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STANDARDLY STRATIFIED SPLIT AND LOWER TRIANGULAR ALGEBRAS *

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To Idun Reiten for her 60th birthday

Abstract

In the first part, we study algebras A such that $A = R \coprod I$, where R is a subalgebra and I a two sided, nilpotent ideal. Under certain conditions on I, we show that A is standardly stratified, if and only if, R is standardly stratified. Next, for $A = \begin{bmatrix} U & 0 \\ M & V \end{bmatrix}$, we show that A is standardly stratified, if and only if, the algebra $R = U \times V$ is standardly stratified and $_VM$ is a good V-module.

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1 Introduction.

In this work k will denote an algebraicaly closed field and algebra always means finite dimensional basic k-algebra.

We consider first the case in which A is a split algebra, that is an algebra with a subalgebra R and a two sided ideal I, such that there is an R-bimodulo decomposition $A = R \coprod I$. The algebra structure in $A = R \coprod I$ is given by

$$(r,i)(r',i') = (rr', ri' + ir' + ii').$$

We prove that, under certain hypothesis, R is standardly stratified, if and only if, A is standardly stratified. We also study relations between the categories of good modules in both algebras.

In the third section we study the case of a lower triangular matrix algebra of the type: $A = \begin{bmatrix} U & 0 \\ M & V \end{bmatrix}$ where U and V are algebras and M is a $V-U$-bimodule. Algebras of this type can be viewed naturally as a split algebra if we let $R = U \times V$ and take M as a R-bimodule in which U acts as zero on the left and V acts as zero on the right. Nevertheless the condition **2** assumed in section 2 is almost never satisfied if we order the idempotents of A in a way that the idempotents of U are smaller than the idempotens of V. So our analysis for lower triangular matrices follows a different approach.

Split algebras have been studied recently in different settings. For instance their Hochschild Cohomology have been study in [3]. Some relations between their almost split sequences have also been study in [1]. Influenced by these works, and others, we studied these algebras from the point of view of stratification.

2 Split by nilpotent algebras

We start this section by reviewing some definitions and fixing the notations.

Let A be an algebra.

Unless otherwise stated, module means finitely generated left module, and A-mod will denote the category of A-modules. Let $\bar{e} = \{e_1, \ldots, e_n\}$ be a complete set of primitive, orthogonal idempotents, and let us give $\bar{e}$, once and for all, the order given by de indices. As usual $P_i = Ae_i$ denotes the projective cover of the simple S_i. For each i we define the standard module $\Delta_A(i)$ to be the maximal quotient of P_i with composition factors S_j with $j \leq i$. Let Δ be the set of all these standard modules $\Delta_A(i)$. An A-module M will be called a Δ-good module, or just a good module, if there is a finite chain of submodules

$$0 = M_0 \subset M_1 \subset \ldots \subset M_t = M$$

such that M_i/M_{i-1} is isomorphic to a module in Δ, for all i. The number t does not depend on the filtration. We will call it the Δ-length of M and denote it by $l(M)$. The full subcategory of A-mod whose objects are the good modules is denoted by $\mathcal{F}_A(\Delta)$. The algebra A is said to be left standardly stratified if A is a good module.

The concept of a stratified algebra appeared for the first time in a lecture by V. Dlab, in 1989, at the AMS Arcata meeting but the corresponding abstract was never published, so the real, explicit appearance of these algebras, under the terminology of Δ-filtered algebras, took place in Dlab's lectures at the Constantza meeting in Romania, and was published as *Quasi-hereditary algebras revisited* in An. St. Univ. Ovidius Constantza, Vol. 4 (1996), pp. 43-54. They have been studied since by quite a few researchers like Dlab, Agoston, Lukacs, Cline-Parshall-Scott and Futorny-König-Mazorchuk.

We come back now to our situation of a split algebra $A = R \coprod I$. We assume in addition, that I is nilpotent, which is equivalent to say that I is contained in the radical of the algebra A, or that there is a complete set of orthogonal primitive idempotents of A, which are in the subalgebra R.

Throughout this section, we assume the following conditions on the R bimodule I:

- (Condition **1**) I_R is a right, projective R-module.
- (Condition **2**) For each i the A-module $I \otimes_R S_i$ has composition factors only of the form S_j with $j \leq i$.

Observe that the conditions above hold for the R-bimodule I, if and and only if, they hold for the R-bimodule A, since $A = R \coprod I$, as an R-bimodule.

Since $1_A \in R$, we can assume that the ordered set $\underline{e}$ is also a set of orthogonal, primitive, idempotents of R. Let us consider the following functors:

$$F : R \otimes_A - : A - mod \to R - mod \quad \text{and} \quad G : A \otimes_R - : R - mod \to A - mod.$$

We have that the functor $R \otimes_A A \otimes_R - \simeq 1_R$ and also that the functor $F : A \otimes_R -$ preserves projectives and projective covers, and applies, for each i, the R-simples $\mathrm{Top}(Re_i)$ onto the A-simples $\mathrm{Top}(Ae_i)$. If there is no danger of misunderstandings, we will call either one S_i. The main result on this section states that, under our conditions **1** and **2**, R is standardly stratified, if and only if, the same holds for A. The following caracterization of the family Δ of standard modules, given by Dlab and Ringel, will be used.

Theorem 1 *For an arbitrary k-algebra Λ, let $D = \{M_1 \dots M_n\}$ be a family of Λ-modules. Then D is the complete, ordered, family of standard modules (up to isomorphism), if and only, the three following conditions hold:*

- *$Top(M_i) \simeq S_i$ for $1 \leq i \leq n$.*
- *For each i, all composition factors of M_i are of the form S_j with $j \leq i$.*
- *For each i, $Ext^1_\Lambda(M_i, S_j) = 0$ for all $j \leq i$.*

Lemma 1 *Under the conditions* **1** *and* **2**, *the following statements are valid:*

1. $\Delta_A(i) \simeq A \otimes_R \Delta_R(i)$
2. $\Delta_R(i) \simeq R \otimes_A \Delta_A(i)$

Proof: : 1) We show that the family $\{A \otimes_R \Delta_R(i)\}$ satisfy the conditions of theorem 1.

- Since the functor $A \otimes_R -$, preserves projective covers one obtains that the top of the A-module $A \otimes_R \Delta_R(i)$ is the simple A-module S_i.
- Let us consider the exact sequence of A-modules $0 \to I \to A \to R \to 0$ which gives for each simple R-simple S_j with $j \leq i$, the exact sequence

  $$0 = Tor^1_R(R, S_j) \to I \otimes_R S_j \to A \otimes_R S_j \to R \otimes_R S_j \to 0$$

  and so $A \otimes_R S_j$ has composition factors only simples S_k with $k \leq i$. From this fact and the exactness of the functor $A \otimes_R -$ we have that all compositions factors of the A-module $A \otimes_R \Delta_R(i)$ are simple of the form S_j, with $j \leq i$.
- Finally we claim that, $\mathrm{Ext}^1_A(A \otimes_R \Delta_R(i), S_j) = 0$ for $j \leq i$. Since A is an R-projective module $\mathrm{Ext}^1_A(A \otimes_R \Delta_R(i), S_j) \simeq \mathrm{Ext}^1_R(\Delta_R(i), Hom_A(A, S_j)) \simeq \mathrm{Ext}^1_R(\Delta_R(i), S_j) = 0$ for $j \leq i$, see[8], exercise 9.21.

Therefore, from theorem 1, we conclude that $A \otimes_R \Delta_R(i) \simeq \Delta_A(i)$.

2) We have that the composition $F \circ G$ is naturally equivalent to Id_{R-mod}, therefore $\Delta_R(i) \simeq R \otimes_A A \otimes_R \Delta_R(i) \simeq R \otimes_A \Delta_A(i)$.

□

Proposition 2 *With the general assumptions made for this section, we have:*

1. *$N \in \mathcal{F}_R(\Delta)$, implies $A \otimes_R N \in \mathcal{F}_A(\Delta)$.*
2. *$N \in \mathcal{F}_A(\Delta)$, implies $R \otimes_A N \in \mathcal{F}_R(\Delta)$*

Proof: : 1) We use induction on $l(N)$. By the previous lemma, the result is true if N is one of the standard modules $\Delta_R(i)$. So, assume that N contains properly one of the $\Delta_R(i)$ for some $i = 1, \ldots n$, then there is a short exact sequence

$$0 \to \Delta_R(i) \to N \to N/\Delta_R(i) \to 0$$

in $\mathcal{F}_R(\Delta)$. Applying the functor $G = A \otimes_R -$ we get

$$0 \to \Delta_A(i) \to A \otimes_R N \to A \otimes_R (N/\Delta_R(i) \to 0$$

by induction and the fact that $\mathcal{F}_A(\Delta)$ is closed by extensions, it follows that $A \otimes_R N$ is in $\mathcal{F}_A(\Delta)$.

2) Again, we use induction on $l(N)$. Here, we get a long exact sequence

$$\dots \to Tor^1_A(R, N/\Delta_A(i)) \to R \otimes_A \Delta_A(i) \to R \otimes_A N \to R \otimes_A N/\Delta_A(i) \to 0.$$

It is enough to show that $Tor^1_A(R, M) = 0$ for all $M \in \mathcal{F}_A(\Delta)$. Using the exact sequence of right A-modules $0 \to I \to A \to R \to 0$, we get, for each $\Delta_A(i)$, an exact sequence:

$$0 \to Tor^1_A(R, \Delta_A(i)) \to I \otimes_A \Delta_A(i) \to \Delta_A(i) \to \Delta_R(i) \to 0.$$

Since $\Delta_A(i) = A \otimes_R \Delta_R(i) \simeq R \otimes_R \Delta_R(i) \coprod I \otimes_R \Delta_R(i)$ (as R-modules) and since $I \otimes_A \Delta_A(i) \simeq I \otimes_A A \otimes_R \Delta_R(i) \simeq I \otimes_R \Delta_R(i)$, we get by counting dimensions that $Tor^1_A(R, \Delta_A(i)) = 0$. Then, it follows by induction that $Tor^1_A(R, M) = 0$ for all $M \in \mathcal{F}_A(\Delta)$.

□

Corollary 3 *For any good A-module M we have that* $Tor^1_A(R, M) = 0$.

□

Corollary 4 *A is a standardly stratified algebra, if and only if, R is a standardly stratified algebra.*

□

Corollary 5 *If the category* $\mathcal{F}_A(\Delta)$ *is of finite representation type, then* $\mathcal{F}_R(\Delta)$ *is of finite representation type.*

Proof: : Let $\{M_1, \dots, M_m\}$ be a complete set of isomorphism class of indecomposable A-modules in $\mathcal{F}_A(\Delta)$ and let $M = M_1 \coprod M_2 \coprod \dots \coprod M_m$, we claim that $\mathcal{F}_R(\Delta)$ =add$(R \otimes_A M)$. Let L in $\mathcal{F}_R(\Delta)$, then $A \otimes_R L \simeq M_1^{t_1} \coprod M_2^{t_2} \coprod \dots \coprod M_m^{t_m}$ so

$M \simeq (R \otimes_A M_1)^{t_1} \coprod \dots \coprod (R \otimes_A M_m)^{t_m}$, which is in add$(R \otimes_A M)$.

□

In [6] Reiten and Platzeck, observed that the subcategory of good modules is always contained in the subcategory of modules of finite projective dimension, they also gave conditions for these subcategories to be equal. We have the following proposition which also relates these concepts. For the statement we use the notation in [6].

Corollary 6 *If $\mathcal{F}_A(\Delta) = P^{<\infty}(A)$, then $\mathcal{F}_R(\Delta) = P^{<\infty}(R)$.*

Proof: : If $M \in P^{<\infty}(R)$, then it has a finite projective resolution of R-modules, $0 \to P^n \to \ldots \to P^1 \to P^0 \to M \to 0$. Applying the functor $G = A \otimes_R -$ we get the following projective resolution of the A-module $A \otimes_R M$.

$$0 \to A \otimes_R P^n \to \ldots \to A \otimes_R P^1 \to A \otimes_R P^0 \to A \otimes_R M \to 0.$$

Hence, $A \otimes_R M$, having finite A projective dimension, is in $\mathcal{F}_A(\Delta)$. Therefore, by proposition 2, $M \cong R \otimes_A (A \otimes_R M) \in \mathcal{F}_R(\Delta)$.

□

Remarks 7 *1. Since the functor $(R \otimes_A (A \otimes_R -))$ is isomorphic to the functor $Id_{R\text{-mod}}$, for M and $N \in R$-mod, we have that, $M \simeq N$, if and only if, $A \otimes_R M \simeq A \otimes_R N$.*

2. Since I is contained in the radical of A, $R \otimes_A M = 0$, if and only if, $M = 0$.

□

Corollary 8 *The R-module M is decomposable (resp. indecomposable), if and only if, the A-module $A \otimes_R M$ is decomposable (resp. indecomposable).*

□

One particular case of a split algebra satisfying our assumptions is the algebra $A = R[x]/(x^2)$, which is isomorphic to $R \coprod R$ with the multiplication given by $(r,s)(r',s') = (rr', rs' + sr')$. The isomorphism is given by $r + sx \mapsto (r,s)$.

It is clear that the quiver Q_R is a subquiver of Q_A. Moreover, Q_A is obtained from Q_R by adding one loop , denoted by l_i, at each vertex v_i. This is a complete description of Q_A.

Now if $J = \{r_1, \ldots, r_n\}$ are relations defining a presentation of R, then a set of relations for a presentation of A is obtained by by adding to J all loops l_i^2 and all differences

$$\{l_i^2 \text{ for each } i, \alpha l_{o(\alpha)} - l_{t(\alpha)}\alpha \text{ for each arrow } \alpha\}$$

We give now some examples.

Example 9

This example shows that it can happen that $\mathcal{F}_R(\Delta)$ is of finite representation type but $\mathcal{F}_A(\Delta)$ is not.

Let $A = k(1 \overset{\alpha}{\underset{\beta}{\rightrightarrows}} 2)$ be the Kronecker algebra, and $R = k(1 \overset{\alpha}{\rightarrow} 2)$. In this case $I = \langle\beta\rangle$ is the two sided ideal generated by β, (which is just the one dimensional vector space $k(\beta)$. As R-bimodule $\langle\beta\rangle$ is the simple $S_2 \otimes_k S_1$ which as a right R-module is isomorphic to S_1 which is a projective right R-module. In this case our algebras are hereditary and $\mathcal{F}_A(\Delta) = A - mod$ and $\mathcal{F}_R(\Delta) = R - mod$. The category $R - mod$ is of finite representation type but $A - mod$ is not.

Example 10

As we showed there is an embedding on the indecomposable objects from the category $\mathcal{F}_R(\Delta)$ into $\mathcal{F}_A(\Delta)$, given by $A \otimes_R -$. We consider the algebra $A = R[x]/(x^2)$ in which $\mathcal{F}_R(\Delta)$ has 3 indecomposables but $\mathcal{F}_A(\Delta)$ has 4 indecomposables, see [4]. In our example, the algebras will be IIP, so that their good modules are the modules of finite projective dimension.

Let $R \simeq k(1 \overset{\alpha}{\rightarrow} 2)$ and $A \simeq kQ_A/I$ where Q_A is obtained from the quiver of Q_R by adding one loop l_i at each vertex i and the relations $\{l_1^2, l_2^2, \alpha l_1 - l_2\alpha\}$ (as described before). We have that $\Delta_R(i) = S_i$ and the indecomposable of the category $R - mod = \mathcal{F}_R(\Delta)$ are S_1, S_2 and P_1. So we have $A \otimes_R S_1 \simeq \Delta_A(1)$, $A \otimes_R S_2 \simeq \Delta_A(2)$ and $A \otimes_R P_1 \simeq P_A(1)$ are indecomposable A-modules in $\mathcal{F}_A(\Delta)$, but in addition we have the following indecomposable A-module

$$\begin{matrix} k & & k \\ \| & \searrow & \| \\ k & & k \end{matrix}$$

which belongs to $\mathcal{F}_A(\Delta)$ and is not of the form $A \otimes_R M$ for any M in $R - mod$.

Example 11

We describe now an example where I is not zero, nevertheless the functors F and G induce bijections between the indecomposable *good* modules.

We take as R the hereditary algebra $k(1 \overset{\alpha}{\rightarrow} 2)$ and $A \simeq kQ_A/(\text{rad})^2$, has radical square zero, where Q_A is obtained from Q_R by adding one loop l at the vertex 1. It is easy to see that in this case both categories have 3 indecomposable objects.

3 Algebras in lower triangular form

In this section we study algebras, which are given as matrix algebras in lower triangular form, with respect to being standardly stratified. We choose the idempotents conveniently. Let us observe again that these are always split algebras. But, with our choice of idempotents, they almost never satisfy all the hypotheses of the former section. So the point of view here is another one and the results that we obtain are of a different nature.

We fix the following notations.

U and V denote finite dimensional k-algebras, M a $V-U$-bimodule and A the finite dimensional k-algebra

$A = \begin{bmatrix} U & 0 \\ M & V \end{bmatrix}$. Also, we take the ordered set $\bar{g} = \{e_1, \ldots, e_t, f_{t+1}, \ldots f_{t+r}\}$ as the complete, ordered set of orthogonal, primitive idempotents of A, where $\bar{e} = \{e_1, \ldots, e_t\} \subset U$ and $\bar{f} = \{f_{t+1}, \ldots, f_{t+r}\} \subset V$ are the fixed complete, ordered sets of orthogonal idempotents of U, V, respectively. (Here, of course, we identify e_i with $\begin{bmatrix} e_i & 0 \\ 0 & 0 \end{bmatrix}$ and f_j with $\begin{bmatrix} 0 & 0 \\ 0 & f_j \end{bmatrix}$..)

Let us begin by quoting two well known results that will be useful.

Given A, a standardly stratified algebra, with respect to $\bar{h} = (h_1, \ldots, h_n)$, let j be such that $1 \leq j \leq n$ and let us denote by ϵ_j the sum $\epsilon = h_j + \ldots + h_n$. We now state two well known results of Dlab and Ringel.

Theorem 2 *1. The algebra $A/A\epsilon_j A$ is standardly stratified and the good-$A/A\epsilon_j A$ modules are the good A-modules anihilated by $A\epsilon_j A$.*

2. The algebra $\epsilon_j A \epsilon_j$ is standardly stratified, with respect to $\{h_j, \ldots, h_n\}$.

We recall the well-known fact that there is an equivalence between the category of A-modules and the category C whose objects are triples (X, Y, f), where X is in U-mod, $Y inV$-mod and $f : M \otimes_U X \to Y$, is a V-module homomorphism. In what follows, by abuse of language, we identify the x of U (resp.of V) with the corresponding triple $(x, 0, 0)$ (resp. $(0, x, 0)$).

The sequence $(A, B, f) \overset{(\alpha,\beta)}{\to} (A', B', f') \overset{(\alpha'\beta')}{\to} (A'', B'', f'')$ is exact, if and only if, the sequences $A \overset{\alpha}{\to} A' \overset{\alpha'}{\to} A''$ and $B \overset{\beta}{\to} B' \overset{\beta'}{\to} B''$ are exact. Moreover the indecomposable A-projective modules are of the form $(P, M \otimes_V P, Id)$ where ${}_UP$ is projective, or of the form $(0, Q, 0)$, where ${}_VQ$ is projective. The indecomposable injective objects in C are objects of the form $(I, 0, 0)$ where I is an indecomposable injective U-module and objects isomorphic to objects of the form $(\mathrm{Hom}_V(M, J), J, \phi)$ where J is an indecomposable injective V-module and $\phi : M \otimes_U \mathrm{Hom}_V(M, J) \to J$ is given by $\phi(m \otimes f) = f(m)$ for $m \in M$ and $f \in \mathrm{Hom}_V(M, J)$.

Remark 12 *It follows, from theorem 2, that the set of standard A-modules is the union of the set of standard U-modules with the set of standard V-modules.*

Lemma 13 *The A-module $(X,Y,f) \in \mathcal{F}_A(\Delta)$, if and only if, $X \in \mathcal{F}_U(\Delta)$ and $Y \in \mathcal{F}_V(\Delta)$.*

Proof: Let $L = (X,Y,f)$ then we have the following filtration $L = A\epsilon_1 L \supseteq A\epsilon_2 L \supseteq \ldots \supseteq A\epsilon_{t-1} L \supseteq (0,Y,0) \supseteq A\epsilon_{t+2} L \supseteq \ldots \supseteq A\epsilon_{t+r+1} L = 0$. Assuming that L is A-good we have that $L/(0,Y,0) \simeq (X,0,0)$ is A-good, and it is annihilated by $\begin{bmatrix} 0 & 0 \\ M & V \end{bmatrix}$. It follows that $X \in \mathcal{F}_U(\Delta)$ and $Y \in \mathcal{F}_V(\Delta)$. The converse is analogous. □

Proposition 14 *The algebra A is standardly stratified with respect to $\bar{g}$, if and only if the following conditions are satisfied.*

a) U is standardly stratified with respect to $\bar{e}$.

b) V is standardly stratified with respect to $\bar{f}$.

c) ${}_VM \in \mathcal{F}_V(\Delta)$.

Proof: Firstly let us assume that A is standardly stratified and prove that the three conditions hold. Since ${}_AA \in \mathcal{F}_A(\Delta)$ and $A = (U,M,1)\coprod(0,V,0)$ then by lemma 13 ${}_UU \in \mathcal{F}_U(\Delta)$ and $M\coprod V \in \mathcal{F}_V(\Delta)$. The converse follows analogously using the other implication of the lemma 13.

□

Corollary 15 *The algebra A is quasi-hereditary, if and only if, U and V are quasi-hereditary and $M \in \mathcal{F}_V(\Delta)$.*

Proof: It follows easily from the remark 12 and the previous proposition.

□

We now want to investigate conditions, for lower triangular matrix algebras, which imply that the category of good modules is the category of modules of finite projective dimension. We write $A = \begin{bmatrix} U & 0 \\ M & V \end{bmatrix}$ and keep the notations above. Then we know that there is an exact, full and faithfull functor V-mod$\to$ A-mod, given by $Y \mapsto (0,Y,0)$, which takes projectives to projectives, and, also $\mathcal{F}_V(\Delta)$ into $\mathcal{F}_A(\Delta)$.

Theorem 3 *[5] Let $A = \begin{bmatrix} U & 0 \\ M & V \end{bmatrix}$ such that ${}_VM$ has finite projective dimension. Then, if $L = (X,Y,f)$, pd $L < \infty$ implies that ${}_VY$ and ${}_UX$ have both finite projective dimension.*

Proof: : It is always true that if L has finite projective dimension then ${}_UX$ also has. (The resolution of L induces a resolution of X).

We show now that ${}_VY$ also has finite projective dimension.

In fact, if L is A-projective then ${}_VY$ is in add $(M \coprod V)$ so it has finite projective dimension.

Let L be any A-module with finite projective resolution of the form:

$$0 \to P_n \to P_n - 1 \ldots \to P_0 \to L \to 0$$

This induces an exact sequence

$$0 \to Y_n \to Y_n - 1 \ldots \to Y_0 \to Y \to 0$$

where all ${}_VY_i$ are in add $(M \coprod V)$ and, so, have finite projective dimension. It follows that Y has finite projective dimension.

□

Theorem 4 *$\mathcal{F}_A(\Delta) = P^{<\infty}(A)$, if and only if, $\mathcal{F}_V(\Delta) = P^{<\infty}(V)$, $\mathcal{F}_U(\Delta) = P^{<\infty}(U)$ and $M \in P^{<\infty}(V)$.*

Proof: : Assume that $\mathcal{F}_A(\Delta) = P^{<\infty}(A)$. Then ${}_AA$ is standardly stratified and it follows that $\mathcal{F}_V(\Delta) = P^{<\infty}(V)$ and, by proposition 14, ${}_VM$ has finite projective dimension. To see that $\mathcal{F}_U(\Delta) = P^{<\infty}(U)$, let us take a U-module X in $P^{<\infty}(U)$. We show by induction on the projective dimension of X that X is A-good and therefore U-good. The hypothesis implies that U is standardly stratified and therefore, the projectives U-modules are good. Assume now that X has projective dimension equal to n and that all U-modules of projective dimension $n-1$ are U-good. We have an exact sequence:

$$0 \to (\Omega_U(X), M \otimes_U P(X), f) \to (P(X), M \otimes_U P(X), Id) \to (X, 0, 0) \to 0,$$

where $P(X)$ denotes the projective cover of X, and $\Omega_U(X)$ the first syzygy of X, which has projective dimension $n-1$. We also have the following exact sequence:

$$0 \to (0, M \otimes_U P(X), 0) \to (\Omega, M \otimes_U P(X), f) \to (\Omega_U X, 0, 0) \to 0.$$

By induction $(\Omega_U X, 0, 0)$ has finite projective dimension and since $(0, M \otimes_U P(X), 0)$ is projective, it follows that $(\Omega, M \otimes_U P(X), f)$ has finite projective dimension too.

Now using the first exact sequence we conclude that $(X, 0, 0)$ has finite projective dimension and therefore it is A-good. Let us assume now that $\mathcal{F}_V(\Delta) = P^{<\infty}(V)$, $\mathcal{F}_U(\Delta) = P^{<\infty}(U)$ and $M \in \mathcal{F}_V(\Delta) = P^{<\infty}(V)$. Then, by proposition 14, A is standardly stratified. Take any A-module (X, Y, f) of finite projective dimension. Using theorem 3 and the fact that $(0, Y, 0)$ is good, we get that X has finite projective dimension and therefore it is U-good. It follows that (X, Y, f) is A-good.

□

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