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Complements to partial
tilting modules

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Let A be a finite-dimensional k -algebra over an algebraically closed field k . We denote by $\text{mod } A$ the category of finitely generated left A -modules. Let n be the number of isomorphism classes of simple A -modules.

A module $T \in \text{mod } A$ is called a (generalized) *tilting module* if the following conditions are satisfied:

- (i) $\text{proj. dim}_A T < \infty$
- (ii) $\text{Ext}_A^i(T, T) = 0$ for all $i > 0$
- (iii) There is a long exact sequence $0 \rightarrow {}_A A \rightarrow T_0 \rightarrow \cdots \rightarrow T_m \rightarrow 0$ with $T_j \in \text{add } T$, the additive category of direct sums of direct summands of T .

We refer to [H1],[H2] and [M] for an outline of tilting theory in this case, and to [B], [HR] and [R] in the classical situation of tilting modules of projective dimension at most one.

An A -module M satisfying the conditions (i) and (ii) of the definition above is called a *partial tilting module*, and if moreover M has $n - 1$ indecomposable direct summands (up to isomorphism), then we call M an *almost complete tilting module*.

Let M be a partial tilting module with $\text{proj. dim}_A M = t$ and let X be an A -module such that $M \oplus X$ is a tilting module and $\text{add } M \cap \text{add } X = 0$. Then X will be called a *complement* to M . It is known that in the classical situation M always admits a complement [B]. However, in the general situation complements do not always exist, as shown in [RS]. Moreover it is shown in [RS] that under suitable finiteness conditions complements exist.

Let M be an almost complete tilting module of projective dimension one. Then it was shown in [HU], [H3] and [RiS] that M has at most two non-isomorphic indecomposable complements, and M has a unique (up to isomorphism) indecomposable complement if and only if M is not faithful.

The aim of this article is to generalize this to arbitrary tilting modules. So we ask for the number of indecomposable complements and their homological properties.

In order to guarantee the existence of complements we have to make the following finiteness assumption. For a partial tilting module M we consider as in [RS] the full subcategory

$$\mathcal{C} = \{X \in \text{mod } A \mid \text{Ext}_A^i(X, M) = 0 \text{ for } i > 0 \text{ and } \text{proj. dim}_A X < \infty\}.$$

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Let M be an almost complete tilting module with $\text{proj.dim}_A M = t$. We assume that \mathcal{C} is contravariantly finite (see section 2 below). This is for example satisfied if \mathcal{C} is finite (i.e. contains only finitely many isomorphism classes of indecomposable modules).

In this case we show in 2.2 that there is an indecomposable complement X_r .

Let r be the maximum of the projective dimensions of modules in \mathcal{C} . It is shown in [AR2] that $r < \infty$, if \mathcal{C} is contravariantly finite.

THEOREM 1. *If M is not faithful, then M has a unique indecomposable complement X and $\text{proj.dim}_A X \leq t$.*

We have to consider also the following subcategories \mathcal{C}_i for $t \leq i \leq r$ with

$$\mathcal{C}_i = \{X \in \text{mod } A \mid \text{Ext}_A^j(X, M) = 0 \text{ for } j > 0 \text{ and } \text{proj.dim}_A X \leq i\}.$$

Note that $\mathcal{C} = \mathcal{C}_r$.

THEOREM 2. *Let M be an almost complete tilting module. If $t < r$ there are indecomposable complements $X_i \in \mathcal{C}_i$ for $t \leq i \leq r$ with $\text{proj.dim}_A X_i = i$ and a long exact sequence*

$$0 \rightarrow X_t \rightarrow E_t \xrightarrow{g_t} E_{t+1} \rightarrow \cdots \rightarrow E_{r-1} \xrightarrow{g_{r-1}} X_r \rightarrow 0$$

with $X_i = \ker g_i$ and $E_i \in \text{add } M$ for $t \leq i < r$. The modules $X_i \in \mathcal{C}_i$ for $t < i \leq r$ are uniquely determined by the property that $\text{proj.dim}_A X_i = i$.

THEOREM 3. *Let M be a faithful almost complete tilting module and assume that there is a complement $Y \in \mathcal{C}_t$ which is generated by M . Then*

- (a) *there is an integer s with $2 \leq s \leq \max(2, t + 1)$ and there are indecomposable complements Y_1, \dots, Y_s to M giving a complete list of non-isomorphic indecomposable complements to M of projective dimension at most t ,*
- (b) *there is a long exact sequence*

$$0 \rightarrow Y_1 \rightarrow F_1 \xrightarrow{f_1} \cdots \rightarrow F_s = E_t \xrightarrow{f_s = g_t} E_{t+1} \rightarrow \cdots \rightarrow E_{r-1} \xrightarrow{g_{r-1}} X_r \rightarrow 0$$

with $Y_i = \ker f_i$ for $1 \leq i \leq s$, $X_i = \ker g_i$ for $t \leq i < r$ and $F_j, E_i \in \text{add } M$ for $1 \leq j \leq s, t \leq i < r$ and X_i

Under additional assumptions we can show that the hypothesis in theorem 3 is superfluous.

PROPOSITION 4. *Let M be a faithful non-projective almost complete tilting module. If \mathcal{C}_i is contravariantly finite for all $t \leq i \leq r$, then the complement X_t is generated by M .*

We refer to section 4 for an example that the value of s in theorem 2 may be strictly bigger than 2.

In section one and two we recall some facts on partial tilting modules and contravariantly finite subcategories. Moreover we will state some useful general facts used in the remaining two sections, which are devoted to the proof of the theorems above. In the proofs of the theorems above we will make use of some of the machinery developed in [AS1] and [AR2].

We denote the composition of morphisms $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ in a given category \mathcal{K} by fg . For unexplained representation-theoretic terminology we refer to [R].

1. Partial tilting modules.

1.1. In this section we collect some preliminary results on partial tilting modules. We keep the basic notation from the introduction. If ${}_A M$ is an A -module we may decompose ${}_A M = \bigoplus_{i=1}^s M_i^{n_i}$ with M_i indecomposable, $M_i \not\cong M_j$ for $i \neq j$ and $n_i > 0$. We denote the number s of non-isomorphic indecomposable direct summands of M by $\delta(M)$. In the classical situation a partial tilting module ${}_A M$ is a tilting module if

$$\delta(M) = \text{rk } K_0(A) = n,$$

where $K_0(A)$ is the Grothendieck group of A . The following lemma is well-known. For completeness we supply a brief proof.

LEMMA. *Let M be a tilting module. Then $\delta(M) = n$.*

PROOF: Let B be the endomorphism algebra of M . Then $\delta(M) = \text{rk } K_0(B)$. By tilting theory (compare [M] or [H2]) we infer that $K_0(A) \simeq K_0(B)$, thus $n = \delta(M)$. Indeed, let $D^b(A)$ be the bounded derived category of $\text{mod } A$. Then $D^b(A)$ is triangle-equivalent to $D^b(B)$. By [G] we know that $K_0(D^b(A)) \simeq K_0(A)$, hence the assertion.

1.2. Let N be a partial tilting module. We denote by $K^b(\text{add } N)$ the homotopy category of bounded complexes over $\text{add } N$. Let $D^b(A)$ be the bounded derived category of $\text{mod } A$. We may consider $K^b(\text{add } N)$ as a full subcategory of $D^b(A)$, whose image is contained in $K^b(\text{add } {}_A A)$ [H1].

LEMMA. *A partial tilting module N is a tilting module if and only if $K^b(\text{add } N)$ is dense in $K^b(\text{add } {}_A A)$.*

PROOF: If N is a tilting module then clearly $K^b(\text{add } N)$ is dense in $K^b(\text{add } {}_A A)$ (see [H1]). For the converse assume that ${}_A A \in K^b(\text{add } N) \subset D^b(A)$, where we identify ${}_A A$ with the complex concentrated in degree zero with stalk being ${}_A A$. So ${}_A A$ is quasi-isomorphic to $N^\bullet = (N^i, d_N^i) \in K^b(\text{add } N)$. There is $r, s \geq 0$ such that $N^t = 0$ for $t > r$ and $N^{-t} = 0$ for $t > s$. Since ${}_A A \in K^b(\text{add } {}_A A)$ the quasi-isomorphism in $D^b(A)$ can be represented as a morphism of complexes

$$f^\bullet : {}_A A \rightarrow N^\bullet$$

which induces an isomorphism for the cohomology groups. Thus

$$H^0(N^\bullet) = \ker d_N^0 / \text{im } d_N^{-1} \simeq {}_A A \text{ and } H^i(N^\bullet) = 0 \text{ for } i \neq 0.$$

Since ${}_A A$ is projective we infer that

$$\ker d_N^0 \simeq {}_A A \oplus \operatorname{im} d_N^{-1}$$

Since $H^i(N^*) = 0$ for $i \neq 0$ we have two exact sequences:

$$0 \rightarrow \ker d_N^0 \rightarrow N^0 \rightarrow \cdots \rightarrow N^r \rightarrow 0$$

and

$$0 \rightarrow N^{-s} \rightarrow \cdots \rightarrow N^{-1} \rightarrow \operatorname{im} d_N^{-1} \rightarrow 0.$$

For $j > 0$ we have that

$$\operatorname{Ext}_A^j(\ker d_N^0, N) = \operatorname{Ext}_A^{j+1}(\ker d_N^1, N) = \cdots = \operatorname{Ext}_A^{j+r}(N^r, N) = 0.$$

Thus it follows from the first sequence that

$$\operatorname{Ext}_A^1(\operatorname{im} d_N^{-1}, \operatorname{im} d_N^{-2}) = \operatorname{Ext}_A^2(\operatorname{im} d_N^{-1}, \operatorname{im} d_N^{-3}) = \cdots = \operatorname{Ext}_A^{s-1}(\operatorname{im} d_N^{-1}, N^{-s}) = 0.$$

Therefore $\operatorname{im} d_N^{-1} = N' \in \operatorname{add} N$. So we have an exact sequence

$$0 \rightarrow {}_A A \oplus N' \rightarrow N^0 \rightarrow \ker d_N^1 \rightarrow 0$$

As above we may show that $\operatorname{Ext}_A^1(\ker d_N^1, N) = 0$. Consider the pushout along the projection $\pi : {}_A A \oplus N' \rightarrow {}_A A$. So we obtain the following commutative diagram of exact rows and columns:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & N' & \xlongequal{\quad} & N' & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & {}_A A \oplus N' & \longrightarrow & N^0 & \longrightarrow & \ker d_N^1 \longrightarrow 0 \\
 & & \pi \downarrow & & \downarrow & & \parallel \\
 0 & \longrightarrow & {}_A A & \longrightarrow & E & \longrightarrow & \ker d_N^1 \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
 \end{array}$$

Applying $\operatorname{Hom}_A(-, N)$ to the last row shows that $\operatorname{Ext}_A^1(E, N) = 0$, hence $E \in \operatorname{add} N$. So we obtain the required exact sequence

$$0 \rightarrow {}_A A \rightarrow E \rightarrow N^1 \rightarrow \cdots \rightarrow N^r \rightarrow 0.$$

This has the following easy consequence which we will use.

COROLLARY. Let M be a partial tilting module and X a complement to M . Let $T = M \oplus Y$ be a partial tilting module and suppose that there exists an exact sequence

$$0 \rightarrow T' \rightarrow T'' \rightarrow X \rightarrow 0$$

with $T', T'' \in \text{add}T$. Then T is a tilting module.

PROOF: In fact, let $T_1 = M \oplus X$. Since T_1 is a tilting module we know that the embedding

$$\mu : K^b(\text{add}T_1) \rightarrow K^b(\text{add}A)$$

is dense. Since T is a partial tilting module we have an embedding

$$\mu' : K^b(\text{add}T) \rightarrow K^b(\text{add}A).$$

The exact sequence shows that μ factors over μ' , hence μ' is dense.

1.3. The next proposition will turn out to be very useful in sections 3 and 4. The proof is inspired by similar arguments in [AR1] and [H3]. Recall that a module Y is said to be *generated* by a module M if there exists a surjective map $f : E \rightarrow Y$ with $E \in \text{add}M$.

PROPOSITION. Let M be an almost complete tilting module. Let Y be an indecomposable complement generated by M . Then

- (1) M is faithful,
- (2) there exists an indecomposable complement X not isomorphic to Y ,
- (3) there exists an exact sequence

$$0 \rightarrow X \xrightarrow{\mu} E \xrightarrow{\pi} Y \rightarrow 0$$

with $E \in \text{add}M$,

- (4) $\text{Ext}_A^i(X, Y) = 0$ for $i > 0$ and $\text{Ext}_A^i(Y, X) = 0$ for $i > 1$, and
- (5) X is uniquely determined by the property (3).

PROOF: (1) It follows from the third property in the definition of a tilting module that $M \oplus Y$ is faithful, since ${}_A A$ is cogenerated by $M \oplus Y$. Let $g : {}_A A \rightarrow F$ be an injective map with $F \in \text{add}(M \oplus Y)$. Since Y is generated by M , there exists a surjective map $h : E \rightarrow F$ with $E \in \text{add}M$. Since ${}_A A$ is projective there exists $f : {}_A A \rightarrow E$ with $fh = g$, hence f is injective and therefore ${}_A A$ is cogenerated by M , thus M is faithful.

(2),(3) Let f_1, \dots, f_r be a k -basis of $\text{Hom}_A(M, Y)$ and let

$$f = \begin{pmatrix} f_1 \\ \vdots \\ f_r \end{pmatrix} : M^r \rightarrow Y$$

be the corresponding map. Let $K = \ker f$. So we obtain an exact sequence

$$0 \rightarrow K \rightarrow M^r \rightarrow Y \rightarrow 0.$$

It is straightforward to show that $K \oplus M$ is a partial tilting module. By construction the sequence above does not split. Thus K is not contained in $\text{add } M$. Using 1.2 we infer that $K \oplus M$ is a tilting module. Therefore there exists an indecomposable module X such that $K \simeq X^s \oplus M'$ for some $s \geq 1$ and some $M' \in \text{add } M$. Let $T = M \oplus Y$ and let $B = \text{End}_A T$. We claim that $s = 1$. Indeed, apply $\text{Hom}_A(T, -)$ to the exact sequence above. This yields the following exact sequence in $\text{mod } B$:

$$\dots \text{Hom}_A(T, M^r) \longrightarrow \text{Hom}_A(T, Y) \longrightarrow \text{Ext}_A^1(T, X^s) \rightarrow 0.$$

But $\text{Hom}_A(T, Y)$ is an indecomposable projective B -module and so has a simple top. But then clearly $s = 1$.

Consider the following commutative diagram with exact rows and columns obtained by constructing the sequence induced by the pushout along the projection

$$\pi : X \oplus M' \longrightarrow X.$$

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & M' & \xlongequal{\quad} & M' & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 0 & \longrightarrow & X \oplus M' & \longrightarrow & M^r & \longrightarrow & Y & \longrightarrow & 0 \\
 & & \pi \downarrow & & \downarrow & & \parallel & & \\
 0 & \longrightarrow & X & \longrightarrow & E & \longrightarrow & Y & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
 \end{array}$$

Applying $\text{Hom}_A(-, M)$ to the lower exact sequence shows that $\text{Ext}_A^1(E, M) = 0$. So

$$0 \rightarrow M' \rightarrow M^r \rightarrow E \rightarrow 0$$

is split exact, thus $E \in \text{add } M$.

(4) This is straightforward.

(5) Suppose there is an indecomposable complement X' and an exact sequence

$$0 \rightarrow X' \xrightarrow{\mu'} E' \xrightarrow{\pi'} Y \rightarrow 0$$

with $E' \in \text{add } M$. Thus there exist $f : E \rightarrow E'$ with $f\pi' = \pi$ and $g : X \rightarrow X'$ with $g\mu' = \mu f$. Also there exist $f' : E' \rightarrow E$ with $f'\pi = \pi'$ and $g' : X' \rightarrow X$ with $g'\mu = \mu' f'$. If $X \not\cong X'$ we infer that gg' is a nilpotent endomorphism, for X is indecomposable. Thus there is $m > 0$ such that $(gg')^m = 0$. But then $0 = (gg')^m \mu = \mu (ff')^m$ shows that there exists $h : Y \rightarrow E$ with $\pi h = (ff')^m$. Since $\pi h \pi = (ff')^m \pi = \pi$ and π is surjective, we infer that $h\pi = \text{id}_Y$, hence the sequence splits, a contradiction. Thus X is uniquely determined by the property (3).

Under the assumptions of the proposition we call X the *complement determined* by Y . The sequence constructed above will be called a *connecting sequence*. Given Y it is uniquely determined up to isomorphism of exact sequences. Note that we have a surjective connecting homomorphism $\delta : \text{End } X \rightarrow \text{Ext}_A^1(Y, X)$ and that the connecting sequence corresponds to a non-zero element in the top of $\text{End } X$.

Let M be an almost complete tilting module and let X an indecomposable complement to M which is cogenerated by M . By the dual of the proposition we obtain an indecomposable complement Z and an exact sequence $0 \rightarrow X \rightarrow E \rightarrow Z \rightarrow 0$ with $E \in \text{add } M$. We call Z the *complement codetermined* by X .

The following is an easy consequence of the uniqueness of the connecting sequence.

COROLLARY. *Let M be an almost complete tilting module. Let Y be an indecomposable complement generated by M . Let X be the complement determined by Y . Then Z is the complement codetermined by X .*

1.4. By the third property in the definition of a tilting module we have an exact sequence

$$0 \rightarrow {}_A A \rightarrow T_0 \rightarrow \cdots \rightarrow T_{m-1} \xrightarrow{\pi} T_m \rightarrow 0$$

with $T_i \in \text{add } T$ for $0 \leq i \leq m$. We may choose m minimal, thus π is not a retraction. For completeness we will provide a proof of the following lemma.

LEMMA. *Let ${}_A T$ be a tilting module with $\text{proj. dim } {}_A T = r$. And let*

$$0 \rightarrow {}_A A \rightarrow T_0 \rightarrow T_1 \rightarrow \cdots \rightarrow T_{m-1} \xrightarrow{\pi} T_m \rightarrow 0$$

be a long exact sequence with $T_i \in \text{add } T$ for $0 \leq i \leq r$ and π not a retraction, then $m = r$.

PROOF: The long exact sequence

$$0 \rightarrow {}_A A \rightarrow T_0 \rightarrow \cdots \rightarrow T_{m-1} \xrightarrow{\pi} T_m \rightarrow 0$$

with $T_i \in \text{add } T$ for $0 \leq i \leq m$ and π is not a retraction gives rise to short exact sequences

$$0 \rightarrow K_i \rightarrow T_i \rightarrow K_{i+1} \rightarrow 0$$

for $0 \leq i < m$ with $K_0 = {}_A A$ and $K_m = T_m$. By assumption we have that $\text{Ext}_A^1(T, K_{m-1}) \neq 0$. Applying $\text{Hom}_A(T, -)$ to these exact sequences shows that $\text{Ext}_A^j(T, K_{m-i}) = 0$ for $0 \leq i < m$ and all $j > i$. Moreover we see that

$$0 \neq \text{Ext}_A^1(T, K_{m-1}) = \cdots = \text{Ext}_A^m(T, {}_A A)$$

Since $\text{proj.dim}_A T = r$, we infer that $\text{Ext}_A^r(T, {}_A A) \neq 0$ and $\text{Ext}_A^j(T, {}_A A) = 0$ for $j > r$. This clearly implies that $m = r$.

2. Cocovers and contravariantly finite subcategories.

2.1. In this section we recall some of the basic notions from [AS1], [AS2], [AR2] and [AR3].

Let \mathcal{D} be a full subcategory of $\text{mod } A$. It is always assumed to be closed under direct sums, direct summands and isomorphisms. An object $D \in \mathcal{D}$ is called a *cocover* of \mathcal{D} if for all $X \in \mathcal{D}$ there exists an injective map $X \rightarrow D'$ for some $D' \in \text{add } D$. A cocover D is called a *minimal cocover* in case no proper direct summand of D is a cocover. If \mathcal{D} is closed under extensions it is shown in [AS1] that a minimal cocover D is \mathcal{D} -*injective* in \mathcal{D} , so satisfies $\text{Ext}_A^1(X, D) = 0$ for all $X \in \mathcal{D}$.

The following example shows that the converse is not true. Consider the algebra A defined by the quiver algebra of

$$\begin{array}{ccccccc} & & \alpha & & \beta & & \gamma \\ & & \circ & \longleftarrow & \circ & \longleftarrow & \circ \\ & & 1 & & 2 & & 3 & & 4 \end{array}$$

bound by $\beta\alpha = \gamma\beta = 0$.

Let $P(i)$, (resp. $I(i)$) be the indecomposable projective (resp. injective) module associated with the vertex i and let $S(i)$ be the top of $P(i)$.

Let $M = P(2) \oplus P(3) \oplus P(4)$ and consider the following subcategory of $\text{mod } A$:

$$\mathcal{D} = \{X \in \text{mod } A \mid \text{Ext}_A^i(X, M) = 0 \text{ for all } i > 0 \text{ and } \text{proj.dim}_A X \leq 2\}.$$

Then \mathcal{D} is closed under extensions and has a minimal cocover C , for it contains only finitely many indecomposable modules up to isomorphism. It is easy to see that $M = C$. On the other hand the simple module $S(3)$ is \mathcal{D} -injective.

The subcategory \mathcal{D} is called *contravariantly finite* in $\text{mod } A$ if every $X \in \text{mod } A$ has a *right \mathcal{D} -approximation*, i.e. there is a morphism $F_X \rightarrow X$ with $F_X \in \mathcal{D}$ such that the induced morphism $\text{Hom}_A(D, F_X) \rightarrow \text{Hom}_A(D, X)$ is surjective for all $D \in \mathcal{D}$. If \mathcal{D} is also closed under extensions and K_X is the kernel of a minimal right \mathcal{D} -approximation of X , then Wakamatsu's Lemma states that $\text{Ext}_A^1(D, K_X) = 0$ for all $D \in \mathcal{D}$ [W].

The subcategory \mathcal{D} is called *resolving* if \mathcal{D} is closed under extensions, kernels of surjective maps and contains ${}_A A$. Note that for a contravariantly finite subcategory which is resolving every right approximation is surjective [AR2].

LEMMA. *Let \mathcal{D} be a resolving subcategory. If $D \in \mathcal{D}$ is \mathcal{D} -injective, then $\text{Ext}_A^i(X, D) = 0$ for all $X \in \mathcal{D}$ and all $i > 0$.*

PROOF: The proof is an easy induction on i , the case $i = 1$ being the fact that D is \mathcal{D} -injective. Let $X \in \mathcal{D}$. Since \mathcal{D} is a resolving subcategory there is an exact sequence $0 \rightarrow X' \rightarrow P \rightarrow X \rightarrow 0$ with terms in \mathcal{D} and P a projective A -module.

Applying $\text{Hom}_A(-, D)$ to this sequence and using the induction hypothesis yields the assertion.

2.2. We recall the construction of a complement as given in [RS] and give a slight generalization of their result. Let M be a partial tilting module and let $\mathcal{C} = \{X \in \text{mod } A \mid \text{Ext}_A^i(X, M) = 0 \text{ for } i > 0 \text{ and } \text{proj.dim}_A X < \infty\}$.

PROPOSITION. *If \mathcal{C} is a contravariantly finite subcategory in $\text{mod } A$, then M admits a complement X .*

PROOF: It is easily seen that \mathcal{C} is a resolving subcategory. By [AS1] we know that \mathcal{C} has a minimal cocover C . In fact C is the minimal right approximation of the minimal injective cogenerator of $\text{mod } A$. By a remark above C is \mathcal{C} -injective. So the lemma above implies that $C \oplus M$ is a partial tilting module.

Let $X \in \mathcal{C}$. Then there exists an exact sequence

$$0 \rightarrow X \rightarrow E \rightarrow X' \rightarrow 0$$

with $X' \in \mathcal{C}$ and $E \in \text{add}(C \oplus M)$. In fact, there is an injective map $\mu : X \rightarrow C'$ with $C' \in \text{add } C$, for C is a cocover. Let f_1, \dots, f_s be a k -basis of $\text{Hom}_A(X, M)$ and let $f : X \rightarrow M^s$ be the map with coordinates the chosen k -basis. We consider $g = (\mu, f) : X \rightarrow C' \oplus M^s$. Then g is injective and $X' = \text{cok } g \in \mathcal{C}$ by construction. We apply these considerations to ${}_A A$. This yields for each integer j a long exact sequence

$$0 \rightarrow {}_A A \xrightarrow{\alpha_0} E_0 \xrightarrow{\alpha_1} E_1 \rightarrow \dots \rightarrow E_{j-1} \xrightarrow{\alpha_j} E_j \rightarrow Q_j \rightarrow 0$$

with $E_i \in \text{add}(C \oplus M)$ and $Q_i = \text{cok } \alpha_i \in \mathcal{C}$ for all $0 \leq i \leq j$. By 3.10 of [AR2] there is an integer r such that $\text{proj.dim}_A X \leq r$ for all $X \in \mathcal{C}$. In particular $\text{Ext}_A^{r+1}(Q_r, {}_A A) = 0$. It is easily seen that $0 = \text{Ext}_A^{r+1}(Q_r, {}_A A) = \text{Ext}_A^1(Q_r, Q_{r-1})$, hence $Q_r \in \text{add}(C \oplus M)$. Thus $C \oplus M$ is a tilting module.

We keep the assumptions and notation of the previous proposition. We may assume that there is $X \in \mathcal{C}$ such that $\text{proj.dim}_A X = r$. Let M be an almost complete tilting module. In this case the proposition gives an indecomposable complement X_r to M which is a direct summand of the minimal cocover C .

COROLLARY. *If $\text{proj.dim}_A M < r$, then $\text{proj.dim}_A X_r = r$. In particular we have that $\text{proj.dim}_A M \oplus X_r = r$.*

PROOF: By assumption there is $X \in \mathcal{C}$ with $\text{proj.dim}_A X = r$. As above we have an exact sequence

$$0 \rightarrow X \rightarrow E \rightarrow X' \rightarrow 0$$

with $E \in \text{add}(M \oplus X_r)$ and $X' \in \mathcal{C}$. There is a module $Z \in \text{mod } A$ with $\text{Ext}_A^r(X, Z) \neq 0$. Applying $\text{Hom}_A(-, Z)$ to the sequence above then yields that $\text{Ext}_A^r(E, Z) \neq 0$. Thus $\text{proj.dim}_A X_r = r$.

The following example shows that the projective dimension of the indecomposable complement constructed in 2.2 may be strictly smaller than r . Consider the algebra A defined by the quiver algebra of

$$\begin{array}{ccccccc} \circ & \xleftarrow{\alpha} & \circ & \xleftarrow{\beta} & \circ & \xleftarrow{\gamma} & \circ \\ 1 & & 2 & & 3 & & 4 \end{array}$$

bound by $\beta\alpha = 0$.

Then $M = P(2) \oplus P(4) \oplus I(3)$ is an almost complete tilting module of projective dimension 2. It admits two non-isomorphic indecomposable complements $S(3)$ and $S(4)$. The simple module $S(4)$ has projective dimension 1 and is the complement constructed in 2.2, for it is injective.

2.3. For the proofs of the main results in the next two sections we will need some elementary results on non-split exact sequences and morphisms. For the convenience of the reader we give a proof.

LEMMA 1. *Let $X \in \text{mod } A$ and let $s \in \mathbb{N}$. If $0 \rightarrow X^s \rightarrow Y \rightarrow Z \rightarrow 0$ is a non-split exact sequence of A -modules with $Z \notin \text{add } Y$, then there is a non-split injective map $X \rightarrow Y'$ with $Y' \in \text{add } Y$.*

PROOF: We prove the assertion by induction on s . If $s = 1$ we are done. Now let

$$0 \rightarrow X^s \xrightarrow{\mu} Y \rightarrow Z \rightarrow 0$$

be a non-split exact sequence with $Z \notin \text{add } Y$ and $s > 1$. Let $\iota_1 : X \rightarrow X^s$ be the canonical inclusion into the first summand. Let $\mu_1 = \iota_1\mu$. This gives the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{id_X} & X & \longrightarrow & 0 & \longrightarrow & 0 \\ & & \downarrow \iota_1 & & \downarrow \mu_1 & & \downarrow & & \\ 0 & \longrightarrow & X^s & \xrightarrow{\mu} & Y & \xrightarrow{\pi} & Z & \longrightarrow & 0 \end{array}$$

where id_X denotes the identity map on X .

Thus we obtain the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & X & \xlongequal{\quad} & X & & & & \\
 & & \iota_1 \downarrow & & \mu_1 \downarrow & & & & \\
 0 & \longrightarrow & X^s & \xrightarrow{\mu} & Y & \xrightarrow{\pi} & Z & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \parallel & & \\
 0 & \longrightarrow & X^{s-1} & \xrightarrow{\mu'} & Y_1 & \xrightarrow{\pi'} & Z & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
 \end{array}$$

We denote the cokernel sequence by η . If μ_1 does not split we are done. Otherwise we obtain that $Y \simeq X \oplus Y_1$, in particular $Y_1 \in \text{add } Y$. Since $Z \notin \text{add } Y$, we infer that η does not split. By the induction hypothesis there is a non-split injective map $X \rightarrow Y'$ with $Y' \in \text{add } Y_1$, hence $Y' \in \text{add } Y$.

COROLLARY. *Let $0 \rightarrow X^s \rightarrow X^{s'} \oplus Y \rightarrow Z \rightarrow 0$ be a non-split exact sequence of A -modules with $Z \notin \text{add}(X \oplus Y)$ and X indecomposable. Then X is cogenerated by Y .*

PROOF: The previous lemma yields a non-split injective map $X \rightarrow X^m \oplus Y_1$ for some integer m and $Y_1 \in \text{add } Y$. The assertion now follows from 2.2 in [AS1].

LEMMA 2. *Let $\eta: 0 \rightarrow X \oplus X' \rightarrow Y \rightarrow Z \rightarrow 0$ be an exact sequence of A -modules and let $\eta': 0 \rightarrow X \rightarrow Y' \rightarrow Z \rightarrow 0$ be the sequence induced from η by the pushout along the projection $\pi_1: X \oplus X' \rightarrow X$. Assume that $\text{Ext}_A^1(Z, X') = 0$. If η does not split, then η' does not split.*

PROOF: Let $\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}: X \oplus X' \rightarrow Y$ be the morphism occurring in η . Assume that η' splits. Then there is $\gamma: Y \rightarrow X$ such that $\mu_1 \gamma = \text{id}_X$ and $\mu_2 \gamma = 0$, hence μ_1 is a section and γ is a retraction. This induces the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccc}
& & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X' & \xrightarrow{\mu'} & \ker \gamma & \xrightarrow{\pi'} & Z \longrightarrow 0 \\
& & \downarrow \iota & & \downarrow \iota' & & \parallel \\
0 & \longrightarrow & X \oplus X' & \xrightarrow{\mu} & Y & \xrightarrow{\pi} & Z \longrightarrow 0 \\
& & \downarrow \pi_1 & & \downarrow \gamma & & \\
& & X & \xlongequal{\quad} & X & & \\
& & \downarrow & & \downarrow & & \\
& & 0 & & 0 & &
\end{array}$$

By assumption the first row splits. By construction the second row is the sequence induced by the pushout along the canonical inclusion $\iota : X' \rightarrow X \oplus X'$. Therefore η splits. In fact, let $\pi'' : Z \rightarrow \ker \gamma$ with $\pi''\pi' = id_Z$. Let $\bar{\pi} = \pi''\iota' : Z \rightarrow Y$. Then $\bar{\pi}\pi = \pi''\iota'\pi = \pi''\pi' = id_Z$.

We refrain from stating but will use the duals of the preceding statements.

3. Complements cogenerated by M .

Let ${}_A M$ be an almost complete tilting module and let

$$\mathcal{C} = \{X \in \text{mod } A \mid \text{Ext}_A^i(X, M) = 0 \text{ for } i > 0 \text{ and } \text{proj.dim}_A X < \infty\},$$

which will be assumed to be contravariantly finite for the rest of this section. Let $r = \max\{\text{proj.dim}_A X \mid X \in \mathcal{C}\}$. In 2.2 we have constructed an indecomposable complement X_r . Note that that a \mathcal{C} -injective module Z lies in $\text{add}(M \oplus X_r)$. Otherwise $T = Z \oplus M \oplus X_r$ is a tilting module with $\delta(T) \geq n + 1$ in contrast to 1.1. We call X_r the \mathcal{C} -injective complement to M . Thus X_r is uniquely determined by this property.

3.1. The following lemma follows from tilting theory (compare [M] or [H2]). For completeness we include a proof.

LEMMA. Let ${}_A T$ be a tilting module. Let ${}_A X$ be a module with $\text{Ext}_A^i(T, X) = 0$ for all $i > 0$. Then there exists a short exact sequence $0 \rightarrow X' \rightarrow T_0 \rightarrow X \rightarrow 0$ with $T_0 \in \text{add} T$ and $\text{Ext}_A^i(T, X') = 0$ for all $i > 0$.

PROOF: Let $B = \text{End}_A T$. We consider the following two subcategories:

$$\mathcal{E}^0 = \{X \in \text{mod } A \mid \text{Ext}_A^i(T, X) = 0 \text{ for all } i > 0\}$$

$$\mathcal{T}_0 = \{Y \in \text{mod } B \mid \text{Tor}_i^B(T, Y) = 0 \text{ for all } i > 0\}$$

Then the restrictions of $\text{Hom}_A(T, -)$ and $T \otimes_B -$ give a pair of inverse equivalences between \mathcal{E}^0 and \mathcal{T}_0 . Let $X \in \mathcal{E}^0$ and let $Y = \text{Hom}_A(T, X) \in \mathcal{T}_0$. Then there is a surjective map $\pi : {}_B B^m \rightarrow Y$ for some integer m . Since $Y \in \mathcal{T}_0$ we infer that $T \otimes_B \pi : {}_A T^m \rightarrow X$ is again surjective, thus X is generated by T . Let f_1, \dots, f_s be a k -basis of $\text{Hom}_A(T, X)$ and let

$$f = \begin{pmatrix} f_1 \\ \vdots \\ f_s \end{pmatrix} : T^s \rightarrow X$$

be the corresponding map. Let $X' = \ker f$. So we obtain an exact sequence

$$0 \rightarrow X' \rightarrow T^s \rightarrow X \rightarrow 0.$$

It is now straightforward to see that $X' \in \mathcal{E}^0$.

3.2. The following result is the main step in proving theorem 1 from the introduction.

THEOREM. *Let Y be an indecomposable complement not isomorphic to X_r . Then Y is cogenerated by M .*

PROOF: We apply 3.1 to $T = M \oplus Y$. Since $T \in \mathcal{C}$ and X_r is \mathcal{C} -injective we have that $\text{Ext}_A^i(T, X_r) = 0$ for all $i > 0$. Therefore repeated application of 3.1 gives a long exact sequence

$$0 \rightarrow K_r \xrightarrow{f_{r+1}} T_r \rightarrow \dots \rightarrow T_0 \xrightarrow{f_0} X_r \rightarrow 0$$

with $T_j \in \text{add } T$ for $0 \leq j \leq r$ and $\text{Ext}_A^i(T, K_j) = 0$ for $0 \leq j \leq r$ and all $i > 0$ with $K_j = \ker f_j$. Since $\text{proj. dim}_A X_r \leq r$ we have that $\text{Ext}_A^{r+1}(X_r, K_r) = 0$. It is easily seen that $\text{Ext}_A^{r+1}(X_r, K_r) = \text{Ext}_A^1(K_{r-1}, K_r)$. Therefore $K_r \in \text{add } T$.

Set $K_{-1} = X_r$. Since Y is not isomorphic to X_r there is an integer $r > s \geq 0$ such that $K_s \in \text{add } T$ and the exact sequence

$$\eta : 0 \rightarrow K_s \rightarrow T_s \rightarrow K_{s-1} \rightarrow 0$$

obtained from the long exact sequence above does not split.

So $0 \neq \text{Ext}_A^1(K_{s-1}, K_s) = \text{Ext}_A^{s+1}(X_r, K_s)$ shows that $K_s \notin \text{add } M$, hence Y is a direct summand of K_s . Hence η is of the form

$$\eta : 0 \rightarrow Y^m \oplus M' \rightarrow Y^{m'} \oplus M'' \rightarrow K_{s-1} \rightarrow 0,$$

with $M', M'' \in \text{add } M$ and $m \geq 1$. According to lemma 2 in 2.3 the sequence $\eta' : 0 \rightarrow Y^m \rightarrow E \rightarrow K_{s-1} \rightarrow 0$ which is induced by the pushout along the projection $\pi : Y^m \oplus M' \rightarrow Y^m$ does not split. So we obtain the following commutative diagram of exact rows and columns:

$$\begin{array}{ccccccc}
& & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \\
& & M' & \xlongequal{\quad} & M' & & \\
& & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Y^m \oplus M' & \longrightarrow & Y^{m'} \oplus M'' & \longrightarrow & K_{s-1} \longrightarrow 0 \\
& & \pi \downarrow & & \downarrow & & \parallel \\
0 & \longrightarrow & Y^m & \longrightarrow & E & \longrightarrow & K_{s-1} \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \\
& & 0 & & 0 & &
\end{array}$$

Since \mathcal{C} is resolving we know that $K_{s-1} \in \mathcal{C}$. Applying $\text{Hom}_A(-, M')$ to η' shows that $\text{Ext}_A^1(E, M') = 0$. Thus the second column splits, hence $E \in \text{add}(M \oplus Y)$. Therefore η' is of the form

$$0 \rightarrow Y^m \rightarrow Y^{m''} \oplus M''' \rightarrow K_{s-1} \rightarrow 0$$

with $M''' \in \text{add } M$. The assertion now follows from the corollary in 2.3.

3.3. The following lemma will be used in the proof of theorem 1.

LEMMA. *Let M be an almost complete tilting module with $\text{proj.dim}_A M = t$. Assume that for some $t < i \leq r$ there is an indecomposable complement X_i to M with $\text{proj.dim}_A X_i = i$. Then X_i is generated by M .*

PROOF: Let $T = M \oplus X_i$ be the corresponding tilting module. By 1.4 there is an exact sequence

$$0 \rightarrow {}_A A \rightarrow T_0 \rightarrow T_1 \rightarrow \cdots \rightarrow T_{i-1} \xrightarrow{\pi} T_i \rightarrow 0$$

with π not a retraction and $T_j \in \text{add } T$ for $0 \leq j \leq i$. Choose such a sequence with T_i of minimal length. We claim that then $T_i \in \text{add } X_i$. In fact, assume that $T_i = M' \oplus X_i^s$ for some $s \geq 0$ and $M' \in \text{add } M$. Let $K = \ker \pi$ and consider the following commutative diagram induced by the pullback of the canonical inclusion $\mu: X_i^s \rightarrow T_i$

4. Complements for M being faithful.

We keep the notation from the preceding sections. In particular M will denote an almost complete tilting module with $\text{proj.dim}_A M = t$. We have to consider also the following subcategories \mathcal{C}_i for $t \leq i \leq r$ with

$$\mathcal{C}_i = \{X \in \text{mod } A \mid \text{Ext}_A^j(X, M) = 0 \text{ for } j > 0 \text{ and } \text{proj.dim}_A X \leq i\}.$$

By the choice of r we infer that $\mathcal{C} = \mathcal{C}_r$. It is easy to see that all these subcategories are resolving. Recall that we assume that \mathcal{C} is a contravariantly finite subcategory.

4.1. In the following proposition we construct indecomposable complements in case $r > t$.

PROPOSITION. *Let M be an almost complete tilting module and $t < r$. Then for all $t < i \leq r$ there is a unique indecomposable complement $X_i \in \mathcal{C}_i$ to M with $\text{proj.dim}_A X_i = i$. Moreover there is an indecomposable complement $X_t \in \mathcal{C}_t$ with $\text{proj.dim}_A X_t = t$. The connecting sequence is of the form*

$$0 \rightarrow X_{i-1} \rightarrow E_{i-1} \rightarrow X_i \rightarrow 0$$

with $E_{i-1} \in \text{add } M$ for $t < i \leq r$.

PROOF: The existence of indecomposable complements $X_i \in \mathcal{C}_i$ for $t \leq i \leq r$ follows by induction from 2.2 and 1.3 by using 3.3.

The connecting sequence then clearly has the form

$$0 \rightarrow X_{i-1} \rightarrow E_{i-1} \rightarrow X_i \rightarrow 0$$

with $E_{i-1} \in \text{add } M$ for $t < i \leq r$.

We prove the uniqueness by induction on i . Let $i = r$ and assume that there is an indecomposable complement Y of $\text{proj.dim}_A Y = r$ and $Y \not\cong X_r$. By 3.2 we know that Y is cogenerated by M . Let $0 \rightarrow Y \rightarrow E \rightarrow Z \rightarrow 0$ be the connecting sequence with $E \in \text{add } M$. Since $t < r$ we infer that the projective dimension of the complement Z is $r+1$, a contradiction. Let $t < i < r$ and assume that there is an indecomposable complement Y with $\text{proj.dim}_A Y = i$. Again we know by 3.2 that Y is cogenerated by M . Let $0 \rightarrow Y \rightarrow E \rightarrow Z \rightarrow 0$ be the connecting sequence with $E \in \text{add } M$. It is easy to see that $Z \in \mathcal{C}_{i+1}$ and that $\text{proj.dim}_A Z = i+1$, hence $Z \simeq X_{i+1}$ by the induction hypothesis. But then $Y \simeq X_i$ by the uniqueness of the connecting sequence

The next result shows theorem 2 from the introduction.

COROLLARY. *Let M be an almost complete tilting module and $t < r$. Then there are indecomposable complements $X_i \in \mathcal{C}_i$ for $t \leq i \leq r$ with $\text{proj.dim}_A X_i = i$ and a long exact sequence*

$$0 \rightarrow X_t \rightarrow E_t \xrightarrow{g_t} E_{t+1} \rightarrow \cdots \rightarrow E_{r-1} \xrightarrow{g_{r-1}} X_r \rightarrow 0$$

with $X_i = \ker g_i$, and $E_i \in \text{add } M$ for $t \leq i < r$. The modules $X_i \in \mathcal{C}_i$ for $t < i \leq r$ are uniquely determined by the property that $\text{proj.dim}_A X_i = i$.

PROOF: The long exact sequence is obtained by splicing together the connecting sequences from the preceding proposition.

4.2. We now turn to the proof of theorem 3. For $t \leq i \leq r$ we denote by X_i the complements determined in the previous corollary.

THEOREM. Let M be a faithful almost complete tilting module and assume that there is a complement $Y \in \mathcal{C}_t$ which is generated by M . Then

- (a) there is an integer s with $2 \leq s \leq \max(2, t+1)$ and there are indecomposable complements Y_1, \dots, Y_s to M giving a complete list of non-isomorphic indecomposable complements to M of projective dimension at most t ,
 (b) there is a long exact sequence

$$0 \rightarrow Y_1 \rightarrow F_1 \xrightarrow{f_1} \dots \rightarrow F_s = E_t \xrightarrow{f_s = g_t} E_{t+1} \rightarrow \dots \rightarrow E_{r-1} \xrightarrow{g_{r-1}} X_r \rightarrow 0$$

with $Y_i = \ker f_i$ for $1 \leq i \leq s$, $X_i = \ker g_i$ for $t \leq i < r$ and $F_j, E_i \in \text{add } M$ for $1 \leq j \leq s, t \leq i < r$ and X_i

PROOF: There is $1 \leq j \leq \max(1, t)$ and an exact sequence

$$0 \rightarrow Z_1 \rightarrow F_1 \xrightarrow{f_1} F_2 \rightarrow \dots \rightarrow F_j \xrightarrow{f_j} Y \rightarrow 0$$

with $F_i \in \text{add } M$, and $Z_i = \ker f_i$ are indecomposable complements in \mathcal{C}_t for $1 \leq i \leq j$. The existence of such a sequence is clear from 1.3. Let $Z_{j+1} = Y$. Since $\text{proj.dim}_A Z_i \leq t$ we infer that $\text{Ext}_A^{t+1}(Z_i, -) = 0$ for $1 \leq i \leq j+1$. But $0 \neq \text{Ext}_A^1(Z_2, Z_1) = \text{Ext}_A^2(Z_3, Z_1) = \dots = \text{Ext}_A^j(Z_{j+1}, Z_1)$ shows $j \leq \max(1, t)$. We construct such a sequence

$$0 \rightarrow Y_1 \rightarrow F_1 \rightarrow \dots \rightarrow F_{s-1} \rightarrow Y_s \rightarrow 0$$

with s maximal. Then Y_1 is not generated by M and $2 \leq s \leq \max(2, t+1)$.

First we show that $Y_s \simeq X_t$. Otherwise we know from 3.2 that Y_s is cogenerated by M . So let

$$0 \rightarrow Y_s \rightarrow E \rightarrow Y' \rightarrow 0$$

be the connecting sequence. By the maximality of s we infer that $Y' \in \mathcal{C}_{t+1}$ and $Y' \notin \mathcal{C}_t$. Thus $Y' \simeq X_{t+1}$ by 4.1, hence $Y_s \simeq X_t$ by 1.3.

Let $Z \in \mathcal{C}_t$ be an indecomposable complement to M . We claim that $Z \simeq Y_i$ for some $1 \leq i \leq s$. Otherwise we know from 3.2 that $Z_1 = Z$ is cogenerated by M . Let Z_2 be the complement codetermined by Z_1 and $0 \rightarrow Z_1 \rightarrow L_1 \rightarrow Z_2 \rightarrow 0$ be the connecting sequence with $L_1 \in \text{add } M$. Clearly $Z_2 \in \mathcal{C}_{t+1}$. If $Z_2 \notin \mathcal{C}_t$, then $Z_2 \simeq X_{t+1}$ by 4.1, where X_{t+1} is the unique indecomposable complement X with $\text{proj.dim}_A X = t+1$. But then it follows from 1.3 that $Z_1 \simeq X_t$. Thus $Z_2 \in \mathcal{C}_t$ and from the dual of the proposition in 1.3 it follows that $Z_2 \not\cong Y_s$. Using again

3.2 we know that Z_2 is cogenerated by M . Thus we obtain for all $j \geq 1$ an exact sequence

$$0 \rightarrow Z_1 \rightarrow L_1 \xrightarrow{h_1} L_2 \rightarrow \cdots \rightarrow L_j \xrightarrow{h_j} Z_{j+1} \rightarrow 0$$

with $Z_i = \ker h_i$; indecomposable complements in \mathcal{C}_t and $L_i \in \text{add } M$. But

$$0 \neq \text{Ext}_A^1(Z_2, Z_1) = \text{Ext}_A^2(Z_3, Z_1) = \cdots = \text{Ext}_A^j(Z_{j+1}, Z_1)$$

in contrast to $\text{proj. dim}_A Z_{j+1} \leq t$.

Part (b) now follows from the corollary in 4.1 by splicing the two long exact sequences together. Note that $Y_s = X_t$.

Note that the assumption that there is a complement $Y \in \mathcal{C}_t$ which is generated by M is equivalent to the condition that there is a complement to M in \mathcal{C}_t which is not isomorphic to X_t .

4.3. The following results are used in the proof of proposition 4 from the introduction.

LEMMA. Let M be a faithful almost complete tilting module. Then

(i) There is an exact sequence

$$(*) \quad 0 \rightarrow {}_A A \rightarrow M' \rightarrow Q \rightarrow 0.$$

with $Q \in \mathcal{C}$ and $M' \in \text{add } M$.

(ii) If X is an indecomposable complement to M , then X is generated by M if and only if $\text{Ext}_A^1(Q, X) = 0$.

PROOF: We first show (i). Let f_1, \dots, f_m be a k -basis of $\text{Hom}_A({}_A A, M)$, and denote by f the map $(f_1, \dots, f_m) : {}_A A \rightarrow M^m$. Then f is injective and let $Q = \text{coker } f$. Consider the exact sequence

$$0 \rightarrow {}_A A \rightarrow M^m \rightarrow Q \rightarrow 0.$$

By construction $\text{Ext}_A^1(Q, M) = 0$, and then $\text{Ext}_A^i(Q, M) = 0$ for all $i > 0$. Hence $Q \in \mathcal{C}$.

Next we show (ii). If X is generated by M we have by 1.3 a connecting sequence

$$0 \rightarrow X' \rightarrow E \rightarrow X \rightarrow 0$$

with $E \in \text{add } M$ and X' the indecomposable complement determined by X . Applying $\text{Hom}_A(-, X')$ to $(*)$ shows that $\text{Ext}_A^i(Q, X') = 0$ for $i \geq 2$. Applying $\text{Hom}_A(Q, -)$ to the connecting sequence then shows the assertion.

For the converse we note that there is $p > 0$ and a surjective map $g : {}_A A^p \rightarrow X$. Applying $\text{Hom}_A(-, X)$ to the exact sequence

$$0 \rightarrow {}_A A^p \xrightarrow{g} M'^p \rightarrow Q^p \rightarrow 0$$

shows that the induced map $\text{Hom}_A(\mu, X) : \text{Hom}_A(M'^p, X) \rightarrow \text{Hom}_A(AA^p, X)$ is surjective. Thus there is $h \in \text{Hom}_A(M'^p, X)$ with $g = h\mu$. Since g is surjective, also h is surjective, thus X is generated by M .

4.4. Under additional assumptions we can construct C_i -injective complements.

LEMMA. Assume that for some $t < i \leq r$ there is a C_i -injective complement J_i to M with $\text{proj.dim}_A J_i = i$. We assume that C_{i-1} is contravariantly finite. Let K_{i-1} be the kernel of the minimal right C_{i-1} -approximation of J_i . Then K_{i-1} is C_{i-1} -injective and has an indecomposable direct summand J_{i-1} which is a complement to M . Moreover, J_{i-1} is cogenerated by M .

PROOF: Let $(*) 0 \rightarrow K_{i-1} \rightarrow F_{i-1} \rightarrow J_i \rightarrow 0$ be the exact sequence with F_{i-1} the minimal right C_{i-1} -approximation of J_i . Since C is resolving we have that $K_{i-1} \in C$. Since $\text{proj.dim}_A J_i = i$ and $\text{proj.dim}_A F_{i-1} \leq i-1$ we infer that $\text{proj.dim}_A K_{i-1} \leq i-1$, hence $K_{i-1} \in C_{i-1}$. According to Wakamatsu's Lemma, K_{i-1} is C_{i-1} -injective. Since J_i is C_i -injective we infer that $\text{Ext}_A^1(C_{i-1}, F_{i-1}) = 0$, hence F_{i-1} is C_{i-1} -injective. It follows from 2.1 that $T = M \oplus K_{i-1} \oplus F_{i-1}$ is a partial tilting module. Using 1.2 we infer that T is a tilting module. Since $(*)$ is non-split we conclude that K_{i-1} contains an indecomposable summand J_{i-1} such that $M \oplus J_{i-1}$ is a tilting module. Using the same arguments as in 3.2 we see that J_{i-1} is cogenerated by M .

COROLLARY. If M is a faithful non-projective almost complete tilting module and if J_i is an indecomposable C_i -injective complement to M for some $t \leq i \leq r$, then J_i is generated by M .

PROOF: Consider the sequence $(*)$ from the lemma in 4.3. Then we infer that $\text{proj.dim}_A Q = \text{proj.dim}_A M$. Therefore, $Q \in C_t$. Hence $\text{Ext}_A^1(Q, J_i) = 0$. The assertion follows from (ii) of the lemma in 4.3.

In 4.1 we have constructed indecomposable complements X_i for $t \leq i \leq r$. In the previous lemma we have constructed indecomposable complements J_i for $t \leq i \leq r$. Next we show that they are isomorphic.

LEMMA. Let M be an almost complete tilting module and assume that the subcategories C_i are contravariantly finite for all $t \leq i \leq r$. Then the complements $X_i \simeq J_i$ for $t \leq i \leq r$.

PROOF: By 4.1 there is nothing to show for $t < i \leq r$. So it remains to show that $X_t \simeq J_t$. We may assume that $t < r$. It follows from 4.1 that

$$0 \rightarrow X_t \rightarrow E_t \rightarrow X_{t+1} = J_{t+1} \rightarrow 0$$

is a connecting sequence with $E_t \in \text{add } M$

By assumption $J_t \not\subseteq X_r$, hence J_t is cogenerated by M .

Let $0 \rightarrow J_t \rightarrow E \rightarrow Z \rightarrow 0$ be the connecting sequence with $E \in \text{add } M$ and Z the indecomposable complement codetermined by J_t . It is easy to see that $Z \in C_{t+1}$. Since J_t is C_t -injective and by construction $\text{Ext}_A^1(Z, J_t) \neq 0$ we obtain

that $Z \notin C_t$. By 4.1 we infer that $Z \simeq X_{t+1}$. But then $X_t \simeq J_t$ by the uniqueness in 1.3.

Combining this with the previous corollary shows proposition 4 from the introduction.

PROPOSITION. *Let M be a faithful non-projective almost complete tilting module. If C_i is contravariantly finite for all $t \leq i \leq r$, then the complement X_t is generated by M .*

4.5. The following corollary is an immediate consequence of the previous results.

COROLLARY. *Let M be an almost complete tilting module and assume that the subcategory C are contravariantly finite. Then there is a unique indecomposable complement Y which is not generated by M and a unique indecomposable complement X which is not cogenerated by M . Moreover, M is faithful if and only if $Y \not\subseteq X$.*

4.6. The following example shows that the integer s in the previous theorem may be strictly bigger than 2.

Let A be given as the quiver algebra of

$$\begin{array}{ccccccc} \circ & \xrightarrow{\alpha} & \circ & \xrightarrow{\beta} & \circ & \xleftarrow{\gamma} & \circ & \xleftarrow{\delta} & \circ \\ 5 & & 3 & & 1 & & 2 & & 4 \end{array}$$

bound by $\alpha\beta = \delta\gamma = 0$.

Then $M = I(1) \oplus I(3) \oplus I(2) \oplus I(5)$ is an almost complete tilting module of projective dimension 2, where we use the notation for modules as introduced in 2.1. The module M is faithful. Then $r = 2$ and the C_2 -injective complement $X_2 = I(4)$. The integer s in theorem 2 is 3 and the complements are given by $Y_1 = P(3)$, $Y_2 = S(2)$ and $Y_3 = X_2$.

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