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Apoio:



## STRONG ALGEBRABILITY ON CERTAIN SET OF ANALYTIC FUNCTIONS

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### Abstract

We show that the set of analytic functions from  $\mathbb{C}^2$  into  $\mathbb{C}^2$ , which are not Lorch-analytic is spaceable and strongly  $\mathfrak{c}$ -algebrable, but is not residual in the space of entire functions from  $\mathbb{C}^2$  into  $\mathbb{C}^2$ .

### 1 Introduction

In the last two decades there has been a crescent interest in the search of nice algebraic-topological structures within sets (mainly sets of functions or sequences) that do not enjoy themselves such structures. Here, we study algebraic structures in certain set of analytic functions. Now we fix the notation. The space of all analytic functions from  $\mathbb{C}^2$  into  $\mathbb{C}^2$  will be denoted by  $\mathcal{H}(\mathbb{C}^2, \mathbb{C}^2)$ . Consider  $\mathbb{C}^2$  as a Banach algebra with the usual product and the  $l_\infty^2$ -norm. We denote the set of all (L)-analytic functions from  $\mathbb{C}^2$  into  $\mathbb{C}^2$  by  $\mathcal{H}_L(\mathbb{C}^2, \mathbb{C}^2)$ . This class of functions was introduced by E. R. Lorch in [3] and has been investigated in [3, 4].

We call by  $\mathcal{G} = \mathcal{H}(\mathbb{C}^2, \mathbb{C}^2) \setminus \mathcal{H}_L(\mathbb{C}^2, \mathbb{C}^2)$ . As we see, there has been different ways to define and also to understand analytic functions. In our work we are interested to see, in a linear/algebraic sense, if these differences are big or not. In this direction, our aim in this note is to establish some structure in the set  $\mathcal{G}$ . Indeed, we show that  $\mathcal{G}$  is spaceable and strongly  $\mathfrak{c}$ -algebrable, but is not topologically large. Research on the theme of describing spaceability, algebrability and residuality has been carried on in recent years. We refer to [1, 2] for a background about these concepts and a good history of the publication on the theme.

### 2 Main Results

E. R. Lorch in [3] introduced a definition of analytic functions (see Definition 2.1), that have for their domains and ranges a complex commutative Banach algebra with identity.

**Definition 2.1.** *Let  $E$  be a commutative Banach algebra over  $\mathbb{C}$  with identity. A mapping  $f : E \rightarrow E$  is **Lorch-analytic** (or **(L)-analytic**) in  $\omega \in E$  if there exists  $\zeta \in E$  such that  $\lim_{h \rightarrow 0} \frac{\|f(\omega + h) - f(\omega) - \zeta \cdot h\|}{\|h\|} = 0$ . We say that  $f$  is (L)-analytic in  $E$  if  $f$  is (L)-analytic in every point of  $E$ .*

It is known that a (L)-analytic function is differentiable in the Fréchet sense and hence holomorphic. However, not every Fréchet-differentiable function on a commutative Banach algebra with identity is analytic in the Lorch sense. Let us give an example. Let  $F : \mathbb{C}^2 \rightarrow \mathbb{C}^2$  be given by  $F(z, w) = (w, z)$ , so  $F$  is analytic but it is not (L)-analytic. Thus the set  $\mathcal{G} = \mathcal{H}(\mathbb{C}^2, \mathbb{C}^2) \setminus \mathcal{H}_L(\mathbb{C}^2, \mathbb{C}^2)$  is not empty and  $\mathcal{G}$  is not a vector space. Then it seems natural to study some algebraic structure inside  $\mathcal{G}$ . In this note we consider  $E = \mathbb{C}^2$  with the usual product and the sup norm.

A function  $\varphi : \mathbb{C}^2 \rightarrow \mathbb{C}^2$  defined by  $\varphi(z, w) = \left( \sum_{j=1}^m a_j e^{b_j z}, \sum_{k=1}^n c_k e^{d_k w} \right)$ , for all  $(z, w) \in \mathbb{C}^2$ ,  $a_j, b_j, c_k, d_k \in \mathbb{C}$ ,  $j = 1, \dots, m$  and  $k = 1, \dots, n$ , such that  $a_j$ 's and  $c_k$ 's are not all zero, and  $b_j$ 's are distinct and  $d_k$ 's are distinct, is called a **two-variable exponential like function**. We will denote by  $\mathcal{E}(\mathbb{C}^2, \mathbb{C}^2)$  the set of all two-variable exponential like functions  $\varphi : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ .

Using the function  $F : \mathbb{C}^2 \rightarrow \mathbb{C}^2$  given in the example above and the functions in  $\mathcal{E}(\mathbb{C}^2, \mathbb{C}^2)$  we present, in the next proposition, a family of functions which belong to  $\mathcal{G}$ , and that will be useful for our results.

**Proposition 2.1.** 1. For each  $\varphi \in \mathcal{E}(\mathbb{C}^2, \mathbb{C}^2)$ , then  $\varphi \circ F \in \mathcal{G}$ .

2. For each  $\alpha > 0$  consider  $f_\alpha : \mathbb{C}^2 \rightarrow \mathbb{C}^2$  given by  $f_\alpha(z, w) = (e^{\alpha w}, e^{\alpha z})$ . Then  $\{f_\alpha\}$  is a linearly independent set in  $\mathcal{H}(\mathbb{C}^2, \mathbb{C}^2)$  and  $\{f_\alpha : \alpha > 0\} \subset \mathcal{G} \cup \{0\}$ .

We remark that as a consequence of Proposition 2.1(1), we have that  $\mathcal{G}$  is maximal lineable.

In [1, Theorem 7.4.1] the authors showed a general theorem, which allowed us to prove the next result.

**Proposition 2.2.**  $\mathcal{G}$  is spaceable.

Naturally, if  $\mathcal{G}$  is spaceable then it implies  $\mathcal{G}$  is lineable. Since  $\mathcal{H}(\mathbb{C}^2, \mathbb{C}^2)$  is a separable Fréchet space, and  $\mathcal{H}_L(\mathbb{C}^2, \mathbb{C}^2)$  is a vector subspace of  $\mathcal{H}(\mathbb{C}^2, \mathbb{C}^2)$ , by [1, Theorem 7.3.3] we have that  $\mathcal{G}$  is dense-lineable.

In [1, Theorem 7.5.1], the authors give a criterion for strong algebraability and as a consequence of this result we have the following.

**Proposition 2.3.**  $\mathcal{G}$  is strongly  $\mathfrak{c}$ -algebraable.

We will finish this note with comments on the non-residualness of the set  $\mathcal{G}$  in  $\mathcal{H}(\mathbb{C}^2, \mathbb{C}^2)$ . As  $\mathcal{H}_L(\mathbb{C}^2, \mathbb{C}^2)$  is of the second category, then the set  $\mathcal{G}$  is not residual in  $\mathcal{H}(\mathbb{C}^2, \mathbb{C}^2)$ . Indeed,  $\mathcal{H}(\mathbb{C}^2, \mathbb{C}^2) \setminus \mathcal{G} = \mathcal{H}_L(\mathbb{C}^2, \mathbb{C}^2)$ , and  $\mathcal{H}_L(\mathbb{C}^2, \mathbb{C}^2)$  is Fréchet space. So  $\mathcal{G}$  is not topologically big.

## References

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