}\right].\end{aligned}$$

This gives that $(n-1)[\mathfrak{D}(e_1)_{11}+\mathfrak{D}(e_1)_{22},a_{12}]=0$ for all $a_{12}\in\mathfrak{R}_{12}$. By condition (♠) of Proposition 3.2 we conclude that $\mathfrak{D}(e_1)_{11}+\mathfrak{D}(e_1)_{22}\in\mathcal{Z}(\mathfrak{R})$. Taking $y=\mathfrak{D}(e_1)_{12}+\mathfrak{D}(e_1)_{21}$ and $z=e_1$ again, we see that $\mathfrak{D}(e_1)-f_{y,z}(e_1)=\mathfrak{D}(e_1)_{11}+\mathfrak{D}(e_1)_{22}\in\mathcal{Z}(\mathfrak{R})$. □

Let us continue our discussions. It is worth noting that $f_{y,z}:=[L_y,L_z]+[L_y,R_z]+[R_y,R_z]$ is a derivation. According to [27, p. 77], we without loss of generality may assume that $\mathfrak{D}(e_1)\in\mathcal{Z}(\mathfrak{R})$.

Remark 3.1. If $\mathfrak{D}(e_1)\in\mathcal{Z}(\mathfrak{R})$, then $\mathfrak{D}(e_2)\in\mathcal{Z}(\mathfrak{R})$. Indeed, since

$$\begin{aligned}0&=\mathfrak{D}(p_n(e_2,e_1,\ldots,e_1)\\&=p_n(\mathfrak{D}(e_2),e_1,\ldots,e_1)+\sum_{i=2}^{n}p_n(e_2,e_1,\ldots,\mathfrak{D}(e_1),\ldots,e_1)\\&=p_n(\mathfrak{D}(e_2),e_1,\ldots,e_1)\\&=\mathfrak{D}(e_2)e_1-e_1\mathfrak{D}(e_2)e_1+(-1)^ne_1\mathfrak{D}(e_2)e_1+(-1)^{n+1}e_1\mathfrak{D}(e_2),\end{aligned}$$

we know that $e_1\mathfrak{D}(e_2)e_2=e_2\mathfrak{D}(e_2)e_1=0$. When n is even, for any $a_{12}\in\mathfrak{R}_{12}$, we have

$$\begin{aligned}\mathfrak{D}(a_{21})&=\mathfrak{D}(p_n(e_2,a_{21},e_2\ldots,e_2))\\&=p_n(\mathfrak{D}(e_2),a_{21},e_2,\ldots,e_2)+p_n(e_2,\mathfrak{D}(a_{21}),e_2,\ldots,e_2)\\&+\sum_{i=3}^{n}p_n(e_2,a_{21},e_2,\ldots,\mathfrak{D}(e_2),\ldots,e_2)\\&=-(-1)^{n-2}a_{21}\mathfrak{D}(e_2)_{11}+(-1)^{n-2}\mathfrak{D}(e_2)_{22}a_{21}\\&+e_2\mathfrak{D}(a_{21})-\mathfrak{D}(a_{21})e_2\\&-(n-2)\left[\mathfrak{D}(e_2)_{11}+\mathfrak{D}(e_2)_{22},a_{21}\right]\\&=-(n-1)\left[\mathfrak{D}(e_2)_{11}+\mathfrak{D}(e_2)_{22},a_{21}\right]+e_2\mathfrak{D}(a_{21})-\mathfrak{D}(a_{21})e_2.\end{aligned}$$

Multiplying by e_2 and e_1 from the left and right sides in the above equation, respectively, we arrive at $-(n-1)[\mathfrak{D}(e_2)_{11}+\mathfrak{D}(e_2)_{22},a_{21}]=0$ for all $a_{21}\in\mathfrak{R}_{21}$. This gives

$$\left[\mathfrak{D}(e_2)_{11}+\mathfrak{D}(e_2)_{22},a_{21}\right]=0$$

for all $a_{21} \in \mathfrak{R}_{21}$, since the characteristic of $\mathfrak{R}$ is not $n-1$. By condition (♣) of Proposition 3.2 it follows that $\mathfrak{D}(e_2) = \mathfrak{D}(e_2)_{11} + \mathfrak{D}(e_2)_{22} \in \mathcal{Z}(\mathfrak{R})$. Now if n is odd, then we have

$$\begin{aligned}\mathfrak{D}(a_{21}) &= \mathfrak{D}(p_n(a_{21}, e_2..., e_2))\\ &= p_n(\mathfrak{D}(a_{21}), e_2, ..., e_2) + \sum_{i=2}^{n} p_n(a_{21}, e_2, ..., \mathfrak{D}(e_2), ..., e_2)\\ &= -2e_2\mathfrak{D}(a_{21})e_2 + e_2\mathfrak{D}(a_{21}) + \mathfrak{D}(a_{21})e_2\\ &- (n-1)\left[\mathfrak{D}(e_2)_{11} + \mathfrak{D}(e_2)_{22}, a_{21}\right].\end{aligned}$$

Multiplying by e_2 and e_1 from the left and right sides in the above equation, respectively, we obtain the same result as n is even.

Lemma 3.6. $\mathfrak{D}(\mathfrak{R}_{ii}) \subseteq \mathfrak{R}_{ii} + \mathcal{Z}(\mathfrak{R})(i = 1, 2)$.

Proof. We only show the case of $i = 1$, because the other case can be treated similarly. For each $a_{11} \in \mathfrak{R}_{11}$, with $\mathfrak{D}(a_{11}) = b_{11} + b_{12} + b_{21} + b_{22}$ we get

$$\begin{aligned}0 &= \mathfrak{D}(p_n(a_{11}, e_1, ...e_1))\\ &= p_n(\mathfrak{D}(a_{11}), e_1, ..., e_1) + \sum_{i=2}^{n} p_n(a_{11}, e_1, ..., \mathfrak{D}(e_1), ..., e_1)\\ &= p_n(\mathfrak{D}(a_{11}), e_1, ..., e_1).\end{aligned}$$

It follows from this that $b_{12} = b_{21} = 0$. By condition (a) of Theorem 3.4 we know that

$$\mathfrak{D}(a_{11}) = b_{11} + e_2\mathfrak{D}(a_{11})e_2 = b_{11} + ze_2 = b_{11} - e_1z + z \in \mathfrak{R}_{11} + \mathcal{Z}(\mathfrak{R}).$$

Lemma 3.7. *$\mathfrak{D}$ is an almost additive mapping. That is, for any $a, b \in \mathfrak{R}, \mathfrak{D}(a+b) - \mathfrak{D}(a) - \mathfrak{D}(b) \in \mathcal{Z}(\mathfrak{R})$.*

Proof. Since $\mathfrak{R}$ is an alternative ring satisfying the conditions (1), (2), and (3), $\mathfrak{R}$ satisfies conditions (♠) and (♣) by Proposition 3.2. Now using Theorem 2.1 we get $\mathfrak{D}$ as an almost additive mapping. □

Now let us define the mappings δ and τ. By the item (c) of Theorem 3.4 and Lemma 3.6 we have

(A) if $a_{ij} \in \mathfrak{R}_{ij}, i \neq j$, then $\mathfrak{D}(a_{ij}) = b_{ij} \in \mathfrak{R}_{ij}$,
(B) if $a_{ii} \in \mathfrak{R}_{ii}$, then $\mathfrak{D}(a_{ii}) = b_{ii} + z, b_{ii} \in \mathfrak{R}_{ii}$, where z is a central element.

It should be remarked that b_{ii} and z in (B) are uniquely determined, Indeed, if $\mathfrak{D}(a_{ii}) = b'_{ii} + z', b'_{ii} \in \mathfrak{R}_{ii}, z' \in \mathcal{Z}(\mathfrak{R})$. Then $b_{ii} - b'_{ii} \in \mathcal{Z}(\mathfrak{R})$. Taking into account the conditions (2) and (3), we assert that $b_{ii} = b'_{ii}$ and $z = z'$. Now let us define a mapping δ of $\mathfrak{R}$ according to the rule $\delta(a_{ij}) = b_{ij}, a_{ij} \in \mathfrak{R}_{ij}$. For each $a = a_{11} + a_{12} + a_{21} + a_{22} \in \mathfrak{R}$, we define $\delta(a) = \sum \delta(a_{ij})$. And a mapping τ of $\mathfrak{R}$ into $\mathcal{Z}(\mathfrak{R})$ is then defined by

$$\begin{aligned}\tau(a) &= \mathfrak{D}(a) - \delta(a)\\ &= \mathfrak{D}(a) - (\delta(a_{11}) + \delta(a_{12}) + \delta(a_{21}) + \delta(a_{22}))\\ &= \mathfrak{D}(a) - (b_{11} + b_{12} + b_{21} + b_{22})\\ &= \mathfrak{D}(a) - (\mathfrak{D}(a_{11}) - z_{a_{11}} + \mathfrak{D}(a_{12}) + \mathfrak{D}(a_{21}) + \mathfrak{D}(a_{22}) - z_{a_{22}})\\ &= \mathfrak{D}(a) - (\mathfrak{D}(a_{11}) + \mathfrak{D}(a_{12}) + \mathfrak{D}(a_{21}) + \mathfrak{D}(a_{22}) - (z_{a_{11}} + z_{a_{22}}))\\ &= \mathfrak{D}(a) - (\mathfrak{D}(a_{11}) + \mathfrak{D}(a_{12}) + \mathfrak{D}(a_{21}) + \mathfrak{D}(a_{22})).\end{aligned}$$

We need to show that δ and τ are the desired mappings.

Lemma 3.8. *δ is an additive mapping.*

Proof. We only need to prove that δ is an additive mapping on $\mathfrak{R}_{ii}$. Let us choose any $a_{ii}, b_{ii} \in \mathfrak{R}_{ii}$,

$$\delta(a_{ii}+b_{ii}) - \delta(a_{ii}) - \delta(b_{ii}) = \mathfrak{D}(a_{ii}+b_{ii}) - \tau(a_{ii}+b_{ii}) - \mathfrak{D}(a_{ii}) + \tau(a_{ii}) \\ - \mathfrak{D}(b_{ii}) + \tau(b_{ii}).$$

Thus, $\delta(a_{ii}+b_{ii}) - \delta(a_{ii}) - \delta(b_{ii}) \in \mathcal{Z}(\mathfrak{R}) \cap \mathfrak{R}_{ii} = \{0\}$. □

Let us next show that $\delta(ab) = \delta(a)b + a\delta(b)$ for all $a, b \in \mathfrak{R}$.

Lemma 3.9. *For any $a_{ii}, b_{ii} \in \mathfrak{R}_{ii}, a_{ij}, b_{ij} \in \mathfrak{R}_{ij}, b_{ji} \in \mathfrak{R}_{ji}$ and $b_{jj} \in \mathfrak{R}_{jj}$ with $i \neq j$, we have*

(I) $\delta(a_{ii}b_{ij}) = \delta(a_{ii})b_{ij} + a_{ii}\delta(b_{ij})$,
(II) $\delta(a_{ij}b_{jj}) = \delta(a_{ij})b_{jj} + a_{ij}\delta(b_{jj})$,
(III) $\delta(a_{ii}b_{ii}) = \delta(a_{ii})b_{ii} + a_{ii}\delta(b_{ii})$,
(IV) $\delta(a_{ij}b_{ij}) = \delta(a_{ij})b_{ij} + a_{ij}\delta(b_{ij})$,
(V) $\delta(a_{ij}b_{ji}) = \delta(a_{ij})b_{ji} + a_{ij}\delta(b_{ji})$.

Proof. Let us begin with (I)

$$\begin{aligned}\delta(a_{ii}b_{ij}) &= \mathfrak{D}(a_{ii}b_{ij}) \\ &= \mathfrak{D}(p_n(a_{ii}, b_{ij}, e_j, \ldots, e_j)) \\ &= p_n(\mathfrak{D}(a_{ii}), b_{ij}, e_j, \ldots, e_j) + p_n(a_{ii}, \mathfrak{D}(b_{ij}), e_j, \ldots e_j) \\ &= p_n(\delta(a_{ii}), b_{ij}, e_j, \ldots, e_j) + p_n(a_{ii}, \delta(b_{ij}), e_j, \ldots, e_j) \\ &= \delta(a_{ii})b_{ij} + a_{ii}\delta(b_{ij}).\end{aligned}$$

Let us see (II)

$$\begin{aligned}\delta(a_{ij}b_{jj}) &= \mathfrak{D}(a_{ij}b_{jj}) \\ &= \mathfrak{D}(p_n(a_{ij}, b_{jj}, e_j, \ldots, e_j)) \\ &= p_n(\mathfrak{D}(a_{ii}), b_{ij}, e_j, \ldots, e_j) + p_n(a_{ii}, \mathfrak{D}(b_{ij}), e_j, \ldots, e_j) \\ &= p_n(\delta(a_{ii}), b_{ij}, e_j, \ldots, e_j) + p_n(a_{ii}, \delta(b_{ij}), e_j, \ldots, e_j) \\ &= \delta(a_{ij})b_{jj} + a_{ij}\delta(b_{jj}).\end{aligned}$$

We next show (III). By linearization of flexible identity and (I) we get

$$\delta((a_{ii}b_{ii})r_{ij}) = \delta(a_{ii}b_{ii})r_{ij} + (a_{ii}b_{ii})\delta(r_{ij}).$$

On the other hand,

$$\delta(a_{ii}(b_{ii}r_{ij})) = \delta(a_{ii})b_{ii}r_{ij} + a_{ii}\delta(b_{ii}r_{ij}) = \delta(a_{ii})b_{ii}r_{ij} + a_{ii}(\delta(b_{ii})r_{ij} + b_{ii}\delta(r_{ij})).$$

Considering the facts $(a_{ii}b_{ii})r_{ij} = a_{ii}(b_{ii}r_{ij})$ and $(a_{ii}b_{ii})\delta(r_{ij}) = a_{ii}(b_{ii}\delta(r_{ij}))$, we obtain

$$(\delta(a_{ii}b_{ii}) - \delta(a_{ii})b_{ii} - a_{ii}\delta(b_{ii}))r_{ij} = 0$$

for all $r_{ij} \in \mathfrak{R}_{ij}$. And hence $\delta(a_{ii}b_{ii}) = \delta(a_{ii})b_{ii} + a_{ii}\delta(b_{ii})$.

Let us prove (IV).

$$\begin{aligned}2\delta(a_{ij}b_{ij}) &= \delta(2a_{ij}b_{ij}) = \mathfrak{D}(2a_{ij}b_{ij})\\ &= \mathfrak{D}(p_n(a_{ij}, b_{ij}, e_i, ..., e_i)) = p_n(\mathfrak{D}(a_{ij}), b_{ij}, e_i, ..., e_i)\\ &+ p_n(a_{ij}, \mathfrak{D}(b_{ij}), e_i, ..., e_i)\\ &= p_n(\delta(a_{ij}), b_{ij}, e_i, ..., e_i) + p_n(a_{ij}, \delta(b_{ij}), e_i, ..., e_i)\\ &= \delta(a_{ij})b_{ij} - b_{ij}\delta(a_{ij}) + a_{ij}\delta(b_{ij}) - \delta(b_{ij})a_{ij}\\ &= 2(\delta(a_{ij})b_{ij} + a_{ij}\delta(b_{ij})).\end{aligned}$$

Since $\mathfrak{R}$ is 2-torsion free, we see that $\delta(a_{ij}b_{ij}) = \delta(a_{ij})b_{ij} + a_{ij}\delta(b_{ij})$. And finally we show (V). We get

$$\begin{aligned}&\tau(p_n(a_{ij}, b_{ji}, c_{ij}, e_j, ..., e_j)) = \mathfrak{D}(p_n(a_{ij}, b_{ji}, c_{ij}, e_j, ..., e_j))\\ &- \delta(p_n(a_{ij}, b_{ji}, c_{ij}, e_j, ..., e_j))\\ &= p_n(\mathfrak{D}(a_{ij}), b_{ji}, c_{ij}, e_j, ..., e_j) + p_n(a_{ij}, \mathfrak{D}(b_{ji}), c_{ij}, e_j, ..., e_j)\\ &\quad + p_n(a_{ij}, b_{ji}, \mathfrak{D}(c_{ij}), e_j, ..., e_j) - \delta((a_{ij}b_{ji})c_{ij} - c_{ij}(b_{ji}a_{ij}))\\ &= p_n(\delta(a_{ij}), b_{ji}, c_{ij}, e_j, ..., e_j) + p_n(a_{ij}, \delta(b_{ji}), c_{ij}, e_j, ..., e_j)\\ &\quad + p_n(a_{ij}, b_{ji}, \delta(c_{ij}), e_j, ..., e_j) - \delta((a_{ij}b_{ji})c_{ij}) - \delta(c_{ij}(b_{ji}a_{ij}))\\ &= (\delta(a_{ij})b_{ji})c_{ij} + c_{ij}(b_{ji}\delta(a_{ij})) + (a_{ij}\delta(b_{ji}))c_{ij} + c_{ij}(\delta(b_{ji})a_{ij}) + (a_{ij}b_{ji})\delta(c_{ij})\\ &\quad + \delta(c_{ij})(a_{ij}b_{ji}) - \delta(a_{ij}b_{ji})c_{ij} - (a_{ij}b_{ji})\delta(c_{ij}) - \delta(c_{ij})(b_{ji})a_{ij}) - c_{ij}\delta(b_{ji}a_{ij})\\ &= \left[(\delta(a_{ij})b_{ji}) + a_{ij}\delta(b_{ji}) - \delta(a_{ij}b_{ji})) + (\delta(b_{ji}a_{ij}) - \delta(b_{ji})a_{ij} - b_{ji}\delta(a_{ij})), c_{ij}\right].\end{aligned}$$

Since $\mathfrak{R}_{ij} \cap \mathcal{Z} = \{0\}$, we know that $[(\delta(a_{ij})b_{ji}) + a_{ij}\delta(b_{ji}) - \delta(a_{ij}b_{ji})) + (\delta(b_{ji}a_{ij}) - \delta(b_{ji})a_{ij} - b_{ji}\delta(a_{ij})), c_{ij}] = 0$ for all $c_{ij} \in \mathfrak{R}_{ij}$. By Proposition 3.2 it follows that

$$\left[\delta(a_{ij})b_{ji} + a_{ij}\delta(b_{ji}) - \delta(a_{ij}b_{ji})\right] + \left[\delta(b_{ji}a_{ij}) - \delta(b_{ji})a_{ij} - b_{ji}\delta(a_{ij})\right] = z \in \mathcal{Z}(\mathfrak{R}).$$

If $z = 0$, then $\delta(a_{ij}b_{ji}) = \delta(a_{ij})b_{ji} + a_{ij}\delta(b_{ji})$. If $z \neq 0$, we multiply by a_{ij} and get

$$a_{ij}\delta(b_{ji}a_{ij}) - a_{ij}\delta(b_{ji})a_{ij} - a_{ij}(b_{ji}\delta(a_{ij})) = a_{ij}z.$$

By (II) we have

$$\delta(a_{ij}b_{ji}a_{ij}) - \delta(a_{ij})(b_{ji}a_{ij}) - a_{ij}\delta(b_{ji})a_{ij} - a_{ij}(b_{ji}\delta(a_{ij})) = a_{ij}z.$$

Now we see that $\delta(a_{ij}b_{ji}a_{ij}) = \delta(a_{ij})(b_{ji}a_{ij}) + a_{ij}\delta(b_{ji})a_{ij} + a_{ij}(b_{ji}\delta(a_{ij}))$. Indeed, note that $p_n(a_{ij}, b_{ji}, a_{ij}, e_j, ..., e_j) = 2a_{ij}b_{ji}a_{ij}$. Thus

$$\begin{aligned}&2\delta(a_{ij}b_{ji}a_{ij}) = \delta(2a_{ij}b_{ji}a_{ij})\\ &= \mathfrak{D}(p_n(a_{ij}, b_{ji}, a_{ij}, e_j, ..., e_j))\\ &= p_n(\mathfrak{D}(a_{ij}), b_{ji}, a_{ij}, e_j, ..., e_j) + p_n(a_{ij}, \mathfrak{D}(b_{ji}), a_{ij}, e_j, ..., e_j)\\ &+ p_n(a_{ij}, b_{ji}, \mathfrak{D}(a_{ij}), e_j, ..., e_j)\\ &= p_n(\delta(a_{ij}), b_{ji}, a_{ij}, e_j, ..., e_j) + p_n(a_{ij}, \delta(b_{ji}), a_{ij}, e_j, ..., e_j)\\ &+ p_n(a_{ij}, b_{ji}, \delta(a_{ij}), e_j, ..., e_j)\\ &= (\delta(a_{ij})b_{ji})a_{ij} + a_{ij}(b_{ji}\delta(a_{ij})) + 2a_{ij}\delta(b_{ji})a_{ij}\\ &+ (a_{ij}b_{ji})\delta(a_{ij}) + \delta(a_{ij})(b_{ji}a_{ij})\\ &= \delta(a_{ij})(b_{ji}a_{ij}) - (a_{ij}b_{ji})\delta(a_{ij}) + a_{ij}(b_{ji}\delta(a_{ij})) + a_{ij}(b_{ji}\delta(a_{ij}))\\ &\quad + 2a_{ij}\delta(b_{ji})a_{ij} + (a_{ij}b_{ji})\delta(a_{ij}) + \delta(a_{ij})(b_{ji}a_{ij})\\ &= 2(\delta(a_{ij})(b_{ji}a_{ij}) + a_{ij}\delta(b_{ji})a_{ij} + a_{ij}(b_{ji}\delta(a_{ij}))).\end{aligned}$$

Applying the fact that $\mathfrak{R}$ is 2-torsion free yields that

$$\delta(a_{ij}b_{ji}a_{ij}) = \delta(a_{ij})(b_{ji}a_{ij}) + a_{ij}\delta(b_{ji})a_{ij} + a_{ij}(b_{ji}\delta(a_{ij})).$$

So $a_{ij}z = 0$. But, by (4) there exist $h \in \mathfrak{R}$ such that $zh = e_1 + e_2$ hence $a_{ij} = 0$, which is a contradiction. Therefore $\delta(a_{ij}b_{ji}) = \delta(a_{ij})b_{ji} + a_{ij}\delta(b_{ji})$. □

Lemma 3.10. *δ is a derivation.*

Proof. For any $a, b \in \mathfrak{R}$, we have

$$\begin{aligned}\delta(ab) &= \delta((a_{11} + a_{12} + a_{21} + a_{22})(b_{11} + b_{12} + b_{21} + b_{22})) \\ &= \delta(a_{11}b_{11}) + \delta(a_{11}b_{12}) + \delta(a_{12}b_{12}) + \delta(a_{12}b_{21}) + \delta(a_{12}b_{22}) \\ &\quad + \delta(a_{21}b_{11}) + \delta(a_{21}b_{12}) + \delta(a_{21}b_{21}) + \delta(a_{22}b_{21}) + \delta(a_{22}b_{22}) \\ &= \delta(a)b + a\delta(b)\end{aligned}$$

by Lemmas 3.8 and 3.9. □

Lemma 3.11. *τ sends the commutators into zero.*

Proof. For any $a_1, a_2, ..., a_n \in \mathfrak{R}$, we get

$$\begin{aligned}\tau(p_n(a_1, a_2, ..., a_n)) &= \mathfrak{D}(p_n(a_1, a_2, ..., a_n)) - \delta(p_n(a_1, a_2, ..., a_n)) \\ &= \sum_{i=1}^{n} p_n(a_1, a_2, ..., a_{i-1}, \mathfrak{D}(a_i), a_{i+1}, ..., a_n) - \delta(p_n(a_1, a_2, ..., a_n)) \\ &= \sum_{i=1}^{n} p_n(a_1, a_2, ..., a_{i-1}, \delta(a_i), a_{i+1}, ..., a_n) - \delta(p_n(a_1, a_2, ..., a_n)) \\ &= 0.\end{aligned}$$

Let us now assume that $\mathfrak{D} : \mathfrak{R} \to \mathfrak{R}$ is a Lie-type derivation of the form $\mathfrak{D} = \delta + \tau$, where δ is a derivation of $\mathfrak{R}$ and τ is a mapping from $\mathfrak{R}$ into its commutative center $\mathcal{Z}(\mathfrak{R})$, such that $\tau(p_n(a_1, a_2, ..., a_n)) = 0$ for all $a_1, a_2, ..., a_n \in \mathfrak{R}$. Then for any $a_{11} \in \mathfrak{R}_{11}$, we see that

$$\begin{aligned}e_2\mathfrak{D}(a_{11})e_2 &= e_2\delta(a_{11})e_2 + e_2\tau(a_{11})e_2 \\ &= e_2\delta(e_1a_{11})e_2 + e_2\tau(a_{11})e_2 \\ &= e_2(\delta(e_1)a_{11} + e_1\delta(a_{11}))e_2 + e_2\tau(a_{11})e_2 \\ &= e_2(\delta(e_1)a_{11})e_2 + e_2(e_1\delta(a_{11}))e_2 + e_2\tau(a_{11})e_2 \\ &= (e_2\delta(e_1))(a_{11}e_2) + (e_2e_1)(\delta(a_{11})e_2) + e_2\tau(a_{11})e_2 \\ &= e_2\tau(a_{11})e_2 \in \mathcal{Z}(\mathfrak{R})e_2.\end{aligned}$$

Now

$$\begin{aligned}e_1\mathfrak{D}(a_{22})e_1 &= e_1\delta(a_{22})e_1 + e_1\tau(a_{22})e_1 \\ &= e_1\delta(e_2a_{22})e_1 + e_1\tau(a_{22})e_1 \\ &= e_1(\delta(e_2)a_{22} + e_2\delta(a_{22}))e_1 + e_1\tau(a_{22})e_1 \\ &= e_1(\delta(e_2)a_{22})e_1 + e_1(e_2\delta(a_{22}))e_1 + e_1\tau(a_{22})e_1 \\ &= (e_1\delta(e_2))(a_{22}e_1) + (e_1e_2)(\delta(a_{22})e_1) + e_1\tau(a_{22})e_1 \\ &= e_1\tau(a_{22})e_1 \in \mathcal{Z}(\mathfrak{R})e_1\end{aligned}$$

for all $a_{22} \in \mathfrak{R}_{22}$. Furthermore,

$$\mathfrak{D}(a_{ij}) = (\delta + \tau)(a_{ij}) = \delta(p_n(a_{ij}, e_j, ..., e_j)) + \tau(p_n(a_{ij}, e_j, ..., e_j))$$
$$= p_n(\delta(a_{ij}), e_j, ..., e_j) \in \mathfrak{R}_{ij}.$$

This shows the items (a), (b), (c) and the proof of the Theorem 3.4 is complete.

Corollary 3.12. *Let $\mathfrak{R}$ be an unital prime alternative ring with nontrivial idempotent satisfying* (4) *and $\mathfrak{D} : \mathfrak{R} \to \mathfrak{R}$ be a multiplicative Lie-type derivation. Then $\mathfrak{D}$ is the form of $\delta + \tau$, where δ is a derivation of $\mathfrak{R}$ and τ is a mapping from $\mathfrak{R}$ into its commutative center $\mathcal{Z}(\mathfrak{R})$, such that $\tau(p_n(a_1, a_2, ..., a_n)) = 0$ for all $a_1, a_2, ..., a_n \in \mathfrak{R}$ if and only if*

(a) $e_2\mathfrak{D}(\mathfrak{R}_{11})e_2 \subseteq \mathcal{Z}(\mathfrak{R})e_2$,
(b) $e_1\mathfrak{D}(\mathfrak{R}_{22})e_1 \subseteq \mathcal{Z}(\mathfrak{R})e_1$,
(c) $\mathfrak{D}(\mathfrak{R}_{ij}) \subseteq \mathfrak{R}_{ij}, 1 \leq i \neq j \leq 2$.

Let us end our work with a direct application to simple alternative rings.

Corollary 3.13. *Let $\mathfrak{R}$ be an unital simple alternative ring with nontrivial idempotent and $\mathfrak{D} : \mathfrak{R} \to \mathfrak{R}$ be a multiplicative Lie-type derivation. Then $\mathfrak{D}$ is the form $\delta + \tau$, where δ is a derivation of $\mathfrak{R}$ and τ is a mapping from $\mathfrak{R}$ into its commutative center $\mathcal{Z}(\mathfrak{R})$, such that $\tau(p_n(a_1, a_2, ..., a_n)) = 0$ for all $a_1, a_2, ..., a_n \in \mathfrak{R}$ if and only if*

(a) $e_2\mathfrak{D}(\mathfrak{R}_{11})e_2 \subseteq \mathcal{Z}(\mathfrak{R})e_2$,
(b) $e_1\mathfrak{D}(\mathfrak{R}_{22})e_1 \subseteq \mathcal{Z}(\mathfrak{R})e_1$,
(c) $\mathfrak{D}(\mathfrak{R}_{ij}) \subseteq \mathfrak{R}_{ij}, 1 \leq i \neq j \leq 2$.

Proof. It is enough to remark that every simple ring is prime and $\mathcal{Z}(\mathfrak{R})$ is a field. □

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