GLOBAL SOLVABILITY AND COHOMOLOGY OF TUBE STRUCTURES ON COMPACT MANIFOLDS

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Abstract. We introduce new techniques to study the differential complexes associated to tube structures on $M \times \mathbb{T}^m$ of corank m, in which M is a compact manifold and $\mathbb{T}^m$ is the m-torus. By systematically employing partial Fourier series, for complex tube structures, we completely characterize global solvability, in a given degree, in terms of a weak form of hypoellipticity, thus generalizing existing results and providing a broad answer to an open problem proposed by Hounie and Zugliani (2017). We also obtain new results on the finiteness of the cohomology spaces in intermediate degrees. In the case of real tube structures, we extend an isomorphism for the cohomology spaces originally obtained by Dattori da Silva and Meziani (2016) in the case $M = \mathbb{T}^n$. Moreover, we establish necessary and sufficient conditions for the differential operator to have closed range in the first degree.

1. Introduction

We investigate the global solvability and the cohomology spaces of differential complexes associated with certain systems of first-order differential operators on compact manifolds. More precisely, let M be a smooth, compact, connected and oriented n-dimensional manifold, and let $\omega_1, \dots, \omega_m$ be smooth, complex, closed 1-forms on M, and let $\mathbb{T}^m \doteq \mathbb{R}^m / 2\pi\mathbb{Z}^m$. On $\Omega \doteq M \times \mathbb{T}^m$, we consider the involutive subbundle $\mathcal{V} \subset \mathbb{C}T\Omega$ whose sections are annihilated by

$$\zeta_k \doteq \mathrm{d}x_k - \omega_k, \quad k \in \{1, \dots, m\}, \tag{1.1}$$

in which $x = (x_1, \dots, x_m)$ denote the usual coordinates on $\mathbb{T}^m$. Such $\mathcal{V}$ gives rise to a complex of vector bundles and first-order differential operators over Ω [10, 5], here constructed as follows: for each $q \in \{0, \dots, n\}$, let Λ^q denote the bundle of q-forms over M and $\underline{\Lambda}^q$ its pullback via the projection $\Omega \to M$. The smooth sections of the latter are locally written as

$$f = \sum_{|J|=q}{}' f_J(t, x) \mathrm{d}t_J, \tag{1.2}$$

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in which $f_J \in \mathscr{C}^\infty(U \times \mathbb{T}^m)$ and $(U; t_1, \dots, t_n)$ is some local chart of M. Denote by d_t the exterior derivative in M and define a differential operator

$$\mathrm{d}' \doteq \mathrm{d}_t + \sum_{k=1}^{m} \omega_k \wedge \frac{\partial}{\partial x_k} : \mathscr{C}^\infty(\Omega; \underline{\Lambda}^q) \longrightarrow \mathscr{C}^\infty(\Omega; \underline{\Lambda}^{q+1}), \tag{1.3}$$

that satisfies $\mathrm{d}' \circ \mathrm{d}' = 0$. Our goal is to investigate global solvability – that is, *closedness of the range* – of (1.3) in any degree $q \in \{0, \dots, n-1\}$, and to provide a better understanding of the smooth cohomology spaces

$$H^q_{\mathrm{d}'}(\mathscr{C}^\infty(\Omega)) \doteq \frac{\ker\{\mathrm{d}' : \mathscr{C}^\infty(\Omega; \underline{\Lambda}^q) \longrightarrow \mathscr{C}^\infty(\Omega; \underline{\Lambda}^{q+1})\}}{\mathrm{ran}\{\mathrm{d}' : \mathscr{C}^\infty(\Omega; \underline{\Lambda}^{q-1}) \longrightarrow \mathscr{C}^\infty(\Omega; \underline{\Lambda}^{q})\}}, \tag{1.4}$$

for $q \in \{0, \dots, n\}$ (with standard conventions regarding the endpoints[1]).

Our motivation to work in this setting is an attempt to encompass the model in [3, 4, 8, 9] where the global solvability of d' in $\Omega = M \times \mathbb{T}^1$ is studied and also the model in [6] where the authors were able to find isomorphisms for the cohomology spaces when $\Omega = \mathbb{T}^n \times \mathbb{T}^m$. Below, we summarize what we understand to be three of the main themes that keep intertwining throughout the paper, and their major consequences.

1.1. The relationship between solvability and regularity.

Our first result is an equivalence between global solvability of d' and a weak notion of hypoellipticity (Theorem 4.4). This is a generalization of [9, Corollary 7.2] that holds in arbitrary degree, for any smooth, complex tube structure of arbitrary corank, and thus provides a broad answer to the Open Problem 2 in [8]. All the steps of this implication are explained in Remark 4.3, Corollary 4.5 and Remark 4.6.

The main statement can be summarized as follows:

Theorem 1.1. *For each $q \in \{0, \dots, n-1\}$, the following are equivalent:*

1. $\mathrm{d}' : \mathscr{C}^\infty(\Omega; \underline{\Lambda}^q) \to \mathscr{C}^\infty(\Omega; \underline{\Lambda}^{q+1})$ *has closed range.*
2. *For every $u \in \mathscr{D}'(\Omega; \underline{\Lambda}^q)$ such that $\mathrm{d}'u \in \mathscr{C}^\infty(\Omega; \underline{\Lambda}^{q+1})$ there exists $v \in \mathscr{C}^\infty(\Omega; \underline{\Lambda}^q)$ such that $\mathrm{d}'v = \mathrm{d}'u$.*

The second property above is what we will call *almost global hypoellipticity* (AGH).

1.2. Isomorphism formulas for the cohomology spaces.

Concerning the cohomology spaces for *real* tube structures, we prove certain isomorphisms (5.10), similar to the main result in [6]. In that work, M is the n-torus, and their result roughly states that, under suitable conditions, we have

$$H^q_{\mathrm{d}'}(\mathscr{C}^\infty(\Omega)) \cong \mathscr{C}^\infty(\mathbb{T}^r) \otimes H^q_{\mathrm{dR}}(M), \tag{1.5}$$

in which r is the rank of a group associated to $\omega_1, \dots, \omega_m$. For a general M, however, our approach (see Section 5.1) generalizes and provides a better understanding of such result by decomposing the action of our operator in convenient subspaces. While the complete statement of our results requires a bit of notation, below we will provide rough versions of them for the convenience of the reader.

[1] All differential complexes presented in this paper are "completed as zero" in negative degrees.

Assume that $\omega_1, \dots, \omega_m$ are closed and real-valued 1-forms on M. We define the following subgroup[2] of $\mathbb{Z}^m$:

$$\Gamma_{\boldsymbol{\omega}} \doteq \Big\{ \xi \in \mathbb{Z}^m \ ; \ \xi \cdot \boldsymbol{\omega} \doteq \sum_{k=1}^{m} \xi_k \omega_k \text{ is an integral 1-form} \Big\}.$$

We interpret the $\xi \in \mathbb{Z}^m$ as frequencies for the partial Fourier series with respect to $x \in \mathbb{T}^m$. Roughly speaking, by applying Fourier series to cohomology classes in $H^q_{\mathrm{d}'}(\mathscr{C}^\infty(\Omega))$ and splitting them into frequencies belonging to $\Gamma_{\boldsymbol{\omega}}$ and to $\mathbb{Z}^m \setminus \Gamma_{\boldsymbol{\omega}}$, one is naturally led to the direct sum decomposition

$$H^q_{\mathrm{d}'}(\mathscr{C}^\infty(\Omega)) \cong H^q_{\mathrm{d}'}(\mathscr{C}^\infty_{\Gamma_{\boldsymbol{\omega}}}(\Omega)) \oplus H^q_{\mathrm{d}'}(\mathscr{C}^\infty_{\mathbb{Z}^m \setminus \Gamma_{\boldsymbol{\omega}}}(\Omega)).$$

(the definition of the summands can be found in Section 5.1). Then, as a result of a series of arguments in Section 5.1, we obtain the following representation of the first factor above (which, again, assume no hypotheses on the given tube structure):

$$H^q_{\mathrm{d}'}(\mathscr{C}^\infty_{\Gamma_{\boldsymbol{\omega}}}(\Omega)) \cong \mathscr{C}^\infty(\mathbb{T}^r) \otimes H^q_{\mathrm{dR}}(M),$$

where r is the rank of $\Gamma_{\boldsymbol{\omega}}$ as a group. In order to reach (1.5), we are necessarily lead to the problem of determining conditions for the vanishing of the second factor $H^q_{\mathrm{d}'}(\mathscr{C}^\infty_{\mathbb{Z}^m \setminus \Gamma_{\boldsymbol{\omega}}}(\Omega))$: this is done in Theorem 5.7, where, besides the obvious requirement of closedness of the range of d', we find an unexpected obstruction (5.11) for the validity of (1.5). Therefore, the main result in [6] does not extend, in general, when M is not an n-torus.

Indeed, for surfaces, this obstruction – which is related both to the nature of the tube structure and the topology of M – is not present in any degree when M is either a 2-torus or a 2-sphere, but is present at $q = 1$ when M has genus $\mathsf{g} \geq 2$. We show in Section 7.2 how to construct tube structures that satisfy all the hypotheses required by [6], but that do not satisfy (1.5). We conjecture that (5.11) holds in any degree when M is a compact Lie group and the tube structure is real, thus providing a complete generalization of the results in [6]. We plan to investigate this topic further in a forthcoming work.

1.3. **Characterization of solvability in the first degree.** Still assuming our 1-forms $\omega_1, \dots \omega_m$ to be real, in Section 5.2 we provide a complete characterization of global solvability for the first degree of our complexes in terms of Diophantine conditions à la [3, 4]. Here, this is our main result, a restatement of Theorem 5.12:

Theorem 1.2. *Assume $\omega_1, \dots, \omega_m$ real-valued. The following are equivalent:*

(1) $\mathrm{d}' : \mathscr{C}^\infty(\Omega) \to \mathscr{C}^\infty(\Omega; \underline{\Lambda}^1)$ *has closed range.*
(2) *Given $\{\beta_\nu\}_{\nu \in \mathbb{N}} \subset \mathscr{C}^\infty(M; \Lambda^1)$ a sequence of closed integral 1-forms and $\{\xi_\nu\}_{\nu \in \mathbb{N}} \subset \mathbb{Z}^m \setminus \Gamma_{\boldsymbol{\omega}}$ such that $|\xi_\nu| \to \infty$, we have that $\{|\xi_\nu|^\nu (\xi_\nu \cdot \boldsymbol{\omega} - \beta_\nu)\}_{\nu \in \mathbb{N}}$ is unbounded in $\mathscr{C}^\infty(M; \Lambda^1)$.*

The second condition above is called the property of *weak non-simultaneous approximability* for the collection $(\omega_1, \dots, \omega_m)$, which is further discussed in Section 5.2.

The systematic use of the key Lemma 4.1 and its implications not only provides a better understanding of the different notions of global solvability present in previous works (that use the so-called compatibility conditions, see Remark 4.3) but also

[2]A word of caution: a different, more general definition of $\Gamma_{\boldsymbol{\omega}}$ will be given later on; one that encompasses the case when $\omega_1, \dots, \omega_m$ are complex-valued (4.3).

results into relatively short and straightforward proofs without the need of usual techniques, such as dualizing with the top degree, or the use of *a priori* inequalities. Indeed, our method automatically supply a formal solution (Definition 4.7), obtained by solving a system of simpler differential equations (with the help of compatibility conditions) that appears after performing the partial Fourier transform. The question then becomes when the formal solution is a true solution, and here the Diophantine condition plays a role.

1.4. **Final results: finiteness theorems and results on surfaces.** In Sections 6 and 7, we revisit complex tube structures, and derive a handful of necessary conditions for finiteness of the cohomology spaces. The results we proved led us to conjecture whether $H^q_{\mathrm{d}'}(\mathscr{C}^\infty(\Omega))$ is finite dimensional only if it is isomorphic with $H^q_{\mathrm{dR}}(M)$ – true, notably, when (5.11) holds and $r = 0$ (Theorem 7.5), as well as in every other situation in which we were able to compute. For example, when M is a surface, we obtain a quite complete description (Section 7.1).

Convention. Except where explicitly stated, the 1-forms $\omega_1, \ldots, \omega_m$ are assumed to be complex-valued.

2. Preliminaries

2.1. **Global solvability in abstract complexes and related concepts.** Let X be a smooth, compact, connected and oriented manifold, and $\mathbb{E}$ be a complex vector bundle over X. The space $\mathscr{C}^\infty(X;\mathbb{E})$ of smooth sections of $\mathbb{E}$ carries its standard Fréchet topology; by endowing X with a Riemannian metric and $\mathbb{E}$ with a Hermitian metric, one may write it as the projective limit of a suitable sequence of Sobolev spaces of sections of $\mathbb{E}$. We also let $\mathscr{D}'(X;\mathbb{E})$ be the space of distribution sections of $\mathbb{E}$, which will be identified with the topological dual of $\mathscr{C}^\infty(X;\mathbb{E}^*)$.

Let $\mathbb{E}, \mathbb{F}$ be vector bundles over X and P a differential operator from $\mathbb{E}$ to $\mathbb{F}$.

Definition 2.1. We say that P is *almost globally hypoelliptic* (AGH) if

$$\forall u \in \mathscr{D}'(X;\mathbb{E}),\ Pu \in \mathscr{C}^\infty(X;\mathbb{F}) \Longrightarrow \exists v \in \mathscr{C}^\infty(X;\mathbb{E}) \text{ such that } Pv = Pu.$$

We have:

Theorem 2.2. *If P is* (AGH)*, then $P : \mathscr{C}^\infty(X;\mathbb{E}) \to \mathscr{C}^\infty(X;\mathbb{F})$ has closed range.*

This result is proved in [2, Theorem 3.5] for scalar operators; its proof extends to vector-valued operators in a straightforward way, hence we omit it. A converse fails to hold even for very simple operators, but is valid for many classes of operators [1, 2], including (1.3), as we will prove in Theorem 4.4 below.

Let $\mathbb{G}$ be a third vector bundle over X and Q a differential operator from $\mathbb{F}$ to $\mathbb{G}$ such that $Q \circ P = 0$. We define two cohomology spaces

$$H_{P,Q}(\mathscr{F}(X)) \doteq \frac{\ker\{Q : \mathscr{F}(X;\mathbb{F}) \longrightarrow \mathscr{F}(X;\mathbb{G})\}}{\operatorname{ran}\{P : \mathscr{F}(X;\mathbb{E}) \longrightarrow \mathscr{F}(X;\mathbb{F})\}}, \quad \mathscr{F} = \mathscr{C}^\infty \text{ or } \mathscr{D}'.$$

Our goal is to understand these two cohomology spaces separately, as well as their relationship. The inclusions $\mathscr{C}^\infty(X;*) \hookrightarrow \mathscr{D}'(X;*)$ $(* = \mathbb{E}, \mathbb{F}, \mathbb{G})$ induce a linear morphism

$$H_{P,Q}(\mathscr{C}^\infty(X)) \longrightarrow H_{P,Q}(\mathscr{D}'(X)) \tag{2.1}$$

which may be neither injective nor surjective. However,

$$P \text{ is (AGH)} \iff (2.1) \text{ is injective},$$

being therefore independent of Q, and a property of P alone. One should recall the traditional notion of global hypoellipticity for complexes, that is:

$$\forall f \in \mathscr{D}'(X;\mathbb{F}),\ Qf \in \mathscr{C}^\infty(X;\mathbb{G}) \Longrightarrow \exists u \in \mathscr{D}'(X;\mathbb{E}) \text{ such that } f - Pu \in \mathscr{C}^\infty(X;\mathbb{F}). \tag{2.2}$$

Since $Q(f - Pu) = Qf$, (2.2) implies that Q is (AGH). Additionally, we have that

$$\forall f \in \mathscr{D}'(X;\mathbb{F}),\ Qf = 0 \Longrightarrow \exists u \in \mathscr{D}'(X;\mathbb{E}) \text{ such that } f - Pu \in \mathscr{C}^\infty(X;\mathbb{F}),$$

which is equivalent to the surjectivity of (2.1). We state this result more precisely:

Proposition 2.3. *Property* (2.2) *holds if and only if Q is* (AGH) *and* (2.1) *is onto.*

The transpose ${}^{\mathrm{t}}P$ is a differential operator from $\mathbb{F}^*$ to $\mathbb{E}^*$, yielding new maps

$${}^{\mathrm{t}}P : \left\{ \begin{array}{ccc} \mathscr{D}'(X;\mathbb{F}^*) & \longrightarrow & \mathscr{D}'(X;\mathbb{E}^*) \\ \mathscr{C}^\infty(X;\mathbb{F}^*) & \longrightarrow & \mathscr{C}^\infty(X;\mathbb{E}^*) \end{array} \right. ,$$

the latter being the restriction of the former. In the presence of a second operator Q satisfying $Q \circ P = 0$, we want to determine necessary conditions on $f \in \mathscr{C}^\infty(X;\mathbb{F})$ so as to be able to solve $Pu = f$ with $u \in \mathscr{C}^\infty(X;\mathbb{E})$. Obviously, we must have

$$f \in \ker\{Q : \mathscr{C}^\infty(X;\mathbb{F}) \longrightarrow \mathscr{C}^\infty(X;\mathbb{G})\}. \tag{2.3}$$

Moreover, if $v \in \mathscr{D}'(X;\mathbb{F}^*)$ is such that ${}^{\mathrm{t}}Pv = 0$, then

$$\langle v, f \rangle = \langle v, Pu \rangle = \langle {}^{\mathrm{t}}Pv, u \rangle = 0,$$

that is,

$$f \in \ker\{{}^{\mathrm{t}}P : \mathscr{D}'(X;\mathbb{F}^*) \longrightarrow \mathscr{D}'(X;\mathbb{E}^*)\}^{\mathrm{o}}. \tag{2.4}$$

However, by Functional Analysis, the annihilator in (2.4) equals the closure of $\operatorname{ran}\{P : \mathscr{C}^\infty(X;\mathbb{E}) \to \mathscr{C}^\infty(X;\mathbb{F})\}$, and since $Q \circ P = 0$, the range of P is contained in $\ker\{Q : \mathscr{C}^\infty(X;\mathbb{F}) \to \mathscr{C}^\infty(X;\mathbb{G})\}$, which is closed in $\mathscr{C}^\infty(X;\mathbb{F})$. We conclude:

$$\ker\{{}^{\mathrm{t}}P : \mathscr{D}'(X;\mathbb{F}^*) \longrightarrow \mathscr{D}'(X;\mathbb{E}^*)\}^{\mathrm{o}} \subset \ker\{Q : \mathscr{C}^\infty(X;\mathbb{F}) \longrightarrow \mathscr{C}^\infty(X;\mathbb{G})\},$$

hence the compatibility condition (2.3) is redundant in light of (2.4). We therefore introduce the following definition in spite of the presence of the operator Q.

Definition 2.4. We say that P is *globally solvable* if for every $f \in \mathscr{C}^\infty(X;\mathbb{F})$ satisfying

$$\langle v, f \rangle = 0 \quad \text{for every } v \in \mathscr{D}'(X;\mathbb{F}^*) \text{ such that } {}^{\mathrm{t}}Pv = 0$$

there exists $u \in \mathscr{C}^\infty(X;\mathbb{E})$ such that $Pu = f$.

In other words, P is globally solvable if and only if the range of P is closed.

2.2. **Partial Fourier series for sections of $\underline{\Lambda}^q$.** Given $f \in \mathscr{C}^\infty(\Omega;\underline{\Lambda}^q)$, for each $\xi \in \mathbb{Z}^m$, we define an element $\hat{f}_\xi \in \mathscr{C}^\infty(M;\Lambda^q)$ as follows: if $U \subset M$ is a coordinate open set in which f is written as (1.2) then

$$\hat{f}_\xi(t) \doteq \sum_{|J|=q}{}' \hat{f}_J(t,\xi)\mathrm{d}t_J = \sum_{|J|=q}{}' \left(\int_{\mathbb{T}^m} e^{-ix\xi} f_J(t,x)\mathrm{d}x \right) \mathrm{d}t_J, \quad t \in U.$$

Note that since $\underline{\Lambda}_{(t,x)} = \Lambda_t$ for every $x \in \mathbb{T}^m$, each $\mathrm{d}t_J$ can be thought as a section of either Λ^q or $\underline{\Lambda}^q$. One can check that the construction above is independent of the choice of coordinates on U, hence defines well a differential form of degree q in M.

It is useful to regard $\hat{f}_\xi$ as a current on M, that is, we define $\hat{f}_\xi : \mathscr{C}^\infty(M;\Lambda^{n-q}) \to \mathbb{C}$ by

$$g \mapsto \langle \hat{f}_\xi, g\rangle_M \doteq \langle f, e^{-ix\xi} \wedge g \wedge \mathrm{d}x\rangle = \int_\Omega f \wedge e^{-ix\xi} \wedge g \wedge \mathrm{d}x, \tag{2.5}$$

with $\mathrm{d}x \doteq \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_m$; both definitions yield the same object. This extends our construction to q-currents, yielding linear maps

$$\mathcal{F}_\xi : \begin{cases} \mathscr{C}^\infty(\Omega;\underline{\Lambda}^q) & \longrightarrow & \mathscr{C}^\infty(M;\Lambda^q) \\ \mathscr{D}'(\Omega;\underline{\Lambda}^q) & \longrightarrow & \mathscr{D}'(M;\Lambda^q) \end{cases}$$

defined by the assignment $f \mapsto \hat{f}_\xi$.

Proposition 2.5. *The map $\mathcal{F}_\xi : \mathscr{C}^\infty(\Omega;\underline{\Lambda}^q) \to \mathscr{C}^\infty(M;\Lambda^q)$ is continuous.*

Going in the other direction we define linear maps

$$\mathcal{E}_\xi : \begin{cases} \mathscr{C}^\infty(M;\Lambda^q) & \longrightarrow & \mathscr{C}^\infty(\Omega;\underline{\Lambda}^q) \\ \mathscr{D}'(M;\Lambda^q) & \longrightarrow & \mathscr{D}'(\Omega;\underline{\Lambda}^q) \end{cases} \tag{2.6}$$

by $f \mapsto (2\pi)^{-m} e^{ix\xi} \wedge f$.

Lemma 2.6. *For every $\xi \in \mathbb{Z}^m$, we have that $\mathcal{F}_\xi \circ \mathcal{E}_\xi$ is the identity on $\mathscr{C}^\infty(M;\Lambda^q)$ and $\mathcal{F}_\xi \circ \mathcal{E}_\eta = 0$ if $\eta \neq \xi$, for every $q \in \{0,\dots,n\}$.*

Proposition 2.7. *Given $f \in \mathscr{C}^\infty(\Omega;\underline{\Lambda}^q)$ we have*

$$f = \frac{1}{(2\pi)^m} \sum_{\xi\in\mathbb{Z}^m} e^{ix\xi} \hat{f}_\xi \tag{2.7}$$

with convergence in $\mathscr{C}^\infty(\Omega;\underline{\Lambda}^q)$. In particular, $f = 0$ if and only if $\hat{f}_\xi = 0$ for all $\xi \in \mathbb{Z}^m$.

2.3. **Fourier analysis of** d'. Let $\boldsymbol{\omega} \doteq (\omega_1,\dots,\omega_m)$ with each ω_k being a complex, closed 1-form on M. Given $\xi \in \mathbb{Z}^m$, we write

$$\xi \cdot \boldsymbol{\omega} \doteq \sum_{k=1}^m \xi_k \omega_k \in \mathscr{C}^\infty(M;\Lambda^1)$$

and define $\mathrm{d}'_\xi : \mathscr{C}^\infty(M;\Lambda^q) \to \mathscr{C}^\infty(M;\Lambda^{q+1})$ by

$$\mathrm{d}'_\xi f \doteq \mathrm{d}f + i(\xi\cdot\boldsymbol{\omega}) \wedge f,$$

a first-order differential operator that satisfies $\mathrm{d}'_\xi \circ \mathrm{d}'_\xi = 0$, thus forming a complex whose smooth cohomology spaces we denote by

$$H^q_\xi(\mathscr{C}^\infty(M)) \doteq \frac{\ker\{\mathrm{d}'_\xi : \mathscr{C}^\infty(M;\Lambda^q) \longrightarrow \mathscr{C}^\infty(M;\Lambda^{q+1})\}}{\operatorname{ran}\{\mathrm{d}'_\xi : \mathscr{C}^\infty(M;\Lambda^{q-1}) \longrightarrow \mathscr{C}^\infty(M;\Lambda^q)\}}, \quad q \in \{1,\dots,n\} \tag{2.8}$$

and, as usual, $H^0_\xi(\mathscr{C}^\infty(M)) \doteq \ker\{\mathrm{d}'_\xi : \mathscr{C}^\infty(M) \longrightarrow \mathscr{C}^\infty(M;\Lambda^1)\}$.

In the next section, we start a deeper study of these zero-order perturbations of the de Rham complex. For the moment, we focus on their formal aspects related to the Fourier analysis of the complex d'. The proofs of the next results are standard.

Lemma 2.8. *The following transposition formulas hold.*

(1) *For $f \in \mathscr{C}^\infty(\Omega;\underline{\Lambda}^q)$ and $g \in \mathscr{C}^\infty(\Omega;\underline{\Lambda}^{n-q-1})$, we have*

$$\int_\Omega \mathrm{d}'f \wedge g \wedge \mathrm{d}x = (-1)^{q+1} \int_\Omega f \wedge \mathrm{d}'g \wedge \mathrm{d}x.$$

(2) *For* $f \in \mathscr{C}^\infty(M;\Lambda^q)$ *and* $g \in \mathscr{C}^\infty(M;\Lambda^{n-q-1})$*, we have*

$$\int_M \mathrm{d}'_\xi f \wedge g = (-1)^{q+1} \int_M f \wedge \mathrm{d}'_{-\xi} g$$

for every $\xi \in \mathbb{Z}^m$*.*

Corollary 2.9. *For each* $\xi \in \mathbb{Z}^m$*, the following commutation relations hold:*

(1) $\mathrm{d}' \circ \mathcal{E}_\xi = \mathcal{E}_\xi \circ \mathrm{d}'_\xi$*;*
(2) $\mathcal{F}_\xi \circ \mathrm{d}' = \mathrm{d}'_\xi \circ \mathcal{F}_\xi$*.*

3. Zero-order perturbations of the de Rham complex

Motivated by the introduction of the operators d'_ξ in Section 2.3, we discuss a general class of operators on M that are perturbations of the exterior derivative. Let ω be a complex, smooth, closed 1-form on M, and define, for $q \in \{0, \dots, n\}$:

$$\mathrm{D}_\omega \doteq \mathrm{d} + i\omega \wedge \cdot : \mathscr{C}^\infty(M;\Lambda^q) \longrightarrow \mathscr{C}^\infty(M;\Lambda^{q+1}). \tag{3.1}$$

Note that $\mathrm{d}'_\xi = \mathrm{D}_\omega$ when $\omega = \xi \cdot \boldsymbol{\omega}$. These are first-order differential operators, satisfying $\mathrm{D}_\omega \circ \mathrm{D}_\omega = 0$ since ω is closed. The operators D_ω define a differential complex, which is elliptic since D_ω has the same principal symbol as d. Such a differential complex, however, does not come from an involutive structure on M (for instance, $\mathrm{D}_\omega(1) = i\omega \neq 0$), hence we cannot apply solvability results from this theory to it. Given any open set $U \subset M$, we define

$$H^q_\omega(\mathscr{F}(U)) \doteq \frac{\ker\{\mathrm{D}_\omega : \mathscr{F}(U;\Lambda^q) \longrightarrow \mathscr{F}(U;\Lambda^{q+1})\}}{\mathrm{ran}\{\mathrm{D}_\omega : \mathscr{F}(U;\Lambda^{q-1}) \longrightarrow \mathscr{F}(U;\Lambda^q)\}}, \quad \mathscr{F} = \mathscr{C}^\infty \text{ or } \mathscr{D}'. \tag{3.2}$$

In particular, $H^q_\xi(\mathscr{C}^\infty(M)) = H^q_\omega(\mathscr{C}^\infty(M))$ when $\omega = \xi \cdot \boldsymbol{\omega}$. If ω is exact in U, say $\mathrm{d}\phi = \omega|_U$ for a $\phi \in \mathscr{C}^\infty(U)$, then $\mathrm{d}e^{i\phi} = ie^{i\phi}\omega|_U$ and

$$\mathrm{d}(e^{i\phi} f) = \mathrm{d}e^{i\phi} \wedge f + e^{i\phi}\mathrm{d}f = e^{i\phi}\mathrm{D}_\omega f$$

whatever $f \in \mathscr{D}'(U;\Lambda^q)$. In particular, for $q = 0$, if U is connected,

$$f \in \mathscr{D}'(U),\ \mathrm{D}_\omega f = 0 \Longrightarrow f = \text{const.} \cdot e^{-i\phi}, \tag{3.3}$$

thus the sheaves of homogeneous solutions of D_ω in smooth functions and in distributions are one and the same; we will denote this sheaf by $\mathscr{S}_\omega$. A crucial feature is that $\mathscr{S}_\omega$ is *locally* isomorphic with the constant sheaf $\mathscr{S}_0$ – its stalks are copies of $\mathbb{C}$ – but not *globally* in general (see below). We have commutative diagrams

$$\begin{array}{ccc} \mathscr{D}'(U;\Lambda^q) & \xrightarrow{\mathrm{D}_\omega} & \mathscr{D}'(U;\Lambda^{q+1}) \\ \downarrow e^{i\phi}\cdot & & \downarrow e^{i\phi}\cdot \\ \mathscr{D}'(U;\Lambda^q) & \xrightarrow{\mathrm{d}} & \mathscr{D}'(U;\Lambda^{q+1}) \end{array}$$

and the same holds for smooth sections. Since $e^{i\phi} \neq 0$ on U, we conclude that

$$H^q_\omega(\mathscr{F}(U)) \cong H^q_{\mathrm{dR}}(U), \quad \mathscr{F} = \mathscr{C}^\infty \text{ or } \mathscr{D}', \tag{3.4}$$

the right-hand side standing for the usual de Rham cohomology space.

Let $\mathscr{F}_q$ denote the sheaf of germs of smooth or distributional sections (depending on the case) of Λ^q, that is, forms and currents. We have the exactness in degree $q \in \{1, \dots, n\}$ of the sequence

$$\cdots \xrightarrow{\mathrm{D}_\omega} \mathscr{F}_q \xrightarrow{\mathrm{D}_\omega} \mathscr{F}_{q+1} \xrightarrow{\mathrm{D}_\omega} \cdots. \tag{3.5}$$

Indeed, by the Poincaré Lemma, every point $t \in M$ has a neighborhood in which ω is exact, thus we can apply the isomorphism (3.4) and use that the exterior derivative d is locally solvable at t in degree $q \in \{1, \dots, n\}$ (this is usually stated for smooth forms, but also holds for currents). In particular, in both cases the sequence (3.5) provides fine resolution of $\mathscr{S}_\omega$, so on *any* open set U, we have

$$H^q_\omega(\mathscr{C}^\infty(U)) \cong H^q_\omega(\mathscr{D}'(U)) \cong H^q(U; \mathscr{S}_\omega),$$

the right-hand side denoting the q-th cohomology space with values in the sheaf $\mathscr{S}_\omega$. Specifically, the natural homomorphism of sheaves $\Phi^0 : \mathscr{C}^\infty_0 \to \mathscr{D}'_0$ induces homomorphisms $\Phi^q : \mathscr{C}^\infty_q \to \mathscr{D}'_q$, forming a homomorphism between resolutions

$$\begin{array}{ccccccccccccc} 0 & \longrightarrow & \mathscr{S}_\omega & \longrightarrow & \mathscr{C}^\infty_0 & \xrightarrow{\mathrm{D}_\omega} & \cdots & \xrightarrow{\mathrm{D}_\omega} & \mathscr{C}^\infty_q & \xrightarrow{\mathrm{D}_\omega} & \mathscr{C}^\infty_{q+1} & \xrightarrow{\mathrm{D}_\omega} & \cdots \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \Phi^0} & & & & \downarrow{\scriptstyle \Phi^q} & & \downarrow{\scriptstyle \Phi^{q+1}} & & \\ 0 & \longrightarrow & \mathscr{S}_\omega & \longrightarrow & \mathscr{D}'_0 & \xrightarrow{\mathrm{D}_\omega} & \cdots & \xrightarrow{\mathrm{D}_\omega} & \mathscr{D}'_q & \xrightarrow{\mathrm{D}_\omega} & \mathscr{D}'_{q+1} & \xrightarrow{\mathrm{D}_\omega} & \cdots \end{array}.$$

Hence, for each $q \in \{0, \dots, n\}$, the map

$$\Phi^q_* : H^q_\omega(\mathscr{C}^\infty(U)) \longrightarrow H^q_\omega(\mathscr{D}'(U)), \quad U \subset M, \tag{3.6}$$

which is precisely the one induced by the inclusion map $\mathscr{C}^\infty(U; \Lambda^q) \hookrightarrow \mathscr{D}'(U; \Lambda^q)$ on the quotients, is an isomorphism[3] – we have just proved a special case of the so-called Atiyah-Bott Lemma. Unwinding quotients in (3.2), one deduces that:

Theorem 3.1. *Given an open set $U \subset M$, we have that:*

(1) *For every $f \in \mathscr{C}^\infty(U; \Lambda^q)$ such that there exists $u \in \mathscr{D}'(U; \Lambda^{q-1})$ with $\mathrm{D}_\omega u = f$, there exists $v \in \mathscr{C}^\infty(U; \Lambda^{q-1})$ such that $\mathrm{D}_\omega v = f$. I.e., $\mathrm{D}_\omega|_U$ is* (AGH).
(2) *For every $f \in \mathscr{D}'(U; \Lambda^q)$ satisfying $\mathrm{D}_\omega f = 0$, there exist $g \in \mathscr{C}^\infty(U; \Lambda^q)$ and $u \in \mathscr{D}'(U; \Lambda^{q-1})$ such that $f - g = \mathrm{D}_\omega u$.*

Proof. Clearly, the first claim is equivalent to the injectivity of (3.6), while the second one is equivalent to its surjectivity – both of them established above. ☐

In particular, D_ω is (AGH) in each degree. On time, we recall that, by elliptic theory, all the cohomology spaces $H^q_\omega(\mathscr{C}^\infty(M))$ are finite dimensional since M is compact. Both properties will be used heavily, often without mention, from here on.

Another consequence of (3.3) is that, given a coordinate ball $U \subset M$, a function in $\mathscr{S}_\omega(U)$ either vanishes identically or is never zero. In particular, the zero set of a global homogeneous solution $f \in \mathscr{S}_\omega(M)$ is both open and closed, so by connectedness, such an f also either vanishes everywhere or not at all.

Lemma 3.2. *Either $\mathscr{S}_\omega(M) = \{0\}$ or $\dim_{\mathbb{C}} \mathscr{S}_\omega(M) = 1$. The latter case happens if and only if $\mathscr{S}_\omega$ is isomorphic with the constant sheaf $\mathscr{S}_0$.*

Proof. Suppose there exists $f \in \mathscr{S}_\omega(M)$ non-zero. By the previous remarks, f never vanishes. We will prove that any $g \in \mathscr{S}_\omega(M)$ is a constant times f. On an open set $U \subset M$ in which ω is exact, we have by (3.3) that $f|_U = c_1 e^{-i\phi}$ and $g|_U = c_2 e^{-i\phi}$ for some constants $c_1, c_2 \in \mathbb{C}$ with $c_1 \neq 0$. Hence $(g/f)|_U = c_1^{-1} c_2$, and, since such open sets cover M, we reach the conclusion that g/f is locally constant: by connectedness, this must be actually a constant function. Now, we prove that multiplication by f

[3] See e.g. [12, Theorem 3.13], but especially the remark by its end concerning naturality.

defines a sheaf isomorphism $\mathscr{S}_0 \to \mathscr{S}_\omega$, a claim that we check locally. Given $t \in M$, take a coordinate open ball $U \subset M$ around it. We have

$$h \in \mathscr{S}_0(U) \Longrightarrow h = \text{const.} \Longrightarrow h \cdot f|_U = \text{const.} \cdot f|_U \in \mathscr{S}_\omega(U),$$

so we have a monomorphism $\mathscr{S}_0(U) \to \mathscr{S}_\omega(U)$. But, since $\omega|_U$ has a primitive, any $g \in \mathscr{S}_\omega(U)$ is a multiple of $f|_U$, hence that monomorphism is also surjective. ☐

Corollary 3.3. *If ω is exact then $\mathscr{S}_\omega \cong \mathscr{S}_0$.*

Proof. If $\phi \in \mathscr{C}^\infty(M)$ is a primitive of ω, then $\mathrm{D}_\omega(e^{-i\phi}) = 0$, i.e., $e^{-i\phi} \in \mathscr{S}_\omega(M)$. ☐

Remark 3.4. The sheaf $\mathscr{S}_\omega$ is always locally isomorphic to $\mathscr{S}_0$; the isomorphism can be made global precisely when $\mathscr{S}_\omega$ admits a non-vanishing global section.

Lemma 3.5. *Given two closed* 1*-forms ω_1, ω_2 on M such that $\mathscr{S}_{\omega_1}(M) \neq \{0\} \neq \mathscr{S}_{\omega_2}(M)$, we have that $\mathscr{S}_{\omega_1+\omega_2}(M) \neq 0$.*

Proof. If f_1 (resp. f_2) is a section of $\mathscr{S}_{\omega_1}$ (resp. $\mathscr{S}_{\omega_2}$) then $f_1 f_2$ is a section of $\mathscr{S}_{\omega_1+\omega_2}$. Indeed:

$$\mathrm{d}(f_1 f_2) = \mathrm{d}f_1 \wedge f_2 + f_1 \wedge \mathrm{d}f_2 = -i\omega_1 \wedge f_1 \wedge f_2 - i f_1 \wedge \omega_2 \wedge f_2 = -i(\omega_1 + \omega_2) \wedge (f_1 f_2).$$

In particular, if $f_1 \in \mathscr{S}_{\omega_1}(M)$ and $f_2 \in \mathscr{S}_{\omega_2}(M)$ are both non-zero then $f_1 f_2 \in \mathscr{S}_{\omega_1+\omega_2}(M)$ is non-zero. ☐

Theorem 3.6. *The set*

$$\mathcal{Z}_M \doteq \{[\omega] \in H^1_{\mathrm{dR}}(M) \ ; \ \mathscr{S}_\omega \cong \mathscr{S}_0\} = \{[\omega] \in H^1_{\mathrm{dR}}(M) \ ; \ \mathscr{S}_\omega(M) \neq \{0\}\}$$

is a group.

Proof. Notice that $\mathcal{Z}_M$ is well-defined: if $\omega, \omega^\bullet$ are two closed 1-forms in the same cohomology class such that $\mathscr{S}_\omega \cong \mathscr{S}_0$ then also $\mathscr{S}_{\omega^\bullet} \cong \mathscr{S}_0$. Indeed, in that case there exists $\phi \in \mathscr{C}^\infty(M)$ such that $\omega^\bullet = \omega + \mathrm{d}\phi$: since $\mathscr{S}_\omega \cong \mathscr{S}_0$ by assumption and $\mathscr{S}_{\mathrm{d}\phi} \cong \mathscr{S}_0$ by Corollary 3.3, it follows from Lemmas 3.5 and 3.2 that $\mathscr{S}_{\omega^\bullet} \cong \mathscr{S}_0$.

The same argument shows that $\mathcal{Z}_M$ is an additive submonoid of $H^1_{\mathrm{dR}}(M)$. Regarding inverses, if $[\omega] \in \mathcal{Z}_M$ then there exists a non-vanishing $f \in \mathscr{S}_\omega(M)$; clearly,

$$\mathrm{d}f + i f \omega = 0 \Longrightarrow \mathrm{d}(1/f) - i(1/f)\omega = 0.$$

Therefore, $1/f$ is a non-vanishing element of $\mathscr{S}_{-\omega}(M)$, so $-[\omega] \in \mathcal{Z}_M$. ☐

Remark 3.7. Recall that a real closed 1-form α is *integral* if $\int_\gamma \alpha \in 2\pi\mathbb{Z}$ for every 1-cycle γ in M. It follows from [4, Lemma 2.1] that

$$\mathcal{Z}_M = \{[\omega] \in H^1_{\mathrm{dR}}(M) \ ; \ \mathsf{Re}\omega \text{ is integral and } \mathsf{Im}\omega \text{ is exact}\}. \tag{3.7}$$

4. Formal characterization of the closure of $\operatorname{ran} \mathrm{d}'$

In this section, we address the issue of formal solvability (i.e., at the level of partial Fourier series) and relate it with the notion of global solvability in degree $q \in \{1, \dots, n\}$. Now that we have all the tools at our disposal, the proofs are pretty straightforward.

Lemma 4.1. *For $f \in \mathscr{C}^\infty(\Omega; \underline{\Lambda}^q)$, the following are equivalent:*

1. *f belongs to the closure of* $\operatorname{ran}\{\mathrm{d}' : \mathscr{C}^\infty(\Omega; \underline{\Lambda}^{q-1}) \to \mathscr{C}^\infty(\Omega; \underline{\Lambda}^q)\}$;
2. *$\hat{f}_\xi$ is d'_ξ-exact for every $\xi \in \mathbb{Z}^m$.*

Remark 4.2. The notion of $\hat{f}_\xi$ being d'_ξ-exact is unambiguous: if $\mathrm{d}'_\xi u_\xi = \hat{f}_\xi$ is solvable in distributions, we can always find a smooth solution since d'_ξ is (AGH) (Theorem 3.1).

Proof. Let $f \in \mathscr{C}^\infty(\Omega; \underline{\Lambda}^q)$ and suppose there exists $\{u_\nu\}_{\nu \in \mathbb{N}} \subset \mathscr{C}^\infty(\Omega; \underline{\Lambda}^{q-1})$ such that $\mathrm{d}' u_\nu \to f$ in $\mathscr{C}^\infty(\Omega; \underline{\Lambda}^q)$. By Proposition 2.5 we have that

$$\mathrm{d}'_\xi \mathcal{F}_\xi(u_\nu) = \mathcal{F}_\xi(\mathrm{d}' u_\nu) \longrightarrow \hat{f}_\xi \quad \text{in } \mathscr{C}^\infty(M; \Lambda^q).$$

Therefore, $\hat{f}_\xi$ belongs to the closure of $\mathrm{ran}\{\mathrm{d}'_\xi : \mathscr{C}^\infty(M; \Lambda^{q-1}) \to \mathscr{C}^\infty(M; \Lambda^q)\}$, which is already closed in $\mathscr{C}^\infty(M; \Lambda^q)$ – for instance, by Theorems 2.2 and 3.1 –, hence:

$$\hat{f}_\xi \in \mathrm{ran}\{\mathrm{d}'_\xi : \mathscr{C}^\infty(M; \Lambda^{q-1}) \longrightarrow \mathscr{C}^\infty(M; \Lambda^q)\}, \quad \forall \xi \in \mathbb{Z}^m. \tag{4.1}$$

Conversely, if $f \in \mathscr{C}^\infty(\Omega; \underline{\Lambda}^q)$ is such that (4.1) holds, then for each $\xi \in \mathbb{Z}^m$ there exists $u_\xi \in \mathscr{C}^\infty(M; \Lambda^{q-1})$ such that $\mathrm{d}'_\xi u_\xi = \hat{f}_\xi$, hence in the topology of $\mathscr{C}^\infty(\Omega; \underline{\Lambda}^q)$ we have

$$f = \frac{1}{(2\pi)^m} \sum_{\xi \in \mathbb{Z}^m} e^{ix\xi} \wedge \mathrm{d}'_\xi u_\xi = \lim_{\nu \to \infty} \mathrm{d}' \left(\frac{1}{(2\pi)^m} \sum_{|\xi| \leq \nu} e^{ix\xi} \wedge u_\xi \right),$$

which proves that f lies in the closure of $\mathrm{ran}\{\mathrm{d}' : \mathscr{C}^\infty(\Omega; \underline{\Lambda}^{q-1}) \to \mathscr{C}^\infty(\Omega; \underline{\Lambda}^q)\}$. ☐

Remark 4.3. Lemma 4.1 helps to settle the question of equivalence between different formulations of compatibility conditions appearing in the literature (hence, the equivalence between different notions of (smooth) global solvability).

Consider, for instance, the space $E \subset \mathscr{C}^\infty(\Omega; \underline{\Lambda}^1)$ determined by the compatibility conditions in [8, 9] (which deal with the case $m = 1$, $q = 1$). Given $f \in \mathscr{C}^\infty(\Omega; \underline{\Lambda}^1)$, it follows easily from de Rham Theorem that, if $\hat{f}_\xi$ is d'_ξ-exact for every $\xi \in \mathbb{Z}^m$, then $f \in E$; whereas the converse follows from results in [8, 9]: if $f \in E$, a solution to $\mathrm{d}'_\xi u_\xi = \hat{f}_\xi$ is obtained for every $\xi \in \mathbb{Z}^m$, first in some covering space of M, and then in M by a convenient choice of initial conditions. In particular, E equals the closure of $\mathrm{ran}\{\mathrm{d}' : \mathscr{C}^\infty(\Omega) \to \mathscr{C}^\infty(\Omega; \underline{\Lambda}^1)\}$ in that case.

The previous lemma yields our first major result; which, together with Theorem 2.2, entails Theorem 1.1.

Theorem 4.4. *If* (1.3) *has closed range, then it is* (AGH).

Proof. Suppose that $\mathrm{d}' : \mathscr{C}^\infty(\Omega; \underline{\Lambda}^q) \to \mathscr{C}^\infty(\Omega; \underline{\Lambda}^{q+1})$ has closed range and let $u \in \mathscr{D}'(\Omega; \underline{\Lambda}^q)$ be such that $f \doteq \mathrm{d}' u \in \mathscr{C}^\infty(\Omega; \underline{\Lambda}^{q+1})$. Then $\hat{f}_\xi = \mathrm{d}'_\xi \hat{u}_\xi$, that is, $\hat{f}_\xi$ is d'_ξ-exact for every $\xi \in \mathbb{Z}^m$. By Lemma 4.1, f belongs to $\mathrm{ran}\{\mathrm{d}' : \mathscr{C}^\infty(\Omega; \underline{\Lambda}^q) \to \mathscr{C}^\infty(\Omega; \underline{\Lambda}^{q+1})\}$, so there exists $v \in \mathscr{C}^\infty(\Omega; \underline{\Lambda}^q)$ such that $\mathrm{d}' v = f = \mathrm{d}' u$. ☐

It generalizes [9, Corollary 7.2] (and preceding results):

Corollary 4.5. *If the operator* $\mathrm{d}' : \mathscr{C}^\infty(\Omega) \to \mathscr{C}^\infty(\Omega; \underline{\Lambda}^1)$ *satisfies the property*

$$\forall u \in \mathscr{D}'(\Omega), \ \mathrm{d}' u = 0 \Longrightarrow u = \text{const.}, \tag{4.2}$$

then it has closed range if and only if it is globally hypoelliptic.

Proof. This is an immediate consequence of Theorems 2.2 and 4.4, since global hypoellipticity is equivalent to (AGH) when (4.2) holds. ☐

Remark 4.6. Notice that (4.2) holds in the case considered in [9, Corollary 7.2] ($m = 1$, $\omega = ib$ with b real, closed and non-exact) thanks to [4, Lemma 2.2]. See also further discussion below about the group $\Gamma_{\boldsymbol{\omega}}$.

Definition 4.7. We say that an $f \in \mathscr{C}^\infty(\Omega; \underline{\Lambda}^q)$ is *formally solvable* if for each $\xi \in \mathbb{Z}^m$ there exists $u_\xi \in \mathscr{C}^\infty(M; \Lambda^{q-1})$ such that $\mathrm{d}'_\xi u_\xi = \hat{f}_\xi$.

A sufficient condition for that to happen is that there exists u in $\mathscr{C}^\infty(\Omega; \underline{\Lambda}^{q-1})$ – or even in $\mathscr{D}'(\Omega; \underline{\Lambda}^{q-1})$ – such that $\mathrm{d}'u = f$: Lemma 4.1 entails the following converse.

Corollary 4.8. *The following are equivalent:*

(1) $\mathrm{d}' : \mathscr{C}^\infty(\Omega; \underline{\Lambda}^{q-1}) \to \mathscr{C}^\infty(\Omega; \underline{\Lambda}^q)$ *has closed range.*
(2) *For every formally solvable* $f \in \mathscr{C}^\infty(\Omega; \underline{\Lambda}^q)$ *there exists* $u \in \mathscr{C}^\infty(\Omega; \underline{\Lambda}^{q-1})$ *such that* $\mathrm{d}'u = f$.

Before we move on, we must introduce an important set of frequencies. For $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_m)$, the mapping $\xi \in \mathbb{Z}^m \mapsto [\xi \cdot \boldsymbol{\omega}] \in H^1_{\mathrm{dR}}(M)$ is a homomorphism of groups, hence

$$\Gamma_{\boldsymbol{\omega}} \doteq \{\xi \in \mathbb{Z}^m \ ; \ [\xi \cdot \boldsymbol{\omega}] \in \mathcal{Z}_M\} \tag{4.3}$$

is a subgroup of $\mathbb{Z}^m$. In particular, either one of the three mutually exclusive alternatives hold:

(1) $\Gamma_{\boldsymbol{\omega}} = \{0\}$;
(2) $\Gamma_{\boldsymbol{\omega}}$ is infinite and proper; or
(3) $\Gamma_{\boldsymbol{\omega}} = \mathbb{Z}^m$.

Using (3.7) one has, for instance, that:

$$\begin{aligned}\Gamma_{\boldsymbol{\omega}} = \{0\} &\iff \mathsf{Re}(\xi \cdot \boldsymbol{\omega}) \text{ non-integral or } \mathsf{Im}(\xi \cdot \boldsymbol{\omega}) \text{ non-exact}, \ \forall \xi \in \mathbb{Z}^m \setminus \{0\} \\ &\implies \mathsf{Re}\omega_k \text{ non-integral or } \mathsf{Im}\omega_k \text{ non-exact}, \ \forall k \in \{1, \ldots, m\};\end{aligned}$$

while

$$\begin{aligned}\Gamma_{\boldsymbol{\omega}} = \mathbb{Z}^m &\iff \mathsf{Re}(\xi \cdot \boldsymbol{\omega}) \text{ integral and } \mathsf{Im}(\xi \cdot \boldsymbol{\omega}) \text{ exact}, \ \forall \xi \in \mathbb{Z}^m \\ &\iff \mathsf{Re}\omega_k \text{ integral and } \mathsf{Im}\omega_k \text{ exact}, \ \forall k \in \{1, \ldots, m\}.\end{aligned}$$

5. Applications to real structures

Throughout this section, $\omega_1, \ldots, \omega_m$ are assumed *real*. In this case,

$$\Gamma_{\boldsymbol{\omega}} = \{\xi \in \mathbb{Z}^m \ ; \ \xi \cdot \boldsymbol{\omega} \text{ is integral}\}.$$

5.1. **Isomorphism theorems.** For $X \subset \mathbb{Z}^m$ we set, for each $q \in \{0, \ldots, n\}$,

$$\mathscr{C}^\infty_X(\Omega; \underline{\Lambda}^q) \doteq \{f \in \mathscr{C}^\infty(\Omega; \underline{\Lambda}^q) \ ; \ \hat{f}_\xi = 0, \ \forall \xi \in \mathbb{Z}^m \setminus X\},$$

and for $f \in \mathscr{C}^\infty(\Omega; \underline{\Lambda}^q)$, we consider its projection on $\mathscr{C}^\infty_X(\Omega; \underline{\Lambda}^q)$:

$$f_X \doteq \frac{1}{(2\pi)^m} \sum_{\xi \in X} e^{ix\xi} \wedge \hat{f}_\xi.$$

It follows that $f = f_X + f_{\mathbb{Z}^m \setminus X}$ and hence

$$\mathscr{C}^\infty(\Omega; \underline{\Lambda}^q) = \mathscr{C}^\infty_X(\Omega; \underline{\Lambda}^q) \oplus \mathscr{C}^\infty_{\mathbb{Z}^m \setminus X}(\Omega; \underline{\Lambda}^q).$$

Notice moreover that

$$\mathrm{d}'\mathscr{C}^\infty_X(\Omega; \underline{\Lambda}^q) \subset \mathscr{C}^\infty_X(\Omega; \underline{\Lambda}^{q+1})$$

and that

$$\mathrm{d}'u = f \iff \mathrm{d}'u_X = f_X \text{ and } \mathrm{d}'u_{\mathbb{Z}^m\setminus X} = f_{\mathbb{Z}^m\setminus X}.$$

It thus makes sense to define

$$H^q_{\mathrm{d}'}(\mathscr{C}^\infty_X(\Omega)) \doteq \frac{\ker\{\mathrm{d}' : \mathscr{C}^\infty_X(\Omega;\underline{\Lambda}^q) \longrightarrow \mathscr{C}^\infty_X(\Omega;\underline{\Lambda}^{q+1})\}}{\mathrm{ran}\{\mathrm{d}' : \mathscr{C}^\infty_X(\Omega;\underline{\Lambda}^{q-1}) \longrightarrow \mathscr{C}^\infty_X(\Omega;\underline{\Lambda}^q)\}},$$

hence, we have

$$H^q_{\mathrm{d}'}(\mathscr{C}^\infty(\Omega)) \cong H^q_{\mathrm{d}'}(\mathscr{C}^\infty_X(\Omega)) \oplus H^q_{\mathrm{d}'}(\mathscr{C}^\infty_{\mathbb{Z}^m\setminus X}(\Omega)).$$

We denote by $\pi : \tilde{M} \to M$ the universal covering of M. For each $k \in \{1,\dots,m\}$ there exists $\psi_k \in \mathscr{C}^\infty(\tilde{M};\mathbb{R})$ such that $\mathrm{d}\psi_k = \pi^*\omega_k$ on $\tilde{M}$ (since the latter form is exact). More generally, for $\xi \in \mathbb{Z}^m$ we set

$$\boldsymbol{\psi}_\xi \doteq \sum_{k=1}^m \xi_k \psi_k \in \mathscr{C}^\infty(\tilde{M};\mathbb{R}), \tag{5.1}$$

hence $\mathrm{d}\boldsymbol{\psi}_\xi = \pi^*(\xi\cdot\boldsymbol{\omega})$. In an open set $U \subset M$ in which $\pi_U : \tilde{U} \to U$ is diffeomorphism ($\tilde{U} \subset \tilde{M}$ being another open set) we have

$$\mathrm{d}e^{i\boldsymbol{\psi}_\xi\circ\pi_U^{-1}} = ie^{i\boldsymbol{\psi}_\xi\circ\pi_U^{-1}}(\xi\cdot\boldsymbol{\omega}) \quad \text{on } U.$$

If $\xi \in \Gamma_{\boldsymbol{\omega}}$, then $\xi\cdot\boldsymbol{\omega}$ is integral, hence by [3, Lemma 2.3], if $P,Q \in \tilde{M}$ satisfy $\pi(P) = \pi(Q)$ then $\boldsymbol{\psi}_\xi(P) - \boldsymbol{\psi}_\xi(Q) \in 2\pi\mathbb{Z}$, we can define for every $\xi \in \Gamma_{\boldsymbol{\omega}}$ a smooth function

$$M \ni t \longmapsto e^{i\boldsymbol{\psi}_\xi\circ\pi^{-1}(t)}, \tag{5.2}$$

even though π^{-1} is not a function. It follows that

$$e^{i\boldsymbol{\psi}_\xi\circ\pi^{-1}} \in \mathscr{S}_{(-\xi\cdot\boldsymbol{\omega})}(M)\setminus\{0\}, \quad \forall \xi \in \Gamma_{\boldsymbol{\omega}}.$$

We define a map $\Psi : \tilde{M}\times\mathbb{T}^m \to \tilde{M}\times\mathbb{T}^m$ by

$$\Psi(\tilde{t},x_1,\dots,x_m) \doteq (\tilde{t},x_1-\psi_1(\tilde{t}),\dots,x_m-\psi_m(\tilde{t})),$$

in which the sum $x_k - \psi_k(\tilde{t})$ takes place in $\mathbb{R}/(2\pi\mathbb{Z})$. In particular, Ψ is a diffeomorphism with inverse

$$\Psi^{-1}(\tilde{t},x_1,\dots,x_m) = (\tilde{t},x_1+\psi_1(\tilde{t}),\dots,x_m+\psi_m(\tilde{t})).$$

Proposition 5.1. *Assume* $\omega_1,\dots,\omega_m$ *real-valued and let* $f \in \mathscr{C}^\infty_{\Gamma_{\boldsymbol{\omega}}}(\Omega;\underline{\Lambda}^q)$. *Then*

$$\Theta(f) \doteq \frac{1}{(2\pi)^m}\sum_{\xi\in\Gamma_{\boldsymbol{\omega}}} e^{ix\xi - i\boldsymbol{\psi}_\xi\circ\pi^{-1}}\hat{f}_\xi \tag{5.3}$$

defines an element in $\mathscr{C}^\infty_{\Gamma_{\boldsymbol{\omega}}}(\Omega;\underline{\Lambda}^q)$. *The map*

$$\Theta : \mathscr{C}^\infty_{\Gamma_{\boldsymbol{\omega}}}(\Omega;\underline{\Lambda}^q) \longrightarrow \mathscr{C}^\infty_{\Gamma_{\boldsymbol{\omega}}}(\Omega;\underline{\Lambda}^q)$$

is a linear isomorphism and satisfies $\mathrm{d}'\circ\Theta = \mathrm{d}_t\circ\Theta$.

Proof. Writing the partial Fourier series of f, we have that

$$f(t,x) = \frac{1}{(2\pi)^m}\sum_{\xi\in\Gamma_{\boldsymbol{\omega}}} e^{ix\xi}\hat{f}_\xi(t).$$

We define $F \doteq (\pi \times \mathrm{id}_{\mathbb{T}^m})^* f$, which is a smooth q-form on $\tilde{M} \times \mathbb{T}^m$. By continuity of the pullback map, it follows that

$$F(\tilde{t}, x) = \frac{1}{(2\pi)^m} \sum_{\xi \in \Gamma_{\boldsymbol{\omega}}} e^{ix\xi} (\pi^* \hat{f}_\xi)(\tilde{t}),$$

with convergence in the space of smooth q-forms on $\tilde{M} \times \mathbb{T}^m$. If we change coordinates using the diffeomorphism Ψ, we obtain

$$(\Psi^* F)(\tilde{t}, x) = \frac{1}{(2\pi)^m} \sum_{\xi \in \Gamma_{\boldsymbol{\omega}}} e^{i(x\xi - \boldsymbol{\psi}_\xi(\tilde{t}))} (\pi^* \hat{f}_\xi)(\tilde{t}),$$

which we will show that descends to $M \times \mathbb{T}^m$ as $\Theta(f)$. The sequence of truncated sums

$$f_\nu(t) \doteq \frac{1}{(2\pi)^m} \sum_{\substack{\xi \in \Gamma_{\boldsymbol{\omega}} \\ |\xi| \leq \nu}} e^{ix\xi} \hat{f}_\xi(t), \quad \nu \in \mathbb{N},$$

which approximates f in $\mathscr{C}^\infty(\Omega; \underline{\Lambda}^q)$, certainly satisfy

$$\Psi^* (\pi \times \mathrm{id}_{\mathbb{T}^m})^* f_\nu \longrightarrow \Psi^* F \quad \text{on } \tilde{M} \times \mathbb{T}^m$$

whereas

$$g_\nu(t) \doteq \frac{1}{(2\pi)^m} \sum_{\substack{\xi \in \Gamma_{\boldsymbol{\omega}} \\ |\xi| \leq \nu}} e^{ix\xi - i\boldsymbol{\psi}_\xi \circ \pi^{-1}(t)} \hat{f}_\xi(t), \quad \nu \in \mathbb{N},$$

satisfy $(\pi \times \mathrm{id}_{\mathbb{T}^m})^* g_\nu = \Psi^* (\pi \times \mathrm{id}_{\mathbb{T}^m})^* f_\nu$ clearly, hence

$$(\pi \times \mathrm{id}_{\mathbb{T}^m})^* g_\nu \longrightarrow \Psi^* F \quad \text{on } \tilde{M} \times \mathbb{T}^m. \tag{5.4}$$

To prove convergence of $\{g_\nu\}_{\nu \in \mathbb{N}}$ in $\mathscr{C}^\infty(\Omega; \underline{\Lambda}^q)$, it suffices to check it on $U \times \mathbb{T}^m$ for a suitably small open set $U \subset M$: its limit will then be automatically a globally defined q-form $\Theta(f)$ on $M \times \mathbb{T}^m$, and can be easily seen to be a section of $\underline{\Lambda}^q$. This is the case if, say, there exists an open set $\tilde{U} \subset \tilde{M}$ such that $\pi_U : \tilde{U} \to U$ is a diffeomorphism. Since the previous convergence (5.4) takes place in $\tilde{U} \times \mathbb{T}^m$ as well by restriction, in which $\pi_U \times \mathrm{id}_{\mathbb{T}^m}$ is invertible, we conclude that $\{g_\nu\}_{\nu \in \mathbb{N}}$ converges in $U \times \mathbb{T}^m$, proving our claim.

It is clear that (5.3) holds (as $\Theta(f)$ is by definition the limit of $\{g_\nu\}_{\nu \in \mathbb{N}}$), hence of course $\Theta(f) \in \mathscr{C}^\infty_{\Gamma_{\boldsymbol{\omega}}}(\Omega; \underline{\Lambda}^q)$: by continuity of each map $\mathcal{F}_\xi$, equation (5.3) gives its Fourier coefficients explicitly. Moreover:

$$\begin{aligned}
(2\pi)^m \mathrm{d}' \Theta(f) &= \left(\mathrm{d}_t + \sum_{k=1}^m \omega_k \wedge \frac{\partial}{\partial x_k} \right) \sum_{\xi \in \Gamma_{\boldsymbol{\omega}}} e^{ix\xi - i\boldsymbol{\psi}_\xi \circ \pi^{-1}} \hat{f}_\xi \\
&= \sum_{\xi \in \Gamma_{\boldsymbol{\omega}}} \left(e^{ix\xi} \mathrm{d}_t (e^{-i\boldsymbol{\psi}_\xi \circ \pi^{-1}} \hat{f}_\xi) + \sum_{k=1}^m \frac{\partial e^{ix\xi}}{\partial x_k} \omega_k \wedge (e^{-i\boldsymbol{\psi}_\xi \circ \pi^{-1}} \hat{f}_\xi) \right) \\
&= \sum_{\xi \in \Gamma_{\boldsymbol{\omega}}} \left(e^{ix\xi - i\boldsymbol{\psi}_\xi \circ \pi^{-1}} \left(\mathrm{d}_t \hat{f}_\xi - i(\xi \cdot \boldsymbol{\omega}) \wedge \hat{f}_\xi \right) + i(\xi \cdot \boldsymbol{\omega}) \wedge e^{ix\xi - i\boldsymbol{\psi}_\xi \circ \pi^{-1}} \hat{f}_\xi \right) \\
&= \sum_{\xi \in \Gamma_{\boldsymbol{\omega}}} e^{ix\xi - i\boldsymbol{\psi}_\xi \circ \pi^{-1}} \widehat{\mathrm{d}_t f_\xi} \\
&= (2\pi)^m \Theta(\mathrm{d}_t f),
\end{aligned}$$

that is: $\Theta^{-1} \circ \mathrm{d}' \circ \Theta = \mathrm{d}_t$ on $\mathscr{C}^\infty_{\Gamma_{\boldsymbol{\omega}}}(\Omega; \underline{\Lambda}^q)$. ☐

A similar calculation shows that, if

$$\Theta_\xi : \mathscr{C}^\infty(M; \Lambda^q) \longrightarrow \mathscr{C}^\infty(M; \Lambda^q), \quad \Theta_\xi(h) \doteq e^{i\boldsymbol{\psi}_\xi \circ \pi^{-1}} h,$$

then $\Theta_\xi^{-1} \circ \mathrm{d}'_\xi \circ \Theta_\xi = \mathrm{d}$, whatever $\xi \in \Gamma_{\boldsymbol{\omega}}$.

Theorem 5.2. *Assume $\omega_1, \dots, \omega_m$ real-valued. Then, $\mathrm{d}' : \mathscr{C}^\infty(\Omega; \underline{\Lambda}^q) \to \mathscr{C}^\infty(\Omega; \underline{\Lambda}^{q+1})$ has closed range if and only if given a formally solvable $f \in \mathscr{C}^\infty_{\mathbb{Z}^m \setminus \Gamma_{\boldsymbol{\omega}}}(\Omega; \underline{\Lambda}^{q+1})$, there exists $u \in \mathscr{C}^\infty(\Omega; \underline{\Lambda}^q)$ such that $\mathrm{d}'u = f$.*

Proof. The direct implication is granted by Corollary 4.8; we prove the converse. Let $f \in \mathscr{C}^\infty(\Omega; \underline{\Lambda}^{q+1})$ be such that for each $\xi \in \mathbb{Z}^m$ there exists $u_\xi \in \mathscr{C}^\infty(M; \Lambda^q)$ satisfying $\mathrm{d}'_\xi u_\xi = \hat{f}_\xi$. Write $f = f_{\Gamma_{\boldsymbol{\omega}}} + f_{\mathbb{Z}^m \setminus \Gamma_{\boldsymbol{\omega}}}$ and notice that both $f_{\Gamma_{\boldsymbol{\omega}}}$ and $f_{\mathbb{Z}^m \setminus \Gamma_{\boldsymbol{\omega}}}$ are also formally solvable. By hypothesis, there exists $v \in \mathscr{C}^\infty(\Omega; \underline{\Lambda}^q)$ such that $\mathrm{d}'v = f_{\mathbb{Z}^m \setminus \Gamma_{\boldsymbol{\omega}}}$: we must solve $\mathrm{d}'u = f_{\Gamma_{\boldsymbol{\omega}}}$.

We claim that $g \doteq \Theta^{-1} f_{\Gamma_{\boldsymbol{\omega}}} \in \mathscr{C}^\infty_{\Gamma_{\boldsymbol{\omega}}}(\Omega; \underline{\Lambda}^{q+1})$ is formally solvable w.r.t. d_t in Ω. Indeed, $\hat{g}_\xi = 0$ for $\xi \in \mathbb{Z}^m \setminus \Gamma_{\boldsymbol{\omega}}$, while for $\xi \in \Gamma_{\boldsymbol{\omega}}$, we have

$$\mathrm{d}'_\xi u_\xi = \hat{f}_\xi \Longrightarrow \mathrm{d}_t \Theta_\xi^{-1} u_\xi = (\Theta_\xi^{-1} \mathrm{d}'_\xi \Theta_\xi) \Theta_\xi^{-1} u_\xi = \Theta_\xi^{-1} \hat{f}_\xi = \mathcal{F}_\xi(\Theta^{-1} f_{\Gamma_{\boldsymbol{\omega}}}) = \hat{g}_\xi.$$

But global solvability of d_t is a general fact – see Lemma 5.3 below – which will also play a role later on. It then follows from Corollary 4.8 applied to d_t that there exists $w \in \mathscr{C}^\infty(\Omega; \underline{\Lambda}^q)$ such that $\mathrm{d}_t w = g$. Note that we can replace w with $w_{\Gamma_{\boldsymbol{\omega}}}$, if necessary, and assume that $w \in \mathscr{C}^\infty_{\Gamma_{\boldsymbol{\omega}}}(\Omega; \underline{\Lambda}^q)$. If we set $u \doteq \Theta w \in \mathscr{C}^\infty_{\Gamma_{\boldsymbol{\omega}}}(\Omega; \underline{\Lambda}^q)$ then

$$\mathrm{d}'u = \mathrm{d}'\Theta w = \Theta \mathrm{d}_t w = \Theta g = f_{\Gamma_{\boldsymbol{\omega}}}$$

and we are done. ☐

Lemma 5.3. *The map $\mathrm{d}_t : \mathscr{C}^\infty(\Omega; \underline{\Lambda}^q) \to \mathscr{C}^\infty(\Omega; \underline{\Lambda}^{q+1})$ has closed range.*

Proof. We endow M with a Riemannian metric and let $\Delta \doteq \mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d}$ be the Laplace-Beltrami operator acting on forms on M; it is elliptic of order 2 and therefore satisfies elliptic estimates: given $k \in \mathbb{Z}_+$ there exists $C_k > 0$ such that

$$\|\psi\|_{\mathscr{H}^{k+2}(M;\Lambda^q)} \leq C_k \left(\|\Delta\psi\|_{\mathscr{H}^k(M;\Lambda^q)} + \|\psi\|_{\mathscr{H}^k(M;\Lambda^q)} \right), \quad \forall \psi \in \mathscr{H}^{k+2}(M; \Lambda^q),$$

where $\mathscr{H}^k$ are Sobolev spaces. A standard argument shows that

$$\|\psi\|_{\mathscr{H}^{k+2}(M;\Lambda^q)} \leq C_k \|\Delta\psi\|_{\mathscr{H}^k(M;\Lambda^q)}, \quad \forall \psi \in \mathscr{H}^{k+2}(M; \Lambda^q), \ L^2\text{-orthogonal to } \ker \Delta.$$

The following orthogonal decomposition w.r.t. the $L^2(M)$ metric is also well-known:

$$\mathscr{C}^\infty(M; \Lambda^q) = \ker \Delta \oplus \operatorname{ran} \mathrm{d} \oplus \operatorname{ran} \mathrm{d}^* \tag{5.5}$$

(all operators acting on smooth forms), hence in particular, for $\psi \in \operatorname{ran} \mathrm{d}^*$,

$$\|\psi\|_{\mathscr{H}^{k+2}(M;\Lambda^q)} \leq C_k \|\mathrm{d}^*\mathrm{d}\psi\|_{\mathscr{H}^k(M;\Lambda^q)} \leq C'_k \|\mathrm{d}\psi\|_{\mathscr{H}^{k-1}(M;\Lambda^{q+1})}$$

since in that case $\Delta\psi = \mathrm{d}^*\mathrm{d}\psi$. We conclude that, for each $k \in \mathbb{Z}_+$, there exist $c_k > 0$ and $j \in \mathbb{Z}_+$ such that

$$\|\psi\|_{\mathscr{H}^k(M;\Lambda^q)} \leq c_k \|\mathrm{d}\psi\|_{\mathscr{H}^j(M;\Lambda^{q+1})}, \quad \forall \psi \in \operatorname{ran} \mathrm{d}^*. \tag{5.6}$$

Now, we obtain our conclusion by means of Theorem 2.2. Let $u \in \mathscr{D}'(\Omega; \underline{\Lambda}^q)$ be such that $\mathrm{d}_t u \in \mathscr{C}^\infty(\Omega; \underline{\Lambda}^{q+1})$. Then, for every $\xi \in \mathbb{Z}^m$, we have that $\mathrm{d}\hat{u}_\xi \in \mathscr{C}^\infty(M; \Lambda^{q+1})$ by Corollary 2.9, hence, by Theorem 3.1, there exists $v_\xi \in \mathscr{C}^\infty(M; \Lambda^q)$

such that $\mathrm{d}v_\xi = \mathrm{d}\hat{u}_\xi$; thanks to (5.5) we can further assume that $v_\xi \in \operatorname{ran}\mathrm{d}^*$. By (5.6), for each $k \in \mathbb{Z}_+$, there exist $c_k > 0$ and $j \in \mathbb{Z}_+$ such that

$$\|v_\xi\|_{\mathscr{H}^k(M;\Lambda^q)} \leq c_k \|\mathrm{d}v_\xi\|_{\mathscr{H}^j(M;\Lambda^{q+1})}, \quad \forall \xi \in \mathbb{Z}^m.$$

Moreover, since $\mathrm{d}_t u$ is smooth and $\mathcal{F}_\xi(\mathrm{d}_t u) = \mathrm{d}\hat{u}_\xi$, for every $j \in \mathbb{Z}_+$ and $s > 0$, there exists a constant $A_{j,s} > 0$ such that

$$\|\mathrm{d}\hat{u}_\xi\|_{\mathscr{H}^j(M;\Lambda^{q+1})} \leq A_{j,s}(1 + |\xi|^2)^{-s}, \quad \forall \xi \in \mathbb{Z}^m,$$

and so, for $c'_{k,s} \doteq c_k A_{j,s}$, we have

$$\|v_\xi\|_{\mathscr{H}^k(M;\Lambda^q)} \leq c'_{k,s}(1 + |\xi|^2)^{-s}, \quad \forall \xi \in \mathbb{Z}^m. \tag{5.7}$$

The latter ensures that the series

$$v \doteq \frac{1}{(2\pi)^m} \sum_{\xi \in \mathbb{Z}^m} e^{ix\xi} \wedge v_\xi$$

converges in $L^2(\Omega; \underline{\Lambda}^q)$ since

$$\frac{1}{(2\pi)^m} \sum_\xi \int_\Omega \|e^{ix\xi} v_\xi(t)\|^2_{\Lambda^q_t} \mathrm{d}V(t,x) = \sum_\xi \int_M \|v_\xi(t)\|^2_{\Lambda^q_t} \mathrm{d}V_M(t)$$
$$\leq (c'_{0,2m})^2 \sum_\xi (1 + |\xi|^2)^{-2m} < \infty.$$

Moreover, for each $\xi \in \mathbb{Z}^m$, we have that $\hat{v}_\xi = v_\xi$, hence $\mathrm{d}\hat{v}_\xi = \mathrm{d}\hat{u}_\xi$, so $\mathrm{d}_t v = \mathrm{d}_t u$. Estimates (5.7) further ensure that $v \in \mathscr{C}^\infty(\Omega; \underline{\Lambda}^q)$. □

Remark 5.4. We have proved, in Theorem 5.2, that d' is always $\Gamma_{\boldsymbol{\omega}}$-*globally solvable*, that is, if $f \in \mathscr{C}^\infty_{\Gamma_{\boldsymbol{\omega}}}(\Omega; \underline{\Lambda}^{q+1})$ is formally solvable, then there exists $u \in \mathscr{C}^\infty(\Omega; \underline{\Lambda}^q)$ such that $\mathrm{d}'u = f$, with no further hypotheses.

5.1.1. *Reduction in cohomology.* Assume $\omega_1, \dots, \omega_m$ real-valued. The map Θ descends to a linear isomorphism

$$H^q_{\mathrm{d}'}(\mathscr{C}^\infty_{\Gamma_{\boldsymbol{\omega}}}(\Omega)) \cong H^q_{\mathrm{d}_t}(\mathscr{C}^\infty_{\Gamma_{\boldsymbol{\omega}}}(\Omega))$$

thanks to its properties deduced above. Now, we provide a more detailed description of those spaces, for which we introduce some notation. Take $\xi^{(1)}, \dots, \xi^{(r)}$ a basis of $\Gamma_{\boldsymbol{\omega}}$ as a $\mathbb{Z}$-module[4], thus, for each $\xi \in \Gamma_{\boldsymbol{\omega}}$, there exists a unique $\eta \in \mathbb{Z}^r$ such that

$$\xi = \eta_1 \xi^{(1)} + \cdots + \eta_r \xi^{(r)} \doteq \eta \cdot \xi^0$$

in which $\xi^0 \doteq (\xi^{(1)}, \dots, \xi^{(r)})$. We also define a smooth map $\theta : \mathbb{T}^m \to \mathbb{T}^r$ by

$$\theta(x_1, \dots, x_m) \doteq \left(x \cdot \xi^{(1)}, \dots, x \cdot \xi^{(r)}\right).$$

Note that

$$\sum_{j=1}^r (x \cdot \xi^{(j)}) \eta_j = x_1 \Big(\sum_{j=1}^r \xi_1^{(j)} \eta_j \Big) + \cdots + x_m \Big(\sum_{j=1}^r \xi_m^{(j)} \eta_j \Big) = x \cdot \xi$$

thus

$$e^{i\theta(x)\cdot\eta} = \prod_{j=1}^r (e^{ix\cdot\xi^{(j)}})^{\eta_j} = e^{ix\cdot(\eta\cdot\xi^0)} = e^{ix\xi}.$$

[4]One should pay special attention to the case $r = 0$, i.e., $\Gamma_{\boldsymbol{\omega}} = \{0\}$. When properly interpreted (e.g. $\mathbb{T}^0 = \{1\}$, $\mathscr{C}^\infty(\mathbb{T}^0) = \mathbb{C}$, etc.), most of the results below are trivial in that case.

Proposition 5.5. *Assume $\omega_1, \ldots, \omega_m$ real-valued. Then the pullback $\theta^* : \mathscr{C}^\infty(\mathbb{T}^r) \to \mathscr{C}^\infty_{\Gamma_{\boldsymbol{\omega}}}(\mathbb{T}^m)$ is a linear isomorphism.*

Proof. We write an $f \in \mathscr{C}^\infty(\mathbb{T}^r)$ as

$$f = \frac{1}{(2\pi)^r} \sum_{\eta \in \mathbb{Z}^r} e^{iy\eta} \hat{f}_\eta$$

with convergence in $\mathscr{C}^\infty(\mathbb{T}^r)$. Hence

$$\theta^* f = \frac{1}{(2\pi)^r} \sum_{\eta \in \mathbb{Z}^r} e^{i\theta(x)\cdot\eta} \hat{f}_\eta = \frac{1}{(2\pi)^r} \sum_{\eta \in \mathbb{Z}^r} e^{ix(\eta\cdot\xi^0)} \hat{f}_\eta$$

meaning that

$$\mathcal{F}_\xi(\theta^* f) = \begin{cases} (2\pi)^{m-r} \hat{f}_\eta, & \text{if } \xi = \eta \cdot \xi^0 \in \Gamma_{\boldsymbol{\omega}}; \\ 0, & \text{if } \xi \notin \Gamma_{\boldsymbol{\omega}}. \end{cases}$$

In particular, $\theta^* f \in \mathscr{C}^\infty_{\Gamma_{\boldsymbol{\omega}}}(\mathbb{T}^m)$. Moreover, θ^* is injective since $\theta^* f = 0$ implies that $\hat{f}_\eta = 0$ for all $\eta \in \mathbb{Z}^r$. As for surjectivity, given $g \in \mathscr{C}^\infty_{\Gamma_{\boldsymbol{\omega}}}(\mathbb{T}^m)$, let

$$f \doteq \frac{1}{(2\pi)^r} \sum_{\eta \in \mathbb{Z}^r} e^{iy\eta} \hat{g}_{\eta\cdot\xi^0} \in \mathscr{C}^\infty(\mathbb{T}^r)$$

which clearly satisfies $\theta^* f = g$.

Note that $|\xi|$ and $|\eta|$ are comparable, so a series converges in $\mathscr{C}^\infty(\mathbb{T}^r)$ if and only if its image by θ^* converges in $\mathscr{C}^\infty_{\Gamma_{\boldsymbol{\omega}}}(\mathbb{T}^m)$. Indeed, we can complete ξ^0 to a basis of $\mathbb{R}^m$, so we can write $A = (\xi^{(1)}, \ldots, \xi^{(r)}, \xi^{(r+1)}, \ldots, \xi^{(m)})$, in which A is a matrix whose j-th column equals $\xi^{(j)}$. It follows that multiplication of A by $(\eta, 0) \in \mathbb{Z}^m$ yields

$$(\eta, 0) \cdot A = \eta \cdot \xi^0 = \xi.$$

Since A is invertible, it follows that $c|\eta| \leq |\xi| \leq C|\eta|$ for some constants $c, C > 0$. ☐

Now given $q \in \{0, \ldots, n\}$, for any $f \in \mathscr{C}^\infty(\Omega; \underline{\Lambda}^q)$ and $x \in \mathbb{T}^m$, we define $f(x) \in \mathscr{C}^\infty(M; \Lambda^q)$ by "fixing the x-variable", i.e. $f(t, x) = f(x)(t)$ for every $t \in M$. As such, if $\mathrm{d}_t f = 0$, then $\mathrm{d} f(x) = 0$, and, if we pick closed forms $\tau_1, \ldots, \tau_{b_q} \in \mathscr{C}^\infty(M; \Lambda^q)$ such that $[\tau_1], \ldots, [\tau_{b_q}]$ is a basis of $H^q_{\mathrm{dR}}(M)$, then for each $x \in \mathbb{T}^m$ we write

$$[f(x)] = \sum_{\ell=1}^{b_q} a_\ell(x)[\tau_\ell] \quad \text{in } H^q_{\mathrm{dR}}(M)$$

for some uniquely determined coefficients $a_1(x), \ldots, a_{b_q}(x)$. Things can be arranged so that $x \in \mathbb{T}^m \mapsto a_\ell(x) \in \mathbb{C}$ are all smooth.

Indeed, by endowing M with a Riemannian metric, we may pick $\tau_1, \ldots, \tau_{b_q}$ as a basis of $\ker \Delta$, the space of harmonic q-forms on M, which is orthonormal w.r.t. the L^2 inner product on $\mathscr{C}^\infty(M; \Lambda^q)$. Then there exists $u_x \in \mathscr{C}^\infty(M; \Lambda^{q-1})$ such that

$$f(x) = \sum_{\ell=1}^{b_q} a_\ell(x)\tau_\ell + \mathrm{d}_t u_x,$$

thus realizing the last sum as the orthogonal projection of $f(x)$ onto $\ker \Delta$ (see (5.5)):

$$a_\ell(x) = \langle f(x), \tau_\ell \rangle_{L^2(M;\Lambda^q)} = \int_M \langle f(t, x), \tau_\ell(t) \rangle_{\Lambda^q_t} \, \mathrm{d}V_M(t), \quad \ell \in \{1, \ldots, b_q\}, \tag{5.8}$$

are therefore smooth w.r.t. x.

Notice also that $a_1(x), \dots, a_{b_q}(x)$ depend only on the cohomology class of f: if $f^\bullet \doteq f + \mathrm{d}_t v$ for some $v \in \mathscr{C}^\infty(\Omega; \underline{\Lambda}^{q-1})$ then

$$f^\bullet(x) = f(x) + \mathrm{d}_t v(x) = \sum_{\ell=1}^{b_q} a_\ell(x)\tau_\ell + \mathrm{d}_t(u_x + v(x)).$$

It follows from (5.8) that, for every $\xi \in \mathbb{Z}^m$,

$$\mathcal{F}_\xi(a_\ell) = \int_{\mathbb{T}^m} e^{-ix\xi} \int_M \langle f(t,x), \tau_\ell(t)\rangle_{\Lambda_t^q} \,\mathrm{d}V_M(t)\mathrm{d}x = \langle \hat{f}_\xi, \tau_\ell\rangle_{L^2(M;\Lambda^q)}. \tag{5.9}$$

If we further assume that $f \in \mathscr{C}^\infty_{\Gamma_{\boldsymbol{\omega}}}(\Omega; \underline{\Lambda}^q)$, then by (5.9) we have $\mathcal{F}_\xi(a_\ell) = 0$ whenever $\xi \notin \Gamma_{\boldsymbol{\omega}}$, that is, $a_\ell \in \mathscr{C}^\infty_{\Gamma_{\boldsymbol{\omega}}}(\mathbb{T}^m)$, so, by Proposition 5.5, there is a unique $a_\ell^\bullet \in \mathscr{C}^\infty(\mathbb{T}^r)$ such that $a_\ell = \theta^*(a_\ell^\bullet)$. We define

$$\mathsf{T} : H^q_{\mathrm{d}_t}(\mathscr{C}^\infty_{\Gamma_{\boldsymbol{\omega}}}(\Omega)) \longrightarrow \mathscr{C}^\infty(\mathbb{T}^r) \otimes H^q_{\mathrm{dR}}(M)$$

by

$$\mathsf{T}([f]) \doteq \sum_{\ell=1}^{b_q} a_\ell^\bullet \otimes [\tau_\ell].$$

Theorem 5.6. *Assume $\omega_1, \dots, \omega_m$ real-valued. Then T is a linear isomorphism.*

Proof. The surjectivity is obvious; given $a_1^\bullet, \dots, a_{b_q}^\bullet \in \mathscr{C}^\infty(\mathbb{T}^r)$, we have

$$f \doteq \sum_{\ell=1}^{b_q} (\theta^* a_\ell^\bullet)\tau_\ell \Longrightarrow \mathsf{T}([f]) = \sum_{\ell=1}^{b_q} a_\ell^\bullet \otimes [\tau_\ell].$$

As for injectivity, if a d_t-closed $f \in \mathscr{C}^\infty_{\Gamma_{\boldsymbol{\omega}}}(\Omega; \underline{\Lambda}^q)$ is such that $\mathsf{T}([f]) = 0$, then $a_1 = \cdots = a_{b_q} = 0$. Since $\mathrm{d}_t f = 0$, it follows that $\mathrm{d}\hat{f}_\xi = 0$ for every $\xi \in \Gamma_{\boldsymbol{\omega}}$, and so we can find $a_\ell^\xi \in \mathbb{C}$ and $u_\xi \in \mathscr{C}^\infty(M; \Lambda^{q-1})$ such that

$$\hat{f}_\xi = \sum_{\ell=1}^{b_q} a_\ell^\xi \tau_\ell + \mathrm{d}u_\xi, \quad \forall \xi \in \Gamma_{\boldsymbol{\omega}}.$$

Note that for $\xi \notin \Gamma_{\boldsymbol{\omega}}$ we have $\hat{f}_\xi = 0$. By (5.9), we have, for each $\ell \in \{1, \dots, b_q\}$ and $\xi \in \Gamma_{\boldsymbol{\omega}}$, using that $\tau_\ell \in \ker \Delta = \ker \mathrm{d} \cap \ker \mathrm{d}^*$:

$$\mathcal{F}_\xi(a_\ell) = a_\ell^\xi + \langle \mathrm{d}u_\xi, \tau_\ell\rangle_{L^2(M;\Lambda^q)} = a_\ell^\xi.$$

By the assumption that $a_\ell = 0$ for every $\ell \in \{1, \dots, b_q\}$, we reach the conclusion that $\hat{f}_\xi$ is exact for every $\xi \in \mathbb{Z}^m$, that is, f is formally d_t-solvable. By Corollary 4.8 and Lemma 5.3, we conclude that f is d_t-exact, i.e. $[f] = 0$ in $H^q_{\mathrm{d}_t}(\mathscr{C}^\infty_{\Gamma_{\boldsymbol{\omega}}}(\Omega))$. ☐

5.1.2. *Summary of the section.* When $\omega_1, \dots, \omega_m$ are real, we always have linear isomorphisms

$$H^q_{\mathrm{d}'}(\mathscr{C}^\infty(\Omega)) \cong H^q_{\mathrm{d}'}(\mathscr{C}^\infty_{\Gamma_{\boldsymbol{\omega}}}(\Omega)) \oplus H^q_{\mathrm{d}'}(\mathscr{C}^\infty_{\mathbb{Z}^m \setminus \Gamma_{\boldsymbol{\omega}}}(\Omega))$$

and

$$H^q_{\mathrm{d}'}(\mathscr{C}^\infty_{\Gamma_{\boldsymbol{\omega}}}(\Omega)) \cong H^q_{\mathrm{d}_t}(\mathscr{C}^\infty_{\Gamma_{\boldsymbol{\omega}}}(\Omega)) \cong \mathscr{C}^\infty(\mathbb{T}^r) \otimes H^q_{\mathrm{dR}}(M); \tag{5.10}$$

recall that r is the dimension of $\Gamma_{\boldsymbol{\omega}}$ as a $\mathbb{Z}$-module. We finish this section with the following theorem, which introduces a condition that will appear later in a more general situation, and that we will prove to hold in several cases (for notation, see (2.8)).

Theorem 5.7. *Assume $\omega_1, \dots, \omega_m$ real-valued. Suppose that* $\mathrm{d}' : \mathscr{C}^\infty(\Omega; \underline{\Lambda}^{q-1}) \to \mathscr{C}^\infty(\Omega; \underline{\Lambda}^q)$ *has closed range and*

$$H^q_\xi(\mathscr{C}^\infty(M)) = 0, \quad \forall \xi \in \mathbb{Z}^m \setminus \Gamma_{\boldsymbol{\omega}}. \tag{5.11}$$

Then

$$H^q_{\mathrm{d}'}(\mathscr{C}^\infty_{\mathbb{Z}^m \setminus \Gamma_{\boldsymbol{\omega}}}(\Omega)) = \{0\}.$$

In particular, in that case (1.5) *holds.*

Proof. Let $f \in \mathscr{C}^\infty_{\mathbb{Z}^m \setminus \Gamma_{\boldsymbol{\omega}}}(\Omega; \underline{\Lambda}^q)$ be such that $\mathrm{d}' f = 0$. Hence $\mathrm{d}'_\xi \hat{f}_\xi = 0$, and by (5.11), there exists $u_\xi \in \mathscr{C}^\infty(M; \Lambda^{q-1})$ such that $\mathrm{d}'_\xi u_\xi = \hat{f}_\xi$ for every $\xi \in \mathbb{Z}^m \setminus \Gamma_{\boldsymbol{\omega}}$. Therefore f is formally solvable, and by Theorem 5.2, there exists $u \in \mathscr{C}^\infty(\Omega; \underline{\Lambda}^{q-1})$ such that $\mathrm{d}' u = f$. Replacing u by $u_{\mathbb{Z}^m \setminus \Gamma_{\boldsymbol{\omega}}}$ yields the desired result. ☐

5.2. **Global solvability in the first degree.** Theorem 5.2 tells us that the obstruction to global solvability of d' is encoded in the frequencies $\xi \in \mathbb{Z}^m \setminus \Gamma_{\boldsymbol{\omega}}$. This fact motivates us to consider the following Diophantine condition. Our approach in this section follows and adapts ideas from [3], hence we omit some proofs.

Definition 5.8. A collection $\boldsymbol{\omega} = (\omega_1, \dots, \omega_m)$ of real closed 1-forms on M is said to be *strongly simultaneously approximable* if there exist a sequence of closed integral 1-forms $\{\beta_\nu\}_{\nu \in \mathbb{N}} \subset \mathscr{C}^\infty(M; \Lambda^1)$ and $\{\xi_\nu\}_{\nu \in \mathbb{N}} \subset \mathbb{Z}^m \setminus \Gamma_{\boldsymbol{\omega}}$ such that $|\xi_\nu| \to \infty$ and

$$\{|\xi_\nu|^\nu (\xi_\nu \cdot \boldsymbol{\omega} - \beta_\nu)\}_{\nu \in \mathbb{N}} \quad \text{is bounded in } \mathscr{C}^\infty(M; \Lambda^1). \tag{5.12}$$

Otherwise, it is said to be *weakly non-simultaneously approximable.*

Such notions depend only on the classes $[\omega_1], \dots, [\omega_m] \in H^1_{\mathrm{dR}}(M)$. Indeed, if for each $k \in \{1, \dots, m\}$ we have $\omega^\bullet_k \doteq \omega_k + \mathrm{d}\gamma_k$ for some $\gamma_k \in \mathscr{C}^\infty(M; \mathbb{R})$ then

$$\xi \cdot \boldsymbol{\omega}^\bullet = \xi \cdot \boldsymbol{\omega} + \xi \cdot \mathrm{d}\boldsymbol{\gamma}, \quad \forall \xi \in \mathbb{Z}^m,$$

in which $\boldsymbol{\omega}^\bullet \doteq (\omega^\bullet_1, \dots, \omega^\bullet_m)$ and $\boldsymbol{\gamma} \doteq (\gamma_1, \dots, \gamma_m)$: by integrating both sides against an arbitrary 1-cycle, it follows that $\xi \cdot \boldsymbol{\omega}^\bullet$ is integral if and only if so is $\xi \cdot \boldsymbol{\omega}$, that is, $\Gamma_{\boldsymbol{\omega}^\bullet} = \Gamma_{\boldsymbol{\omega}}$. Moreover,

$$|\xi_\nu|^\nu (\xi_\nu \cdot \boldsymbol{\omega} - \beta_\nu) = |\xi_\nu|^\nu (\xi_\nu \cdot \boldsymbol{\omega}^\bullet - (\beta_\nu + \xi_\nu \cdot \mathrm{d}\boldsymbol{\gamma}))$$

and since each $\beta_\nu + \xi_\nu \cdot \mathrm{d}\boldsymbol{\gamma}$ is integral, it follows that $\boldsymbol{\omega}^\bullet$ is strongly simultaneously approximable whenever $\boldsymbol{\omega}$ is.

As a finitely generated Abelian group, $H_1(M; \mathbb{Z})$ admits a primary decomposition $\mathbb{Z}^d \oplus \mathbb{Z}_{p_1} \oplus \dots \oplus \mathbb{Z}_{p_t}$. We consider $\sigma_1, \dots, \sigma_d$ smooth 1-cycles whose homological classes form a basis of its free part $\mathbb{Z}^d$, and the map $I : H^1_{\mathrm{dR}}(M; \mathbb{R}) \to \mathbb{R}^d$ given by

$$I([\alpha]) \doteq \frac{1}{2\pi} \left(\int_{\sigma_1} \alpha, \dots, \int_{\sigma_d} \alpha \right).$$

Since $\int_\sigma \alpha = 0$ if σ belongs to some $\mathbb{Z}_{p_i}$, it follows that a closed 1-form α is integral (resp. rational) if and only if $I([\alpha]) \in \mathbb{Z}^d$ (resp. $I[\alpha] \in \mathbb{Q}^d$). By de Rham Theorem, we conclude that I is a linear isomorphism and that the classes of $\sigma_1, \dots, \sigma_d$, regarded as elements of $H_1(M; \mathbb{R}) \cong H^1_{\mathrm{dR}}(M; \mathbb{R})^*$, form an $\mathbb{R}$-basis for the latter vector space. We associate to $\boldsymbol{\omega}$ the following family of vectors:

$$v_\ell \doteq \frac{1}{2\pi} \left(\int_{\sigma_\ell} \omega_1, \dots, \int_{\sigma_\ell} \omega_m \right) \in \mathbb{R}^m, \quad \ell \in \{1, \dots, d\}. \tag{5.13}$$

These, as well, depend only on $[\omega_1], \dots, [\omega_m] \in H^1_{\mathrm{dR}}(M)$ by Stokes Theorem.

Proposition 5.9. *The collection* $\boldsymbol{\omega}$ *is strongly simultaneously approximable if and only if there are sequences* $\{\eta_\nu\}_{\nu\in\mathbb{N}} \subset \mathbb{Z}^d$ *and* $\{\xi_\nu\}_{\nu\in\mathbb{N}} \subset \mathbb{Z}^m \setminus \Gamma_{\boldsymbol{\omega}}$ *such that* $|\xi_\nu| \to \infty$ *and*

$$\{|\xi_\nu|^\nu(\xi_\nu \cdot v_\ell - \eta_{\nu\ell})\}_{\nu\in\mathbb{N}} \quad \textit{is bounded in } \mathbb{R} \textit{ for every } \ell \in \{1,\dots,d\}. \tag{5.14}$$

Proof. Let $Z^1 \subset \mathscr{C}^\infty(M;\Lambda^1)$ be the space of real, closed 1-forms, endowed with the subspace topology, and define $J : Z^1 \to \mathbb{R}^d$ by $J(\alpha) \doteq I([\alpha])$. It is continuous – as the composition of I (a linear map between finite dimensional spaces) and the projection $Z^1 \to H^1_{\mathrm{dR}}(M;\mathbb{R})$ – hence maps bounded sets to bounded sets. If $\boldsymbol{\omega}$ is strongly simultaneously approximable, there exist a sequence of closed integral 1-forms $\{\beta_\nu\}_{\nu\in\mathbb{N}} \subset \mathscr{C}^\infty(M;\Lambda^1)$ and $\{\xi_\nu\}_{\nu\in\mathbb{N}} \subset \mathbb{Z}^m \setminus \Gamma_{\boldsymbol{\omega}}$ such that $|\xi_\nu| \to \infty$ such that (5.12) holds. Letting $\eta_\nu \doteq I([\beta_\nu]) \in \mathbb{Z}^d$, we have, for each $\ell \in \{1,\dots,d\}$, that

$$J(|\xi_\nu|^\nu(\xi_\nu \cdot \boldsymbol{\omega} - \beta_\nu))_\ell = |\xi_\nu|^\nu(J(\xi_\nu \cdot \boldsymbol{\omega}) - J(\beta_\nu))_\ell = |\xi_\nu|^\nu(\xi \cdot v_\ell - \eta_{\nu\ell}) \tag{5.15}$$

must be the ℓ-th coordinate of a bounded sequence in $\mathbb{R}^d$, yielding our first claim.

For the converse, we start with a digression. Let $\ker\Delta \subset Z^1$ denote the space of harmonic 1-forms w.r.t. some Riemannian metric, which is well-known to be finite dimensional and hence has a well-defined norm topology; we have more, the map $\alpha \in \ker\Delta \mapsto [\alpha] \in H^1_{\mathrm{dR}}(M;\mathbb{R})$ is a linear isomorphism. Moreover, $\ker\Delta$ inherits a Fréchet topology from $\mathscr{C}^\infty(M;\Lambda^1)$, which matches the former one, by [11, Theorem 9.1]. Therefore the restriction $J : \ker\Delta \to \mathbb{R}^d$ is a linear isomorphism, hence a topological one by finiteness; in particular, an $S \subset \ker\Delta$ is bounded if and only if $J(S) \subset \mathbb{R}^d$ is bounded. We further pick $\vartheta_1,\dots,\vartheta_d$ a basis of $\ker\Delta$ dual to $[\sigma_1],\dots,[\sigma_d] \in H_1(M;\mathbb{R})$, in the sense that

$$\frac{1}{2\pi}\int_{\sigma_\ell} \vartheta_{\ell'} = \delta_{\ell\ell'}, \quad \forall \ell,\ell' \in \{1,\dots,d\}.$$

Such a choice makes $\vartheta_1,\dots,\vartheta_d$ integral.

Now, suppose there are sequences $\{\eta_\nu\}_{\nu\in\mathbb{N}} \subset \mathbb{Z}^d$ and $\{\xi_\nu\}_{\nu\in\mathbb{N}} \subset \mathbb{Z}^m \setminus \Gamma_{\boldsymbol{\omega}}$ such that $|\xi_\nu| \to \infty$ and (5.14) holds. Then

$$\beta_\nu \doteq \sum_{\ell=1}^d \eta_{\nu\ell}\vartheta_\ell \in \ker\Delta$$

is integral and $J(\beta_\nu) = \eta_\nu$ for every $\nu \in \mathbb{N}$. Since (5.15) holds once more for each $\ell \in \{1,\dots,d\}$, we have from (5.14) that $\{J(|\xi_\nu|^\nu(\xi_\nu \cdot \boldsymbol{\omega} - \beta_\nu))\}_{\nu\in\mathbb{N}}$ must be a bounded sequence in $\mathbb{R}^d$: the conclusion follows from our digression, since we may assume w.l.o.g. $\omega_1,\dots,\omega_m \in \ker\Delta$. □

Corollary 5.10. *The collection* $\boldsymbol{\omega}$ *is weakly non-simultaneously approximable if and only if there exist* $C,\rho > 0$ *such that (see* (5.13)*)*

$$\max_{1\le\ell\le d} |\xi \cdot v_\ell - \eta_\ell| \ge C|\xi|^{-\rho}, \quad \forall \eta \in \mathbb{Z}^d,\ \forall \xi \in \mathbb{Z}^m \setminus \Gamma_{\boldsymbol{\omega}}. \tag{5.16}$$

Remark 5.11.

(1) Notice that

$$\Gamma_{\boldsymbol{\omega}} = \{\xi \in \mathbb{Z}^m \ ; \ \xi \cdot v_\ell \in \mathbb{Z},\ \forall \ell \in \{1,\dots,d\}\};$$

(2) The inequality (5.16) is equivalent to condition (DC) in [6, Section 2] for the $d \times m$ matrix whose rows are $v_1,\dots,v_d$. When $\Gamma_{\boldsymbol{\omega}} = \{0\}$, it recovers the standard notion of non-simultaneous approximability for collections of

vectors in [7, Definition 1.1] (see further connections with previous conditions in the literature there);

(3) If $v_1, \dots, v_d \in \mathbb{Q}^m$, which corresponds to case when $\omega_1, \dots, \omega_m$ are all rational 1-forms, we pick a non-zero $\lambda \in \mathbb{Z}_+$ such that $\lambda v_\ell \in \mathbb{Z}^m$ for every $\ell \in \{1, \dots, d\}$. Hence, given $\xi \in \mathbb{Z}^m \setminus \Gamma_{\boldsymbol{\omega}}$, there must exist an $\ell_0 \in \{1, \dots, d\}$ for which $\xi \cdot v_{\ell_0} \notin \mathbb{Z}$, thus, for any $\eta \in \mathbb{Z}^d$, we must have

$$\begin{aligned} \xi \cdot v_{\ell_0} - \eta_{\ell_0} \neq 0 &\Longrightarrow \lambda(\xi \cdot v_{\ell_0} - \eta_{\ell_0}) \in \mathbb{Z} \setminus 0 \\ &\Longrightarrow \lambda \max_{1 \leq \ell \leq d} |\xi \cdot v_\ell - \eta_\ell| \geq 1, \end{aligned}$$

that is, (5.16) holds with $C \doteq \lambda^{-1}$ and $\rho > 0$ arbitrary.

Now, we can state our characterization of solvability in the first degree in terms of the Diophantine condition just introduced (reworded as Theorem 1.2 in the Introduction).

Theorem 5.12. *Assume $\omega_1, \dots, \omega_m$ real-valued. The map $\mathrm{d}' : \mathscr{C}^\infty(\Omega) \to \mathscr{C}^\infty(\Omega; \underline{\Lambda}^1)$ has closed range if and only if the collection $\boldsymbol{\omega}$ is weakly non-simultaneously approximable.*

Proof. ($\Longrightarrow$): Suppose there exist a sequence of integral 1-forms $\{\beta_\nu\}_{\nu \in \mathbb{N}} \subset \mathscr{C}^\infty(M; \Lambda^1)$ and $\{\xi_\nu\}_{\nu \in \mathbb{N}} \subset \mathbb{Z}^m \setminus \Gamma_{\boldsymbol{\omega}}$ such that $|\xi_\nu| \to \infty$ and (5.12) holds. We may assume that $\xi_\nu \neq \xi_{\nu'}$ whenever $\nu \neq \nu'$. Let $\varphi_\nu \in \mathscr{C}^\infty(\tilde{M}; \mathbb{R})$ be such that $\mathrm{d}\varphi_\nu = \pi^* \beta_\nu$ and define

$$f(t, x) \doteq \frac{1}{(2\pi)^m} \sum_{\nu=1}^{\infty} e^{ix\xi_\nu - i\varphi_\nu \circ \pi^{-1}(t)} (\xi_\nu \cdot \boldsymbol{\omega} - \beta_\nu)(t),$$

which, we claim, is smooth and formally solvable. As we saw in (5.2), we have that

$$M \ni t \longmapsto e^{-i\varphi_\nu \circ \pi^{-1}(t)}$$

is a well-defined smooth function since β_ν is integral. Therefore, we define, for every $\xi \in \mathbb{Z}^m$, a smooth 1-form on M

$$f_\xi \doteq \begin{cases} e^{-i\varphi_\nu \circ \pi^{-1}} (\xi_\nu \cdot \boldsymbol{\omega} - \beta_\nu), & \text{if } \xi = \xi_\nu, \\ 0, & \text{otherwise.} \end{cases}$$

Note that, in order to prove that f is a well-defined smooth 1-form, it is enough to prove that every point of M belongs to a coordinate system $(U; t_1, \dots, t_n)$, with $\alpha \in \mathbb{Z}_+^n$ and $s > 0$, there exists $C > 0$ such that

$$\sup_U \|\partial_t^\alpha f_{\xi_\nu}\| \leq C(1 + |\xi_\nu|)^{-s}, \quad \forall \nu \in \mathbb{N},$$

in which the norm $\| \cdot \|$ in the left-hand side is the sum of the absolute values of the coordinate components of the 1-form w.r.t. the local frame $\mathrm{d}t_1, \dots, \mathrm{d}t_n$. By hypothesis (5.12), we may assume that, for each $\gamma \in \mathbb{Z}_+^n$, there exists $B > 0$ such that

$$\sup_M \|\partial_t^\gamma (\xi_\nu \cdot \boldsymbol{\omega} - \beta_\nu)\| \leq B |\xi_\nu|^{-\nu}, \quad \forall \nu \in \mathbb{N},$$

It remains to prove that the derivatives of $e^{-i\varphi_\nu \circ \pi^{-1}}$ are bounded by some constant times a power of $|\xi_\nu|$. We can assume that U is small enough in order to have a diffeomorphism $\pi_U : \tilde{U} \to U$ and so

$$e^{-i\varphi_\nu \circ \pi^{-1}(t)} = e^{-i\varphi_\nu \circ \pi_U^{-1}(t)}$$

for every $t \in U$ and so, since φ_ν is real, our claim follows from the identity

$$\sum_{j=1}^{n} \frac{\partial(\varphi_\nu \circ \pi_U^{-1})}{\partial t_j} \mathrm{d}t_j = \beta_\nu = -(\xi_\nu \cdot \boldsymbol{\omega} - \beta_\nu) + \xi_\nu \cdot \boldsymbol{\omega},$$

since the right-hand of the equality above clearly is bounded by a constant times $|\xi_\nu|$, hence proving that $f \in \mathscr{C}^\infty(\Omega; \underline{\Lambda}^1)$. The next step is to prove that f is formally solvable. Since $\hat{f}_\xi = 0$, if $\xi \notin \{\xi_\nu\}_{\nu \in \mathbb{N}}$, it follows that $\hat{f}_\xi$ is d'_ξ-exact for such values of ξ; for $\xi = \xi_\nu$ we have

$$\mathrm{d}'_{\xi_\nu} e^{-i\varphi_\nu \circ \pi^{-1}} = \mathrm{d}e^{-i\varphi_\nu \circ \pi^{-1}} + ie^{-i\varphi_\nu \circ \pi^{-1}}(\xi_\nu \cdot \boldsymbol{\omega}) = -ie^{-i\varphi_\nu \circ \pi^{-1}}\beta_\nu + ie^{-i\varphi_\nu \circ \pi^{-1}}(\xi_\nu \cdot \boldsymbol{\omega}) = \hat{f}_{\xi_\nu}.$$

It remains to prove that there is no $u \in \mathscr{C}^\infty(\Omega)$ such that $\mathrm{d}'u = f$. If such an u exists, then $\mathrm{d}'_\xi \hat{u}_\xi = \hat{f}_\xi$ for every $\xi \in \mathbb{Z}^m$, in particular $\hat{u}_{\xi_\nu} - e^{-i\varphi_\nu \circ \pi^{-1}} \in \ker \mathrm{d}'_{\xi_\nu}$ for every $\nu \in \mathbb{N}$. But since $\xi_\nu \notin \Gamma_{\boldsymbol{\omega}}$, we have by [4, Lemma 2.1] that $\ker \mathrm{d}'_{\xi_\nu} = \{0\}$, hence $\hat{u}_{\xi_\nu} = e^{-i\varphi_\nu \circ \pi^{-1}}$ for every $\nu \in \mathbb{N}$ and then

$$\|\hat{u}_{\xi_\nu}\|^2_{L^2(M)} = \int_M \mathrm{d}V_M, \quad \nu \in \mathbb{N},$$

does not decrease fast, contradicting the smoothness of u.

($\Longleftarrow$): Suppose that $f \in \mathscr{C}^\infty(\Omega; \underline{\Lambda}^1)$ is formally solvable, i.e., for every $\xi \in \mathbb{Z}^m$, there exists $u_\xi \in \mathscr{C}^\infty(M)$ such that $\mathrm{d}'_\xi u_\xi = \hat{f}_\xi$. Thanks to Theorem 5.2, we only need to prove that there exists $u \in \mathscr{C}^\infty(\Omega)$ such that $\mathrm{d}'u = f_{\mathbb{Z}^m \setminus \Gamma_{\boldsymbol{\omega}}}$. The natural candidate is

$$u \doteq \frac{1}{(2\pi)^m} \sum_{\xi \in \mathbb{Z}^m \setminus \Gamma_{\boldsymbol{\omega}}} e^{ix\xi} u_\xi,$$

and it suffices to prove that the series above converges in $\mathscr{C}^\infty(\Omega)$. This claim is local, so we must verify that every point of M belongs to a coordinate ball U enjoying the following property: for each $\alpha \in \mathbb{Z}^n_+$ and $s > 0$, there exists $C > 0$ such that

$$|\partial_t^\alpha u_\xi(t)| \leq C(1 + |\xi|)^{-s}, \quad \forall \xi \in \mathbb{Z}^m \setminus \Gamma_{\boldsymbol{\omega}}, \ \forall t \in U. \tag{5.17}$$

We write

$$\omega_k = \sum_{j=1}^{n} \omega_{kj} \mathrm{d}t_j, \quad \omega_{kj} \in \mathscr{C}^\infty(U),$$

on U for $k \in \{1, \dots, m\}$, and

$$f = \sum_{j=1}^{n} f_j \ \mathrm{d}t_j, \quad f_j \in \mathscr{C}^\infty(U \times \mathbb{T}^m),$$

on $U \times \mathbb{T}^m$. We may assume ω_{kj} is bounded in the sup norm in U.

Lemma 5.13. *Suppose that given $s > 0$ there exists $C > 0$ such that*

$$|u_\xi(t)| \leq C(1 + |\xi|)^{-s}, \quad \forall \xi \in \mathbb{Z}^m \setminus \Gamma_{\boldsymbol{\omega}}, \quad \forall t \in U. \tag{5.18}$$

Then (5.17) holds true.

Proof. We proceed by induction on $|\alpha|$; the base $|\alpha| = 0$ is (5.18). For the general case, the equality $\mathrm{d}'_\xi u_\xi = \hat{f}_\xi$ can be written in U as

$$\frac{\partial u_\xi}{\partial t_j} + i \sum_{k=1}^{m} \xi_k \omega_{kj} u_\xi = \mathcal{F}_\xi(f_j), \quad j \in \{1, \dots, n\}.$$

If $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_j > 0$ and $\beta \doteq \alpha - e_j$, in which e_j is the j-th vector of the canonical basis of $\mathbb{R}^n$, then

$$\partial_t^\alpha u_\xi = \partial_t^\beta \partial_{t_j} u_\xi = \partial_t^\beta \left(\mathcal{F}_\xi(f_j) - i \sum_{k=1}^{m} \xi_k \omega_{kj} u_\xi \right),$$

and we use that f is smooth and $s > 0$ in (5.17) is arbitrary. □

We fix $t_0 \in M$ and prove that (5.18) holds in a coordinate ball U centered at t_0. As in Section 5.1, we assume that $\pi_U : \tilde{U} \to U$ is a diffeomorphism and consider the functions $\boldsymbol{\psi}_\xi \in \mathscr{C}^\infty(\tilde{M}; \mathbb{R})$. Then on U we have, for all $\xi \in \mathbb{Z}^m$,

$$\mathrm{d}(e^{i\boldsymbol{\psi}_\xi \circ \pi_U^{-1}} u_\xi) = e^{i\boldsymbol{\psi}_\xi \circ \pi_U^{-1}} \mathrm{d}'_\xi u_\xi = e^{i\boldsymbol{\psi}_\xi \circ \pi_U^{-1}} \hat{f}_\xi$$

which, integrating along any curve γ in U connecting t_0 to t yields, by Stokes Theorem,

$$u_\xi(t) = e^{i(\boldsymbol{\psi}_\xi \circ \pi_U^{-1}(t_0) - \boldsymbol{\psi}_\xi \circ \pi_U^{-1}(t))} u_\xi(t_0) + e^{-i\boldsymbol{\psi}_\xi \circ \pi_U^{-1}(t)} \int_\gamma e^{i\boldsymbol{\psi}_\xi \circ \pi_U^{-1}} \hat{f}_\xi.$$

Since f is smooth and $\boldsymbol{\psi}_\xi$ is real-valued, the last equality shows that (5.18) holds true provided that we prove that, given $s > 0$, there exists $B > 0$ such that

$$|u_\xi(t_0)| \leq B(1 + |\xi|)^{-s}, \quad \forall \xi \in \mathbb{Z}^m \setminus \Gamma_{\boldsymbol{\omega}},$$

which we do next.

By the Hurewicz's Theorem, we may assume that $\sigma_1, \ldots, \sigma_d$ all have base point t_0. We lift them to curves $\tilde{\sigma}_\ell : [0, 1] \to \tilde{M}$ whose endpoints we denote by

$$Q_0 \doteq \tilde{\sigma}_\ell(0), \quad Q_\ell \doteq \tilde{\sigma}_\ell(1), \quad \ell \in \{1, \ldots, d\}.$$

We integrate along $\tilde{\sigma}_\ell$ both sides of the equality

$$\mathrm{d}(e^{i\boldsymbol{\psi}_\xi} \pi^* u_\xi) = e^{i\boldsymbol{\psi}_\xi} \pi^* \hat{f}_\xi$$

and use Stokes Theorem again to conclude that, since $\pi(Q_\ell) = \pi(Q_0) = t_0$,

$$\left(e^{i(\boldsymbol{\psi}_\xi(Q_\ell) - \boldsymbol{\psi}_\xi(Q_0))} - 1\right) u_\xi(t_0) = e^{i\boldsymbol{\psi}_\xi(Q_0)} \int_{\tilde{\sigma}_\ell} e^{i\boldsymbol{\psi}_\xi} \pi^* \hat{f}_\xi.$$

Moreover,

$$\boldsymbol{\psi}_\xi(Q_\ell) - \boldsymbol{\psi}_\xi(Q_0) = \int_{\partial\tilde{\sigma}_\ell} \boldsymbol{\psi}_\xi = \int_{\tilde{\sigma}_\ell} \mathrm{d}\boldsymbol{\psi}_\xi = \int_{\tilde{\sigma}_\ell} \pi^*(\xi \cdot \boldsymbol{\omega}) = \int_{\sigma_\ell} \xi \cdot \boldsymbol{\omega}, \tag{5.19}$$

so, for $\xi \in \mathbb{Z}^m \setminus \Gamma_{\boldsymbol{\omega}}$, there exists ℓ such that $\boldsymbol{\psi}_\xi(Q_\ell) - \boldsymbol{\psi}_\xi(Q_0) \notin 2\pi\mathbb{Z}$, hence

$$u_\xi(t_0) = \frac{e^{i\boldsymbol{\psi}_\xi(Q_0)}}{\left(e^{i(\boldsymbol{\psi}_\xi(Q_\ell) - \boldsymbol{\psi}_\xi(Q_0))} - 1\right)} \int_{\tilde{\sigma}_\ell} e^{i\boldsymbol{\psi}_\xi} \pi^* \hat{f}_\xi$$

for any such ℓ. Using again that f is smooth and $\boldsymbol{\psi}_\xi$ is real-valued, it suffices to prove that there exist $c, \rho > 0$ such that

$$\max_{1 \leq \ell \leq d} \left| e^{i(\boldsymbol{\psi}_\xi(Q_\ell) - \boldsymbol{\psi}_\xi(Q_0))} - 1 \right| \geq c|\xi|^{-\rho}, \quad \forall \xi \in \mathbb{Z}^m \setminus \Gamma_{\boldsymbol{\omega}}. \tag{5.20}$$

To conclude, we use the following technical lemma, whose proof we omit.

Lemma 5.14. *There exists $\epsilon > 0$ such that, for each $\xi \in \mathbb{Z}^m \setminus \Gamma_{\boldsymbol{\omega}}$, at least one of the following conditions holds:*

(1) *For every $\ell \in \{1, \dots, d\}$ there exists $p_\ell \in \mathbb{Z}$ such that*
$$\left|e^{i(\boldsymbol{\psi}_\xi(Q_0)-\boldsymbol{\psi}_\xi(Q_\ell))} - 1\right| \geq \frac{1}{2}\left|\boldsymbol{\psi}_\xi(Q_0) - \boldsymbol{\psi}_\xi(Q_\ell) - 2\pi p_\ell\right|;$$
(2) *There exists $\ell \in \{1, \dots, d\}$ such that*
$$\left|e^{i(\boldsymbol{\psi}_\xi(Q_0)-\boldsymbol{\psi}_\xi(Q_\ell))} - 1\right| \geq \epsilon.$$

By Lemma 5.14, we may assume that, for every $\xi \in \mathbb{Z}^m \setminus \Gamma_{\boldsymbol{\omega}}$, there are $p_1, \dots, p_d \in \mathbb{Z}$ such that
$$\max_{1\leq\ell\leq d}\left|e^{i(\boldsymbol{\psi}_\xi(Q_\ell)-\boldsymbol{\psi}_\xi(Q_0))} - 1\right| \geq \frac{1}{2}\max_{1\leq\ell\leq d}\left|\boldsymbol{\psi}_\xi(Q_\ell) - \boldsymbol{\psi}_\xi(Q_0) - 2\pi p_\ell\right|.$$
It follows from (5.19) and (5.13) that
$$\max_{1\leq\ell\leq d}\left|e^{i(\boldsymbol{\psi}_\xi(Q_\ell)-\boldsymbol{\psi}_\xi(Q_0))} - 1\right| \geq \pi \max_{1\leq\ell\leq d}|\xi \cdot v_\ell - p_\ell|.$$
Since we are assuming that $\boldsymbol{\omega}$ is weakly non-simultaneously approximable, we can use Corollary 5.10 to conclude that (5.20) holds. $\square$

6. Formal Fourier analysis of cohomology spaces

Allowing again $\omega_1, \dots, \omega_m$ to be complex-valued, we add a third characterization of solvability to our list in Corollary 4.8.

Proposition 6.1. *The following are equivalent:*

(1) *For every formally solvable $f \in \mathscr{C}^\infty(\Omega; \underline{\Lambda}^q)$ there exists $u \in \mathscr{C}^\infty(\Omega; \underline{\Lambda}^{q-1})$ such that $\mathrm{d}'u = f$.*
(2) *The map*
$$H^q_{\mathrm{d}'}(\mathscr{C}^\infty(\Omega)) \longrightarrow \prod_{\xi\in\mathbb{Z}^m} H^q_\xi(\mathscr{C}^\infty(M)) \tag{6.1}$$
given by $[f] \mapsto ([\hat{f}_\xi])_{\xi\in\mathbb{Z}^m}$ is injective.

Proof. First, we prove that solvability implies the injectivity of (6.1). Let $[f] \in H^q_{\mathrm{d}'}(\mathscr{C}^\infty(\Omega))$ be such that $[\hat{f}_\xi] = 0$ in $H^q_\xi(\mathscr{C}^\infty(M))$ for every $\xi \in \mathbb{Z}^m$, i.e., $\hat{f}_\xi$ is d'_ξ-exact for each $\xi \in \mathbb{Z}^m$, i.e., f is formally solvable. By the solvability, it follows that $[f] = 0$ in $H^q_{\mathrm{d}'}(\mathscr{C}^\infty(\Omega))$, proving the injectivity.

Conversely, a formally solvable $f \in \mathscr{C}^\infty(\Omega; \underline{\Lambda}^q)$ is d'-closed, hence it determines a class $[f] \in H^q_{\mathrm{d}'}(\mathscr{C}^\infty(\Omega))$. Formal solvability of f ensures that $[\hat{f}_\xi] = 0$ in $H^q_\xi(\mathscr{C}^\infty(M))$ for every $\xi \in \mathbb{Z}^m$. Thus, if (6.1) is injective, we have $[f] = 0$ in $H^q_{\mathrm{d}'}(\mathscr{C}^\infty(\Omega))$, which is precisely what we wanted to prove. $\square$

Next, we study the map induced by $\mathcal{E}_\xi$ in cohomology. Unlike in the previous proposition for the map induced by $\mathcal{F}_\xi$, its injectivity always holds true and requires no extra hypotheses.

Proposition 6.2. *For each $\xi \in \mathbb{Z}^m$, $\mathcal{E}_\xi$ induces an injection $H^q_\xi(\mathscr{C}^\infty(M)) \hookrightarrow H^q_{\mathrm{d}'}(\mathscr{C}^\infty(\Omega))$. Their direct sum induces the following injection:*
$$\bigoplus_{\xi\in\mathbb{Z}^m} H^q_\xi(\mathscr{C}^\infty(M)) \longrightarrow H^q_{\mathrm{d}'}(\mathscr{C}^\infty(\Omega)). \tag{6.2}$$

Proof. Let $[f] \in H^q_\xi(\mathscr{C}^\infty(M))$. Then, by Corollary 2.9,

$$\mathrm{d}'\mathcal{E}_\xi f = \mathcal{E}_\xi \mathrm{d}'_\xi f = 0$$

and the class of $\mathcal{E}_\xi f$ in $H^q_{\mathrm{d}'}(\mathscr{C}^\infty(\Omega))$ is well-defined. Indeed, if $[f^\bullet] = [f]$ in $H^q_\xi(\mathscr{C}^\infty(M))$, then there exists $v \in \mathscr{C}^\infty(M; \Lambda^{q-1})$ such that $\mathrm{d}'_\xi v = f - f^\bullet$, hence

$$\mathcal{E}_\xi f - \mathcal{E}_\xi f^\bullet = \mathcal{E}_\xi \mathrm{d}'_\xi v = \mathrm{d}'\mathcal{E}_\xi v.$$

Thus $[\mathcal{E}_\xi f^\bullet] = [\mathcal{E}_\xi f]$ in $H^q_{\mathrm{d}'}(\mathscr{C}^\infty(\Omega))$. Moreover, if $[\mathcal{E}_\xi f] = 0$ in $H^q_{\mathrm{d}'}(\mathscr{C}^\infty(\Omega))$, then there exists $u \in \mathscr{C}^\infty(\Omega; \underline{\Lambda}^{q-1})$ such that $\mathrm{d}'u = \mathcal{E}_\xi f$, hence, by Lemma 2.6, we have

$$f = \mathcal{F}_\xi \mathcal{E}_\xi f = \mathcal{F}_\xi \mathrm{d}'u = \mathrm{d}'_\xi \hat{u}_\xi$$

hence the assignment $[f] \mapsto [\mathcal{E}_\xi f]$ is injective. More generally, if

$$\sum_{\xi \in \mathbb{Z}^m} [f_\xi] \in \bigoplus_{\xi \in \mathbb{Z}^m} H^q_\xi(\mathscr{C}^\infty(M))$$

is such that

$$\sum_{\xi \in \mathbb{Z}^m} \mathcal{E}_\xi f_\xi \in \mathscr{C}^\infty(\Omega; \underline{\Lambda}^q) \quad \text{is } \mathrm{d}'\text{-exact}$$

(the smoothness is possible by taking 0 as representative whenever $[f_\xi] = 0$), then for some $u \in \mathscr{C}^\infty(\Omega; \underline{\Lambda}^{q-1})$, we have

$$\mathrm{d}'u = \sum_{\xi \in \mathbb{Z}^m} \mathcal{E}_\xi f_\xi = \frac{1}{(2\pi)^m} \sum_{\xi \in \mathbb{Z}^m} e^{ix\xi} \wedge f_\xi \Longrightarrow \mathrm{d}'_\xi \hat{u}_\xi = f_\xi, \quad \forall \xi \in \mathbb{Z}^m,$$

that is, $[f_\xi] = 0$ in $H^q_\xi(\mathscr{C}^\infty(M))$ for every $\xi \in \mathbb{Z}^m$. ☐

7. General finiteness theorems

Assume $\omega_1, \dots, \omega_m$ complex-valued. The next result is the heart of our forthcoming analysis.

Theorem 7.1. *Given $q \in \{0, \dots, n\}$, the following are equivalent:*

(1) *$H^q_{\mathrm{d}'}(\mathscr{C}^\infty(\Omega))$ is finite dimensional;*
(2) *$\mathrm{d}' : \mathscr{C}^\infty(\Omega; \underline{\Lambda}^{q-1}) \to \mathscr{C}^\infty(\Omega; \underline{\Lambda}^q)$ has closed range and there exists a finite set $F \subset \mathbb{Z}^m$ such that*

$$H^q_\xi(\mathscr{C}^\infty(M)) = \{0\}, \quad \forall \xi \in \mathbb{Z}^m \setminus F.$$

In this case:

$$H^q_{\mathrm{d}'}(\mathscr{C}^\infty(\Omega)) \cong \bigoplus_{\xi \in F} H^q_\xi(\mathscr{C}^\infty(M)).$$

Proof. It is a standard argument in Functional Analysis that, if $H^q_{\mathrm{d}'}(\mathscr{C}^\infty(\Omega))$ is finite dimensional, then the denominator in (1.4) is a closed subspace of $\mathscr{C}^\infty(\Omega; \underline{\Lambda}^q)$. Moreover, since the map (6.2) is injective, finiteness of $H^q_{\mathrm{d}'}(\mathscr{C}^\infty(\Omega))$ entails that only finitely many terms in that direct sum are non-zero. Conversely, if $\mathrm{d}' : \mathscr{C}^\infty(\Omega; \underline{\Lambda}^{q-1}) \to \mathscr{C}^\infty(\Omega; \underline{\Lambda}^q)$ has closed range, then the map (6.1) is injective (by Corollary 4.8 and Proposition 6.1), i.e. $H^q_{\mathrm{d}'}(\mathscr{C}^\infty(\Omega))$ injects into

$$\prod_{\xi \in F} H^q_\xi(\mathscr{C}^\infty(M))$$

which is finite dimensional since so is every factor and F is finite. It follows that $\dim H^q_{\mathrm{d}'}(\mathscr{C}^\infty(\Omega)) < \infty$, and we have a sequence of injections (6.2)-(6.1) between finite dimensional spaces

$$\bigoplus_{\xi\in F} H^q_\xi(\mathscr{C}^\infty(M)) \longrightarrow H^q_{\mathrm{d}'}(\mathscr{C}^\infty(\Omega)) \longrightarrow \prod_{\xi\in F} H^q_\xi(\mathscr{C}^\infty(M))$$

with isomorphic endpoints. The conclusion follows. □

Below we will use that, by (3.4) and the definition of $\Gamma_{\boldsymbol{\omega}}$ (4.3):

$$\xi \in \Gamma_{\boldsymbol{\omega}} \Longrightarrow \mathscr{S}_{\xi\cdot\boldsymbol{\omega}} \cong \mathscr{S}_0 \Longrightarrow H^q_\xi(\mathscr{C}^\infty(M)) \cong H^q_{\mathrm{dR}}(M), \quad \forall q \in \{0,\dots,n\}. \tag{7.1}$$

Proposition 7.2. *Assume $\Gamma_{\boldsymbol{\omega}} \neq \{0\}$. Given $q \in \{0,\dots,n\}$, if $H^q_{\mathrm{d}'}(\mathscr{C}^\infty(\Omega))$ is finite dimensional, then $H^q_\xi(\mathscr{C}^\infty(M)) = \{0\}$ for every $\xi \in \mathbb{Z}^m$ – in particular, $H^q_{\mathrm{dR}}(M) = \{0\}$ –, hence $H^q_{\mathrm{d}'}(\mathscr{C}^\infty(\Omega)) = \{0\}$.*

Proof. Pick a non-zero $\eta \in \Gamma_{\boldsymbol{\omega}}$ and a non-vanishing function $f \in \mathscr{S}_\eta(M)$. Given $u \in \mathscr{C}^\infty(M;\Lambda^q)$ and $\xi \in \mathbb{Z}^m$, we have that

$$\mathrm{d}(fu)+i((\xi+\eta)\cdot\boldsymbol{\omega})\wedge(fu) = \mathrm{d}f\wedge u + f\mathrm{d}u + if(\xi\cdot\boldsymbol{\omega}+\eta\cdot\boldsymbol{\omega})\wedge u) = f(\mathrm{d}u + i(\xi\cdot\boldsymbol{\omega})\wedge u)$$

hence multiplication by f maps $H^q_\xi(\mathscr{C}^\infty(M))$ to $H^q_{\xi+\eta}(\mathscr{C}^\infty(M))$; since multiplication by $1/f$ reverses this job we have $H^q_\xi(\mathscr{C}^\infty(M)) \cong H^q_{\xi+\eta}(\mathscr{C}^\infty(M))$ for every $\xi \in \mathbb{Z}^m$. Thus, if some $H^q_\xi(\mathscr{C}^\infty(M))$ is non-zero, then so are infinitely many of them, and the left-hand side of (6.2) must contain infinitely many copies of it: injectivity of (6.2) would then lead us to a contradiction, hence proving our first claim. Now, it follows from Theorem 7.1 (with $F = \emptyset$) that $H^q_{\mathrm{d}'}(\mathscr{C}^\infty(\Omega))$ vanishes. □

Corollary 7.3. *If $0 < \dim H^q_{\mathrm{d}'}(\mathscr{C}^\infty(\Omega)) < \infty$ for some q, then $\Gamma_{\boldsymbol{\omega}} = \{0\}$.*

Interesting special cases of the results above are obtained for $q \in \{0, n\}$, for then $H^q_{\mathrm{dR}}(M)$ is one-dimensional (compare with [4, Lemma 2.2]).

Proposition 7.4. *Suppose that:*

(1) $\Gamma_{\boldsymbol{\omega}} = \mathbb{Z}^m$*;*
(2) $H^q_{\mathrm{dR}}(M) = \{0\}$*; and*
(3) $\mathrm{d}' : \mathscr{C}^\infty(\Omega;\underline{\Lambda}^{q-1}) \to \mathscr{C}^\infty(\Omega;\underline{\Lambda}^q)$ *has closed range.*

Then $H^q_{\mathrm{d}'}(\mathscr{C}^\infty(\Omega)) = \{0\}$.

Proof. The first hypothesis ensures that all the factors in the direct product in (6.1) are copies of $H^q_{\mathrm{dR}}(M)$, which is zero by the second hypothesis; the third one implies that $H^q_{\mathrm{d}'}(\mathscr{C}^\infty(\Omega))$ embeds there, which is therefore also zero. □

Theorem 7.5. *If (5.11) holds, then*

$$H^q_{\mathrm{d}'}(\mathscr{C}^\infty(\Omega)) \text{ is finite dimensional} \Longleftrightarrow H^q_{\mathrm{d}'}(\mathscr{C}^\infty(\Omega)) \cong H^q_{\mathrm{dR}}(M). \tag{7.2}$$

Proof. We have by (5.11) and (7.1):

$$\bigoplus_{\xi\in\mathbb{Z}^m} H^q_\xi(\mathscr{C}^\infty(M)) = \bigoplus_{\xi\in\Gamma_{\boldsymbol{\omega}}} H^q_\xi(\mathscr{C}^\infty(M)) \cong \bigoplus_{\xi\in\Gamma_{\boldsymbol{\omega}}} H^q_{\mathrm{dR}}(M)$$

which is isomorphic with $H^q_{\mathrm{d}'}(\mathscr{C}^\infty(\Omega))$ by Theorem 7.1 provided the latter space is finite dimensional. Two possibilities arise: either $H^q_{\mathrm{dR}}(M)$ is zero (in which case $H^q_{\mathrm{d}'}(\mathscr{C}^\infty(\Omega))$ vanishes too) or not; in the latter case, there must be at most finitely many indices in the last direct sum (by finite dimensionality) i.e. $\Gamma_{\boldsymbol{\omega}}$ is a finite set.

Since this is a subgroup of $\mathbb{Z}^m$, we have that $\Gamma_{\boldsymbol{\omega}} = \{0\}$ i.e. there is a single term in that direct sum, namely $H^q_0(\mathscr{C}^\infty(M)) = H^q_{\mathrm{dR}}(M)$. In both cases we get (7.2). ☐

7.1. **Calculations on surfaces.** In the case $\dim M = 2$, we know that

$$\dim H^q_{\mathrm{dR}}(M) = \begin{cases} 1, & \text{if } q = 0, 2, \\ 2\mathsf{g}, & \text{if } q = 1, \end{cases}$$

in which g is the genus of M. It follows from the Atiyah-Singer Index Theorem that the index of the elliptic complex d'_ξ on M, that is,

$$\sum_{q=0}^{2} (-1)^q \dim H^q_\xi(\mathscr{C}^\infty(M)), \tag{7.3}$$

depends only on the principal symbol of d'_ξ, which is the same as that of the exterior derivative. In particular, (7.3) does not depend on ξ, hence equals

$$\sum_{q=0}^{2} (-1)^q \dim H^q_\xi(\mathscr{C}^\infty(M)) = \sum_{q=0}^{2} (-1)^q \dim H^q_{\mathrm{dR}}(M) = \chi(M) = 2 - 2\mathsf{g},$$

the Euler characteristic of M. We already knew this for $\xi \in \Gamma_{\boldsymbol{\omega}}$ by (7.1); the extra information comes for $\xi \notin \Gamma_{\boldsymbol{\omega}}$, in which case by definition

$$H^0_\xi(\mathscr{C}^\infty(M)) = \mathscr{S}_{\xi\cdot\boldsymbol{\omega}}(M) = \{0\}.$$

Also, it follows from the second part of Lemma 2.8 that

$$H^2_\xi(\mathscr{C}^\infty(M)) \cong H^0_{-\xi}(\mathscr{D}'(M))^* \cong H^0_{-\xi}(\mathscr{C}^\infty(M))^*$$

also vanishes, since $-\xi \notin \Gamma_{\boldsymbol{\omega}}$. We conclude that for $\xi \notin \Gamma_{\boldsymbol{\omega}}$ we have

$$\dim H^q_\xi(\mathscr{C}^\infty(M)) = \begin{cases} 0, & \text{if } q = 0, 2, \\ 2\mathsf{g} - 2, & \text{if } q = 1. \end{cases}$$

Case $\mathsf{g} = 0$. In the case M is the 2-sphere, we always have $\Gamma_{\boldsymbol{\omega}} = \mathbb{Z}^m$, since every closed 1-form is exact by simply connectedness. Hence:

- $H^0_{\mathrm{d}'}(\mathscr{C}^\infty(\Omega))$ and $H^2_{\mathrm{d}'}(\mathscr{C}^\infty(\Omega))$ are infinite dimensional (Proposition 7.2);
- $H^1_{\mathrm{d}'}(\mathscr{C}^\infty(\Omega))$ is finite dimensional if and only if $\mathrm{d}' : \mathscr{C}^\infty(\Omega) \to \mathscr{C}^\infty(\Omega; \underline{\Lambda}^1)$ has closed range, in which case (Theorem 7.1)

$$H^1_{\mathrm{d}'}(\mathscr{C}^\infty(\Omega)) \cong \bigoplus_{\xi\in\mathbb{Z}^m} H^1_\xi(\mathscr{C}^\infty(M)) \cong \bigoplus_{\xi\in\mathbb{Z}^m} H^1_{\mathrm{dR}}(M) = \{0\}.$$

Case $\mathsf{g} = 1$. In the case M is the 2-torus, since no de Rham cohomology space vanishes, we must have that every $H^q_{\mathrm{d}'}(\mathscr{C}^\infty(\Omega))$ is infinite dimensional, unless $\Gamma_{\boldsymbol{\omega}} = \{0\}$ (Proposition 7.2). In this case we have that

$$\bigoplus_{\xi\in\mathbb{Z}^m} H^q_\xi(\mathscr{C}^\infty(M)) = H^q_{\mathrm{dR}}(M) \oplus \bigoplus_{\xi\notin\Gamma_{\boldsymbol{\omega}}} H^q_\xi(\mathscr{C}^\infty(M)) = H^q_{\mathrm{dR}}(M)$$

is in particular finite dimensional, hence (7.2) holds for each $q \in \{0, 1, 2\}$ (finiteness granted when $q = 0$ thanks to Theorem 7.1).

Case $\mathsf{g} \geq 2$**.** As in the previous case, every $H^q_{\mathrm{d}'}(\mathscr{C}^\infty(\Omega))$ is infinite dimensional, except when $\Gamma_{\boldsymbol{\omega}} = \{0\}$, in which case:

- for $q = \{0, 2\}$, we have
$$\bigoplus_{\xi \in \mathbb{Z}^m} H^q_\xi(\mathscr{C}^\infty(M)) = H^q_{\mathrm{dR}}(M) \oplus \bigoplus_{\xi \notin \Gamma_{\boldsymbol{\omega}}} H^q_\xi(\mathscr{C}^\infty(M)) = H^q_{\mathrm{dR}}(M)$$
hence (7.2) holds, with finiteness ensured at least for $q = 0$;
- for $q = 1$, we have that
$$\bigoplus_{\xi \in \mathbb{Z}^m} H^1_\xi(\mathscr{C}^\infty(M)) = H^1_{\mathrm{dR}}(M) \oplus \bigoplus_{\xi \notin \Gamma_{\boldsymbol{\omega}}} H^1_\xi(\mathscr{C}^\infty(M))$$
is infinite dimensional, hence so is $H^1_{\mathrm{d}'}(\mathscr{C}^\infty(\Omega))$ by Proposition 6.2.

7.2. **On the existence of an isomorphism under global solvability.** Finally, back again to the case when $\omega_1, \dots, \omega_m$ are real, the main result in [6] – in which $M = \mathbb{T}^n$ – essentially states that, under a hypothesis that is equivalent to the global solvability in degree 1 of the operator d', we have
$$H^q_{\mathrm{d}'}(\mathscr{C}^\infty(\mathbb{T}^{m+n})) \cong \mathscr{C}^\infty(\mathbb{T}^r) \otimes H^q_{\mathrm{dR}}(\mathbb{T}^m). \tag{7.4}$$
This inspired us to prove our isomorphism (5.10), which considers only the cohomology of q-forms associated with the cluster $\Gamma_{\boldsymbol{\omega}}$. We will show below that an isomorphism similar to (7.4) is not true for general M, even assuming solvability in the first degree.

Indeed, back to the case $\dim M = 2$, $\mathsf{g} \geq 2$, let $\boldsymbol{\omega} \doteq \{\lambda \vartheta_1\}$ (corank 1) in which ϑ_1 is as in the proof of Proposition 5.9 ($d = 2\mathsf{g} > 0$ there). Therefore, if $\lambda \in \mathbb{R} \setminus \mathbb{Q}$, then $\Gamma_{\boldsymbol{\omega}} = \{0\}$ thus $r = 0$, hence $\mathscr{C}^\infty(\mathbb{T}^r) \otimes H^1_{\mathrm{dR}}(M) \cong H^1_{\mathrm{dR}}(M)$ cannot be isomorphic with the infinite dimensional $H^1_{\mathrm{d}'}(\mathscr{C}^\infty(\Omega))$, even when $\mathrm{d}' : \mathscr{C}^\infty(\Omega) \to \mathscr{C}^\infty(\Omega; \underline{\Lambda}^1)$ has closed range (for instance, if λ is a non-Liouville number, by Theorem 5.12).

Conflict of interest statement

On behalf of all authors, the corresponding author states that there is no conflict of interest.

Data availability statement

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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