RT-MAT 2000-26

Addition Formulas for Relative Reidemeister Numbers

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Setembro 2000

ADDITION FORMULAS FOR RELATIVE REIDEMEISTER NUMBERS

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ABSTRACT. The relative Reidemeister number $\mathrm{R}(f;X,A)$ is an upper bound for the relative Nielsen number $\mathrm{N}(f;X,A)$. If f, f_A is a pair of fibre-preserving maps of a pair of Hurewicz fibrations, then under certain conditions, the relative Reidemeister number can be calculated in terms of those on the base and on the fiber. In this paper, we give addition formula of these numbers and for the relative Reidemeister numbers on the complement. As an application, we give estimation of the asymptotic Nielsen type number.

1. INTRODUCTION

For a selfmap $f\colon X \longrightarrow X$ of a compact connected polyhedron, the Nielsen number $\mathrm{N}(f)$ provides a lower bound for the minimal number of fixed points in the homotopy class of f. If X is a manifold of dimension at least 3, this lower bound is sharp. The computation of $\mathrm{N}(f)$ is in general difficult and therefore is one of the central problems in Nielsen fixed point theory.

Let $p\colon E \longrightarrow B$ be a Hurewicz fibration and $f\colon E \longrightarrow E$ a fiber preserving map with induced map $\overline{f}\colon B \longrightarrow B$. R. Brown [1] initiated the study of the Nielsen fixed point theory for fiber-preserving maps and gave conditions for which $\mathrm{N}(f) = \mathrm{N}(f_b)\mathrm{N}(\overline{f})$, where $\overline{f}$ denotes the induced map in the base and f_b the restriction of f on the fiber over

Date: September 19, 2000.

1991 *Mathematics Subject Classification.* Primary 55M20; Secondary 57T15.

Key words and phrases. Relative Nielsen number, relative Reidemeister number, fixed points of pairs, fiber-preserving maps, asymptotic Nielsen type number.

This work was conducted in part during the first author's visit to Bates College, supported by a grant from FAPESP of Brasil.

$b \in \mathrm{Fix}(\overline{f})$. Such a product formula was further studied by Fadell [4]; necessary and sufficient conditions for the validity of this naïve product formula were given by You [16].

One of the standing assumptions in these works is that the fibration p be orientable. This turns out to be rather restrictive. By relaxing the orientability of p, a naïve addition formula, rather than a product formula has been obtained by Heath, Keppelmann and Wong [7]. The main objective of these formulas is to compute the Nielsen number of f in terms of possibly simpler Nielsen type invariants of $\overline{f}$ and of f_b.

In 1986, H. Schirmer [13] introduced a Nielsen type theory for maps of pairs. More precisely, for any map $f\colon (X,A) \longrightarrow (X,A)$ of a compact polyhedral pair (X,A), a Nielsen type number $\mathrm{N}(f;X,A)$ was introduced and a Wecken type theorem was proven. This relative Nielsen theory has since prompted many subsequent works by Schirmer and others. By definition, $\mathrm{N}(f;X,A)$ is defined in terms of f and of $f_A\colon A \longrightarrow A$ so the computation of $\mathrm{N}(f;X,A)$ is more difficult than that of the (absolute) Nielsen number. In his thesis [14], A. Schusteff established the naïve product formula for the relative Nielsen number of a fiber preserving map of pairs. More precisely, given a commutative diagram

$$\begin{array}{ccc} (E,E_0) & \xrightarrow{f,f_0} & (E,E_0) \\ {\scriptstyle p}\downarrow{\scriptstyle p_0} & & {\scriptstyle p}\downarrow{\scriptstyle p_0} \\ (B,B_0) & \xrightarrow{\overline{f},\overline{f_0}} & (B,B_0) \end{array}$$

conditions were given to ensure that the naïve product holds, i.e.,

$$\mathrm{N}(f;E,E_0) = \mathrm{N}(f_b;F_b,F_{0b})\,\mathrm{N}(\overline{f};B,B_0) \quad ,$$

where $E_0 \xrightarrow{p_0} B_0$ is a sub-fibration of a Hurewicz fibration $E \xrightarrow{p} B$ and F_b, F_{0b} are the fibers of p and p_0, respectively, over $b \in \mathrm{Fix}(\overline{f})|_{B_0}$. Furthermore, a relative Reidemeister number $\mathrm{R}(f;E,E_0)$ was introduced in [14] to give computational results when the spaces are Jiang spaces. Subsequently in [2], the basic properties of the relative Reidemeister number were established and a relative Reidemeister number

on the complement, analogous to the Nielsen type invariant of Zhao [17], was introduced.

The main objective of this paper is to compute the relative Reidemeister number $\mathrm{R}(f; E, E_0)$ and the relative Reidemeister number on the complement $\mathrm{R}(f; E - E_0)$ of fiber-preserving maps of pairs. We obtain addition formulas for these homotopy invariants, generalizing previously known formulas. Another motivation for computing $\mathrm{R}(f; E - E_0)$ is its connections with an asymptotic Nielsen type invariant introduced by Jiang (see [15]). As an application, we estimate the asymptotic Nielsen type number $\mathrm{NI}^\infty(f)$ (see [9]) when f is a fiber-preserving map on a compact polyhedron.

This paper is organized as follows. In section 2, we give an algebraic formulation of the relative Reidemeister numbers for fiber preserving maps. We present an addition formula for the absolute mod K Reidemeister number in section 3. The main results of the paper are contained in section 4 in which $\mathrm{R}(f; E, E_0)$ and $\mathrm{R}(f; E - E_0)$ are computed or estimated in terms of the relative Reidemeister numbers of $\overline{f}$ and of f_b. We deduce in section 5 Schusteff's results as special cases of the addition formulas obtained in section 4. In the final section, we make use of the relative Reidemeister number on the complement and equivariant fixed point theory to give an estimate for $\mathrm{NI}^\infty(f)$.

The authors wish to thank Robert F. Brown for his careful reading of an earlier version of the manuscript. In particular, the first author wishes to thank Robert F. Brown for his help and comments during the first months of her visit to the US, as well as the staff of the Math Dept. at UCLA. Also, she could not let this Introduction finish without thanking the invitation to Bates College and all the people in the College that made the rest of her trip a very productive one.

2. Algebraic formulation of the relative Reidemeister numbers for pairs of fiber-preserving maps

Let X be a compact, connected polyhedron, and $A \subset X$ a finite subpolyhedron, also connected. Let $f\colon (X,A) \longrightarrow (X,A)$ be a map of pairs, that is, a map $f\colon X \longrightarrow X$ such that $f(A) \subset A$. Let $\eta_Y\colon \widetilde{Y} \longrightarrow Y$ be the universal cover of Y and $Cov(\eta_Y)$ be its group of deck transformations.

Let $\widetilde{f}$ be a lifting of f, then for every $\sigma \in Cov(\eta_X)$ (since $Cov(\eta_X)$ is identified with the fundamental group $\pi_1(X)$, we will denote it by π_X), there is a unique element $\varphi(\sigma) \in \pi_X$ such that $\widetilde{f}\sigma = \varphi(\sigma)\widetilde{f}$. The map thus defined, $\varphi\colon \pi_X \longrightarrow \pi_X$ is a group homomorphism. Similarly, we can define $\varphi_A\colon \pi_A \longrightarrow \pi_A$ a group homomorphism, where $\pi_A = Cov(\eta_A)$.

If $[\alpha\widetilde{f}] = [\beta\widetilde{f}]$ then $\beta\widetilde{f} = \sigma\alpha\widetilde{f}\sigma^{-1} = \sigma\alpha\varphi(\sigma^{-1})\widetilde{f}$ and, therefore, $\beta = \sigma\alpha\varphi(\sigma^{-1})$. This gives a one-to-one correspondence between Reidemeister classes and lifting classes.

Given $\varphi\colon \pi_X \longrightarrow \pi_X$, we have the so-called *Reidemeister action* of π on π, given by $\sigma\cdot\alpha = \sigma\alpha\varphi(\sigma^{-1})$. Therefore, the Reidemeister classes are the orbits of this action, and the set of Reidemeister classes is denoted by $\mathcal{R}(\varphi,\pi_X)$. From the discussion above, we have $\mathrm{R}(f) = \#\mathcal{R}(\varphi,\pi_X)$, where $\#$ denotes the cardinality. Similarly, we have $\mathcal{R}(\varphi_A,\pi_A)$ and $\mathrm{R}(f_A) = \#\mathcal{R}(\varphi_A,\pi_A)$.

Now, we will proceed to define those sets in the pair of fibrations settings. Let (E,p,B) be a Hurewicz fibration, with all spaces being $0-$connected. Suppose E_0 is a subspace of E; let $B_0 = p(E_0)$ and $p_0 = p|_{E_0}$. Also, suppose (E_0,p_0,B_0) is a Hurewicz fibration, with all spaces being $0-$connected.

Let $(f,f_0)\colon (E,E_0) \longrightarrow (E,E_0)$ be a fiber preserving map pair, with $f_0 = f|_{E_0}$. Let $b_0 \in B_0$ be a fixed point of $\overline{f_0}$, then choose $e_0 \in p_0^{-1}(b_0)$ and a lifting $\widetilde{f}_0$ of f_0. Given $\alpha \in Cov(\eta_{E_0})$, there is a unique element $\varphi_0(\alpha) \in Cov(\eta_{E_0})$ such that $\widetilde{f}_0\alpha = \varphi_0(\alpha)\widetilde{f}_0$. This defines a homomorphism $\varphi_0\colon \pi_{E_0} \longrightarrow \pi_{E_0}$, where $\pi_{E_0} \equiv Cov(\eta_{E_0})$. Observe

that $\pi_{E_0} \equiv Cov(\eta_{E_0}) \cong \pi_1(E_0, e_0)$ and also $Cov(\eta_{E_0}) \cong \pi_1(E_0, f_0(e_0))$. While $f_{0\#}\colon \pi_1(E_0, e_0) \longrightarrow \pi_1(E_0, f_0(e_0))$, the map $\varphi_0\colon \pi_{E_0} \longrightarrow \pi_{E_0}$ does not depend on the choice of base point; as a matter of fact, it does not depend on the lifting $\widetilde{f}_0$ chosen, either. It only depends on the map f_0 itself. (See [10].)

Similarly we construct the maps $\overline{\varphi_0}$ for (B_0, b_0) and φ_0' for $(F_0 = p^{-1}(b_0), e_0)$. At this point we have made all the necessary constructions to define the Reidemeister number, through the algebraic approach, for the maps on the subspaces involved. For the spaces themselves, we perform the same type of constructions, noticing that we have to use the same base points as the subspaces.

Observe that, for both the spaces and subspaces in the fibrations we have the commutative diagrams:

(1)

$$\begin{array}{ccc} (F_0, e_0) & \xrightarrow{f_0'} & (F_0, f(e_0)) \\ {\scriptstyle i_0}\downarrow & & \downarrow{\scriptstyle i_0} \\ (E_0, e_0) & \xrightarrow{f_0} & (E_0, f(e_0)) \\ {\scriptstyle p_0}\downarrow & & \downarrow{\scriptstyle p_0} \\ (B_0, b_0) & \xrightarrow{\overline{f_0}} & (B_0, b_0) \end{array} \qquad \begin{array}{ccc} (F, e_0) & \xrightarrow{f'} & (F, f(e_0)) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i} \\ (E, e_0) & \xrightarrow{f} & (E, f(e_0)) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ (B, b_0) & \xrightarrow{\overline{f}} & (B, b_0) \end{array}$$

Between these two diagrams we have inclusions and every possible square commutes. Beginning with (E, E_0) we have a morphism of self-maps, so we can choose a lifting $\widetilde{i_{E,E_0}}$ of i_{E,E_0} such that the diagram

$$\begin{array}{ccc} \widetilde{E_0} & \xrightarrow{\widetilde{i_{E,E_0}}} & \widetilde{E} \\ {\scriptstyle \eta_{E_0}}\downarrow & & \downarrow{\scriptstyle \eta_E} \\ E_0 & \xrightarrow{i_{E,E_0}} & E \end{array}$$

commutes. As in [3], this $\widetilde{i_{E,E_0}}$ determines a correspondence $\widetilde{i_{\{E,E_0\},lift}}$ from the liftings of f_0 to those of f. This correspondence induced by $\widetilde{i_{\{E,E_0\},lift}}$ is independent of the choice of the lift $\widetilde{i_{E,E_0}}$ of i_{E,E_0} and

depends only on the map i_{E,E_0} itself; similarly we can do the same for $\widetilde{i_{B,B_0}}$ and $\widetilde{i_{F,F_0}}$.

Remark 1. We should not expect to have commutative diagrams on the corresponding liftings of diagrams in (1), since commutativity would imply that we had already a correspondence between the liftings chosen for the subspaces and those for the spaces. And that is not always true, neither is it necessary for our purposes.

Now, we can go through the main step to define the relative Reidemeister numbers. The map $i\colon A \hookrightarrow X$, $A \subset X$, induces a homomorphism $i_\#\colon \pi_1(A) \longrightarrow \pi_1(X)$ and, because of the identification $\pi_1(Y) \equiv Cov(\eta_Y)$, we have a well-defined map

$$\begin{aligned} \widehat{i}\colon \mathcal{R}(\varphi_A, \pi_A) &\longrightarrow \mathcal{R}(\varphi, \pi) \\ [\alpha] &\longrightarrow [i_\#\alpha] \end{aligned}$$

(for more details, see [10] and [17]). In [2], we defined the set of *weakly common lifting classes of f and f_A*, and denoted it by $\mathrm{r}(f, f_A)$, as the set of liftings $\widetilde{f}$ of f that are in the image of any lifting $\widetilde{f_A}$ of f_A, by the map $\widetilde{i}_{\mathrm{lift}}$. It is only natural that we have the corresponding set, from an algebraic point of view. That is, the set of classes of loops of π under φ that are in the image of a class of loops of π_A under φ_A by the map $\widehat{i}$. Denoting it by $\mathcal{R}(\varphi, \varphi_A)$, we have

$$\mathcal{R}(\varphi, \varphi_A) = \{[\alpha] \in \mathcal{R}(\varphi, \pi) \mid [\alpha] = \widehat{i}([\beta]) \text{ for some } [\beta] \in \mathcal{R}(\varphi_A, \pi_A)\} \quad .$$

The Reidemeister number of f on the complement $X - A$ is thus defined as

$$\mathrm{R}(f; X - A) = \mathrm{R}(f) - \mathrm{R}(f, f_A)$$

where

$$\begin{aligned} \mathrm{R}(f) &= \#\mathcal{R}(\varphi, \pi) \\ \mathrm{R}(f, f_A) &= \#\mathcal{R}(\varphi, \varphi_A) \end{aligned}$$

and the cardinalities are all finite. Otherwise, if any of them is not finite, then we define $\mathrm{R}(f; X - A) = \infty$.

The relative Reidemeister number of f on the pair (X, A) is thus defined as

$$\mathrm{R}(f; X, A) = \mathrm{R}(f_A) + \mathrm{R}(f) - \mathrm{R}(f, f_A)$$

where

$$\mathrm{R}(f) = \#\mathcal{R}(\varphi, \pi)$$
$$\mathrm{R}(f_A) = \#\mathcal{R}(\varphi_A, \pi_A)$$
$$\mathrm{R}(f, f_A) = \#\mathcal{R}(\varphi, \varphi_A)$$

and we define $\mathrm{R}(f; X, A) = \infty$ if any of them is infinite .

Remark 2. Note that if $\mathrm{R}(f, f_A)$ is infinite then both $\mathrm{R}(f)$ and $\mathrm{R}(f_A)$ are necessarily infinite. (For more details see [3].)

Remark 3. Notice that while we can have $\mathrm{R}(f; X - A) = 0$ for the Reidemeister number of the complement, the relative Reidemeister number of the pair $\mathrm{R}(f; X, A) > 0$.

Remark 4. Because of the one-to-one correspondence between the sets in both definitions, all results proved in [2] hold true for the algebraic definitions of $\mathrm{R}(f; X, A)$ and $\mathrm{R}(f; X - A)$. In particular, both the relative Reidemeister number $\mathrm{R}(f; X, A)$ and the relative Reidemeister number on the complement $\mathrm{R}(f; X - A)$ satisfy the basic properties such as (relative) homotopy invariance, homotopy type invariance, and commutativity.

Now we are able to define the Reidemeister numbers on the complement and the relative Reidemeister numbers, for each one of the pair of spaces in the fibration.

3. Addition formulas for the mod K Reidemeister number of fiber-preserving maps

In [6] an addition formula for the Reidemeister number for coincidences was given via an algebraic approach. We will adapt what was done there for our fixed-point settings.

Let (E, p, B) be a Hurewicz fibration, with all spaces being 0-connected. Let $f\colon E \longrightarrow E$ be a fiber preserving map. Suppose $K = \ker i$,

where $i_{\#}\colon \pi_F \longrightarrow \pi_E$ is induced by $i\colon F \hookrightarrow E$. We will denote by $i_{\#K}$ the induced map on the quotient, $i_{\#K}\colon \pi_F/K \longrightarrow \pi_E$. Also, the set of orbits of the Reidemeister action of φ' in π_F/K will be denoted by $\mathcal{R}_K(\varphi', \pi_F)$. Moreover, we can suppose, without lost of generality, that $i_{\#K}$ is the inclusion map (notice that $\pi_F/K \cong \operatorname{Im} i_{\#K} < \pi_E$).

We have the following commutative diagram of homomorphisms and groups, whose rows are short exact sequences.

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_F/K & \xrightarrow{i_{\#K}} & \pi_E & \xrightarrow{p_{\#}} & \pi_B & \longrightarrow & 1 \\ & & \varphi'\downarrow & & \varphi\downarrow & & \overline{\varphi}\downarrow & & \\ 1 & \longrightarrow & \pi_F/K & \xrightarrow{i_{\#K}} & \pi_E & \xrightarrow{p_{\#}} & \pi_B & \longrightarrow & 1 \end{array}$$

This induces the following sequence, which is exact at $\mathcal{R}(\varphi, \pi_E)$ and $\mathcal{R}(\overline{\varphi}, \pi_B)$,

$$\mathcal{R}_K(\varphi', \pi_F) \xrightarrow{\widehat{i_K}} \mathcal{R}(\varphi, \pi_E) \xrightarrow{\widehat{p}} \mathcal{R}(\overline{\varphi}, \pi_B) \longrightarrow 1$$

i.e., $\widehat{p}$ is surjective and $\operatorname{Im}\widehat{i_K} = \widehat{p}^{\,-1}[e]$, where e is the identity element of π_B.

Now, we are able to give the addition formula for the Reidemeister number for a fiber-preserving map. In what follows we will indicate the conjugation map by $\tau_\alpha(\beta) = \alpha\beta\alpha^{-1}$, without being explicit where it is defined, since the context will make it clear.

Proposition 3.1. Let (E, p, B) be a fibration as described above. Suppose $\operatorname{Fix}(\tau_{\overline{\alpha}}\overline{\varphi}) = \{e\}$ for all $[\overline{\alpha}] \in \mathcal{R}(\overline{\varphi}, \pi_B)$. There is a one-to-one correspondence between the sets

$$\mathcal{R}(\varphi, \pi_E) \leftrightarrow \coprod_{[\overline{\alpha}]\in\mathcal{R}(\overline{\varphi},\pi_B)} \mathcal{R}_K(\tau_\alpha\varphi', \pi_F) \quad .$$

Also, $\mathrm{R}(f) = \#\mathcal{R}(\varphi, \pi_E) < \infty$ if, and only if, $\#\mathcal{R}_K(\tau_\alpha\varphi', \pi_F) < \infty$ for all $[\alpha] \in \mathcal{R}(\varphi, \pi_E)$. In this case, we have

$$\mathrm{R}(f) = \#\mathcal{R}(\varphi, \pi_E) = \sum_{[\overline{\alpha}]\in\mathcal{R}(\overline{\varphi},\pi_B)} \#\mathcal{R}_K(\tau_\alpha\varphi', \pi_F) \quad .$$

If, in addition, $\#\mathcal{R}_K(\tau_\alpha\varphi', \pi_F)$ is *mod K fiber uniform*, i.e., it is independent of the choice of $[\alpha] \in \mathcal{R}(\varphi, \pi_E)$, then we have a product

formula

$$\#\mathcal{R}(\varphi,\pi_E) = \#\mathcal{R}_K(\varphi',\pi_F)\cdot\#\mathcal{R}(\overline{\varphi},\pi_B) \quad .$$

Proof. Similar to what we have in the beginning of this section, we have the following commutative diagram of homomorphisms and groups, whose rows are short exact sequences.

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_F/K & \xrightarrow{i_{\#K}} & \pi_E & \xrightarrow{p_\#} & \pi_B & \longrightarrow & 1 \\ & & \tau_\alpha\varphi'\downarrow & & \tau_\alpha\varphi\downarrow & & \tau_{\overline{\alpha}}\overline{\varphi}\downarrow & & \\ 1 & \longrightarrow & \pi_F/K & \xrightarrow{i_{\#K}} & \pi_E & \xrightarrow{p_\#} & \pi_B & \longrightarrow & 1 \end{array}$$

It induces the following short exact sequence,

$$1 \longrightarrow \mathcal{R}_K(\tau_\alpha\varphi',\pi_F) \xrightarrow{\widehat{i_{\alpha K}}} \mathcal{R}(\tau_\alpha\varphi,\pi_E) \xrightarrow{\widehat{p_\alpha}} \mathcal{R}(\tau_{\overline{\alpha}}\overline{\varphi},\pi_B) \longrightarrow 1$$

i.e., $\widehat{i_{\alpha K}}$ is injective (due to the hypothesis $\mathrm{Fix}(\tau_{\overline{\alpha}}\overline{\varphi}) = \{e\}$ for all $[\overline{\alpha}] \in \mathcal{R}(\overline{\varphi},\pi_B)$), $\widehat{p_\alpha}$ is surjective and $\mathrm{Im}\,\widehat{i_{\alpha K}} = \widehat{p_\alpha}^{-1}[e]$, where e is the identity element of π_B.

It follows from [6] that there is a one-to-one correspondence between the sets $\widehat{p_\alpha}^{-1}[e]$ and $\widehat{p}^{-1}[\overline{\alpha}]$

Therefore we have the one-to-one correspondence

$$\mathcal{R}(\varphi,\pi_E) \leftrightarrow \coprod_{[\overline{\alpha}]\in\mathcal{R}(\overline{\varphi},\pi_B)} \mathcal{R}_K(\tau_\alpha\varphi',\pi_F) \quad .$$

Now it is straightforward to verify the assertions from this bijection. □

We will adapt the *local Fix group index* (see [7]) for the more geometric approach to our settings, in order to give a more general formula, without the hypothesis $\mathrm{Fix}(\tau_{\overline{\alpha}}\overline{\varphi}) = \{e\}$. To see that everything is equivalent under algebraic settings, we refer the reader to work done by Gonçalves for coincidences, see [5], where, in the first section, we can find all the necessary elements to construct the bridge between the geometric and algebraic approach.

Theorem 3.2. Let (E,p,B) be a fibration as defined as above; let f be a fiber-preserving map. Then, there is a one-to-one correspondence

between the sets

$$\mathcal{R}(\varphi,\pi_E) \leftrightarrow \coprod_{[\overline{\alpha}]\in\mathcal{R}(\overline{\varphi},\pi_B)} \widehat{i_{\alpha K}}\mathcal{R}_K(\tau_\alpha\varphi',\pi_F) \quad ,$$

for any $[\alpha] \in \widehat{p}^{\,-1}([\overline{\alpha}])$. If the cardinalities of the sets involved are finite, we have

$$\mathrm{R}(f) = \sum_{[\overline{\alpha}]\in\mathcal{R}(\overline{\varphi},\pi_B)} \sum_{[\alpha]} \frac{1}{[\mathrm{Fix}(\tau_{\overline{\alpha}}\overline{\varphi}) : p_\#(\mathrm{Fix}(\tau_{\beta\alpha}\varphi))]}$$

where $[\beta] \in \mathcal{R}_K(\tau_\alpha\varphi',\pi_F)$ and $[\alpha] \in \widehat{p}^{\,-1}([\overline{\alpha}])$.

Proof. From the proof of Proposition 3.1 we have the 1-1 bijections $\mathcal{R}(\varphi,\pi_E) \leftrightarrow \coprod_{[\overline{\alpha}]\in\mathcal{R}(\overline{\varphi},\pi_B)} \widehat{p}^{\,-1}([\overline{\alpha}]) \leftrightarrow \coprod_{[\overline{\alpha}]\in\mathcal{R}(\overline{\varphi},\pi_B)} \widehat{i_{\alpha K}}(\mathcal{R}(\tau_\alpha\varphi',\pi_F))$, for any $[\alpha] \in \widehat{p}^{\,-1}[\overline{\alpha}]$. Therefore,

$$\mathrm{R}(f) = \#\mathcal{R}(\varphi,\pi_E) = \sum_{[\overline{\alpha}]\in\mathcal{R}(\overline{\varphi},\pi_B)} \#\widehat{p}^{\,-1}([\overline{\alpha}]).$$

Consider in $\mathcal{R}_K(\tau_\alpha\varphi',\pi_F)$ the equivalence relation given by $\beta_1 \sim \beta_2 \Longrightarrow \widehat{i_{\alpha K}}(\beta_1) = \widehat{i_{\alpha K}}(\beta_2)$. Notice that $\#\widehat{p}^{\,-1}([\overline{\alpha}]) = \#\mathcal{R}_K(\tau_\alpha\varphi',\pi_F)/_\sim$, for any $[\alpha] \in \widehat{p}^{\,-1}[\overline{\alpha}]$. Given $\overline{\theta} \in \mathrm{Fix}(\tau_{\overline{\alpha}}\overline{\varphi})$ and $[\beta] \in \mathcal{R}_K(\tau_\alpha\varphi',\pi_F)$ define $\overline{\theta}[\beta] = [\theta\beta\tau_\alpha\varphi(\theta^{-1})]$, where $\beta \in [\beta]$ and $p_\#(\theta) = \overline{\theta}$. This is a well-defined action

$$\begin{aligned} \mathrm{Fix}(\tau_{\overline{\alpha}}\overline{\varphi}) \times \mathcal{R}_K(\tau_\alpha\varphi',\pi_F) &\longrightarrow \mathcal{R}_K(\tau_\alpha\varphi',\pi_F) \\ (\overline{\theta},[\beta]) &\mapsto \overline{\theta}[\beta] \end{aligned}$$

of a group on a set. (Notice that $\theta\beta\tau_\alpha\varphi(\theta^{-1}) \in \ker p_\# = \mathrm{Im}\, i_\#$). Two classes in $\mathcal{R}_K(\tau_\alpha\varphi',\pi_F)$ are in the same class of $\mathcal{R}(\varphi,\pi_E)$ if, and only if, they are in the same orbit by the action just defined (see [5, Props. 1.5, 1.6 and 1.7]).

The isotropy groups of this action are $p_\#(\mathrm{Fix}(\tau_{\beta\alpha}\varphi))$, where $[\beta] \in \mathcal{R}_K(\tau_\alpha\varphi',\pi_E)$ and $p_\#(\beta\alpha) = \overline{\alpha}$, therefore the cardinality of the orbits is given by the local index $[\mathrm{Fix}(\tau_{\overline{\alpha}}\overline{\varphi}) : p_\#(\mathrm{Fix}(\tau_{\beta\alpha}\varphi))]$. (See [5, Prop. 1.6].)

Therefore

$$\#\,\widehat{p}^{\,-1}([\overline{\alpha}]) = \sum_{[\beta]\in\mathcal{R}_K(\tau_\alpha\varphi',\pi_F)} \frac{1}{[\mathrm{Fix}(\tau_{\overline{\alpha}}\overline{\varphi}) : p_\#(\mathrm{Fix}(\tau_{\beta\alpha}\varphi))]},$$

for any $[\alpha] \in \widehat{p}^{\,-1}([\overline{\alpha}])$. □

Remark 5. The existence of the natural map $p_\#^{-1}(\overline{\alpha}) \longrightarrow \widehat{p}^{\,-1}([\overline{\alpha}])$ allows us to take $[\alpha] \in \widehat{p}^{\,-1}([\overline{\alpha}])$ as in the above formula.

Remark 6. Note that if $[\alpha_1] = [\alpha_2]$ as elements of $\mathcal{R}(\varphi, \pi_E)$ then

$$[\mathrm{Fix}(\tau_{\overline{\alpha_1}}\overline{\varphi}) : p(\mathrm{Fix}(\tau_{\alpha_1}\varphi))] = [\mathrm{Fix}(\tau_{\overline{\alpha_2}}\overline{\varphi}) : p(\mathrm{Fix}(\tau_{\alpha_2}\varphi))].$$

Observe that there is $\gamma \in \pi_E$ such that $\alpha_1 = \gamma\alpha_2\varphi(\gamma^{-1})$ and the map

$$\begin{aligned} \mathrm{Fix}(\tau_{\overline{\alpha_1}}\overline{\varphi}) &\longrightarrow \mathrm{Fix}(\tau_{\overline{\alpha_2}}\overline{\varphi}) \\ b &\mapsto \tau_{p(\gamma^{-1})}\, b \end{aligned}$$

is an isomorphism on the respective cosets.

Remark 7. To see that the sum above yields an integer, remember that within the orbit of $[\beta] \in \mathcal{R}_K(\tau_\alpha\varphi', \pi_F)$ the index $[\mathrm{Fix}(\tau_{\overline{\alpha}}\overline{\varphi}) : p_\#(\mathrm{Fix}(\tau_{\beta\alpha}\varphi))]$ (which counts the "size" of the orbit) remains the same for every $[\gamma] \in \mathcal{R}_K(\tau_\alpha\varphi', \pi_F)$ in the orbit of $[\beta]$; and after summing up these fractions over all of $\mathcal{R}_K(\tau_\alpha\varphi', \pi_F)$, we will get in return exactly 1 for each orbit. Thus we end up obtaining the number of orbits.

Let $\alpha \in \mathcal{R}(\varphi, \pi_E)$. The *index of* α is simply $\mathrm{index}(f, \eta_E\mathrm{Fix}(\alpha\widetilde{f}))$, the usual fixed point index. So, if index of α is nonzero then α is said to be *essential*. Denote by $\mathcal{N}(\varphi, \pi_E)$ the set of essential $\alpha \in \mathcal{R}(\varphi, \pi_E)$. Therefore, $\mathrm{N}(f) = \#\mathcal{N}(\varphi, \pi_E)$. Similarly, we denote by $\mathcal{N}_K(\varphi', \pi_F)$ the set of essential $\alpha' \in \mathcal{R}_K(\varphi', \pi_F)$, and $\mathrm{N}_K(f') = \#\mathcal{N}_K(\varphi', \pi_F)$.

Definition 1. Let f be a fiber-preserving map, let φ and $\overline{\varphi}$ be the homomorphisms induced by f and $\overline{f}$. We say that f is *locally* (resp. *essentially locally*) *Fix group uniform* if

$$[\mathrm{Fix}(\tau_{\overline{\alpha}}\overline{\varphi}) : p_\#(\mathrm{Fix}(\tau_\alpha\varphi))]$$

does not depend on $[\alpha] \in \mathcal{R}(\varphi, \pi_E)$ (resp.$[\alpha] \in \widehat{p}^{\,-1}([\overline{\alpha}]) \cap \mathcal{N}(\varphi, \pi_E)$).

Then as a corollary of Theorem 3.2, we have

Corollary 3.3. Let (E,p,B) be a fibration as described above and $f\colon E \longrightarrow E$ a fiber-preserving map. If f is essentially locally Fix group uniform, then

$$\mathrm{N}(f) = \sum_{[\overline{\alpha}]\in\mathcal{N}(\overline{\varphi},\pi_B)} \frac{\#\mathcal{N}_K(\tau_\alpha\varphi',\pi_F)}{[\mathrm{Fix}(\tau_{\overline{\alpha}}\overline{\varphi}) : p_\# (\mathrm{Fix}(\tau_\alpha\varphi))]}$$

where $[\alpha]$ is chosen in $\widehat{p}^{\,-1}([\overline{\alpha}]) \cap \mathcal{N}(\varphi,\pi_E)$.

Proof. Just remember there is a "local product formula" for the index, i.e. $[\alpha] \in \mathcal{R}(\varphi,\pi_E)$ is essential iff $\widehat{p}\,[\alpha] \in \mathcal{R}(\overline{\varphi},\pi_B)$ is essential and $\alpha' \in \mathcal{R}(\varphi',\pi_F)$, such that $\widehat{i_K}[\alpha'] = [\alpha]$, is essential. The result follows from the fact that if f is essentially locally Fix group uniform then

$$\#\widehat{i_{\alpha K}}(\mathcal{N}(\tau_\alpha\varphi',\pi_F)) = \frac{\#\mathcal{N}(\tau_\alpha\varphi',\pi_F)}{[\mathrm{Fix}(\tau_{\overline{\alpha}}\overline{\varphi}) : p_\# (\mathrm{Fix}(\tau_\alpha\varphi))]}$$

for any $[\alpha]$ chosen in $\widehat{p}^{\,-1}([\overline{\alpha}]) \cap \mathcal{N}(\varphi,\pi_E)$. □

Definition 2. Let f be a fiber-preserving map, φ and $\overline{\varphi}$ be the homomorphisms induced by f and $\overline{f}$. We say that f is *globally* (resp. *essentially globally*) *Fix group uniform* if

$$[\mathrm{Fix}(\tau_{\overline{\alpha}}\overline{\varphi}) : p_\# (\mathrm{Fix}(\tau_\alpha\varphi))]$$

does not depend on $[\overline{\alpha}] \in \mathcal{R}(\overline{\varphi},\pi_B)$ (resp. $\mathcal{N}(\overline{\varphi},\pi_B)$).

Corollary 3.4. Let (E,p,B) be a fibration as described above and $f\colon E \longrightarrow E$ a fiber-preserving map. If f is globally Fix group uniform, then

$$[\mathrm{Fix}(\overline{\varphi}) : p_\# \mathrm{Fix}(\varphi)]\mathrm{R}(f) = \sum_{[\overline{\alpha}]\in\mathcal{R}(\overline{\varphi},\pi_B)} \#\mathcal{R}_K(\tau_\alpha\varphi',\pi_F) \quad ,$$

for any $[\alpha] \in \widehat{p}^{\,-1}([\overline{\alpha}])$.

If, in addition, f is *mod K fiber uniform* (i.e., $\#\mathcal{R}_K(\tau_\alpha\varphi',\pi_F)$ is independent of $[\alpha]$ in $\widehat{p}^{\,-1}([\overline{\alpha}])$), then

$$[\mathrm{Fix}(\overline{\varphi}) : p_\# \mathrm{Fix}(\varphi)]\mathrm{R}(f) = \#\mathcal{R}(\overline{\varphi},\pi_B)\cdot\#\mathcal{R}_K(\varphi',\pi_F) \quad ,$$

i.e.,

$$[\mathrm{Fix}(\overline{\varphi}) : p_{\#}\,\mathrm{Fix}(\varphi)]\mathrm{R}(f) = \mathrm{R}(\overline{f})\mathrm{R}_K(f') \quad .$$

As a consequence of the above definition, we had the previous corollary, which is a "kind of" generalization of [14, Theorem II.1.18]. To the point: while in Schusteff's result we have that each fraction is the same under the hypothesis of eventually commutativity of f; in our case, if f is globally Fix group uniform we get that each denominator is the same, coupled that with the hypothesis of f being *mod K fiber uniform* we also have the same numerator, thus getting the same results, with less stringent hypotheses.

Remark 8. The above result is a generalization of Corollary 4.3 of [7]; it is enough to have f globally essentially Fix group uniform for the first part of Corollary 4.3 of [7], and for the second part, we need also f to be mod K uniform.

4. Addition formulas for Reidemeister numbers of pairs of fiber preserving maps

For the rest of this paper by a *fiber map of the pair* we mean a pair of fiber preserving maps $(f, f_0)\colon (E, E_0) \longrightarrow (E, E_0)$ with $f_0 = f|_{E_0}$ where (E_0, p_0, B_0) is a Hurewicz (sub-)fibration of a Hurewicz fibration (E, p, B). Furthermore, we assume that E, E_0, B, B_0 and the typical fibers are all 0−connected spaces.

Suppose $K_0 = \ker i_0$, where $i_0\colon \pi_{F_0} \longrightarrow \pi_{E_0}$ is induced by $i_0\colon F_0 \hookrightarrow E_0$. Denote by i_{K_0} the induced map on the quotient, the set of orbits of the respective Reidemeister action by $\mathcal{R}_{K_0}(\varphi_0', \pi_{F_0})$, and the respective cardinality, $\#\mathcal{R}_{K_0}(\varphi_0', \pi_{F_0})$, by $\mathrm{R}_{K_0}(f_0')$. We will denote the set of orbits of the Reidemeister action of φ' which are in the image of the orbits of the Reidemeister action of φ_0' under $i_{F,F_0}\colon \pi_{F_0}/K_0 \longrightarrow \pi_F/K$ by $\mathcal{R}_{K,K_0}(\varphi', \varphi_0')$, and the respective cardinality, $\#\mathcal{R}_{K,K_0}(\varphi', \varphi_0')$, by $\mathrm{R}_{K,K_0}(f', f_0')$. Moreover, we can suppose, without lost of generality, that i_K is the inclusion map (notice that $\pi_F/K \cong \mathrm{Im}\, i_K < \pi_E$).

We have the following

Proposition 4.1. Let (f, f_0) be a fiber map of the pair, $K = \ker i$ and $K_0 = \ker i_0$. Then we have the inclusion

$$\mathcal{R}(\varphi, \varphi_0) \subseteq \widehat{p}^{\,-1}\mathcal{R}(\overline{\varphi}, \overline{\varphi_0}) \quad .$$

Also, we have the inequalities

$$\mathrm{R}(f; E - E_0) \geq \sum_{[\overline{\alpha}] \in \mathcal{R}(\overline{\varphi}, \pi_B) - \mathcal{R}(\overline{\varphi}, \overline{\varphi_0})} \sum_{[\beta]} \frac{1}{[\mathrm{Fix}(\tau_{\overline{\alpha}}\overline{\varphi}) : p_{\#}\,(\mathrm{Fix}(\tau_{\beta\alpha}\varphi))]} \quad ,$$

where $[\beta] \in \mathcal{R}_K(\tau_\alpha \varphi', \pi_F)$ and $[\alpha] \in \widehat{p}^{\,-1}([\overline{\alpha}])$ and,

$$\mathrm{R}(f; E, E_0) \geq \sum_{[\overline{\alpha_0}] \in \mathcal{R}(\overline{\varphi_0}, \pi_{B_0})} \sum_{[\beta_0]} \frac{1}{[\mathrm{Fix}(\tau_{\overline{\alpha_0}}\overline{\varphi_0}) : p_{0\#}(\mathrm{Fix}(\tau_{\beta_0\alpha_0}\varphi_0))]}$$

$$+ \sum_{[\overline{\alpha}] \in \mathcal{R}(\overline{\varphi}, \pi_B) - \mathcal{R}(\overline{\varphi}, \overline{\varphi_0})} \sum_{[\beta]} \frac{1}{[\mathrm{Fix}(\tau_{\overline{\alpha}}\overline{\varphi}) : p_{\#}\,(\mathrm{Fix}(\tau_{\beta\alpha}\varphi))]} \quad ,$$

where $[\beta] \in \mathcal{R}_K(\tau_\alpha \varphi', \pi_F)$, $[\beta_0] \in \mathcal{R}_{K_0}(\tau_{\alpha_0} \varphi_0', \pi_{F_0})$, $[\alpha] \in \widehat{p}^{\,-1}([\overline{\alpha}])$ and $[\alpha_0] \in \widehat{p_0}^{\,-1}([\overline{\alpha_0}])$

Proof. Consider the following commutative diagram

$$\begin{array}{ccc} \mathcal{R}(\varphi_0, \pi_{E_0}) & \xrightarrow{\widehat{i_{E,E_0}}} & \mathcal{R}(\varphi, \pi_E) \\ \Big\downarrow{\scriptstyle \widehat{p_0}} & & \Big\downarrow{\scriptstyle \widehat{p}} \\ \mathcal{R}(\overline{\varphi_0}, \pi_{B_0}) & \xrightarrow{\widehat{i_{B,B_0}}} & \mathcal{R}(\overline{\varphi}, \pi_B) \end{array}$$

where $\widehat{p}$ and $\widehat{p_0}$ are surjective.

Let $[\alpha] \in \mathcal{R}(\varphi, \varphi_0)$. There is $[\overline{\alpha}] = \widehat{p}\,[\alpha]$ such that $[\overline{\alpha}] \in \mathcal{R}(\overline{\varphi}, \pi_B)$ and also there is $[\alpha_0] \in \mathcal{R}(\varphi_0, \pi_{E_0})$ such that $\widehat{i_{E,E_0}}[\alpha_0] = [\alpha]$. By the commutativity of the diagram above, we have $[\overline{\alpha}] \in \widehat{i_{B,B_0}}(\mathcal{R}(\overline{\varphi_0}, \pi_{B_0}))$. Therefore $[\alpha] \in \widehat{p}^{\,-1}(\widehat{i_{B,B_0}}\mathcal{R}(\overline{\varphi_0}, \pi_{B_0}))$, and the desired inclusion follows.

Since $\mathrm{R}(f; E - E_0) = \mathrm{R}(f) - \mathrm{R}(f, f_0)$, using Theorem 3.2, the inclusion proven above, plus the algebraic definition of the Reidemeister

number and properties of the sequence (3), we have

$$\begin{aligned}
&\mathrm{R}(f; E - E_0) \\
&= \sum_{[\overline{\alpha}]\in\mathcal{R}(\overline{\varphi},\pi_B)} \sum_{[\beta]} \frac{1}{[\mathrm{Fix}(\tau_{\overline{\alpha}}\,\overline{\varphi}) : p_{\#}\,(\mathrm{Fix}(\tau_{\beta\alpha}\,\varphi))]} - \#\mathcal{R}(\varphi, \varphi_0) \\
&\geq \sum_{[\overline{\alpha}]\in\mathcal{R}(\overline{\varphi},\pi_B)} \sum_{[\beta]} \frac{1}{[\mathrm{Fix}(\tau_{\overline{\alpha}}\,\overline{\varphi}) : p_{\#}\,(\mathrm{Fix}(\tau_{\beta\alpha}\,\varphi))]} - \#\,\widehat{p}^{\,-1}(\mathcal{R}(\overline{\varphi}, \overline{\varphi_0})) \\
&= \sum_{[\overline{\alpha}]\in\mathcal{R}(\overline{\varphi},\pi_B)} \sum_{[\beta]} \frac{1}{[\mathrm{Fix}(\tau_{\overline{\alpha}}\,\overline{\varphi}) : p_{\#}\,(\mathrm{Fix}(\tau_{\beta\alpha}\,\varphi))]} \\
&\qquad - \sum_{[\overline{\alpha}]\in\mathcal{R}(\overline{\varphi},\overline{\varphi_0})} \sum_{[\beta]} \frac{1}{[\mathrm{Fix}(\tau_{\overline{\alpha}}\,\overline{\varphi}) : p_{\#}\,(\mathrm{Fix}(\tau_{\beta\alpha}\,\varphi))]} \\
&= \sum_{[\overline{\alpha}]\in\mathcal{R}(\overline{\varphi},\pi_B)-\mathcal{R}(\overline{\varphi},\overline{\varphi_0})} \sum_{[\beta]} \frac{1}{[\mathrm{Fix}(\tau_{\overline{\alpha}}\,\overline{\varphi}) : p_{\#}\,(\mathrm{Fix}(\tau_{\beta\alpha}\,\varphi))]}
\end{aligned}$$

Again, using the definition of $\mathrm{R}(f; E, E_0)$, Theorem 3.2 and the inequality just proven for the relative Reidemeister number on the complement, we obtain the desired inequality for the relative Reidemeister number. □

We could have equality for $\mathrm{R}(f; E - E_0)$ provided the reversed inclusion in Proposition 4.1 were true. But the following example shows that a class in the complement may project to a common class.

Example 4.1. Let (E, p, B) be a product fibration with $B = D^2 \times S^1$ and $F = S^1 \times S^1$. Let $\overline{f}\colon B \longrightarrow B$ be $\overline{f} = \overline{f_1} \times \overline{f_2}$ where $\overline{f_1} = id_{D^2}$ and $\overline{f_2}$ be the flipping across the x-axis. Let $f'\colon F \longrightarrow F$ be $f' = f_1' \times f_2'$ where f_1' is the flipping across the x-axis and $f_2' = id_{S^1}$.

Let (E_0, p_0, B_0) also be a product fibration with $B_0 = D^2 \times I_1$, where I_1 is the segment in the circle that goes counterclockwise from $-i$ to

i, and $F_0 = I_2 \times S^1$ where I_2 is the segment in the circle that goes counterclockwise from i to $-i$. Let $\overline{f_0} = \overline{f}|_{B_0}$ and let $f_0' = f'|_{B_0}$.

The base space B has two distinct classes ($[(d,-1)]$ and $[(d,1)]$) and only $[(d,1)]$ is weakly common. In the total space E we have four distinct classes ($[1,1]$, $[1,-1]$, $[-1,1]$ and $[-1,-1]$) and only one of them is weakly common, namely, $[1,-1]$. Both $[1,1]$ and $[1,-1]$ project to the weakly common class of B so $\widehat{p}^{\,-1}(\mathcal{R}(\overline{\varphi},\overline{\varphi_0})) \supset \mathcal{R}(\varphi,\varphi_0)$, but $\widehat{p}^{\,-1}(\mathcal{R}(\overline{\varphi},\overline{\varphi_0})) \not\subset \mathcal{R}(\varphi,\varphi_0)$.

However, even without the reversed inclusion of Proposition 4.1, under suitable hypotheses, we may still have equality in the above formulas. To begin, let us assume that $\mathrm{Fix}(\tau_{\overline{\alpha}}\,\overline{\varphi}) = \mathrm{Fix}(\tau_{\overline{\alpha_0}}\,\overline{\varphi_0}) = \{e\}$, for all $\overline{\alpha} \in \mathcal{R}(\overline{\varphi},\pi_B)$, respectively $\overline{\alpha_0} \in \mathcal{R}(\overline{\varphi_0},\pi_{B_0})$.

Let $[\beta] \in \mathcal{R}(\varphi,\varphi_0)$ be a weakly common class. This means $[\beta] \in \mathcal{R}(\varphi,\pi_E)$ and there exists a $[\beta_0] \in \mathcal{R}(\varphi_0,\pi_{E_0})$ such that $\widehat{i_{E,E_0}}[\beta_0] = [\beta]$. We have already seen that there are correspondences

$$\mathcal{R}(\varphi,\pi_E) \leftrightarrow \amalg_{[\overline{\alpha}]\in\mathcal{R}(\overline{\varphi},\pi_B)} \widehat{p}^{\,-1}([\overline{\alpha}]) \leftrightarrow \amalg_{[\overline{\alpha}]\in\mathcal{R}(\overline{\varphi},\pi_B)} \widehat{p_\alpha}^{\,-1} < \overline{e} >= \amalg_{[\overline{\alpha}]\in\mathcal{R}(\overline{\varphi},\pi_B)} \widehat{i_{\alpha K}}\mathcal{R}(\tau_\alpha\,\varphi',\pi_F) \quad .$$

And similar ones for $\mathcal{R}(\varphi_0,\pi_{E_0})$.

Also due to the hypotheses $\mathrm{Fix}(\tau_{\overline{\alpha}}\,\overline{\varphi}) = \{e\}$, for all $\overline{\alpha} \in \mathcal{R}(\overline{\varphi},\pi_B)$ and, respectively $\mathrm{Fix}(\tau_{\overline{\alpha_0}}\,\overline{\varphi_0}) = \{e\}$, for all $\overline{\alpha_0} \in \mathcal{R}(\overline{\varphi_0},\pi_{B_0})$, the maps induced by the inclusions are, in fact, injective and the diagram below is commutative.

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathcal{R}_{K_0}(\varphi_0',\pi_{F_0}) & \xrightarrow{\widehat{i_{K_0}}} & \mathcal{R}(\varphi_0,\pi_{E_0}) & \xrightarrow{\widehat{p_0}} & \mathcal{R}(\overline{\varphi_0},\pi_{B_0}) & \longrightarrow & 1 \\
 & & \downarrow \widehat{i_{F,F_0}} & & \downarrow \widehat{i_{E,E_0}} & & \downarrow \widehat{i_{B,B_0}} & & \\
1 & \longrightarrow & \mathcal{R}_K(\varphi',\pi_F) & \xrightarrow{\widehat{i_K}} & \mathcal{R}(\varphi,\pi_E) & \xrightarrow{\widehat{p}} & \mathcal{R}(\overline{\varphi},\pi_B) & \longrightarrow & 1
\end{array}$$

In the way it was constructed, to each such $[\beta] \in \mathcal{R}(\varphi,\varphi_0)$ there is a corresponding class $[\beta'] \in \mathcal{R}_{K,K_0}(\tau_\beta\,\varphi',\tau_{\beta_0}\,\varphi_0')$. So we have,

$$\#\mathcal{R}(\varphi,\varphi_0) = \sum_{\overline{\alpha}\in\mathcal{R}(\overline{\varphi},\overline{\varphi_0})} \#\mathcal{R}_{K,K_0}(\tau_\alpha\,\varphi',\tau_{\alpha_0}\,\varphi_0')$$

where $\widehat{p}\,([\alpha]) = [\overline{\alpha}]$ and $\widehat{i_{F,F_0}}[\alpha_0] = [\alpha]$.

Also, if instead of having the hypotheses $\mathrm{Fix}(\tau_{\overline{\alpha}}\overline{\varphi}) = \mathrm{Fix}(\tau_{\overline{\alpha_0}}\overline{\varphi_0}) = \{e\}$, for all $[\overline{\alpha}] \in \mathcal{R}(\overline{\varphi}, \pi_B)$, respectively $[\overline{\alpha_0}] \in \mathcal{R}(\overline{\varphi_0}, \pi_{B_0})$, we have that f is globally fix group uniform then

$$(2) \qquad \#\mathcal{R}(\varphi, \varphi_0) = \sum_{[\overline{\alpha}] \in \mathcal{R}(\overline{\varphi}, \overline{\varphi_0})} \frac{\#\mathcal{R}_{K,K_0}(\tau_\alpha \varphi', \tau_{\alpha_0} \varphi_0')}{[\mathrm{Fix}(\overline{\varphi}) : p_{\#}\ \mathrm{Fix}(\varphi)]}$$

Although $\widehat{p}^{-1}\mathcal{R}(\overline{\varphi}, \overline{\varphi_0}) \supset \mathcal{R}(\varphi, \varphi_0)$, but not necessarily $\mathcal{R}(\overline{\varphi}, \overline{\varphi_0}) = \mathcal{R}(\varphi, \varphi_0)$, we still have $\mathcal{R}(\varphi, \varphi_0)$ mapped surjectively onto $\mathcal{R}(\overline{\varphi}, \overline{\varphi_0})$. This is the reason why some of the terms in the summand in (2) may be equal to zero.

Similar to $\mathcal{N}_K$ and N_K, we can define $\mathcal{N}_{K,K_0}(\varphi', \varphi_0')$ as the set of essential $\alpha' \in \mathcal{R}_{K,K_0}(\varphi', \varphi_0')$, and $\mathrm{N}_{K,K_0}(f', f_0') = \#\mathcal{N}_{K,K_0}(\varphi', \varphi_0')$.

With the equality (2) and the calculation of $\mathrm{R}(f; E - E_0)$ from Proposition 4.1, we obtain the following

Theorem 4.2. Let (f, f_0) be a fiber map of the pair. Let $K = \ker i$ and $K_0 = \ker i_0$. Suppose f is globally Fix group uniform and let $s = [\mathrm{Fix}(\overline{\varphi}) : p_{\#}\ (\mathrm{Fix}(\varphi))]$, then

$$\mathrm{R}(f; E - E_0) =$$

$$\frac{1}{s}\left\{ \sum_{[\overline{\alpha}] \in \mathcal{R}(\overline{\varphi}, \pi_B)} \#\mathcal{R}_K(\tau_\alpha \varphi', \pi_F) - \sum_{[\overline{\alpha}] \in \mathcal{R}(\overline{\varphi}, \overline{\varphi_0})} \#\mathcal{R}_{K,K_0}(\tau_\alpha \varphi', \tau_{\alpha_0} \varphi_0') \right\}$$

and,

$$\mathrm{R}(f; E, E_0) = \sum_{[\overline{\alpha_0}] \in \mathcal{R}(\overline{\varphi_0}, \pi_{B_0})} \frac{\#\mathcal{R}_{K_0}(\tau_{\alpha_0} \varphi_0', \pi_{F_0})}{[\mathrm{Fix}(\tau_{\overline{\alpha_0}} \overline{\varphi_0}) : p_{0\#}(\mathrm{Fix}(\tau_{\alpha_0} \varphi_0))]}$$

$$+ \frac{1}{s}\left\{ \sum_{[\overline{\alpha}] \in \mathcal{R}(\overline{\varphi}, \pi_B)} \#\mathcal{R}_K(\tau_\alpha \varphi', \pi_F) - \sum_{[\overline{\alpha}] \in \mathcal{R}(\overline{\varphi}, \overline{\varphi_0})} \#\mathcal{R}_{K,K_0}(\tau_\alpha \varphi', \tau_{\alpha_0} \varphi_0') \right\}.$$

Also, if (E, E_0) is a Jiang-type pair with nonzero Lefschetz numbers, $\mathrm{L}(f) \cdot \mathrm{L}(f_0) \neq 0$, then we have the respective Nielsen numbers

$$\mathrm{N}(f; E - E_0) =$$

$$\frac{1}{s}\left\{ \sum_{[\overline{\alpha}] \in \mathcal{N}(\overline{\varphi}, \pi_B)} \#\mathcal{N}_K(\tau_\alpha \varphi', \pi_F) - \sum_{[\overline{\alpha}] \in \mathcal{N}(\overline{\varphi}, \overline{\varphi_0})} \#\mathcal{N}_{K,K_0}(\tau_\alpha \varphi', \tau_{\alpha_0} \varphi_0') \right\}$$

and,

$$\mathrm{N}(f; E, E_0) = \sum_{[\overline{\alpha_0}] \in \mathcal{N}(\overline{\varphi_0}, \pi_{B_0})} \frac{\#\mathcal{N}_{K_0}(\tau_{\alpha_0} \varphi_0', \pi_{F_0})}{[\mathrm{Fix}(\tau_{\overline{\alpha_0}} \overline{\varphi_0}) : p_{0\#}(\mathrm{Fix}(\tau_{\alpha_0} \varphi_0))]}$$

$$+ \frac{1}{s}\left\{ \sum_{[\overline{\alpha}] \in \mathcal{N}(\overline{\varphi}, \pi_B)} \#\mathcal{N}_K(\tau_\alpha \varphi', \pi_F) - \sum_{[\overline{\alpha}] \in \mathcal{N}(\overline{\varphi}, \overline{\varphi_0})} \#\mathcal{N}_{K,K_0}(\tau_\alpha \varphi', \tau_{\alpha_0} \varphi_0') \right\}.$$

5. Product formulas for Nielsen numbers of pairs of fiber preserving maps

C. You in his paper [16], and also J. Pak in [12], came up with a "generalization" of the naïve product formula for Nielsen numbers that was introduced by Brown in [1]. More recently, Schusteff (see [14, Theorem III.2.3]) was able to obtain similar results for the relative Reidemeister numbers, but still with very strong hypotheses. The theorem that follows is a generalization of such results.

Theorem 5.1. Let (f, f_0) be a fiber map of the pair. Suppose that $\mathrm{R}(f; E, E_0)$ is finite, f is mod K fiber uniform and globally Fix group uniform, with $[\mathrm{Fix}(\overline{\varphi}) : p_{\#}\ \mathrm{Fix}(\varphi)]$ finite.

(1) If the inclusion $i_{B,B_0} \colon B_0 \longrightarrow B$ is a homotopy equivalence, f_0 is mod K_0 fiber uniform and

$$(\mathcal{C}_1) \qquad p_{0\#}\mathrm{Fix}(\tau_{\overline{\alpha}} \varphi_0) = p_{\#}\ \mathrm{Fix}(\tau_\alpha \varphi) \quad ,$$

where $\alpha \in \widehat{p}^{\,-1}[\overline{\alpha}]$ and $\overline{\alpha} \in \mathcal{R}(\overline{\varphi}, \pi_B)$, then

$$[\mathrm{Fix}(\overline{\varphi}) : p_{\#}\ \mathrm{Fix}(\varphi)]\, \mathrm{R}(f; E, E_0) = \mathrm{R}(\overline{f}; B, B_0)\, \mathrm{R}_{K,K_0}(f'; F, F_0)$$

where

$$\mathrm{R}_{K,K_0}(f';F,F_0) := \mathrm{R}_K(f') + \mathrm{R}_{K_0}(f_0') - \mathrm{R}_{K,K_0}(f',f_0').$$

(2) If the inclusion $i_{F,F_0}: F_0 \longrightarrow F$ is a homotopy equivalence and

$$(\mathcal{C}_2)\quad \frac{\#\mathcal{R}_{K_0}(\tau_{\alpha_0}\varphi_0')}{[\mathrm{Fix}(\tau_{\overline{\alpha_0}}\overline{\varphi_0}) : p_{0\#}\mathrm{Fix}(\tau_{\alpha_0}\varphi_0)]} = \frac{\#\mathcal{R}_K(\tau_\alpha\varphi')}{[\mathrm{Fix}(\tau_{\overline{\alpha}}\overline{\varphi}) : p_\# \mathrm{Fix}(\tau_\alpha\varphi)]}\quad,$$

where $\widehat{i_{E,E_0}}[\alpha_0] = [\alpha] \in \widehat{p}^{\,-1}[\overline{\alpha}]$ and $[\overline{\alpha}] \in \mathcal{R}(\overline{\varphi},\pi_B)$, then

$$[\mathrm{Fix}(\overline{\varphi}) : p_\# \mathrm{Fix}(\varphi)]\,\mathrm{R}(f;E,E_0) = \mathrm{R}(\overline{f};B,B_0)\,\mathrm{R}_K(f')\quad.$$

If, in addition, we have $\mathrm{R}_{K_0}(f_0') = \mathrm{R}_K(f')$ (for instance, if $K = K_0$), then we also have $\mathrm{R}_{K,K_0}(f';F,F_0) = \mathrm{R}_K(f')$ and the product formulas above are equivalent.

Proof. 1. From the homotopy equivalence of the bases and condition $(\mathcal{C}_1)$ we have that $[\mathrm{Fix}(\tau_{\overline{\alpha}}\overline{\varphi_0}) : p_0\#\,\mathrm{Fix}(\tau_\alpha\varphi_0)] = [\mathrm{Fix}(\tau_{\overline{\alpha}}\overline{\varphi}) : p_\#\,\mathrm{Fix}(\tau_\alpha\varphi)]$. Now, the fact that f is globally Fix group uniform implies that both are equal to $[\mathrm{Fix}(\overline{\varphi}) : p_\#\,\mathrm{Fix}(\varphi)]$.

The homotopy equivalence of the bases also implies that all Reidemeister numbers related to the bases are equal, i.e. $\mathrm{R}(\overline{f}) = \mathrm{R}(\overline{f_0}) = \mathrm{R}(\overline{f},\overline{f_0}) = \mathrm{R}(\overline{f};B,B_0)$. It follows from Theorem 4.2 and the fiber uniformity of f and of f_0 that

$$\begin{aligned}\mathrm{R}(f;E,E_0) &= \frac{1}{s}\Bigg\{\sum_{[\overline{\alpha_0}]\in\mathcal{R}(\overline{\varphi_0},\pi_{B_0})} \#\mathcal{R}_{K_0}(\tau_{\alpha_0}\varphi_0',\pi_{F_0})\\ &\quad + \sum_{[\overline{\alpha}]\in\mathcal{R}(\overline{\varphi},\pi_B)} \#\mathcal{R}_K(\tau_\alpha\varphi',\pi_F) - \sum_{[\overline{\alpha}]\in\mathcal{R}(\overline{\varphi},\overline{\varphi_0})} \#\mathcal{R}_{K,K_0}(\tau_\alpha\varphi',\tau_{\alpha_0}\varphi_0')\Bigg\}\\ &= \frac{1}{s}\Bigg\{\sum_{\mathfrak{A}} \mathrm{R}_{K_0}(\varphi_0') + \sum_{\mathfrak{A}} \mathrm{R}_K(\varphi') - \sum_{\mathfrak{A}} \mathrm{R}_{K,K_0}(\varphi',\varphi_0')\Bigg\}\\ &= \frac{1}{s}\sum_{\mathfrak{A}} \mathrm{R}_{K,K_0}(f';F,F_0)\quad.\end{aligned}$$

where $|\mathfrak{A}| = \mathrm{R}(\overline{f};B,B_0)$ and $s = [\mathrm{Fix}(\overline{\varphi}) : p_\#\,\mathrm{Fix}(\varphi)]$.

2. The homotopy equivalence of the fibers implies that we do have a mapping of $\mathcal{R}_{K_0}(\tau_{\alpha_0}\varphi_0', \pi_{F_0})$ onto $\mathcal{R}_K(\tau_\alpha\varphi', \pi_F)$. Therefore, all weakly common classes of f are weakly common classes mod K, K_0, i.e.,

$$\mathcal{R}_{K,K_0}(\tau_\alpha\varphi', \tau_{\alpha_0}\varphi_0') = \mathcal{R}_K(\tau_\alpha\varphi').$$

Now, under Condition $(\mathcal{C}_2)$ together with the hypothesis of f being mod K fiber uniform, Theorem 4.2 implies that

$$\begin{aligned}
\mathrm{R}(f; E, E_0) &= \sum_{[\overline{\alpha_0}]\in\mathcal{R}(\overline{\varphi_0},\pi_{B_0})} \frac{\#\mathcal{R}_K(\tau_\alpha\varphi')}{[\mathrm{Fix}(\tau_{\overline{\alpha}}\overline{\varphi}) : p_\sharp\,\mathrm{Fix}(\tau_\alpha\varphi)]} \\
&+ \frac{1}{s}\left\{\sum_{[\overline{\alpha}]\in\mathcal{R}(\overline{\varphi},\pi_B)} \#\mathcal{R}_K(\tau_\alpha\varphi', \pi_F) - \sum_{[\overline{\alpha}]\in\mathcal{R}(\overline{\varphi},\overline{\varphi_0})} \#\mathcal{R}_{K,K_0}(\tau_\alpha\varphi', \tau_{\alpha_0}\varphi_0')\right\} \\
&= \frac{1}{s}\left\{\sum_{[\overline{\alpha_0}]\in\mathcal{R}(\overline{\varphi_0},\pi_{B_0})} \#\mathcal{R}_K(\varphi', \pi_F)\right. \\
&\left.+ \sum_{[\overline{\alpha}]\in\mathcal{R}(\overline{\varphi},\pi_B)} \#\mathcal{R}_K(\varphi', \pi_F) - \sum_{[\overline{\alpha}]\in\mathcal{R}(\overline{\varphi},\overline{\varphi_0})} \#\mathcal{R}_K(\varphi', \pi_F)\right\} \\
&= \frac{1}{s}\,\mathrm{R}_K(f')\,\mathrm{R}(\overline{f}; B, B_0).
\end{aligned}$$

□

Remark 9. In [14, Theorem III.2.3], Schusteff proved Theorem 5.1 under the assumptions that both f and f_0 be eventually commutative. Here, we assume weaker hypotheses as we shall see in Example 5.1.

Suppose B, B_0, F and F_0 are Jiang-type spaces. We proved in [3] that, if $\mathrm{L}(\overline{f}) \neq 0$ and $\mathrm{L}(f_b) \neq 0$ then $\mathrm{N}(\overline{f}; B, B_0) = \mathrm{R}(\overline{f}; B, B_0)$ and $\mathrm{N}(f_b; F, F_0) = \mathrm{R}(f_b; F, F_0)$. This coupled with the fact that the index has "nice" multiplicative properties (meaning that any essential class on the total space must project to an essential class in the base space and also must be in the image of an essential class of the fiber), lead us to the following

Corollary 5.2. Let (f, f_0) be a fiber map of the pair, where both maps have non-zero Lefschetz number. Let B, B_0, F and F_0 be Jiang-type

spaces. Suppose $\mathrm{R}(f; E, E_0)$ is finite, f is mod K fiber uniform and globally Fix group uniform, with $[\mathrm{Fix}(\overline{\varphi}) : p_{\#}\ \mathrm{Fix}(\varphi)]$ finite. Suppose also both maps restricted to the fibers (f', f_0') have non-zero Lefschetz number.

(1) If the inclusion $i_{B,B_0}\colon B_0 \longrightarrow B$ is a homotopy equivalence, f_0 is mod K_0 fiber uniform and

$$(\mathcal{C}_1) \qquad p_{0\#}\mathrm{Fix}(\tau_{\overline{\alpha}}\varphi_0) = p_{\#}\ \mathrm{Fix}(\tau_\alpha\,\varphi)$$

where $\alpha \in \widehat{p}^{\ -1}[\overline{\alpha}]$ and $\overline{\alpha} \in \mathcal{R}(\overline{\varphi}, \pi_B)$, then

$$[\mathrm{Fix}(\overline{\varphi}) : p_{\#}\ \mathrm{Fix}(\varphi)]\,\mathrm{N}(f; E, E_0) = \mathrm{N}(\overline{f}; B, B_0)\,\mathrm{N}_{K,K_0}(f'; F, F_0) \quad .$$

(2) If the inclusion $i_{F,F_0}\colon F_0 \longrightarrow F$ is a homotopy equivalence and

$$(\mathcal{C}_2') \qquad \frac{\#\mathcal{N}_{K_0}(\tau_\alpha\,\varphi_0')}{[\mathrm{Fix}(\tau_{\overline{\alpha}}\,\overline{\varphi_0}) : p_{0\#}\mathrm{Fix}(\tau_\alpha\,\varphi_0)]} = \frac{\#\mathcal{N}_K(\tau_\alpha\,\varphi')}{[\mathrm{Fix}(\tau_{\overline{\alpha}}\,\overline{\varphi}) : p_{\#}\ \mathrm{Fix}(\tau_\alpha\,\varphi)]} \quad ,$$

where $\widehat{i_{E,E_0}}[\alpha_0] = [\alpha] \in \widehat{p}^{\ -1}[\overline{\alpha}]$ and $[\overline{\alpha}] \in \mathcal{N}(\overline{\varphi}, \pi_B)$, then

$$[\mathrm{Fix}(\overline{\varphi}) : p_{\#}\ \mathrm{Fix}(\varphi)]\,\mathrm{N}(f; E, E_0) = \mathrm{N}(\overline{f}; B, B_0)\,\mathrm{N}_K(f') \quad .$$

If, in addition, we have $\mathrm{N}_{K_0}(f_0') = \mathrm{N}_K(f')$ (for instance, if $K = K_0$), then we also have $\mathrm{N}_{K,K_0}(f'; F, F_0) = \mathrm{N}_K(f')$ and the product formulas above are equivalent.

Remark 10. Again, Schusteff [14] proved the same results as above in his Theorem III.2.19, but under the more demanding hypotheses of f and f_0 being eventually commutative, and the factor spaces B, B_0, F and F_0 being Jiang spaces. Also, in Theorem III.2.23, he proved that for nilmanifolds we do get a naïve product formula for the relative Reidemeister number. In the above corollary we managed to prove both results and more, since the Jiang-type spaces not only include both Jiang spaces and nilmanifolds, but also certain classes of solvmanifolds and of coset spaces of compact Lie groups.

The example below shows that, especially with nilmanifolds, the hypothesis of f being eventually commutative is too restrictive.

Example 5.1. Consider the subgroup of $GL(3,\mathbb{R})$,

$$G = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \middle| \; x,\, y,\, z \in \mathbb{R} \right\} .$$

and the subgroup of $GL(3,\mathbb{Z})$,

$$\Gamma = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \middle| \; x,\, y,\, z \in \mathbb{Z} \right\} .$$

The coset space $E = G/\Gamma$ is a nilmanifold, which fibers over the 2-torus $B = T^2$, with the circle $F = S^1$ as a typical fiber. We have a subfibration with $E_0 = T^2 \subset E$,

$$E_0 = \left\{ \begin{pmatrix} 1 & 0 & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \Gamma \middle| \; y,\, z \in \mathbb{Z} \right\} ,$$

considering the usual fibration of T^2 by S^1.

Consider the fiber map $f\colon (E, E_0) \longrightarrow (E, E_0)$ defined by

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & -a & -2b \\ 0 & 1 & 2c \\ 0 & 0 & 1 \end{pmatrix} ,$$

which has Nielsen numbers $\mathrm{N}(f) = 6$, $\mathrm{N}(\overline{f}) = 2$ and $\mathrm{N}(f_b) = 3$, for any $b \in B$.

On the subfibration, we have Nielsen numbers $\mathrm{N}(f_0) = 3$, $\mathrm{N}(f_{0_b}) = 3$ and $\mathrm{N}(\overline{f_0}) = 1$. The respective Nielsen numbers for the essentially common classes (see [13]) are $\mathrm{N}(f, f_0) = 3$, $\mathrm{N}(f_b, f_{0_b}) = 3$ and $\mathrm{N}(\overline{f}, \overline{f_0}) = 1$. Therefore we obtain the relative Nielsen numbers $\mathrm{N}(f; E, E_0) = 6$, $\mathrm{N}(f_b; F, F_0) = 3$ and $\mathrm{N}(\overline{f}; B, B_0) = 2$, which satisfy statement 2 of Corollary 5.2. Note that this is the case where $\mathrm{N}(f_b) = \mathrm{N}_{K,K_0}(f_b; F, F_0)$ since $K = K_0 = \{e\}$.

The map f is not eventually commutative. To see that, observe

$$f_\#^n \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & (-1)^n a & (-1)^n 2^n b \\ 0 & 1 & 2^n c \\ 0 & 0 & 1 \end{pmatrix}.$$

So, for any two elements

$$P = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}, \qquad Q = \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}$$

of $\pi_1(E)(\equiv G)$, for any $n \geq 1$ we have that $f_\#^n(P)$ commutes with $f_\#^n(Q)$ iff $az = cx$.

6. Estimation of Asymptotic Nielsen type numbers

In this final section, we make use of the computational techniques for calculating or approximating the relative Reidemeister number on the complement for a fiber-preserving map of pairs to estimate the asymptotic Nielsen type number introduced by Jiang [9].

In dynamics, one is interested in the asymptotic behavior of the growth of the number of periodic points. N. Ivanov [8] introduced the asymptotic Nielsen number to be the growth rate of $\{\mathcal{N}(f^n)\}$, i.e., $\mathrm{N}^\infty(f) = \mathrm{Growth}_{n\to\infty}\mathrm{N}(f^n) := \max\{1, \limsup \mathrm{N}(f^n)^{1/n}\}$ for any selfmap $f : X \to X$ on a compact connected polyhedron X. He showed that $h(f) \geq \log \mathrm{N}^\infty(f)$ where $h(f)$ denotes the topological entropy of f. While the Nielsen number $\mathrm{N}(f^n)$ gives a lower bound for the number of n-periodic points of f, it is often a poor estimate. On the other hand, the Nielsen type number $\frac{1}{n}NP_n(f)$ (see [9]), which measures the number of periodic orbits of least period n, yields better information so that the growth rate of $\{\frac{1}{n}NP_n(f)\}$, denoted by $\mathrm{NI}^\infty(f)$, was introduced by Jiang to study the asymptotic behavior of the growth of the periodic orbits.

In [15], it was shown that

$$\mathrm{NI}^\infty(f) \leq \mathrm{Growth}_{n\to\infty}\mathrm{R}(g_n(f); Y_{n(1)}).$$

Here, $Y_n = X \times ... \times X$ is the n-fold product with a cyclic action of $\mathbf{Z}_n$; $g_n(f)$ is the associated $\mathbf{Z}_n$-map given by $g_n(f)(x_1, ..., x_n) = (f(x_n), f(x_1), ..., f(x_{n-1}))$; $Y_{n_{(1)}} = Y_n - Y_n^{>(1)}$ where $Y_n^{>(1)}$ is the closed subspace consisting of points of non-trivial isotropy and $\mathrm{R}(g_n(f); Y_{n_{(1)}})$ is the relative Reidemeister number on the complement.

Suppose $\xi : X \to \overline{X}$ is a Hurewicz fibration and $f : X \to X$ is a fiber-preserving map with induced map $\overline{f} : \overline{X} \to \overline{X}$. Let p be a prime. Then $(g_{p^r}(f), g_{p^{r-1}}(f))$ is a fiber map of the pair $(Y_{p^r}, Y_{p^{r-1}})$, in the sense of section 4, with induced maps $(g_{p^r}(\overline{f}), g_{p^{r-1}}(\overline{f}))$. We have the following commutative diagram

$$\begin{array}{ccc} Y_{p^r}, Y_{p^{r-1}} & \xrightarrow{g_{p^r}(f)} & Y_{p^r}, Y_{p^{r-1}} \\ {\scriptstyle q}\downarrow{\scriptstyle q_0} & & {\scriptstyle q}\downarrow{\scriptstyle q_0} \\ \overline{Y}_{p^r}, \overline{Y}_{p^{r-1}} & \xrightarrow{g_{p^r}(\overline{f})} & \overline{Y}_{p^r}, \overline{Y}_{p^{r-1}} \end{array}$$

where $q = \xi \times ... \times \xi$ (p^r-fold product) and q_0 is the p^{r-1}-fold product of ξ. As a consequence of Proposition 4.1, we have the following

Theorem 6.1. *Suppose* $\mathrm{N}(f^n) = \mathrm{R}(f^n)$ *for all* $n \geq 1$. *Then, for any prime* p,

$$\begin{aligned} &\mathrm{NI}^\infty(f) \\ \geq &\mathrm{Growth}_{r\to\infty} \mathrm{R}(g_{p^r}(f); Y_{p^r{}_{(1)}}) \\ \geq &\mathrm{Growth}_{r\to\infty} \sum_{[\overline{\alpha}]\in \mathcal{R}(\overline{\varphi_r}, \pi_{B_r}) - \mathcal{R}(\overline{\varphi_r}, \overline{\varphi_{0_r}})} \sum_{[\beta]} \frac{1}{[\mathrm{Fix}(\tau_{\overline{\alpha}}\overline{\varphi_r}) : q_\#(\mathrm{Fix}(\tau_{\beta\alpha}\varphi_r))]} \end{aligned}$$

where φ_r *and* $\overline{\varphi_r}$ *are the homomorphisms induced by* $g_{p^r}(f)$ *and* $g_{p^r}(\overline{f})$ *respectively at the fundamental group level. Here,* $B_r = \overline{Y_{p^r}}$ *and* $F_r = q^{-1}(b)$, $b \in \mathrm{Fix}(g_{p^r}(\overline{f})) \cap (\overline{Y_{p^r}} - \overline{Y_{p^{r-1}}})$.

Corollary 6.2. If X is a solvmanifold and $\mathrm{R}(f^n) < \infty$ for all n, then

$$\mathrm{NI}^\infty(f) \geq \mathrm{Growth}_{r\to\infty} \sum_{[\overline{\alpha}]\in \mathcal{R}(\overline{\varphi_r}, \pi_{B_r}) - \mathcal{R}(\overline{\varphi_r}, \overline{\varphi_{0_r}})} \#\mathcal{R}(\tau_\alpha \varphi'_r, \pi_{F_r})$$

where f is a fiber-preserving map of a Mostow fibration of X.

Proof. Since $\mathrm{R}(f^n) < \infty$, then $\mathrm{R}(\overline{f}^n) < \infty$. Since the base of the fibration of X is a torus, it follows that $\mathrm{Fix}(\tau_{\overline{\alpha}}\overline{\varphi_n}) = \mathrm{Fix}(\overline{\varphi_n}) = \{e\}$. Moreover, K is trivial. Now the inequality follows from the Theorem 6.1. □

Remark 11. In Corollary 6.2 the fiber is a nilmanifold, thus $\#\mathcal{R}(\tau_\alpha \varphi'_r, \pi_{F_r})$ coincides with the corresponding Lefschetz number on the fiber.

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