DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Volume **22**, Number **3**, November **2008**

Website: http://aimSciences.org

pp. **807–815**

FIBONACCI BIMODAL MAPS

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(Communicated by Sebastian van Strien)

Abstract. We introduce the Fibonacci bimodal maps on the interval and show that their two turning points are both in the same minimal invariant Cantor set. Two of these maps with the same orientation have the same kneading sequences and, among bimodal maps without central returns, they exhibit turning points with the strongest recurrence as possible.

1. **Introduction.** A *bimodal map* is a continuous map f from the interval $[0,1]$ to itself which leaves the set $\{0,1\}$ invariant and has exactly one local maximum and one local minimum in $(0,1)$. If the points of $\{0,1\}$ are fixed we say that the bimodal map is *positive* and in the case that these points are permuted we say that it is *negative*. Each one of the 2 parameters families $P^+_{\alpha\beta}$ and $P^-_{\alpha\beta}$ of real cubic polynomials given by $P^+_{\alpha\beta}(x) = \alpha x^3 + \beta x^2 + (1-\alpha-\beta)x$ and $P^-_{\alpha\beta}(x) = 1-\alpha x^3 - \beta x^2 - (1-\alpha-\beta)x$ contains many bimodal maps. The bimodal maps from $P^+_{\alpha\beta}$ are positive and those from $P^-_{\alpha\beta}$ are negative. We are interested in bimodal maps which have neither wandering intervals nor periodic attractors and exhibit *turning points* (the local maximum and the local minimum) with the *Fibonacci combinatorics* (defined later).

The Fibonacci combinatorics appeared before in the context of unimodal maps related to a question posed by J. Milnor [8] about the classification of the measure theoretical attractors in one dimensional dynamics. Among the quadratic polynomials $Q_\alpha(x) = \alpha x(1-x)$ there is one whose turning point has this combinatorics and, because it implies a strong recurrence of the turning point, it was considered in [3] as a candidate to exhibit a compact invariant minimal Cantor set (called *wild attractor*) whose basin of attraction is a meager subset of $[0,1]$ with full Lebesgue measure. Later it was shown in [6] and [5] (see also [2]) that real quadratic polynomials do not have wild attractors. On the other hand, it was proved in [1] that a real unimodal polynomial with a turning point of order high enough and the Fibonacci combinatorics exhibits such attractor.

Here we deal with combinatorial (topological) aspects of bimodal maps with the Fibonacci combinatorics defined below.

2000 *Mathematics Subject Classification.* Primary: 37E05; Secondary: 37E15.
Key words and phrases. Interval dynamics, bimodal maps, Fibonacci, kneading sequence.
The author is partially supported by CNPq-Brasil, Grant 304517/2005-4.

2. **Some concepts and main results.** Let $\mathcal{F}^+$ and $\mathcal{F}^-$ denote, respectively, the set of positive and the set of negative bimodal maps of $[0,1]$ which have neither wandering intervals nor periodic attractors. If a bimodal map $f \in \mathcal{F}^+ \cup \mathcal{F}^-$ has a fixed point p between its two turning points (say c and d) we define the open intervals $I_1 \ni c$ and $J_1 \ni d$ such that $\partial I_1 \cap \partial J_1 = \{p\}$ and $f(\partial I_1) = f(\partial J_1) = p$. If this fixed point does not exist f is in $\mathcal{F}^-$ and it has only one fixed point $q \in (0,1)$. In this case we define the minimal open intervals, $I_1 \ni c$ and $J_1 \ni d$, possibly coincident, such that $f(\partial I_1) = f(\partial J_1) = q$.

We assume that $c < d$ are recurrent and define

$$I_1 \supset I_2 \supset I_3 \supset \ldots \supset \{c\} \quad \text{and} \quad J_1 \supset J_2 \supset J_3 \supset \ldots \supset \{d\}$$

such that, for $k \geq 1$, the intervals I_{k+1} and J_{k+1} are components of the domain of the first return map ϕ_k to $I_k \cup J_k$. The *critical return times* r_k and s_k are defined by $\phi_k(c) = f^{r_k}(c)$ and $\phi_k(d) = f^{s_k}(d)$. If $\phi_k(c) \in I_{k+1} \cup J_{k+1}$ or $\phi_k(d) \in I_{k+1} \cup J_{k+1}$, for some $k \geq 1$, we say that f has a *central return* . We also say that f has a central return if $f(c) \in I_1 \cup J_1$ or $f(d) \in I_1 \cup J_1$.

We call $f \in \mathcal{F}^+ \cup \mathcal{F}^-$ a *Fibonacci bimodal map* iff the infinite sequences of critical return times r_k and s_k are well defined (that is: c and d are recurrent) and coincide with the Fibonacci sequence $2, 3, 5, \ldots$. Observe that a Fibonacci bimodal map has no central return and there exists the fixed point p between its turning points.

Our main result gives combinatorial (topological) information about the dynamics of Fibonacci bimodal maps $f \in \mathcal{F}^+ \cup \mathcal{F}^-$.

Theorem 2.1. *Both $\mathcal{F}^+$ and $\mathcal{F}^-$ contains Fibonacci bimodal maps and the following properties hold true:*

1. *If $f \in \mathcal{F}^+ \cup \mathcal{F}^-$ is a Fibonacci bimodal map then there is a minimal invariant Cantor set which contains its two turning points.*
2. *Two Fibonacci bimodal maps in $\mathcal{F}^+$ (or in $\mathcal{F}^-$) are topologically conjugate.*
3. *Any bimodal map which has at least one of the sequences of critical return times smaller, with respect to the lexicographic order, than the Fibonacci sequence exhibits central returns.*

According to [7] the families of cubic polynomials $P^+_{\alpha\beta}$ and $P^-_{\alpha\beta}$ are both complete. This together with Theorem 2.1 assure that both of them exhibit Fibonacci bimodal maps. In fact, from a rigidity result in [4], it follows that in each one of these families there is at most one Fibonacci bimodal map.

3. **Combinatorial aspects of the Fibonacci bimodal dynamics.** To prove that $\mathcal{F}^+$ (and $\mathcal{F}^-$) contains Fibonacci bimodal maps and study their dynamics we introduce some additional concepts. Given $f \in \mathcal{F}^+ \cup \mathcal{F}^-$ the components of the domain of the corresponding first return map ϕ_k which contain c and d are called *critical domains*. Moreover, there are components (not necessarily different) C_{k+1} and D_{k+1} of this domain such that $\phi_k(c) \in C_{k+1}$ and $\phi_k(d) \in D_{k+1}$ which are called *post critical domains*. Note that for a map without central returns, which is the case of Fibonacci bimodal maps, the post critical branches $\phi_{k|C_{k+1}}$ and $\phi_{k|D_{k+1}}$ are diffeomorphisms. Following [2], a branch of ϕ_k is called an *immediate branch* iff it is a diffeomorphic branch which is a restriction of a critical branch of ϕ_{k-1}. Here we define $\phi_0 := f$ and make the convention that a restriction of ϕ_0 to I_1 or J_1 is also an immediate branch.

The Fibonacci combinatorics, as we will see below, implies some constraints on the position of the post critical domains C_{k+1}, D_{k+1} and their images. This leads us to consider the 3 types of first return maps ϕ_k below:

- **Type $\mathcal{A}$:** if $C_{k+1} \subset J_k$, $D_{k+1} \subset I_k$, $\phi_k(C_{k+1}) = I_k$ and $\phi_k(D_{k+1}) = J_k$.
- **Type $\mathcal{B}$:** if $C_{k+1} \subset I_k$, $D_{k+1} \subset J_k$, $\phi_k(C_{k+1}) = J_k$ and $\phi_k(D_{k+1}) = I_k$.
- **Type $\mathcal{C}$:** if $C_{k+1} \subset J_k$, $D_{k+1} \subset I_k$, $\phi_k(C_{k+1}) = J_k$ and $\phi_k(D_{k+1}) = I_k$.

Observe that in these 3 types the precise position and orientation of the critical and post critical branches are not specified yet. Figure 1 illustrate a possible position but still without the orientation.

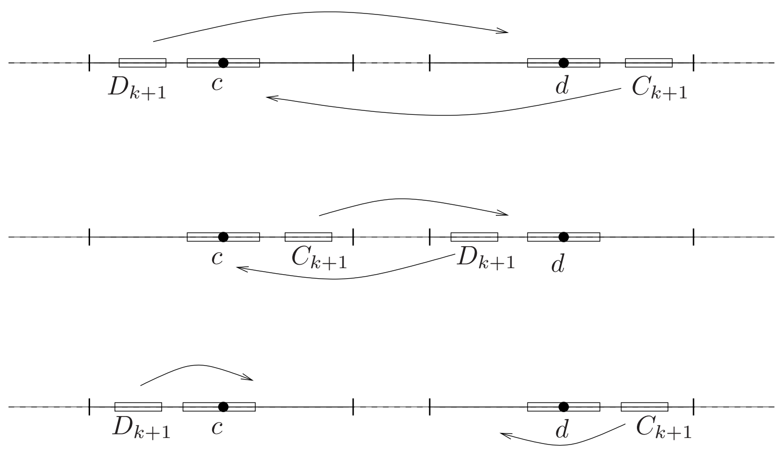


FIGURE 1. Examples of types $\mathcal{A}$ $\mathcal{B}$ $\mathcal{C}$

The Fibonacci combinatorics implies that the sequence $\phi_1,\ \phi_2,\ \phi_3,\ \ldots$ of first return maps exhibits a specific sequence of types as described in the next lemma.

Lemma 3.1. *If $f \in \mathcal{F}^+ \cup \mathcal{F}^-$ is a Fibonacci bimodal map then the following holds for all $k \geq 1$:*

1. *The branches of ϕ_k which are not an immediate branch have return times at least as big as $r_k = s_k$, the critical return times. Moreover, the post critical branches of ϕ_k are immediate branches and $\phi_{k+1}(x) = \phi_{k-1} \circ \phi_k(x)$ for all $x \in I_{k+2} \cup J_{k+2}$.*
2. *The sequence $\phi_1,\ \phi_2,\ \phi_3,\ \ldots$ of first return maps exhibits the sequence $\mathcal{A}\ \mathcal{B}\ \mathcal{C}\ \mathcal{A}\ \ldots$ or $\mathcal{C}\ \mathcal{A}\ \mathcal{B}\ \mathcal{C}\ \ldots$ of types depending respectively on f being positive or negative.*
3. *The image of the critical domain $I_{k+1} \cup J_{k+1}$ by ϕ_k contains itself, that is: $I_{k+1} \cup J_{k+1} \subset \phi_k(I_{k+1} \cup J_{k+1})$.*

Proof. Philosophically, the Fibonacci combinatorics controls the critical return times and this implies that the post critical branches are immediate branches. So these immediate branches must exist in the right place which implies the lemma. To be more precise we take the open intervals $I_1,\ J_1 \subset (0,1)$ as before and by assumption $\{f(c), f(d)\} \cap (I_1 \cup J_1) = \emptyset$. From $r_1 = s_1 = 2$ and the possible orientations of f we have that $f(c) \in J^{-1}$ and $f(d) \in I^{-1}$ where I^{-1} and J^{-1} are, respectively, connected components of $f^{-1}(I_1)$ and $f^{-1}(J_1)$. If f is positive then $I^{-1} \subset (0,c)$ and $J^{-1} \subset (d,1)$. If f is negative then $I^{-1} \subset (d,1)$ and $J^{-1} \subset (0,c)$.

The domain of ϕ_1 has 6 connected components which are the 2 critical domains I_2, J_2 and more 4 intervals which are connected components of $f^{-1}(I_1 \cup J_1)$. See Figure 2 and Figure 3 where we draw only the critical and the post critical branches of ϕ_1.

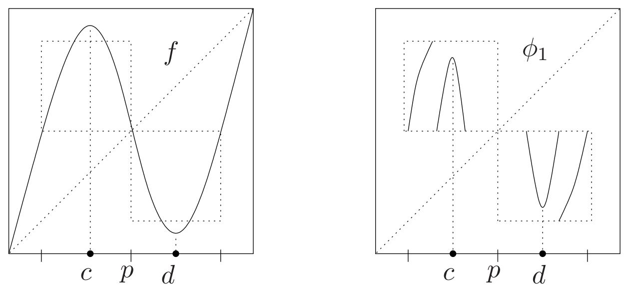


FIGURE 2. graph of $f \in \mathcal{F}^+$ and ϕ_1 of type $\mathcal{A}$

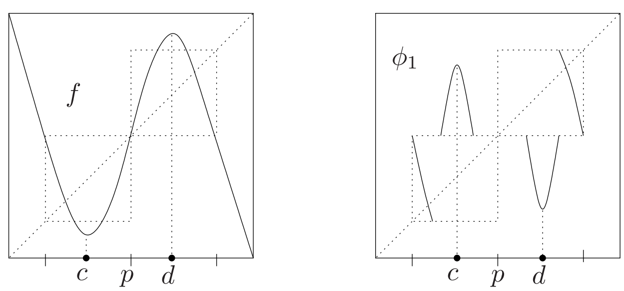


FIGURE 3. graph of $f \in \mathcal{F}^-$ and ϕ_1 of type $\mathcal{C}$

The return times to $I_1 \cup J_1$ of the critical branches in I_2 and J_2 are both the Fibonacci number 2 and the return times of all the others branches are the Fibonacci number 1. This implies that the return times of all the branches of ϕ_2 which are not immediate are at least 3 which is the return time of its critical branches.

We will proceed by induction. First we note that Statement 1 of the lemma for ϕ_1 follows from the hypothesis that $r_2 = s_2 = 3$. Observe that ϕ_1 has type $\mathcal{A}$ or $\mathcal{C}$ depending respectively on f being positive or negative. This is Statement 2 for ϕ_1. Let us prove Statement 3 for ϕ_1: we already know that $C_2 \subset J_1$, $D_2 \subset I_1$ and f has no central returns. If $J_2 \subset \phi_1(I_2)$ does not hold we have that C_2 is on the left of d and $\phi_1(I_2) \subset C_2$. Then $\phi_2(I_3) \subset I_2$ if f is positive and $\phi_2(I_3) \subset J_2$ if f is negative. In both cases the critical return time $r_2 = 3$ but the return times of all the others branches of ϕ_2 inside I_2 are at least 4. In the positive case this implies that $r_3 > 5$ and in the negative case this implies that $s_3 > 5$. This is not compatible with the Fibonacci combinatorics. The same reasoning applies to prove that $I_2 \subset \phi_1(J_2)$ and we are done.

Now we assume that the lemma holds for ϕ_k and prove that it holds for ϕ_{k+1}. The return time of the immediate branches of ϕ_k is $r_{k-1} = s_{k-1}$, the return time of its critical branches is $r_k = s_k$ and the return time of the others branches is at least $r_k = s_k$. Then the post critical branches of ϕ_k are immediate branches and the first part of Statement 1, for ϕ_{k+1}, follows easily. The assumption that $r_{k+2} = s_{k+2} = r_{k+1} + r_k$ implies the second part of this statement. The map ϕ_k has type $\mathcal{A}$, $\mathcal{B}$ or $\mathcal{C}$ and satisfy Statement 3. Then one can see that the type of ϕ_{k+1} is, respectively, $\mathcal{B}$, $\mathcal{C}$ or $\mathcal{A}$ and Statement 2 follows. Now let us prove that Statement 3 holds for ϕ_{k+1}. We assume that ϕ_{k+1} has type $\mathcal{A}$, the same reasoning works for the others types. In this case we have that $\phi_{k+1}(I_{k+2}) \subset J_{k+1}$ and $\phi_{k+1}(J_{k+2}) \subset I_{k+1}$. We remember that the return time of a branch of ϕ_{k+1} which are not an immediate branch is at least $r_{k+1} = s_{k+1}$. Then we claim that $J_{k+2} \subset \phi_{k+1}(I_{k+2})$ and $I_{k+2} \subset \phi_{k+1}(J_{k+2})$. If this first inclusion does not hold we have $r_{k+2} = r_{k+1} + r_k$ and $\phi_{k+2}(I_{k+3}) \subset I_{k+2}$. But the return times of all non-critical branches of ϕ_{k+2} inside I_{k+2} are at least $2r_{k+1}$. This implies that the return time r_{k+3} is at least $r_{k+2} + 2r_{k+1}$ which is not possible with the Fibonacci combinatorics. The same reasoning can show that the second inclusion of our claim holds and the proof is finished. □

Corollary 1. *Both turning points of a Fibonacci bimodal map are in a minimal Cantor set.*

Proof. For $k \geq 1$, the union of compact intervals

$$\Lambda_k := (\cup_{i=1}^{r_k} f^i(\overline{I_{k+1}} \cup \overline{J_{k+1}})) \cup (\cup_{j=1}^{r_{k-1}} f^j(\overline{C_{k+1}} \cup \overline{D_{k+1}}))$$

contains the forward orbit of $\{c, d\}$. Lemma 3.1 and the non-existence of wandering intervals imply that $\cap_{k=1}^{\infty} \Lambda_k$ is a minimal invariant Cantor set which contains $\{c, d\}$ as stated. □

The next lemma assures the existence of positive and negative Fibonacci bimodal maps in $\mathcal{F}^+$ and in $\mathcal{F}^-$.

Lemma 3.2. *The sets $\mathcal{F}^+$ and $\mathcal{F}^-$ contain Fibonacci bimodal maps.*

Proof. We are going to construct just a positive Fibonacci bimodal map f since the negative case follows the same reasoning. This bimodal map f is obtained as the limit of a sequence of smooth positive bimodal maps $f_m : [0, 1] \to [0, 1]$ which have two turning points $c,\ d \in (0, 1)$ and satisfy the following:

- There are nested open intervals

$$I_1 \supset I_2 \supset \ \dots \ \supset I_{m+1} \supset \{c\} \quad \text{and} \quad J_1 \supset J_2 \supset \ \dots \supset J_{m+1} \supset \{d\}$$

  such that, for $1 \leq k \leq m$, the intervals I_{k+1} and J_{k+1} are components of the domain of the first return map ϕ_k to $I_k \cup J_k$ associate to f_m.
- For $1 \leq k \leq m$, the maps f_{k+1} and f_k coincide in $[0, 1] \setminus (I_{k+1} \cup J_{k+1})$.
- The sequence $\phi_1, \dots, \phi_m$ of first return maps of f_m exhibits the sequence $\mathcal{A}\ \mathcal{B}\ \mathcal{C}\ \mathcal{A}\ \dots$ of types.
- For $1 \leq k \leq m$, we have that the first return map ϕ_k of f_m satisfies $I_{k+1} \cup J_{k+1} \subset \phi_k(I_{k+1} \cup J_{k+1})$.
- For $1 \leq k \leq m$, the critical return times $r_k,\ s_k$ are well defined and coincide with the first m Fibonacci numbers, starting with $r_1 = s_1 = 2$.

To construct this sequence f_m we proceed by induction on m starting with a smooth positive bimodal map $f_1 : [0, 1] \to [0, 1]$ with turning points $c,\ d \in (0, 1)$ which has exactly one fixed point p between its turning points c and d. Then we take the open intervals I_1 and J_1 as before and choose f_1 and f_2 so that the five properties above are satisfied for $m = 1$, see Figure 2.

Now we assume that the bimodal maps $f_1, \dots, f_m$ are defined and construct the bimodal map f_{m+1}. We assume that $m = 1 \mod 3$ (the other cases are similar and will be omitted) which implies that ϕ_m has type $\mathcal{A}$. Then there exist $\tilde{C}_{m+1} \subset C_{m+1} \subset J_m$ and $\tilde{D}_{m+1} \subset D_{m+1} \subset I_m$ which are mapped by ϕ_m diffeomorphically onto I_{m+1} and J_{m+1}, respectively. As ϕ_m is a first return map we can modify f_m in $I_{m+1} \cup J_{m+1}$ so that the non-critical branches of ϕ_m stay unchanged and the critical values $\phi_m(c)$ and $\phi_m(d)$ move independently to convenient positions inside $\tilde{C}_{m+1}$ and $\tilde{D}_{m+1}$, respectively. Then it is clear that with this modification we can choose f_{m+1} satisfying our needs.

From the construction we have that $f_{m+j} = f_m$ $(m, j \geq 1)$ outside $I_{m+1} \cup J_{m+1}$. Then taking f_m of class C^∞ we conclude that there exists a map f, of class C^∞ in $[0,1] \setminus \{c, d\}$, such that $f_m \to f$ when $m \to \infty$. In order to conclude that f is a continuous positive bimodal map we just need to verify the continuity of f at c and d. This follows from the fact that f_m can be defined so that $\bigcap_{m=1}^{\infty} I_m = \{c\}$, $\bigcap_{m=0}^{\infty} J_m = \{d\}$ and $f_m(\partial I_m)$ and $f_m(\partial J_m)$ are two monotone and convergent sequences of points. □

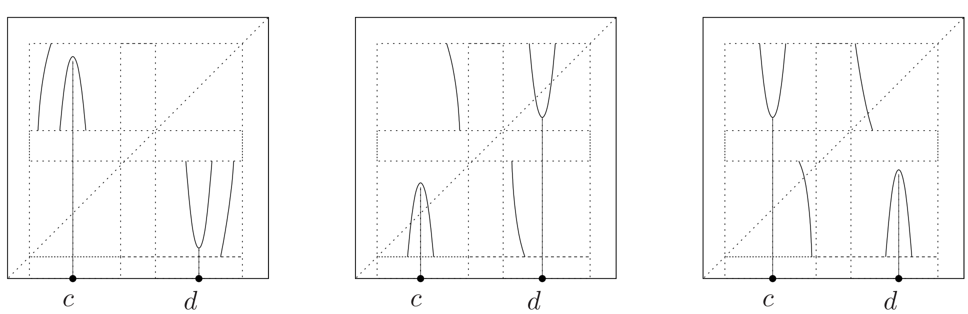
4. **The kneading sequence of Fibonacci bimodal maps.** The Fibonacci combinatorics implies that the sequence $\phi_1, \ \phi_2, \ \phi_3, \ \dots$ of first return maps exhibits a specific sequence of types as in Lemma 3.1. This together with an analysis of the orientation and the precise positions of the branches of these first return maps will determine the kneading sequences and the class of topological conjugacy of a Fibonacci bimodal map. Let us subdivide each type $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ (respectively) in subtypes $\mathcal{A}^{ij}$, $\mathcal{B}^{ij}$ and $\mathcal{C}^{ij}$ with $i, j \in \{+\,,\,-\}$. Here $i = +$ or $i = -$ if the monotone branch near c is increasing or decreasing and $j = +$ or $j = -$ if the critical branch at c has a maximum or a minimum, respectively. With this we have that the sequence of first return maps of a positive Fibonacci bimodal map f exhibits the following sequence of subtypes (see Figure 4):

$$\mathcal{A}^{++}\mathcal{B}^{-+}\mathcal{C}^{--}\mathcal{A}^{-+}\mathcal{B}^{+-}\mathcal{C}^{+-}\mathcal{A}^{+-}\mathcal{B}^{--}\mathcal{C}^{-+}\mathcal{A}^{--}\mathcal{B}^{++}\mathcal{C}^{++}\mathcal{A}^{++} \quad \dots$$

For negative Fibonacci bimodal maps f the sequence of types of first return maps is $\mathcal{C}\ \mathcal{A}\ \mathcal{B}\ \mathcal{C}\ \dots$ as stated in Lemma 3.1. It starts with $\mathcal{C}^{-+}$ and then follows the pattern in the Figure 4. Then, in this case, the sequence of first return maps exhibits the following sequence of subtypes:

$$\mathcal{C}^{-+}\mathcal{A}^{--}\mathcal{B}^{++}\mathcal{C}^{++}\mathcal{A}^{++}\mathcal{B}^{-+}\mathcal{C}^{--}\mathcal{A}^{-+}\mathcal{B}^{+-}\mathcal{C}^{+-}\mathcal{A}^{+-}\mathcal{B}^{--}\mathcal{C}^{-+} \quad \dots$$

The best way to be convinced of these facts is by the sequence of graphs of the first return maps in the Figure 4. Observe that the 2 post critical branches have the same orientation while the critical branches have opposite orientation.



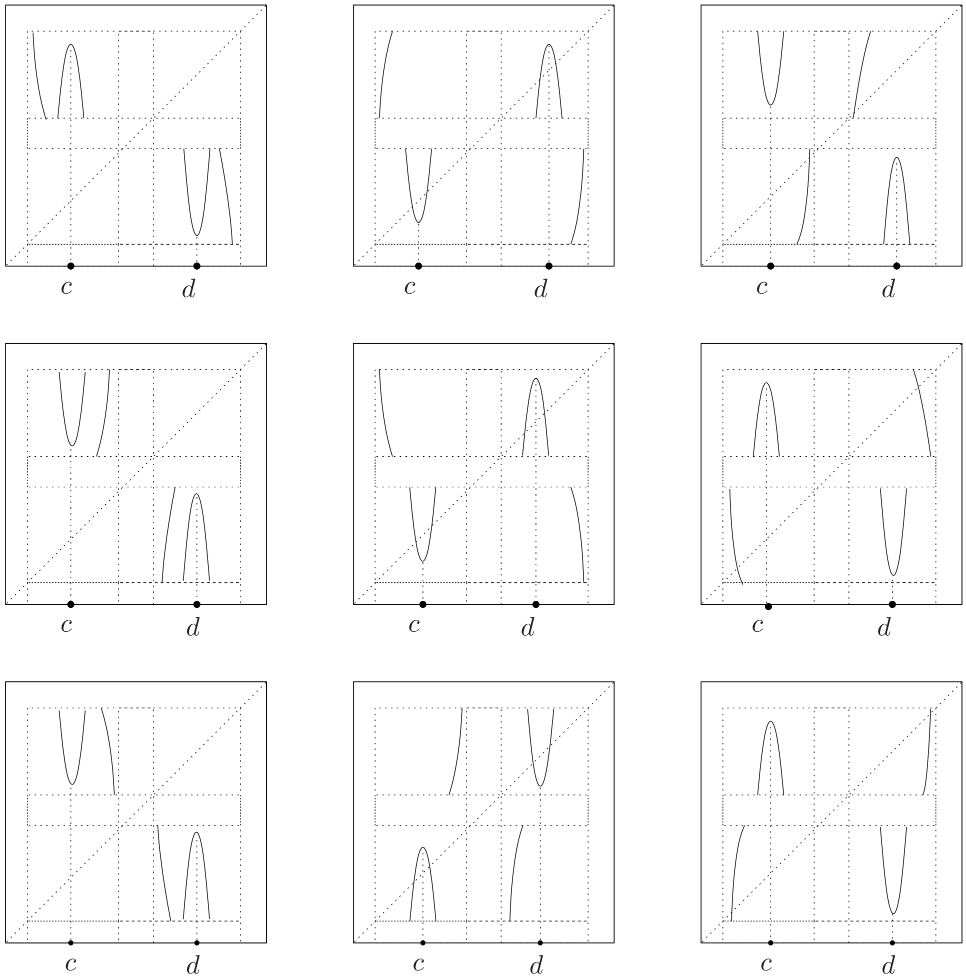


FIGURE 4. Types $\mathcal{A}^{++}$ $\mathcal{B}^{-+}$ $\mathcal{C}^{--}$ $\mathcal{A}^{-+}$ $\mathcal{B}^{+-}$ $\mathcal{C}^{+-}$ $\mathcal{A}^{+-}$ $\mathcal{B}^{--}$ $\mathcal{C}^{-+}$ $\mathcal{A}^{--}$ $\mathcal{B}^{++}$ $\mathcal{C}^{++}$

In one dimensional dynamics the main tool to determine the class of topological conjugacy of a map is the well known kneading theory of Milnor-Thurston [9]. The *kneading sequences* of a bimodal map f are the itineraries $I(f(c))$ and $I(f(d))$, where c and d are the turning points of f. Remember that the *itinerary* $I(x)$ of a point $x \in [0,1]$ is $I(x) = (i_0(x),\ i_1(x),\ i_2(x),\ \cdots)$ with $i_j(x)$ given by:

$$i_j(x) = \begin{cases} c, & \text{if} \quad f^j(x) = c \\ d, & \text{if} \quad f^j(x) = d \\ 1, & \text{if} \quad f^j(x) < c \\ 2, & \text{if} \quad c < f^j(x) < d \\ 3, & \text{if} \quad f^j(x) > d \end{cases}$$

It follows from Figure 4 that the following properties hold:

- **Positions of critical returns.**

$$(i_{r_1}(c),\ \ i_{r_2}(c), \ldots,\ \ i_{r_{12}}(c)) = (3,\ 2,\ 2,\ 3,\ 1,\ 2,\ 2,\ 1,\ 3,\ 2,\ 2,\ 3).$$

- **Invariance of positions of critical returns.** For $m \geq 1$, we have that $i_{r_{m+12}}(c) = i_{r_m}(c)$.
- **Relation between positions of critical returns.** $i_{r_j}(c) = i_{r_j}(d)$ iff both of them are equal to 2. Otherwise one of them is 1 and the other is 3.
- **Position of critical returns between Fibonacci times.** If $C_{k+1} \subset I_k$ or $C_{k+1} \subset J_k$ we have, respectively, that

$$(i_{r_k+1}(c), \dots,\ i_{r_{k+1}-1}(c)) = (i_1(c), \dots,\ i_{r_{k-1}-1}(c))$$

or

$$(i_{r_k+1}(c), \dots,\ i_{r_{k+1}-1}(c)) = (i_1(d), \dots,\ i_{r_{k-1}-1}(d)).$$

These properties together with the sequence of types given by Figure 4 and the fact that in the case of positive maps $i_1(c) = 3$ and $i_1(d) = 1$ and in the case of negative maps $i_1(c) = 1$ and $i_1(d) = 3$ determine the kneading sequences of f, they are the following:

- If f is a positive Fibonacci bimodal map then

$$I(f(c)) = (3,\ \ 3,\ \ 2,\ \ 3,2,\ \ 1,1,3,\ \ 1,1,2,1,1,\ \ 3,3,2,3,2,1,1,2,\ \ \dots)$$

and

$$I(f(d)) = (1,\ \ 1,\ \ 2,\ \ 1,2,\ \ 3,3,1,\ \ 3,3,2,3,3,\ \ 1,1,2,1,2,3,3,2,\ \ \dots)$$

- If f is a negative Fibonacci bimodal map then

$$I(f(c)) = (1,\ \ 3,\ \ 2,\ \ 1,2,\ \ 1,3,3,\ \ 3,1,2,3,3,\ \ 3,1,2,3,2,1,2,2,\ \ \dots)$$

and

$$I(f(d)) = (3,\ \ 1,\ \ 2,\ \ 3,2,\ \ 3,1,1,\ \ 1,3,2,1,1,\ \ 1,3,2,1,2,3,2,2,\ \ \dots)$$

5. **The Fibonacci combinatorics and critical recurrence.** Here we use the sequences of critical return times to measure the recurrence between the turning points of a bimodal map. To be precise we consider the space

$$\Sigma^{\mathbb{N}} = \{\mu = (\mu_1,\ \mu_2,\ \mu_3,\ \dots) : \mu_i \in \mathbb{N},\ \ i \geq 1\}.$$

Endow $\Sigma^{\mathbb{N}}$ with the *lexicographic order* $\prec$ defined as follows: given 2 different sequences $\mu = (\mu_1,\ \mu_2,\ \mu_3,\ \dots)$ and $\nu = (\nu_1,\ \nu_2,\ \nu_3,\ \dots)$ we say that μ is smaller than ν and write $\mu \prec \nu$ iff $\mu_k < \nu_k$ for the smallest $k \geq 1$ such that $\mu_k \neq \nu_k$. We also use the notation $\mu \preceq \nu$ which means that μ is smaller or equal to ν.

Given a bimodal map f with two recurrent turning points c and d we consider the sequences $r = (r_1,\ r_2,\ r_3,\ \dots)$ and $s = (s_1,\ s_2,\ s_3,\ \dots)$, where r_k and s_k are the critical return times defined in Section 2. According to the next lemma, if r or s is smaller than the Fibonacci sequence $\mathrm{Fib} = (2,\ 3,\ 5,\ \dots)$ then f has central returns.

Lemma 5.1. *If one of the sequences of critical return times of a bimodal map f is smaller (with respect to the lexicographic order) than the Fibonacci sequence Fib then f exhibits central returns.*

Proof. Given a bimodal map f with two recurrent turning points c and d we consider its sequences of critical return times $r = (r_1,\ r_2,\ r_3,\ \dots)$ and $s = (s_1,\ s_2,\ s_3,\ \dots)$. We assume that f has no central return and prove by induction that $\mathrm{Fib} \preceq r$ and $\mathrm{Fib} \preceq s$. Then $f(c)$ and $f(d)$ are not in $I_1 \cup J_1$, the smallest possible values for the critical returns times r_1 and s_1 is 2 and they are realized in the case that

$\phi_1(c) \in (f^{-1}(J_1)) \setminus (I_1 \cup J_1)$ and $\phi_1(d) \in f^{-1}(I_1) \setminus (I_1 \cup J_1)$. Moreover, the return times of the branches of ϕ_1 which are not a critical branch is 1. Now we assume that the the critical returns times $r_1, \ldots, r_k$ and $s_1, \ldots, s_k$ coincide with the corresponding Fibonacci numbers starting with $r_1 = s_1 = 2$. So, as in the proof of Lemma 3.1, we know that the return times of the branches of ϕ_k which are not an immediate branch are at least $r_k = s_k$. In this case, if f has no central returns as we are assuming, the next critical return times r_{k+1} and s_{k+1} are at least as big as the corresponding Fibonacci number. □

Proof of Theorem 2.1. Lemma 3.2, Corollary 1, Lemma 5.1 together with the construction of the kneading sequences in the previous section imply this theorem. □

Acknowledgement. The author would like to thank the referees for their comments.

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Received August 2007; revised March 2008.

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