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RIEMANNIAN FOLIATION

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EQUIFOCALITY OF A SINGULAR RIEMANNIAN FOLIATION

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ABSTRACT. A singular foliation on a complete riemannian manifold M is said to be riemannian if each geodesic that is perpendicular at one point to a leaf remains perpendicular to every leaf it meets. We prove that the regular leaves are equifocal, i.e., the end point map of a normal foliated vector field has constant rank. This implies that we can reconstruct the singular foliation by taking all parallel submanifolds of a regular leaf with trivial holonomy. In addition, the end point map of a normal foliated vector field on a leaf with trivial holonomy is a covering map. These results generalize previous results of the authors on singular riemannian foliations with sections.

1. INTRODUCTION

In this section, we will recall some definitions and state our main results as Theorem 1.5 and Corollary 1.6.

We start by recalling the definition of a singular riemannian foliation (see the book of Molino [9]).

Definition 1.1 (s.r.f.). A partition \mathcal{F} of a complete riemannian manifold M by connected immersed submanifolds (the *leaves*) is called a *singular riemannian foliation* (s.r.f. for short) if it verifies condition (1) and (2):

- (1) \mathcal{F} is a *singular foliation*, i.e., the module $\mathcal{X}_{\mathcal{F}}$ of smooth vector fields on M that are tangent at each point to the corresponding leaf acts transitively on each leaf. In other words, for each leaf L and each $v \in TL$ with footpoint p , there is $X \in \mathcal{X}_{\mathcal{F}}$ with $X(p) = v$.
- (2) \mathcal{F} *transnormal*, i.e., every geodesic that is perpendicular at one point to a leaf remains perpendicular to every leaf it meets.

Let \mathcal{F} be a singular riemannian foliation on a complete riemannian manifold M . A leaf L of \mathcal{F} (and each point in L) is called *regular* if the dimension of L is maximal, otherwise L is called *singular*.

Typical examples of s.r.f. are the partition by orbits of an isometric action, by leaf closures of a Riemannian foliation, examples constructed by suspension of homomorphisms (see [2, 4]) and examples constructed by changes of metric and surgery (see [5]).

A particular class of s.r.f. are the one which admits sections, i.e., for each regular point p the set $\Sigma := \exp(\nu_p L_p)$ is a complete immersed submanifold that meets each leaf orthogonally.

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The concept of singular riemannian foliations with sections (s.r.f.s. for short) was introduced in [2] and continued to be studied by the authors in [1, 3, 4, 11, 12, 5], by Lytchak and Thorbergsson in [8] and recently by Gorodski and the first author in [6]. In [7] Boualem dealt with a singular riemannian foliation \mathcal{F} on a complete manifold M such that the distribution of normal spaces of the regular leaves is integrable. It was proved in [4] that such an \mathcal{F} must be a s.r.f.s.

S.r.f.s. include the partitions by orbits of a polar action and the well known class of isoparametric foliations on space forms, some of them with inhomogeneous leaves.

In [10], Terng and Thorbergsson introduced the concept of equifocal submanifolds with flat sections in compact symmetric spaces in order to generalize the definition of isoparametric submanifolds in euclidean space.

A connected immersed submanifold L of a complete riemannian manifold M is called *equifocal* if it satisfies the following conditions:

- (1) The normal bundle $\nu(L)$ is flat.
- (2) For each parallel normal field ξ on a neighborhood $U \subset L$, the derivative of the map $\eta_\xi : U \rightarrow M$ defined by $\eta_\xi(x) := \exp_x(\xi)$ has constant rank.
- (3) L has sections, i.e. for each $p \in L$, the set $\Sigma := \exp_p(\nu_p L_p)$, called section, is a complete immersed totally geodesic submanifold.

There is almost an equivalence between the notions of a s.r.f.s. and equifocal submanifolds that is worked out in the authors' works [2] and [11].

On the one hand it was proved that a closed embedded equifocal submanifold induces a s.r.f.s. by taking all its parallel submanifolds ([11], [3]) if and only if there is exactly one section through every regular value of the normal exponential map of the equifocal submanifold. The global structure inherent to a s.r.f.s. was then used to generalize some results known for isoparametric submanifolds in euclidean space.

On the other hand, it was proved in [2] that the leaves of a s.r.f.s. are equifocal (see [11] for an alternative proof). In converse direction to above the equifocality of a s.r.f.s. is also a very important tool in the theory of s.r.f.s. For example, it allows us to have a Slice Theorem, singular holonomy, Weyl pseudogroups, a relation of s.r.f.s. to normal maps and an extension of Weyl-invariant forms to basic forms.

While the existence of sections has interesting structural implications it naturally restricts the number of cases that are covered. This can be best seen in the case of homogenous s.r.f., when comparing an arbitrary isometric actions with a polar action. The latter is best exemplified by the action of a compact Lie group on itself by conjugation. In this paper we want to drop the condition on the existence of sections and prove that regular leaves of a s.r.f. are also equifocal. In order to make this statement precise, we will drop the first and third condition in the definition of equifocal submanifold and we will also need to change the concept of parallel normal fields to foliated vector fields. Note that the restriction \mathcal{F}_r of \mathcal{F} to the regular stratum of M is a regular foliation. We recall that a vector field ξ in the normal bundle of the foliation over an open subset U in the regular stratum is called *foliated*, if for each vector field $Y \in \mathcal{X}_F$ the Lie bracket $[\xi, Y]$ also belongs to \mathcal{X}_F . If we consider a local submersion π which describes the plaques of \mathcal{F} in a neighborhood of a point of L , then a normal foliated vector field is a normal projectable/basic vector field with respect to π .

Remark 1.2. A Bott or basic connection ∇ of a foliation \mathcal{F} is a connection of TM with $\nabla_X Y = [X, Y]^{\nu\mathcal{F}}$ whenever $X \in \mathcal{X}_F$ and Y is vector field of the normal bundle $\nu\mathcal{F}$ of the foliation. Here the superscript $\nu\mathcal{F}$ denotes projection onto $\nu\mathcal{F}$. A foliated vector field clearly is parallel with respect to the Bott connection. This connection can be restricted to the normal bundle of a leaf.

Definition 1.3. Let L be a regular leaf of a s.r.f. A normal vector field along L is said to be *foliated*, if it is Bott-parallel, or in other words, if it is locally the restriction of a foliated vector field of \mathcal{F}_r to a neighborhood $U \subset L$.

Remark 1.4. Note that if the s.r.f. admits sections then a normal foliated vector field is a parallel normal field along each regular leaf L with respect to the induced Levi-Civita connection on νL and vice versa. In other words in the case of sections the induced Levi-Civita connection is a Bott-connection.

We are finally ready to state our result precisely.

Theorem 1.5. *Let \mathcal{F} be a s.r.f. on a complete riemannian manifold M . Then for each regular point p there exists a neighborhood U of p in L_p such that*

- (1) *For each normal foliated vector field ξ along U the derivative of the map $\eta_\xi : U \rightarrow M$, defined as $\eta_\xi(x) := \exp_x(\xi)$, has constant rank.*
- (2) *$W := \eta_\xi(U)$ is an open set of $L_{\eta_\xi(p)}$.*

Corollary 1.6. *Let L_p be a regular leaf with trivial holonomy and Ξ denote the set of all normal foliated vector fields along L_p .*

- (1) *Let $\xi \in \Xi$. Then $\eta_\xi : L_p \rightarrow L_q$ is a covering map if $q = \eta_\xi(p)$ is a regular point.*
- (2) *$\mathcal{F} = \{\eta_\xi(L_p)\}_{\xi \in \Xi}$, i.e., we can reconstruct the singular foliation by taking all parallel submanifolds of the regular leaf L_p .*

This paper is organized as follows. In Section 2 we present the propositions need to prove the theorem. In particular we prove two propositions which contain some improvements of Molino's results on the local analysis of a s.r.f. More precisely, we review a local decomposition result and a product theorem due to Molino (see Proposition 2.2 and Proposition 2.3). In Section 3 we prove Theorem 1.5 and in Section 4 we prove Corollary 1.6.

2. PROPERTIES OF A S.R.F.

In this section we will present the propositions needed to prove Theorem 1.5. Throughout this section we assume that \mathcal{F} is a s.r.f. on a complete riemannian manifold M .

We start by recalling the so called *Homothetic Transformation Lemma* of Molino (see Lemma 6.2 [9]).

By conjugating the homothetic transformations of the normal bundle of a plaque P via the normal exponential map, one defines for small strictly positive real numbers λ , a homothetic transformation h_λ with proportionality constant λ with respect to the plaque P .

Proposition 2.1 ([9]). *The homothetic transformation h_λ sends plaque to plaque and therefore respects the singular foliation \mathcal{F} in the tubular neighborhood $\text{Tub}(P)$ where it is defined.*

The next two propositions contain some improvements of Molino's results (compare with Theorem 6.1 and Proposition 6.5 of [9]).

Proposition 2.2. *Let g be the original metric on M and $q \in M$. Then there exists a tubular neighborhood $\text{Tub}(P_q)$ and a new metric \tilde{g} on $\text{Tub}(P_q)$ with the following properties.*

- (a) *For each $x \in \text{Tub}(P_q)$ the normal space of the leaf L_x is tangent to the slice $S_{\tilde{q}}$ which contains x , where $\tilde{q} \in P_q$.*
- (b) *Let $\pi : \text{Tub}(P_q) \rightarrow P_q$ be the orthogonal projection. Then the restriction $\pi|_{P_x}$ is a riemannian submersion.*
- (c) *$\mathcal{F} \cap \text{Tub}(P_q)$ is a s.r.f.*
- (d) *$\mathcal{F} \cap S_{\tilde{q}}$ is a s.r.f. for each $\tilde{q} \in P_q$.*
- (e) *The associated transverse metric is not changed, i.e., the distance between the plaques with respect to g is the same distance between the plaques with respect to \tilde{g} .*
- (f) *If a curve γ is a geodesic orthogonal to P_q with respect to the original metric g , then γ is a geodesic orthogonal to P_q with respect to the new metric \tilde{g} .*

Proof. Let $X_1, \dots, X_r \in \mathcal{X}_F$ (i.e. vector fields that are always tangent to the leaves) so that $\{X_i(q)\}_{i=1, \dots, r}$ is a linear basis of $T_q P_q$. Let $\varphi_1^1, \dots, \varphi_r^1$ denote the associated one parameter groups and define $\varphi(t_1, \dots, t_r, y) := \varphi_1^1 \circ \dots \circ \varphi_r^1$ where $y \in S_q$ and (t_1, \dots, t_r) belongs to a neighborhood U of $0 \in \mathbb{R}^r$. Then, reducing U and $\text{Tub}(P_q)$ if necessary, one can guarantee the existence of a regular foliation \mathcal{F}^2 with plaques $P_y^2 = \varphi(U, y)$. We note that the plaques $P_x^2 \subset P_x$ and each plaque P^2 cuts each slice at exactly one point. Using the fact that $\pi|_{P_y^2} : P_y^2 \rightarrow P_q$ is a diffeomorphism, we can define a metric on each plaque P_y^2 as $\tilde{g}^2 := (\pi|_{P_y^2})^* g$.

Now we want to define a metric \tilde{g}^1 on each slice $S \in \{S_{\tilde{q}}\}_{\tilde{q} \in P_q}$. Set $D_p := \nu_p L_p^2$ and define $\Pi : T_p M \rightarrow D_p$ as the orthogonal projection with respect to g . The fact that each plaque P^2 cuts each slice at one point implies that $\Pi|_{T_p S} : T_p S \rightarrow D_p$ is an isomorphism. Finally we define $\tilde{g}^1 := (\Pi|_{T_p S})^* g$ and $\tilde{g} := \tilde{g}^1 + \tilde{g}^2$, meaning that \mathcal{F}^2 and the slices meet orthogonally. Items (a) and (b) follow directly from the definition of \tilde{g} .

To prove Item (c) it suffices to prove that the plaques of \mathcal{F} are locally equidistant to each other. Let $x \in S_{\tilde{q}}$, P_x a plaque of \mathcal{F} . We know that the plaques of \mathcal{F} are contained in the leaves of the foliation by distance-cylinders $\{C\}$ with axis P_x with respect to g . We will prove that each C is also a distance-cylinder with axis P_x with respect to the new metric \tilde{g} . These facts and the arbitrary choice of x will imply that the plaques of \mathcal{F} are locally equidistant to each other.

First we recall that a smooth function $f : M \rightarrow \mathbb{R}$ is called a *transnormal function* with respect to the metric g if there exists a $C^2(f(M))$ function b such that $g(\text{grad } f, \text{grad } f) = b \circ f$. Let $f : \text{Tub}(P_x) \rightarrow \mathbb{R}$ be a smooth transnormal function with respect to the metric g so that each regular level set $f^{-1}(c)$ is a cylinder C with axis P_x , e.g. $f(y) = d(y, P_x)^2$. Let $\widetilde{\text{grad}} f$ denote the gradient of f with respect to the metric \tilde{g} . It follows from the construction of \tilde{g} that

$$(2.1) \quad \widetilde{\text{grad}} f = \text{grad } f + l$$

where l is a vector tangent to a plaque of \mathcal{F}^2 and in particular to a plaque of \mathcal{F} .

Indeed, let $v \in D_p$ and $w := (\Pi|_{T_p S})^{-1}(v)$. Then

$$\begin{aligned} g(\text{grad } f, v) &= df(v) \\ &= df(w) \\ &= \tilde{g}(\widetilde{\text{grad } f}, w) \\ &= \tilde{g}^{-1}(\widetilde{\text{grad } f}, w) \\ &= g(\Pi \widetilde{\text{grad } f}, \Pi w) \\ &= g(\Pi \widetilde{\text{grad } f}, v) \end{aligned}$$

We conclude from the arbitrary choice of $v \in D_p$, that $\text{grad } f = \Pi \widetilde{\text{grad } f}$, and hence $\widetilde{\text{grad } f} = \text{grad } f + l$.

Equation 2.1 implies that f is also a transnormal function with respect to the metric \tilde{g} , i.e.,

$$(2.2) \quad \tilde{g}(\widetilde{\text{grad } f}, \widetilde{\text{grad } f}) = b \circ f,$$

Indeed,

$$\begin{aligned} \tilde{g}(\widetilde{\text{grad } f}, \widetilde{\text{grad } f}) &= df(\widetilde{\text{grad } f}) \\ &= df(\text{grad } f) \\ &= g(\text{grad } f, \text{grad } f) \\ &= b \circ f \end{aligned}$$

Using a local version of Q-M Wang's theorem [13], we conclude that each regular level set of f (i.e. C) is a distance cylinder around P_x with respect to the metric \tilde{g} .

Now we want to prove Item (d). Set $P_x^s = P_x \cap S_{\tilde{g}}$ and $C^s := C \cap S_{\tilde{g}}$. It suffices to note that the singular foliation $\{C^s\}$ is a foliation by cylinders with axis P_x^s with respect to the new metric \tilde{g} . This follows from the fact that $\nu_x P_x \subset T_x S_{\tilde{g}}$ and that each geodesic orthogonal to P_x at x is contained in $S_{\tilde{g}}$ (see Item (a)).

In particular we conclude that the distance between C and P_x and the distance between C^s and P_x^s with respect to the metric \tilde{g} are the same.

To prove Item (e) we have to prove that the distance between the cylinder C and the plaque P_x is the same for both metrics. Let f be the transnormal function (with respect to g) defined above. According to Q-M Wang [13] for $k = f(P_x)$ and a regular value c we have $d(P_x, f^{-1}(c)) = \int_c^k \frac{ds}{\sqrt{b(s)}}$. Since f is also a transnormal function with respect to \tilde{g} (see Equation (2.2)), we conclude that $d(P_x, C) = \tilde{d}(P_x, C)$, for $C = f^{-1}(c)$.

Finally we prove Item (f). We consider the transnormal function f above with $x = q$. In this case, Equation (2.1) and the fact that $\text{grad } f \in D_p \cap T_p S$ imply that $\text{grad } f = \widetilde{\text{grad } f}$. On the other hand, the integral curves of the gradient of a transnormal function are geodesic segments up to reparametrization (see e.g. [13]). Therefore the radial geodesics of P_q coincide in both metrics. This finishes the proof. \square

Proposition 2.3. *Let \tilde{g} be the metric defined in Proposition 2.2. Then there exists a new metric g_0 on $\text{Tub}(P_q)$ so that,*

- (a) Consider the tangent space $T_{\tilde{q}}S_{\tilde{q}}$ with the metric \tilde{g} and $S_{\tilde{q}}$ with the metric g_0 . Then $\exp_{\tilde{q}}: T_{\tilde{q}}S_{\tilde{q}} \rightarrow S_{\tilde{q}}$ is an isometry.
- (b) For this new metric g_0 we have that $\mathcal{F} \cap S_{\tilde{q}}$ and \mathcal{F} restricted to $\text{Tub}(P_{\tilde{q}})$ are also s.r.f.
- (c) For each $x \in \text{Tub}(P_{\tilde{q}})$ the normal space of the leaf L_x is tangent to the slice $S_{\tilde{q}}$ which contains x , where $\tilde{q} \in P_{\tilde{q}}$.

Remark 2.4. Clearly a curve γ which is a geodesic orthogonal to $P_{\tilde{q}}$ with respect to the original metric, remains a geodesic orthogonal to $P_{\tilde{q}}$ with respect to the new metric g_0 .

Proof. Let Π_1 be the orthogonal projection to the slices, recall that $\tilde{g}^1 = \tilde{g} \circ \Pi_1$ and $\tilde{g}^2 = \tilde{g} \circ d\pi$. Let h_λ denote the homothetic transformation with respect to $P_{\tilde{q}}$. Define $g_\lambda = \frac{1}{\lambda^2} h_\lambda^* \tilde{g}^1 + \tilde{g}^2$. Note that the metric g_λ tends uniformly to a metric g_0 for $\lambda \rightarrow 0$. This metric g_0 restricted to $S_{\tilde{q}}$ is the induced metric on $\nu P_{\tilde{q}}$, where $\tilde{q} \in P_{\tilde{q}}$.

This implies that L_λ tends uniformly to L_0 , where L_λ is the length function. It follows then that

$$(2.3) \quad \lim_{\lambda \rightarrow 0} d_\lambda(x, P) = d_0(x, P)$$

where P is a plaque.

Now we claim that \mathcal{F} is a s.r.f. with respect to g_λ . Indeed, since $h_\lambda^* \tilde{g}^2 = \tilde{g}^2$ and the homothetic transformation h_λ sends plaque to plaque (see Proposition 2.1) it suffices to prove that \mathcal{F} is a s.r.f. with respect to $\frac{1}{\lambda^2} \tilde{g}^1 + \tilde{g}^2$. Let $f: \text{Tub}(P_x) \rightarrow \mathbb{R}$ be a smooth transnormal function with respect to the metric \tilde{g} so that each regular level set $f^{-1}(c)$ is a cylinder with axis P_x . Note that f is also a transnormal function with respect to the metric $\frac{1}{\lambda^2} \tilde{g}^1 + \tilde{g}^2$, because $\widetilde{\text{grad}} f$ is tangent to the slice. Using a local version of Q-M Wang's theorem [13], we conclude that each regular level set of f is a tube over P_x with respect to the metric $\frac{1}{\lambda^2} \tilde{g}^1 + \tilde{g}^2$. Therefore the plaques are equidistant to P_x and hence we conclude that \mathcal{F} is a s.r.f. with respect to $\frac{1}{\lambda^2} \tilde{g}^1 + \tilde{g}^2$.

Finally let x and y be points which belong to the same plaque. Using Equation 2.3 and the fact that \mathcal{F} is a s.r.f. with respect to g_λ we conclude that

$$\begin{aligned} 0 &= \lim_{\lambda \rightarrow 0} (d_\lambda(x, P) - d_\lambda(y, P)) \\ &= d_0(x, P) - d_0(y, P) \end{aligned}$$

The above equation implies that the plaques are locally equidistant and hence that the singular foliation \mathcal{F} is riemannian.

Now we want to prove Item (c). Let P_x be a plaque with $x \in S$. Note that for each metric g_λ the normal space H_λ of P_x at x (with respect to the metric g_λ) is tangent to S . This fact will imply that the normal space of P_x at x with respect to g_0 is also tangent to S . Indeed, we can find a sequence of normal spaces $H_{1/n}$ such that $H_{1/n}$ converge to a subspace H_0 tangent to S at x . Then we can find a subsequence of frames $\{e_i^n\}$ which converge to a frame $\{e_i\}$ such that $\{e_i^n\}$ and $\{e_i\}$ are bases of $H_{1/n}$ and H_0 respectively. Since

$$\frac{1}{\lambda^2} h_\lambda^* \tilde{g}^1(d(\exp_q)_\nu Y, d(\exp_q)_\nu Z) = \tilde{g}^1(d(\exp_q)_\lambda \nu Y, d(\exp_q)_\lambda \nu Z),$$

we conclude that

$$g_0(e_i, l) = \lim_{n \rightarrow \infty} g_{1/n}(e_i^n, l) = 0$$

where l is tangent to the plaque. The last equation implies that H_0 is the normal space of P_x at x with respect to g_0 . \square

Proposition 2.5. *Let S_q be a slice at q and $\varphi : S_q \rightarrow S_q$ be the geodesic symmetry at q , i.e., $\varphi = \exp_q \circ (-\text{id}) \circ \exp_q^{-1}$. Then the map φ is $\mathcal{F} \cap S_q$ foliated, i.e. the foliation $\mathcal{F} \cap S_q$ is invariant by the involution φ .*

Proof. It follows from Proposition 2.3 and Remark 2.4 that we can lift \mathcal{F} via the exponential map in a neighborhood of q to a s.r.f. of $T_q S_q$. Therefore we assume that \mathcal{F} is a s.r.f. of \mathbb{R}^n with euclidean metric which has $\{0\}$ as a leaf.

Lemma 2.6. *The induced singular foliation on the unit sphere $\mathcal{F}' := \mathcal{F}|S^{n-1}$ is a s.r.f.*

Proof. First note that every leaf of \mathcal{F} that has a point in S^{n-1} lies in S^{n-1} . Clearly \mathcal{F}' is a singular foliation. We now want to show transversality. Let $v \in S^{n-1}$ and $\xi \in \nu_v L_v \cap T_v S^{n-1}$ a unit vector. We denote by γ_ξ the geodesic in S^{n-1} with initial vector ξ . We want to show that $\xi(t) := \dot{\gamma}_\xi(t) \in \nu_w L_w$, where $w = \gamma_\xi(t)$. First we assume $t \in (0, \pi)$. Then the two unit radial vectors of S^{n-1} in v and w span a 2-plane of \mathbb{R}^n containing the origin. As it contains the straight line from v to w , it lies in $\nu_w L_w$ by transversality of \mathcal{F} . The intersection of this 2-plane with S^{n-1} is exactly the geodesic γ_ξ . Therefore $\xi(t) \in \nu_w L_w$. This shows that $\gamma_\xi|_{[0, t]}$ and consequently $\gamma_\xi|_{[0, \pi]}$ is transversal. To prove transversality of $\gamma_\xi|_{[0, 2\pi]}$ we repeat the argument with w respectively $\xi(t)$ as our new v respectively ξ . Since the geodesic γ_ξ is closed of period 2π only a third step is needed to show its transversality. \square

Now let $v \in S^{n-1}$ and let L_v be leaf through v . Here we denote by $\nu_v L_v$ the normal space of L_v in S^{n-1} . For any $\xi \in \nu_v^1 L_v$ the geodesic γ_ξ in S^{n-1} meets the leaf L_{-v} in the antipodal point $-v$ orthogonally, i.e. $-\xi = \dot{\gamma}_\xi(\pi) \in \nu_{-v} L_{-v}$. So as vector spaces in \mathbb{R}^n we have $\nu_v L_v \subset \nu_{-v} L_{-v}$ and by symmetry we have equality for every $v \in S^{n-1}$. In other words $-\text{id}$ respects the normal bundle and therefore also the tangent bundle of \mathcal{F} . From this we conclude that $-\text{id}$ respects \mathcal{F} on S^{n-1} . \square

Corollary 2.7. *Let γ be a geodesic orthogonal to a regular leaf of a s.r.f. Then the singular points are isolated on γ .*

3. PROOF OF THE THEOREM

In this section we will apply the above propositions to prove the theorem. We start by proving a local version of Theorem 1.5.

Proposition 3.1. *Let $\text{Tub}(P_q)$ be a tubular neighborhood of a plaque P_q , $x_0 \in \text{Tub}(P_q)$, a regular point and $\xi \in \nu_{P_{x_0}}$ such that $\exp_{x_0}(\xi) = q$. Then we can find a neighborhood U of x_0 in P_{x_0} with the following properties:*

- 1) *We can extend ξ to a foliated normal vector field ξ on U .*
- 2) *The geodesic segment that is orthogonal to P_q and contains a point $x \in U$ is $\gamma_x(t) := \exp_x((t+1)\xi)$ where $t \in [-1, 1]$.*
- 3) *$\eta_{(t+1)\xi}(U)$ is an open subset of $L_{\gamma_{x_0}(t)}$.*

- 4) $\eta_{t\xi} : U \rightarrow \eta_{t\xi}(U)$ is a diffeomorphism for $t \neq 1$.
 5) $\dim \text{rank } D\eta_{t\xi}$ is constant on U .

Proof. The proof of 1) is straightforward. The proof of 2) follows from the Homothetic Transformation Lemma by Molino (Proposition 2.1).

Using Proposition 2.1 and Proposition 2.5 we can conclude the following lemma.

Lemma 3.2. *Let $\alpha(s)$ be a curve in U . Define $f(s, t) = \exp_{\alpha(s)}(t\xi)$ and $J(t) = \frac{\partial f}{\partial s}(0, t)$. Then to prove item 3), 4) and 5) it suffices to prove that the Jacobi field J is always tangent to the leaves.*

In what follows we will prove that the Jacobi field J defined above is always tangent to the leaves.

Let g_0 be the metric defined in Proposition 2.3. Then Remark 2.4 and Item 2) imply that the Jacobi field J defined in Lemma 3.2 has not been changed when the metric was modified.

Now consider a geodesic segment γ orthogonal to the leaves of \mathcal{F} so that $\gamma(0) = q$ and $\gamma(1)$ is a regular point contained in S_q . It follows from Corollary 2.7 that $\gamma(t)$ is always regular for $-1 \leq t < 0$ and $0 < t \leq 1$.

We define σ as the submanifold contained in S_q which is the image by \exp_q of a subspace and so that σ is orthogonal to L_x at x .

By Proposition 2.5, Proposition 2.3 and Proposition 2.1 we have that the plaques $P_{\gamma(t)} \cap S_q$ are orthogonal to σ for $-1 \leq t \leq 1$. Then it follows from Proposition 2.3 that the plaques $P_{\gamma(t)}$ are orthogonal to σ for $-1 \leq t \leq 1$.

Consider a geodesic segment β so that $\beta(0) = \gamma(t)$ and β is orthogonal to $P_{\gamma(t)}$. Then Proposition 2.3 imply that β is contained in S_q . Since S_q is identified with $T_q S_q$ we can consider β as a straight line. Since $P_{\gamma(t)} \cap S_q$ is orthogonal to σ , and σ is identified with a subspace, we conclude that β is contained in σ .

Therefore $\exp_{\gamma(t)}(\nu(P)_{\gamma(t)} \cap B_\epsilon(0))$ is an open set of σ . A standard argument from riemannian geometry implies that the second form is null at $\gamma(t)$, i.e., σ is geodesic at $\gamma(t)$. In particular the curvature tensor R of σ is the same as the ambient space at $\gamma(t)$. This fact and the fact that $R(\gamma', \cdot)\gamma'$ is self-adjoint imply that $T_{\gamma(t)}\sigma$ as well $(T_{\gamma(t)}\sigma)^\perp$ are families of parallel subspace along γ which are invariant by $R(\gamma', \cdot)\gamma'$.

Finally consider the L_x -Jacobi field J defined in Lemma 3.2. This Jacobi field has initial conditions at $(T_{\gamma(1)}\sigma)^\perp$ and satisfies the Jacobi equation. So $J(t) \in (T_{\gamma(t)}\sigma)^\perp$ for $-1 \leq t \leq 1$.

As remarked above plaques $P_{\gamma(t)}$ are orthogonal to σ for $-1 \leq t \leq 1$. Since $P_{\gamma(t)}$ are regular plaques for $t \neq 0$ (see Corollary 2.7) we conclude that $J(t)$ is always tangent to $P_{\gamma(t)}$. □

We are finally ready to prove Theorem 1.5.

Let L be a leaf of \mathcal{F} , and ξ be a normal foliated vector field along a neighborhood U of L . Let $p \in U$. Since singular points are isolated along $\gamma_p(t) = \exp_p(t\xi)|_{[-\epsilon, 1+\epsilon]}$, there exists a partition $0 = t_0 < \dots < t_n = 1$ such that $\gamma(t_i)$ are the only possible singular points.

Let $P_{\gamma_p(r_i)}$ be regular plaques that belong to $\text{Tub}(P_{\gamma_p(t_{i-1})}) \cap \text{Tub}(P_{\gamma_p(t_i)})$, where $t_{i-1} < r_i < t_i$. Applying Proposition 3.1 we can find an open set $U_0 \subset P_p$, of the plaque P_p , an open set U_{n+1} of $P_{\gamma_p(1)}$, open sets $U_i \subset P_{\gamma_p(r_i)}$ of the plaques $P_{\gamma_p(r_i)}$

(for $1 \leq i \leq n$) and normal foliated vector fields ξ_i along U_i , (for $0 \leq i \leq n$) with the following properties:

- 1) For each U_i , the normal foliated vector field ξ_i is tangent to the geodesics $\gamma_x(t)$, where $x \in U_0$;
- 2) $\eta_{\xi_i} : U_i \rightarrow U_{i+1}$ is surjective and for $i < n$ a diffeomorphism.
- 3) $\eta_{\xi}|_{U_0} = \eta_{\xi_n} \circ \eta_{\xi_{n-1}} \circ \cdots \circ \eta_{\xi_0}|_{U_0}$

Because $\dim \text{rank } d\eta_{\xi_i}$ is constant on U_i , it follows that $\dim d\eta_{\xi}$ is constant on U_0 . Since this hold for each $p \in U$, $\dim d\eta_{\xi}$ is constant on U . It also follows that $\eta_{\xi}(U)$ is an open set of $L_{\eta_{\xi}(U)}$.

4. PROOF OF COROLLARY 1.6

Let L_p be a regular leaf with trivial holonomy and ξ a normal foliated vector fields along L_p . It follows from Theorem 1.5 that $\eta_{\xi}(L_p)$ is an open set of L_q , where $q = \eta_{\xi}(p)$. In this section we will prove that $\eta_{\xi}(L_p)$ is also a closed set in L_q and hence that $\eta_{\xi}(L_p) = L_q$. In addition, when q is a regular point, we will also prove that $\eta_{\xi} : L_p \rightarrow L_q$ is a covering map.

At first suppose that L_q is a regular leaf.

For a point $z_0 \in L_q$ assume that there exists a point $z_1 \in \eta_{\xi}(L_p)$ which also belongs to the plaque P_{z_0} . Let x_{α} be a point in L_p such that $\eta_{\xi}(x_{\alpha}) = z_1$. Let $\hat{\xi}_{\alpha}$ be the vector in $T_{z_1}M$ tangent to the geodesic $\exp_{x_{\alpha}}(t\xi)$ so that $\exp_{z_1}(\hat{\xi}_{\alpha}) = x_{\alpha}$. We can extend $\hat{\xi}_{\alpha}$ along the plaque P_{z_0} . Theorem 1.5 implies that $\eta_{\hat{\xi}_{\alpha}} : P_{z_0} \rightarrow L_p$. Let A be the set of points $x \in P_{z_0}$ such that $\hat{\xi}_{\alpha}(x)$ is tangent to the geodesic $\exp_x(t\xi)$ and $\exp_x(\hat{\xi}_{\alpha}) = x$ for $x \in L_p$. The fact $\eta_{\xi} : L_p \rightarrow L_q$ is a local diffeomorphism (see Theorem 1.5) implies that A is an open set of P_{z_0} . On the other hand, the fact that $\eta_{\hat{\xi}_{\alpha}} : P_{z_0} \rightarrow L_p$ is a local diffeomorphism implies that A is a closed set of P_{z_0} . Therefore $A = P_{z_0}$. This means that $z_0 \in \eta_{\xi}(L_p)$ and hence that $\eta_{\xi}(L_p)$ is a closed set in L_q .

Now we want to prove that $\eta_{\xi} : L_p \rightarrow L_q$ is a covering map, for a regular point q . For a plaque P_z consider all points $x_{\alpha} \in L$ so that $\eta_{\xi}(x_{\alpha}) = z$. For each x_{α} let $\hat{\xi}_{\alpha}$ be the vector in T_xM tangent to the geodesic $\exp_{x_{\alpha}}(t\xi)$ so that $\exp_z(\hat{\xi}_{\alpha}) = x_{\alpha}$. As proved above, we can extend each vector $\hat{\xi}_{\alpha}$ to a vector field along the plaque P_z and we can show that the map $\eta_{\xi} : W_{\alpha} \rightarrow P_z$ is a diffeomorphism, where $W_{\alpha} = \eta_{\hat{\xi}_{\alpha}}(P_z)$. Note that $\eta_{\xi}^{-1}(P_z) = \cup_{\alpha} W_{\alpha}$. We conclude that $\eta_{\xi} : L_p \rightarrow L_q$ is a covering map.

At last, suppose that L_q is a singular leaf.

For a point $z_0 \in L_q$ assume that there exists a point $z_1 \in \eta_{\xi}(L_p)$ which also belongs to the plaque P_{z_0} . There exists $x_1 \in L_p$ such that $z_1 = \eta_{\xi}(x_1) \in P_{z_0}$. We can find a $s < 1$ such that $y_1 = \eta_{s\xi}(x_1)$ is a regular point. Since y_1 is a regular point, the plaque P_{y_1} is an open set of $\eta_{s\xi}(L_p)$. There exists a parallel normal field $\hat{\xi}$ along P_{y_1} such that $\eta_{\hat{\xi}} \circ \eta_{s\xi} = \eta_{\xi}$. It follows that $\eta_{\hat{\xi}}(P_{y_1}) \subset P_{z_0}$. On the other hand, since the foliation is singular, the plaque P_{y_1} intersect the slice S_{z_0} . These two facts imply that $z_0 \in \eta_{\hat{\xi}}(P_{y_1})$. Therefore $z_0 \in \eta_{\xi}(L_p)$ and hence $\eta_{\xi}(L_p)$ is a closed set in L_q .

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